Fast Last-Iterate Convergence of Learning in Games Requires Forgetful Algorithms

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Main objective:
Understanding the last-iterate behavior of learning algorithms

Why it’s interesting?
Recent applications: Poker [Brown and Sandholm, 2018], Stratego [Perolat et al., 2022], LLMs [Munos et al., 2023]...

Main result:
Optimistic FTRL does not admit last-iterate convergence rate that depends “nicely” on the players’ dimensions and the payoff matrix
Regret minimization

Let $R : \Delta^d \rightarrow \mathbb{R}$ be 1-strongly convex.

Bregman divergence:

$$D_R(x, x') = R(x) - R(x') - \langle \nabla R(x'), x - x' \rangle.$$ 

**Online Mirror Descent:**

$$x^t = \arg\min_{x \in \Delta^d} \left\{ \langle \ell^{t-1}, x \rangle + \frac{1}{\eta} D_R(x, x^{t-1}) \right\} \quad \text{(OMD)}$$

**Follow-The-Regularized-Leader:**

$$x^t = \arg\min_{x \in \Delta^d} \left\{ \sum_{k=1}^{t-1} \ell^k, x \right\} + \frac{1}{\eta} R(x) \quad \text{(FTRL)}$$

Note: OMD and FTRL are the same for Legendre regularizers.
Optimistic algorithms [Rakhlin and Sridharan, 2013, Syrgkanis et al., 2015]

**Optimistic Online Mirror Descent:**

\[
\hat{x}^t = \arg\min_{x \in \Delta^d} \{ \langle \ell^{t-1}, x \rangle + \frac{1}{\eta} D_R(x, \hat{x}^{t-1}) \} \\
x^t = \arg\min_{x \in \Delta^d} \{ \langle \ell^{t-1}, x \rangle + \frac{1}{\eta} D_R(x, \hat{x}^t) \}
\]

(OOMD)

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(OFTRL)
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**Optimistic Follow-The-Regularized-Leader:**

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(OFTRL)

Two important algorithms:

- OGD: OOMD with \( R = \frac{1}{2} \| \cdot \|_2^2 \)
- OMWU: OFTRL/OOMD with \( R = \) negative entropy
Matrix games

Optimization problem:

$$\min_{x \in \Delta^{d_1}} \max_{y \in \Delta^{d_2}} x^\top Ay$$

Goal: Compute \((x^*, y^*)\) with \(\text{DualityGap}(x^*, y^*) = 0\), where

$$\text{DualityGap}(x^*, y^*) := \max_{y \in \Delta^{d_2}} (x^*)^\top Ay - \min_{x \in \Delta^{d_1}} x^\top Ay^*.$$  

Self-play: both x-player and y-player use regret minimizers, using

$$\ell^t_x = Ay^t, \ell^t_y = -A^\top x^t.$$
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$$\ell_x^t = Ay^t, \ell_y^t = -A^\top x^t.$$ 

Two important algorithms:

- OGDA: both players use OGD
- “OMWU”: both players use OMWU
Advantages of OMWU over OGDA:

- Logarithmic dependence on the size of payoff matrix
- Closed-form updates:

\[ x^t[i] \propto x^1[i] \times \exp \left( -\eta \left( \sum_{k=1}^{t-1} \ell^k[i] + \ell^{t-1}[i] \right) \right). \]

- $\tilde{O}(1/T)$ ergodic convergence to (coarse) correlated equilibrium in general-sum games [Daskalakis et al., 2021, Anagnostides et al., 2022]
Notions of convergence

Consider the sequences \( \{x^t\}, \{y^t\} \) computed by self-play.

We have **ergodic convergence** when

\[
\lim_{T \to +\infty} \text{DualityGap}(\bar{x}^T, \bar{y}^T) = 0.
\]

[Rakhlin and Sridharan, 2013, Syrgkanis et al., 2015]: \( O \left( \frac{1}{T} \right) \) ergodic convergence rate for OGDA/OMWU.
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We have **last-iterate convergence** when

$$\lim_{T \to +\infty} \text{DualityGap}(x^T, y^T) = 0.$$  

Advantage: less computation (no need to average)
Convergence of optimistic algorithms

Last-iterate dynamics of OGDA:

- Unconstrained setting [Daskalakis et al., 2018, Hsieh et al., 2019, Liang and Stokes, 2019, Golowich et al., 2020]
- Matrix games: linear convergence with metric subregularity constants [Wei et al., 2021]
- Matrix games: convergence in $O(1/\sqrt{T})$ [Cai et al., 2022, Gorbunov et al., 2022]

Last-iterate dynamics of OMWU:

- Unique N.E.: convergence with fixed step sizes (could be exponentially small), no rate [Daskalakis and Panageas, 2019]
- Unique N.E.: linear convergence with metric subregularity constants [Wei et al., 2021]
- Convergence for adaptive step sizes, no rate [Mertikopoulos et al., 2019, Hsieh et al., 2021]
- This work: rate of convergence of OMWU for matrix games?
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For matrix games:

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Note: \(C_1, C_2, C_3, C_4\) obtained from metric subregularity:

\[ \text{DualityGap}(x, y) \geq c \cdot \text{dist}((x, y), \text{set of N.E.}) \]

\[ \implies \text{may be arbitrarily bad even with fixed } d_1, d_2, \max_{i,j} |A_{ij}|. \]
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Our main theorem

Theorem (Informal)

Consider two-player zero-sum games with matrix entries in $[0, 1]$, and $d_1$ and $d_2$ are the number of actions.
For OMWU with constant step size, no function $f$ can satisfy

1. $\text{DualityGap}(x^T, y^T) \leq f(d_1, d_2, T)$ for all $T$.
2. $\lim_{T \to \infty} f(d_1, d_2, T) \to 0$. 

Holds for OFTRL with regularizer = entropy/log/squared $L^2$ norm/Tsallis entropy

Idea of the proof:

• We construct a $2 \times 2$ matrix game $A_{\delta}$ parametrized by $\delta > 0$.
• After $\Omega(1/\eta \delta)$ iterations of OFTRL, the duality gap is a constant $c$, a universal constant depending on the regularizer.
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Holds for OFTRL with regularizer = entropy/log/squared \(L_2\) norm/Tsallis entropy

Idea of the proof:

- We construct a 2x2 matrix game \(A_\delta\) parametrized by \(\delta > 0\).
- After \(\Omega(1/\eta \delta)\) iterations of OFTRL, the duality gap is a constant \(c\), a universal constant depending on the regularizer.
Consider the matrix game $A_\delta$ with $\delta \in (0, 1/2]$:

$$A_\delta := \begin{bmatrix} \frac{1}{2} + \delta & \frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$ 

$A_\delta$ has a unique Nash equilibrium: $x^*[1] = \frac{1}{1+\delta}$, $y^*[1] = \frac{1}{2(1+\delta)}$.

Bad region: $x[1] \geq \frac{1}{1+\delta}$, $y[1] \geq \frac{1}{2} + c \Rightarrow \text{DualityGap}(x, y) \geq c.$
A difficult matrix game for OFTRL

\[
\begin{align*}
\text{DualityGap}(x, y) & \geq c \\
x : 1 & \preceq 2 \\
y : 1 & \preceq 2 \\
x : 1 & \preceq 2 \\
y : 1 & \succeq 2
\end{align*}
\]
Reformulating OFTRL

2x2 game ⇒ we focus on \( x[1], y[1] \). Define

\[
F_{\eta,R}(e) := \arg\min_{x \in [0,1]} \left\{ x \cdot e + \frac{1}{\eta} R(x) \right\}
\]

\[
e_x^t := \ell_x^t[1] - \ell_x^t[2]
\]

\[
E_x^t := \sum_{k=1}^{t} e_x^k
\]

We can rewrite OFTRL:

\[
x^t[1] = F_{\eta,R} \left( E_x^{t-1} + e_x^{t-1} \right)
\]

\[
y^t[1] = F_{\eta,R} \left( E_y^{t-1} + e_y^{t-1} \right)
\]

(OFTRL)
Assumptions on the regularizers

Define
\[ F_{\eta,R}(e) := \arg\min_{x \in [0,1]} \{ x \cdot e + \frac{1}{\eta} R(x) \}. \]

Important: \( F_{\eta,R} : \mathbb{R} \to [0,1] \) is non-increasing.

Assumption

We assume that the regularizer \( R \) satisfies the following properties: the function \( F_{\eta,R} : \mathbb{R} \to [0,1] \) is

1. **Unbiased**: \( F_{\eta,R}(0) = \frac{1}{2} \).
2. **Rational**: \( \lim_{E \to -\infty} F_{\eta,R}(E) = 1 \) and \( \lim_{E \to +\infty} F_{\eta,R}(E) = 0 \).
3. **Lipschitz continuous**: There exists \( L \geq 0 \) such that \( F_{1,R} \) is \( L \)-Lipschitz.

More assumptions here: (1).
Reformulating OFTRL

\[ x^t[1] = F_{\eta,R} (E_x^{t-1} + e_x^{t-1}) \]
\[ y^t[1] = F_{\eta,R} (E_y^{t-1} + e_y^{t-1}) \]

(OFTRL)

Important note: \( e^t_y \in [-\delta, 1] \)
\( \Rightarrow \) if \( E_y^t \) is large, it takes \( \Omega(1/\delta) \) iterations to make it close to 0.
Numerical experiments

Figure: Dynamics produced by OMWU and OGDA in the same game $A_\delta$. 
Numerical experiments

Figure: Dynamics produced by variants of OFTRL with different regularizers and OGDA in the same game $A_\delta$.

\begin{align*}
x^t[1] &= F_{\eta,R} \left( E_x^{t-1} + e_x^{t-1} \right) \\
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\end{align*}  

(OFTRL)
Numerical experiments

Figure: Influence of $\delta > 0$ on the duality gaps of OMWU after $10^4$ iterations.
\[ x^t[1] = F_{\eta,R} \left( E_x^{t-1} + e_x^{t-1} \right) \]
\[ y^t[1] = F_{\eta,R} \left( E_y^{t-1} + e_y^{t-1} \right) \]

(OFTRL)

**Stage I:** Starting at \( x^1[1] = y^1[1] = 1/2 \), we prove
- \( x^t[1] \) increases until \( T_1 \) s.t. \( x^{T_1}[1] \geq \frac{1}{1+\delta} \).
Stage I: Starting at $x^1[1] = y^1[1] = 1/2$, we prove

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- $e_x^t \geq 0$: action 1 $\prec$ action 2 for the y-player, $y^t[1]$ decreases.
\[ x^t[1] = F_{\eta,R} \left( E_x^{t-1} + e_x^{t-1} \right) \]
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- \( x^t[1] \) increases until \( T_1 \) s.t. \( x^{T_1}[1] \geq \frac{1}{1+\delta} \).
- \( e^t_y \geq 0 \): action 1 \( \prec \) action 2 for the y-player, \( y^t[1] \) decreases.
- At the last period \( T_1 \), we have \( y^{T_1}[1] \leq \frac{1}{2} - c_1 \).
\[ x^t[1] = F_{\eta,R} \left( E_{x}^{t-1} + e_{x}^{t-1} \right) \]
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**Stage II:**

- \([-\delta \leq e_{y}^{t} < 0 \Rightarrow y^t[1]\] increases, at most by \(\eta L\delta\).
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**Stage II:**

- \(-\delta \leq e_{y}^{t} < 0 \Rightarrow y^t[1] \) increases, at most by \( \eta L \delta \).
- Stage II lasts until \( T_2 \) s.t. \( y^{T_2}[1] \geq 1/2(1 + \delta) \).
- Thus \( T_2 - T_1 = \Omega \left( c_1 / \eta L \delta \right) \).
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- Thus \(T_2 - T_1 = \Omega(c_1/\eta L \delta)\).
- \(e_x^t < 0,\) and \(x^t[1]\) keeps growing closer to 1: \(E_x^{T_2} \leq E_x^{T_1} - \Omega(1/\eta L \delta)\).
\[
x^t[1] = F_{\eta,R} \left( E_{x}^{t-1} + e_{x}^{t-1} \right)
\]
\[
y^t[1] = F_{\eta,R} \left( E_{y}^{t-1} + e_{y}^{t-1} \right)
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(OFTRL)

**Stage III:**
- \(y^t[1]\) keeps increasing until \(x^t[1] \leq 1/(1 + \delta)\).
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- \( x^t[1] \leq 1/(1 + \delta) \) \( \Rightarrow \) \( E^{T_1}_x \leq E^t_x \).
\( x^t[1] = F_{\eta,R} \left( E_x^{t-1} + e_x^{t-1} \right) \)

\( y^t[1] = F_{\eta,R} \left( E_y^{t-1} + e_y^{t-1} \right) \)  \hspace{1cm} (OFTRL)

**Stage III:**

- \( y^t[1] \) keeps increasing until \( x^t[1] \leq 1/(1 + \delta) \).
- \( x^t[1] \leq 1/(1 + \delta) \Rightarrow E_x^{T_1} \leq E_x^t \).
- But at \( T_2 \), we have \( E_x^{T_2} \leq E_x^{T_1} - \Omega(1/\eta L \delta) \). Since \( e_x^t \leq 1 \), we still have \( x^t[1] \geq 1/(1 + \delta) \) after \( \Omega(1/\eta L \delta) \) steps.
\[ x^t[1] = F_{\eta,R}(E_{x}^{t-1} + e_{x}^{t-1}) \]

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- Until \( T_3 = T_2 + \Omega(1/\eta L \delta) \), \( e_{y}^t \leq 0 \), and \( y^{T_3}[1] \geq \frac{1}{2} + c_2 \).
Main theorem

We proved:

There is a universal constant $c$ (dependent on the regularizer) such that for any $\delta > 0$, we can find a game $A_\delta$ where the duality gap for OFTRL is at least $c$ after $\Omega(1/\delta)$ rounds.
Main theorem

We proved:

*There is a universal constant* \( c \) (dependent on the regularizer) *such that for any* \( \delta > 0 \), *we can find a game* \( A_\delta \) *where the duality gap for OFTRL is at least* \( c \) *after* \( \Omega(1/\delta) \) *rounds.*

Formal statement:

**Theorem**

Assume the regularizer \( R \) satisfies our assumptions with universal constant \( c_1, c_2, \hat{\delta}, L > 0 \). Let \( \delta \in (0, \hat{\delta}) \)

*The OFTRL dynamics on* \( A_\delta \) *with any step size* \( \eta \leq \frac{1}{4L} \) *satisfies the following: there exists an iteration* \( t \geq \frac{c_1}{3\eta L \delta} \) *with a duality gap of at least* \( c_2 \).
Conclusion

- **Main result**: Negative result for the rate of convergence of OMWU (OFTRL).

- **Next steps**:
  Universal best-iterate convergence rate?
  Rate for adaptive step sizes?
  Mixing OGD/OMWU?

  Slides/code on my website
Conclusion

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• Preprint: https://arxiv.org/abs/2406.10631
  Slides/code on my website

Thank you!


References II


Assumptions on the regularizers

Define

\[ F_{\eta,R}(e) := \arg\min_{x \in [0,1]} \{ x \cdot e + \frac{1}{\eta} R(x) \}. \]

Assumption (Informal)

There are some constants such that

\[
F_{1,R}(E) \geq \frac{1}{1+\delta} \Rightarrow \quad F_{1,R} \left( E - \Omega \left( \frac{1}{\delta} \right) \right) \geq F_{1,R}(E) + \Omega(\delta)
\]

(1)

\[
F_{1,R}(E) \geq \frac{1}{2(1+\delta)} \Rightarrow \quad F_{1,R} \left( E - \Omega(\delta) \right) \geq \frac{1}{2} + \Omega(1)
\]

(2)

Both our assumption hold for the negative entropy, squared Euclidean norm, the log barrier, and the Tsallis entropy regularizers. Link to main presentation: (2).
Assumptions on the regularizers

Define

\[ F_{\eta,R}(e) := \arg\min_{x \in [0,1]} \{ x \cdot e + \frac{1}{\eta} R(x) \}. \]

Assumption

Let \( L \) be the Lipschititness constant of \( F_{1,R} \). Denote constant \( c_1 = \frac{1}{2} - F_{1,R}(\frac{1}{20L}) \). There exist universal constants \( \delta', c_2 > 0 \) and \( c_3 \in (0, \frac{1}{2}] \) such that for any \( 0 < \delta \leq \delta' \),

1. If \( F_{1,R}(E) \geq \frac{1}{1+\delta} \), then \( F_{1,R}(\frac{-c_2^2}{30L\delta} + E) \geq \frac{1+c_3}{1+c_3+c_3+\delta} \)

2. If \( F_{1,R}(E) \geq \frac{1}{2(1+\delta)} \), then \( F_{1,R}(\frac{-c_3c_1^2}{120L} + \frac{\delta}{4L} + E) \geq \frac{1}{2} + c_2 \).

Both our assumption hold for the negative entropy, squared Euclidean norm, the log barrier, and the negative Tsallis entropy regularizers. Link to main presentation: (2)
Other convergence rates

Following [Wei et al., 2021]:

**Corollary**

Let $\delta \in (0, \frac{1}{2})$. For OMWU with step size $\eta \leq \frac{1}{8}$ on $A_\delta$ satisfies

$$\text{DualityGap}(x^T, y^T) \leq \frac{1200e^{\frac{10}{\delta}}}{\eta} \cdot \frac{1}{\sqrt{T}}, \forall ~ T \geq 1.$$ 

$\Rightarrow$ Problem-constant independent best-iterate rate for OMWU:

**Theorem**

Let $\delta \in (0, \frac{1}{32})$. For OMWU with step size $\eta \leq \frac{1}{8}$

$$\min_{t \in [T]} \text{DualityGap}(x^t, y^t) \leq O\left(\frac{1}{\eta \ln T}\right), \forall ~ T \geq 2.$$
Adaptive stepsizes [Duchi et al., 2011]: \( \eta_t = \frac{1}{\sqrt{\epsilon + \sum_{k=1}^{t-1} \|\ell_k\|_k^2}} \)

Figure: Here \( \delta := 10^{-2} \) and adaptive step size with \( \epsilon = 0.1 \).
Figure: Dynamics and duality gap when the x-player uses OGD. We choose $\delta = 0.01$ and $\eta = 0.1$ in all figures.
Figure: Dynamics and duality gap when the y-player uses OGD. We choose $\delta = 0.01$ and $\eta = 0.1$ in all figures.