Unanimity and the aggregation of multiple prior opinions*

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Abstract

In a situation of decision under uncertainty, a decision maker wishes to choose according to the maxmin expected utility rule, and he can observe the preferences of a set of experts who all share his utility function and all use the maxmin EU rule. This paper considers rules for aggregating the experts' sets of priors into a set that the decision maker can use. It is shown that, in a multi profile setting, among the rules that allow the decision maker's evaluation of an act to depend only on the experts' evaluations of that act, the only rule satisfying the standard unanimity or Pareto condition on preferences is the "set of weights" aggregation rule, according to which the decision maker's set of priors is the set of weighted averages of the priors in the experts' sets, with the weights taken from a set Λ of probability vectors over the experts. An analogous characterisation is obtained for variational preferences.

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1 Introduction

Consider a decision maker who, prior to making her decision, can consult a collection of experts who may have different opinions on the facts pertinent to the decision in question. In other words, the experts may have different beliefs – and hence different preferences over the options available – and the decision maker would like to use a rational or justified method to aggregate these beliefs, and so inform her preferences. What method should she use?

This question can be considered in a specific or a general form. The specific form asks, for a given set of expert beliefs, what beliefs the decision maker could admissibly hold. The general form requires as an answer a rule or method for aggregating any set of expert beliefs into a belief for the decision maker. This distinction is well known in the literature on social choice and preference aggregation, where one distinguishes between the single profile setting and the more common multi profile setting. The latter in particular is a natural choice when one is interested in a method for aggregating any sets of beliefs or preferences, rather than in a single case of aggregation.

Under the standard assumption for decision making under uncertainty – the Bayesian assumption that all involved have beliefs represented by probability measures and form preferences by maximising expected utility – the question of belief aggregation has been studied in both single and multi profile settings. A popular rule for belief aggregation is the linear pooling rule, according to which the decision maker's probability measure is a weighted average of those of the experts (Stone, 1961). This rule has been axiomatically derived both in the single profile setting (Mongin, 1995) and in the multi profile setting (McConway, 1981;

Wagner, 1982; Genest and Zidek, 1986), although not necessarily in a preference setup. However, the Bayesian paradigm has been forcefully challenged, and several have argued that nothing should force the experts and the decision maker to have a single probability measure (Gilboa et al., 2009, 2012). In this paper, we consider the problem of belief aggregation in a case where each agent's beliefs cannot necessarily be represented by a single probability measure, but rather by a set of probability measures, or priors. Moreover, since such sets are generally not observable, we consider the problem from the point of view of preferences; we assume that all the agents form preferences according to arguably the simplest model using sets of priors, namely the maxmin expected utility model axiomatised by Gilboa and Schmeidler (1989).[1] To focus on the problem of belief aggregation, and avoid well-known difficulties with simultaneous aggregation of beliefs and utilities (Mongin, 1995; Gilboa et al., 2004; Gajdos et al., 2008), we assume that the agents all share the same utility function; they only differ in their respective sets of priors.

This problem has been considered in the single profile setup in a recent paper by Crès et al. (2011). They axiomatise and defend the following rule for aggregating sets of priors: there exists a set of weight vectors $\Lambda \subseteq \Delta(\{1, \ldots, n\})$ such that, for given expert sets of priors C_i, the decision maker has the set of priors

$$\left\{ p = \sum_{i=1}^{n} \lambda_i p_i \middle|\ \lambda \in \Lambda,\ p_i \in C_i \right\} \tag{1}$$

Just like the weight vector in the linear pooling rule in the Bayesian case, the set Λ can be thought of as representing the decision maker's confidence in

[1] In this paper, we generally retain the interpretation of the sets of priors as the experts' beliefs or objective information. However, given that this interpretation is complicated in the maxmin EU model by the inability of this model to distinguish belief from attitude to uncertainty, one may prefer an alternative reading of the main result as a step towards aggregation results for models using sets of priors but allowing a distinction between beliefs and uncertainty attitudes, such as Hill (2010).

the different experts. Moreover, by allowing a set of weight vectors rather than a single vector, one can represent more uncertainty on the part of the decision maker concerning the "value" of the different experts. (We refer to Crès et al. (2011) for more discussion of this interpretation, as well as for another interpretation of the rule.) However, whereas in the Bayesian case, a unanimity or Pareto condition on preferences is necessary and sufficient in the single profile setting for the linear pooling rule (under appropriate richness conditions), the work of Crès et al. (2011) indicates that this is not the case for maxmin EU decision makers and the rule (1). Indeed, they introduce a new condition, Expert Uncertainty Aversion, which requires a sort of unanimity with respect to mixes of certainty equivalents, and which is motivated precisely by the possibility of uncertainty on the part of the decision maker and an assumption of aversion to that uncertainty. They show that this condition, which is stronger than the standard unanimity condition, is equivalent to aggregation rule (1).

In the main result of the paper, we show that things are in a certain sense "cleaner" in the multi profile setting. In this setting, one explicitly considers rules for aggregating sets of priors, and hence for aggregating the maxmin expected utility preferences over acts generated by these sets. As long as one restricts oneself to rules that, in determining the value of a particular act for the decision maker, only take into account how the experts value that act, the standard unanimity condition is necessary and sufficient for (1). Moreover, the set Λ in (1) is unique. As well as providing a conceptually simpler derivation of the aggregation rule (1), this result can be thought of as another argument in its favour. There is no need to appeal to intuitions about the decision maker's attitude to his uncertainty concerning the experts; if one seeks a general rule for aggregating maxmin expected utility preferences where the decision maker's evaluation of an act is simply a function of the experts' evaluations, then the unanimity condition alone implies that one must use (1).

The paper is organised as follows. The formal framework is given in Section

2.1, the conditions are given and discussed in Section 2.2, and the aggregation rule is defined in Section 2.3. The result is stated in Section 2.4. Section 3 contains a brief discussion of some properties of the result, a related condition, an extension to the case of variational preferences, and other relevant literature. Proofs are relegated to the appendices.

2 Multi profile multi prior aggregation

2.1 Setup

We use the standard Anscombe-Aumann framework (Anscombe and Aumann, 1963), as adapted by Fishburn (1970). Let S be a set of states, containing at least three elements, with a σ-algebra Σ of *events*. $\Delta(\Sigma)$ is the set of finitely-additive probability measures on (S, Σ). X is a nonempty set of outcomes; a *consequence* is a von Neumann-Morgenstern lottery on X, that is, a probability measure on X with finite support. L is the set of consequences. *Acts* are Σ-measurable functions from states to consequences; F is the set of acts. So, for an act f, and a state s, $f(s)$ is a lottery over X with finite support; for a utility function u over X, we will denote the expected utility of this lottery by $u(f(s)) = \sum_{x \in supp(f(s))} f(s)(x)u(x)$. F is a mixture set with the mixture relation defined pointwise: for $f, h, \in F$ and $\alpha \in \Re$, $0 \leq \alpha \leq 1$, the mixture $\alpha f + (1-\alpha)h$ is defined by $(\alpha f + (1-\alpha)h)(s, x) = \alpha f(s, x) + (1-\alpha)h(s, x)$. A constant act is an act yielding the same consequence in every state; F_c is the set of constant acts. With slight abuse of notation, for any $c \in L$, we shall use c to denote the constant act yielding c in every state.

We consider a decision maker who can rely on the advice of n experts. The decision maker's and the experts' preferences are represented by binary relations on F, denoted by $\preceq_i$ (with appropriate subscripts). $\sim_i$ and $\prec_i$ are the symmetric and asymmetric components of $\preceq_i$, defined as usual. $\preceq_i$ can be extended to L as standard. We assume that the decision maker's and experts' preferences satisfy the axioms of Gilboa and Schmeidler (1989). Moreover, as noted in the introduction, to focus on the case of aggregation of sets of priors, we assume that the experts and the decision maker have the same non-degenerate utility u. In other words, their preference relations agree on F_c (and so on L). Let $\mathcal{P}_u$ denote the set of preference relations satisfying the Gilboa-Schmeidler axioms and such that the restriction to L is represented by u; the decision maker's and experts' preference relations all belong to $\mathcal{P}_u$.

By the Gilboa and Schmeidler (1989) result, $\preceq_i \in \mathcal{P}_u$ if and only if there exists a convex and weak* closed set of finitely additive measures on (S, Σ), C_i, such that $f \preceq_i g$ iff $J_i(f) \leq J_i(g)$, where

$$J_i(f) = \min_{p \in C_i} \int u(f(s))dp(s) \tag{2}$$

Let $\mathcal{J}_u$ be the set of functionals of this form with utility u. We denote the subset of $\mathcal{J}_u$ containing J_i where C_i is a singleton by $\mathcal{EU}_u$; evidently, these are expected utility functionals with utility function u.

We seek an aggregation rule that, for any set of expert preference relations of the sort just specified, yields a preference relation for the decision maker. That is, we seek a function $A : \mathcal{P}_u^n \to \mathcal{P}_u$, which we call the *aggregation function*. Following the terminology in social choice, we call elements of $\mathcal{P}_u^n$ *profiles*, and denote them by $\preceq$ (using superscripts to indicate different profiles and subscripts to indicate different members of a given profile). We write $\preceq_{A(\preceq)}$ instead of $A(\preceq)$. In the light of the previous remark concerning the representation of preference relations in $\mathcal{P}_u$, A generates a (unique) function from $\mathcal{J}_u^n$ to $\mathcal{J}_u$; with slight of abuse of notation, we also denote this function by A. As for preferences, n-tuples of functionals in $\mathcal{J}_u$ are called profiles, and are denoted by J.

2.2 The conditions

We consider two properties of the relationship between the experts' preferences and those of the decision maker. The first, which can only be formulated in a multi profile setting, is the following.

Weak independence For every pair of profiles $\preceq^1, \preceq^2 \in \mathcal{P}_u^n$ and every $f \in F$, if, for all $1 \leq i \leq n$ and all $c \in F_c$, $f \preceq_i^1 c$ iff $f \preceq_i^2 c$, then, for all $d \in F_c$, $f \preceq_{A(\preceq^1)} d$ iff $f \preceq_{A(\preceq^2)} d$.

Recall that all the agents involved have the same utility function, and hence the same preferences over F_c; the set of constant acts thus provides "universal" scale on which the other acts can be evaluated. Weak independence states that if every expert evaluates a given act in the same way in two different profiles – in the sense that he ranks the act in the same way in both profiles relative to every constant act – then the decision maker evaluates the act in the same way for both profiles. Although this property is stated in terms of observables, namely preferences, it is equivalent to the following condition on the functionals: for every pair of profiles of functionals, $J^1, J^2 \in \mathcal{J}_u^n$ and every $f \in F$,

$$\text{if, for all } 1 \leq i \leq n,\ J_i^1(f) = J_i^2(f), \text{ then } A(J^1)(f) = A(J^2)(f).$$

It is hence clear that this condition basically states that the value assigned to an act by the decision maker's functional depends only on the values assigned to the act by the experts' functionals, and not, say, on other properties of their functionals. Weak independence is a natural condition when one is searching for an aggregation rule: one would want that rule to only take into account the evaluations of the experts, and not other aspects of their preferences. It has the flavour of the independence of irrelevant alternatives axioms in social choice, though it is evidently much weaker. Moreover, it is analogous to the weak setwise function property (McConway, 1981), also called irrelevance of alternatives (Wagner, 1982), in the literature on pooling Bayesian beliefs, though it is a condition on preferences, whereas the cited conditions are formulated directly in terms of probability measures.

The other property involved is the standard unanimity or Pareto weak-preference property, which requires no further comment.

Unanimity For any $\preceq = (\preceq_1, \ldots, \preceq_n) \in \mathcal{P}_u$ and $f, g \in F$, if $f \preceq_i g$ for all $1 \leq i \leq n$, then $f \preceq_{A(\preceq)} g$.

2.3 The aggregation rule

We consider the aggregation rule proposed in Crès et al. (2011), namely, for a profile $\preceq = (\preceq_1, \ldots, \preceq_n) \in \mathcal{P}_u^n$ represented according to (2) by sets of priors $C_1, \ldots, C_n$ respectively, $\preceq_{A(\preceq)}$ is represented (according to (2)) by the set:

$$C_{A(\preceq)} = \left\{ p = \sum_{i=1}^{n} \lambda_i p_i | \; \lambda \in \Lambda, \; p_i \in C_i \right\} \tag{1}$$

for some closed and convex set $\Lambda \subseteq \Delta(\{1, \ldots, n\})$.

As noted in the cited paper, this rule is equivalent to a condition on the relationship between the decision maker's and experts' functionals: the former is the minimum over a set of mixes of the latter, with mixing weights taken in Λ. This is recalled in the following fact.

Fact 1 (Crès et al. (2011), Proposition 1). For a closed and convex set $\Lambda \subseteq \Delta(\{1, \ldots, n\})$, and maxmin EU functionals $J_A, J_1, \ldots, J_n$ represented by sets of priors $C_A, C_1, \ldots C_n$ respectively,

$$J_A(f) = \min_{\lambda \in \Lambda} \sum_{i=1}^{n} \lambda_i J_i(f) \qquad \text{iff} \qquad C_A = \left\{ p = \sum_{i=1}^{n} \lambda_i p_i | \; \lambda \in \Lambda, \; p_i \in C_i \right\}$$

2.4 Result

We can now state the main result.

Theorem 1. *The following are equivalent:*

(i) *A satisfies unanimity and weak independence*

(ii) *There exists a closed and convex set $\Lambda \subseteq \Delta(\{1,\ldots,n\})$ such that, for each $J = (J_1,\ldots,J_n) \in \mathcal{J}_u^n$, and every $f \in F$,*

$$A(J)(f) = \min_{\lambda\in\Lambda} \sum_{i=1}^{n} \lambda_i J_i(f) \tag{3}$$

(iii) *There exists a closed and convex set $\Lambda \subseteq \Delta(\{1,\ldots,n\})$ such that, for each $\preceq = (\preceq_1,\ldots,\preceq_n) \in \mathcal{P}_u^n$, represented by the sets $C_1,\ldots,C_n$ respectively, $\preceq_{A(\preceq)}$ is represented by the set of priors*

$$C_{A(\preceq)} = \left\{ p = \sum_{i=1}^{n} \lambda_i p_i |\ \lambda \in \Lambda,\ p_i \in C_i \right\} \tag{1}$$

Moreover, there is a unique closed convex set Λ satisfying (ii) and (iii).

3 Discussion

1. Size of the state space. The requirement that the state space has at least three states is necessary. If there are only two states s_1, s_2, then the rule which, given $\preceq_1,\ldots,\preceq_n \in \mathcal{P}_u$ represented by sets of priors $C_1,\ldots,C_n$, yields the maxmin EU preference represented by the set of priors

$$C' = \left\{ p \;\middle|\; \begin{array}{l} p(s_1) \geq \frac{1}{2} \min_{1 \leq i \leq n} \min_{p_j \in C_i} p_j(s_1) + \frac{1}{2} \max_{1 \leq i \leq n} \min_{p_j \in C_i} p_j(s_1) \ \& \\ p(s_1) \leq \frac{1}{2} \min_{1 \leq i \leq n} \max_{p_j \in C_i} p_j(s_1) + \frac{1}{2} \max_{1 \leq i \leq n} \max_{p_j \in C_i} p_j(s_1) \end{array} \right\}$$

satisfies weak independence and unanimity. To understand this example, it suffices to consider the functional $J'(f) = \frac{1}{2} \min_{1 \leq i \leq n} J_i(f) + \frac{1}{2} \max_{1 \leq i \leq n} J_i(f)$. Whereas the preferences generated by this functional evidently satisfy weak independence and unanimity, this functional is guaranteed to be a maxmin EU functional only when there are two states, and in this case, it is straightforward to show that it is represented according to (2) by the set of priors C'.

2. Weak and strong independence. The weak independence condition states that the decision maker's evaluation of an act depends only on the experts' evaluations of that act, but allows the nature of this dependence to differ from act to act. A stronger condition, which is analogous to stronger conditions considered in the literature on pooling Bayesian beliefs (such as the strong setwise function property of McConway (1981), also called strong label neutrality by Wagner (1982)) is the following.

Strong independence For every pair of profiles $\preceq^1, \preceq^2 \in \mathcal{P}_U^n$ and every $f, g \in F$, if, for all $1 \leq i \leq n$ and all $c \in F_c$, $f \preceq_i^1 c$ iff $g \preceq_i^2 c$, then, for all $d \in F_c$, $f \preceq_{A(\preceq^1)} d$ iff $g \preceq_{A(\preceq^2)} d$.

This property basically states that the decision maker's evaluation of an act is a function solely of the evaluations of the experts, independently of the act in question. In particular, it is equivalent to demanding that, for every pair of profiles $J^1, J^2 \in \mathcal{J}_u^n$ and every pair of acts $f, g \in F$, if, for all $1 \leq i \leq n$, $J_i^1(f) = J_i^2(g)$, then $A(J^1)(f) = A(J^2)(g)$. It turns out that this condition is equivalent to the conjunction of weak independence and unanimity.

Proposition 1. *A satisfies strong independence if and only if it satisfies weak independence and unanimity.*

Similar equivalences have been obtained in the literature, but only for cases where all agents are Bayesian (McConway, 1981; Wagner, 1982) or have belief functions (Wagner, 1989); this appears to be the first such result for sets of priors and maxmin EU preferences. Of course, it follows immediately from Theorem 1 that strong independence is necessary and sufficient for aggregation rule (1).

3. Extension to variational preferences. Similar methods to those developed in the proof of Theorem 1 may prove useful for obtaining characterisations of aggregation rules for other classes of preferences. To illustrate, consider the problem of finding an aggregation rule for the case where both the decision maker and the experts have variational preferences, and where they share the same unbounded utility function. Variational preferences (Maccheroni et al., 2006) involve the representation of preferences by a functional of the following form:

$$J_i(f) = \min_{p \in \Delta(\Sigma)} \left(\int u(f(s)) dp(s) + c_i(p) \right) \tag{4}$$

where $c_i : \Delta(\Sigma) \to [0, \infty]$ is a grounded, convex and lower semicontinuous function. This representation is a generalisation of maxmin EU preferences, which correspond to the special case where c is an indicator function. Let $\mathcal{P}_u^{var}$ be the set of preferences satisfying the axioms of Maccheroni et al. (2006), agreeing on F_c, and being represented on F_c by an unbounded utility function u,[2] and let $\mathcal{J}_u^{var}$ be the set of functionals of the form (4) with utility u.

Consider aggregation rules for variational preferences, that is, functions A^{var} : $(\mathcal{P}_u^{var})^n \to \mathcal{P}_u^{var}$; as for the maxmin EU case considered previously, each such rule generates a unique rule for aggregating functionals, which we call A^{var} as well.

[2] See Maccheroni et al. (2006, Axiom A7) for an axiom characterising unboundedness. The assumption of unboundedness has the advantage of improving the uniqueness properties of the representation (Maccheroni et al., 2006, Proposition 6) and facilitating the proof of Theorem 2 below; it remains to be ascertained whether similar results hold in the absence of this assumption.

As is clear from the representation (4), such rules are characterised by how they operate on the functions c_i. An interesting aggregation rule is the following: for a profile $\preceq = (\preceq_1, \ldots, \preceq_n) \in (\mathcal{P}_u^{var})^n$ represented according to (4) by functions $c_1, \ldots, c_n$ respectively, $\preceq_{A^{var}(\preceq)}$ is represented (according to (4)) by the function:

$$c_{A^{var}(\preceq)}(p) = \min_{\substack{(\lambda, p_1, \ldots, p_n) \in \Delta(\{1,\ldots,n\}) \times \Delta(\Sigma)^n \\ s.t. \ \sum \lambda_i p_i = p}} \left(\sum_{i=1}^{n} \lambda_i c_i(p_i) + e(\lambda) \right) \tag{5}$$

for some grounded, convex and lower semicontinuous function $e : \Delta(\{1, \ldots, n\}) \to [0, \infty]$. Just like the set Λ in rule (1), the function e can be thought of as representing the decision maker's confidence in the different experts: uncertainty as to the calibre of the experts is represented by the different values assigned to the possible weight vectors over them. Indeed (5) incorporates the decision maker's confidence according to the variational preference functional (4): that is, by taking the minimum sum of the mix according to a weight vector and the value that e gives to that weight vector. Rule (1) is the special case where e is an indicator function, with the set of weights assigned the value 0 being used in maxmin EU rule. As shown in the theorem below, rule (5) is the rule that one must use if one wishes to satisfy weak independence and unanimity. Moreover, and analogously with clauses (ii) and (iii) in Theorem 1, the rule corresponds to using a variational preference representation over the experts' functionals, with function e.

Theorem 2. *The following are equivalent:*

(i) $A^{var} : (\mathcal{P}_u^{var})^n \to \mathcal{P}_u^{var}$ *satisfies unanimity and weak independence*

(ii) There exists a grounded, convex and lower semicontinuous function $e : \Delta(\{1, \ldots, n\}) \to [0, \infty]$ *such that, for each* $J = (J_1, \ldots, J_n) \in (\mathcal{J}_u^{var})^n$*, and every* $f \in F$*,*

$$A^{var}(J)(f) = \min_{\lambda \in \Delta(\{1,\ldots,n\})} \left(\sum_{i=1}^{n} \lambda_i J_i(f) + e(\lambda) \right) \tag{6}$$

(iii) There exists a grounded, convex and lower semicontinuous function $e : \Delta(\{1,\ldots,n\}) \to [0,\infty]$ *such that, for each* $\preceq = (\preceq_1, \ldots, \preceq_n) \in (\mathcal{P}_u^{var})^n$*, represented by the functions* $c_1, \ldots, c_n$ *respectively,* $\preceq_{A^{var}(\preceq)}$ *is represented by the function*

$$c_{A^{var}(\preceq)}(p) = \min_{\substack{(\lambda, p_1, \ldots, p_n) \in \Delta(\{1,\ldots,n\}) \times \Delta(\Sigma)^n \\ s.t.\ \sum \lambda_i p_i = p}} \left(\sum_{i=1}^{n} \lambda_i c_i(p_i) + e(\lambda) \right) \tag{5}$$

Moreover, there is a unique nonnegative, grounded, convex, lower semicontinuous function e *satisfying (ii) and (iii).*

4. Other related literature. Apart from Crès et al. (2011), the closest paper to the current one is undoubtedly Wagner (1989), who considers the aggregation of Dempster-Shafer belief functions in a non-preference setup (he assumes that the agents' belief functions are given). Using a condition similar to our weak independence, and a weaker version of unanimity (on the values taken by belief functions, rather than on preferences), he derives an equivalent of the linear pooling rule, where the aggregated belief function is a mix of the experts' belief functions, with "precise" weights. Belief functions correspond to a special sub-class of the class of sets of priors, and the rule he finds is a special case of rule (1), where the set Λ is a singleton. As such, the present paper can be thought of as a generalisation of his result to sets of priors in general.

Among other related papers, Gajdos and Vergnaud (2011) study a rule similar to (1), the sole difference being that the sets of priors of the experts (interpreted as their information) may be transformed before being aggregated. They use a different setup, with only two experts and preferences defined over triples consisting of acts and the sets of priors of the two experts. Nascimento (2012) obtains a rule similar to (5) in a single profile setting with a possibly infinite set of experts using a richer domain of preferences, namely lotteries of acts.

A Proof of Theorem 1

Proof. Fact 1 implies that (ii) and (iii) are equivalent. That (ii) implies (i) is immediate. We consider the direction (i) implies (ii).

Let B be the range of $J(f)$ over $J \in \mathcal{J}_u^n$ and $f \in F$, and assume without loss of generality that 0 is in the interior of the range of u. Moreover, where required, we use ba to denote the space of bounded signed charges on Σ, $\Delta(\Sigma)$ to denote the subspace of probability measures, and $B(S)$ to denote the space of bounded Σ-measurable real-valued functions on S. The topology used is the weak*-topology.

The proof proceeds as follows. Firstly, in Lemmas 1–3, we establish that, for any profile J and act f, $A(J)(f)$ is a function of $J(f)$ alone. That is, there exists $\phi : B \to \Re$ such that $A(J(f)) = \phi(J(f))$. Then we show that any pair of elements in the positive quadrant of $\Re^n$ and whose sum is bounded above by the unit element are in the range of some $\bar{J} \in \mathcal{EU}_u$. It follows in particular that $cone(B)$ is convex and has full-dimensionality. Borrowing results from Crès et al. (2011), it can be shown that ϕ is homogeneous on B, that its extension to $cone(B)$ is homogeneous and constant additive, and that the cone is closed under addition of constant vectors. Hence $cone(B)$ is the whole space $\Re^n$. Moreover, using the property established previously, ϕ can be shown to be monotonic and concave. Finally, the argument in Gilboa and Schmeidler (1989) is applied to show the existence of a closed and convex set Λ and the representation (3).

We begin by noting that $A(J(f))$ is a function of f and $J(f)$. To this end, let $\widehat{F \times B} = \{(f, b) |\ f \in F, b = J(f) \text{ for some } J \in \mathcal{J}_u^n\}$.

Lemma 1. *There is a function* $\widehat{\phi} : \widehat{F \times B} \to \Re$ *such that, for every* $J \in \mathcal{J}_u^n$ *and*

$f \in F$,

$$A(J)(f) = \widehat{\phi}(f, J(f))$$

Proof. Let $f \in F$ and $J^1, J^2 \in \mathcal{J}_u^n$ such that $J^1(f) = J^2(f)$. So, for all $c \in L$ and $1 \leq i \leq n$, $J_i^1(f) \leq u(c)$ iff $J_i^2(f) \leq u(c)$. By weak independence, it follows that, for any $d \in L$, $A(J^1)(f) \leq u(d)$ iff $A(J^2)(f) \leq u(d)$, and hence, by the continuity of the maxmin EU functional, that $A(J^1)(f) = A(J^2)(f)$; hence $A(J)(f)$ is a function of f and $J(f)$, ie. $A(J)(f) = \widehat{\phi}(f, J(f))$.

□

The following lemma shall prove useful.

Lemma 2. *Suppose that* $(f, b) \in \widehat{F \times B}$ *and let* $z \in F_c$ *be such that* $u(z(s)) = 0$ *for every* $s \in S$*. Then, for every* $0 < \alpha < 1$, $(\alpha f + (1-\alpha)z, \alpha b) \in \widehat{F \times B}$ *and* $\widehat{\phi}(\alpha f + (1-\alpha)z, \alpha b) = \alpha \widehat{\phi}(f, b)$.

Proof. Let $J \in \mathcal{J}_u^n$ be such that $J(f) = b$, and consider the act $g = \alpha f + (1 - \alpha)z$. Since u is affine, $u(g(s)) = \alpha u(f(s))$ for all $s \in S$; it follows from the homogeneity of the maxmin EU functional that $J(g) = \alpha J(f)$ and $A(J)(g) = \alpha A(J)(f)$. Hence $(\alpha f + (1-\alpha)z, \alpha b) = (g, J(g)) \in \widehat{F \times B}$, and moreover, $\widehat{\phi}(\alpha f + (1-\alpha)z, \alpha b) = A(J)(g) = \alpha A(J)(f) = \alpha \widehat{\phi}(f, b)$, as required. □

We now establish that $A(J)(f)$ is a function of $J(f)$ alone, independently of f.

Lemma 3. *There is a function* $\phi : B \to \Re$ *such that, for every* $J \in \mathcal{J}_u^n$ *and* $f \in F$,

$$A(J)(f) = \phi(J(f))$$

Proof. We show that $\widehat{\phi}$ (from Lemma 1) is independent of f: that is, for any $f, g \in F$ and $b \in \Re^n$ with $(f, b), (g, b) \in \widehat{F \times B}$, $\widehat{\phi}(f, b) = \widehat{\phi}(g, b)$. If $f, g \in F_c$, this is trivial, so we suppose that this is not the case. Consider $[\int u(f)dp = b_i] = \{p \in \Delta(\Sigma) | \int u(f)dp = b_i\}$ (i.e. the restriction of the hyperplane in ba defined by $\int u(f)d\nu = b_i$ to $\Delta(\Sigma)$) and $[\int u(g)dp = b_i]$, defined similarly. Note that, since there exists $J^1, J^2 \in \mathcal{J}_u^n$ with $J^1(f) = J^2(g) = b$, it follows from the form of the maxmin EU functional that each of the $[\int u(f)dp = b_i]$ and $[\int u(g)dp = b_i]$ are non-empty. As a point of notation, let $\overline{b} = \max_{1\leq i\leq n} b_i$ and $\underline{b} = \min_{1\leq i\leq n} b_i$. We now proceed by considering three cases.

Case (i). For each $1 \leq i \leq n$, $[\int u(f)dp = b_i] \cap [\int u(g)dp = b_i] \neq \emptyset$. It follows that, for each $1 \leq i \leq n$, there exists $p_i \in \Delta(\Sigma)$ with $\int u(f)dp_i = b_i$ and $\int u(g)dp_i = b_i$. Letting ${J_i}'$ be the expected utility functional in $\mathcal{EU}_u$ with probability measure p_i and $J' = ({J_1}', \ldots, {J_n}')$, we have by construction that $J'(f) = J'(g) = b$. By unanimity, it follows that $A(J')(f) = A(J')(g)$, so $\widehat{\phi}(f, b) = \widehat{\phi}(g, b)$, as required.

Case (ii). For all $s \in S$, $u(g(s)) = \gamma u(f(s)) + c$ for some $\gamma > 0$ and $c \in \Re$, and $[\int u(f)dp = b_i] \cap [\int u(g)dp = b_i] = \emptyset$ for some $1 \leq i \leq n$. Note that $[\int u(f)dp = b_i]$ is disjoint from the (relative) interior of $\Delta(\Sigma)$ (considered as a subset of the space $\{\nu | \int d\nu = 1\} \subset ba$) for at most one b_i, and similarly for $[\int u(g)dp = b_j]$. Furthermore, if both $[\int u(f)dp = b_i]$ and $[\int u(g)dp = b_j]$ are disjoint from the (relative) interior of $\Delta(\Sigma)$, then $[\int u(f)dp = b_i] = [\int u(g)dp = b_j]$ and, moreover, either $b_i = b_j = \overline{b}$ or $b_i = b_j = \underline{b}$. It follows from these observations and the fact that $B(S)$ is dense in the dual of ba (Aliprantis and Border, 2007, Theorem 6.24) that there exists $r \in B(S)$ and points $\overline{p}^1 \in [\int u(f)dp = \overline{b}]$, $\overline{p}^2 \in [\int u(g)dp = \overline{b}]$, $\underline{p}^1 \in [\int u(f)dp = \underline{b}]$, $\underline{p}^2 \in [\int u(g)dp = \underline{b}]$, with $\int r d\overline{p}^1 = \int r d\overline{p}^2 = x \neq y = \int r d\underline{p}^1 = \int r d\underline{p}^2$, for some $x, y \in \Re$. Taking $r' = \frac{\overline{b}-\underline{b}}{x-y} r + (\underline{b} - \frac{\overline{b}-\underline{b}}{x-y} y)$, we have a function with $\int r' d\overline{p}^1 = \int r' d\overline{p}^2 = \overline{b}$ and $\int r' d\underline{p}^1 = \int r' d\underline{p}^2 = \underline{b}$. By continuity and monotonicity of the linear functional, there exists, for each $1 \leq i \leq n$, $p_i^1, p_i^2 \in \Delta(\Sigma)$ such that $\int u(f)dp_i^1 = \int r' dp_i^1 = b_i = \int r' dp_i^2 = \int u(g)dp_i^2$. Take $0 < \alpha \leq 1$ such that $\alpha r'(s)$ is in the range of u for all $s \in S$, and take $z \in F_c$ such that $u(z(s)) = 0$ for all $s \in S$. Then there exists $h \in F$ with $u(h(s)) = r'(s)$ for all $s \in S$. Let J_i^1 be the expected utility functional in $\mathcal{EU}_u$ with probability measure p_i^1 and $J^1 = (J_1^1, \ldots, J_n^1)$; similarly for J^2, with

J_i^2 being the expected utility functionals in $\mathcal{EU}_u$ with probability measures p_i^2. It follows by construction that $J_i^1(h) = J_i^1(\alpha f + (1-\alpha)z) = \alpha b_i$ for all $1 \leq i \leq n$, whence, by unanimity, $A(J^1)(h) = A(J^1)(\alpha f + (1-\alpha)z)$. Similarly, $J_i^2(h) = J_i^2(\alpha g + (1-\alpha)z) = \alpha b_i$ for all $1 \leq i \leq n$, whence, by unanimity, $A(J^2)(h) = A(J^2)(\alpha g + (1-\alpha)z)$. So $\widehat{\phi}(\alpha f + (1-\alpha)z, \alpha b) = A(J^1)(\alpha f + (1-\alpha)z) = A(J^1)(h) = A(J^2)(h) = A(J^1)(\alpha g + (1-\alpha)z) = \phi(\alpha g + (1-\alpha)z, \alpha b)$ (where the middle equality holds by Lemma 1), whence, by Lemma 2, $\phi(f,b) = \phi(g,b)$ as required.

Case (iii). Now consider the case where neither (i) or (ii) apply: $[\int u(f)dp = b_i] \cap [\int u(g)dp = b_i] = \emptyset$ for some $1 \leq i \leq n$, but it is not the case that for all $s \in S$, $u(g(s)) = \gamma u(f(s)) + c$ for some $\gamma > 0$ and $c \in \Re$. Define $r, r' \in B(S)$ by $r(s) = u(f(s))$ and $r'(s) = u(g(s))$ for all $s \in S$. Pick $\gamma, \delta > 0$, $c, d \in \Re$ such that, for each $1 \leq i \leq n$, $[\int r dp = \frac{1}{\gamma}(b_i - c)] \cap [\int r' dp = \frac{1}{\delta}(b_i - d)] \neq \emptyset$. (Such γ, δ, c, d exist since the set of $[\int r dp = x]$ covers $\Delta(\Sigma)$ and similarly for $[\int r' dp = x]$.) Hence, given the linearity of the functionals, $[\int(\gamma r + c)dp = b_i] \cap [\int(\delta r' + d)dp = b_i] \neq \emptyset$ for each $1 \leq i \leq n$. Take $0 < \alpha \leq 1$ such that $\alpha(\gamma r + c)(s), \alpha(\delta r' + d)(s)$ belong to the range of u for all $s \in S$. Hence, there exist $h, h' \in F$ with $u(h(s)) = \alpha(\gamma r + c)(s)$ and $u(h'(s)) = \alpha(\delta r' + d)(s)$ for all $s \in S$. For $z \in F_c$ with $u(z(s)) = 0$ for all $s \in S$, we have $\phi(\alpha f + (1-\alpha)z, \alpha b) = \phi(h, \alpha b) = \phi(h', \alpha b) = \phi(\alpha g + (1-\alpha)z, \alpha b)$, where the first and last equalities are obtained by applying case (ii), and the middle equality is an application of case (i). By Lemma 2, we have $\phi(f,b) = \phi(g,b)$ as required.

□

Now we show that ϕ can be extended to a monotonic, homogeneous, concave, constant additive function on $\Re^n$. To this end, the following lemma is central.

Lemma 4. *For each $a, b \in \Re^n$ such that $\overrightarrow{0} \leq a, b$ and $\overrightarrow{1} \geq a + b$, there exists $\bar{J} = (\bar{J}_1, \ldots, \bar{J}_n) \in \mathcal{EU}_u^n$ and $f, g \in F$ with $\bar{J}(f) = a$ and $\bar{J}(g) = b$.*

Proof. Take any three disjoint non-empty events E_1, E_2, E_3 with $\bigcup E_j = S$.

(Such events exist since S contains at least three states.) Members of $\mathcal{EU}_u$ are fully characterised by a single probability measure. For each $1 \leq i \leq n$, take $\bar{J}_i$ to be the expected utility functional with probability measure p_i defined as follows:

$$p_i(E_1) = a_i$$
$$p_i(E_2) = b_i$$
$$p_i(E_3) = 1 - a_i - b_i$$

Such measures exist since $a_i + b_i \leq 1$ for all i. Let $x, z \in L$ be such that $u(x) = 1$, $u(z) = 0$, and define $f, g \in F$ as follows: $f(s) = x$ for $s \in E_1$ and z otherwise, and $g(s) = x$ for $s \in E_2$ and z otherwise. It is clear by construction that $\bar{J}(f) = a$ and $\bar{J}(g) = b$. □

Lemma 5. *For all $b \in B$ and $0 < \alpha < 1$, $\alpha b \in B$. Moreover ϕ is homogenous on B. Furthermore, the extension of ϕ to $cone(B)$ by homogeneity is homogeneous and constant additive (that is, for all $c \in \Re$, $\phi(b + \overrightarrow{c}) = \phi(b) + c$). Finally, $cone(B)$ is closed under addition of constant acts (ie. $cone(B) + \overrightarrow{c} \subseteq cone(B)$ for all $c \in \Re$).*[3]

Proof. The first two clauses are an immediate consequence of Lemma 2; as concerns the latter clauses, it is straightforward to check that the arguments in the proof in Crès et al. (2011, Lemma 4) apply in the current setting. □

It follows from Lemmas 4 and 5 that $cone(B) = \Re^n$. Henceforth, we use ϕ to denote the extension of ϕ to $\Re^n$.

Lemma 6. *ϕ is monotonic.*

[3] Crès et al. (2011) use the terminology "Shift" instead of constant additivity.

Proof. First consider $a, b \in \Re^n$ with $a \leq b$ and suppose that $\overrightarrow{0} \leq a, b$ and $\overrightarrow{1} \geq a + b$. By Lemma 4, there exists $\bar{J} \in \mathcal{EU}_u^n$ and $f, g \in F$ with $\bar{J}(f) = a$ and $\bar{J}(g) = b$. Since $a \leq b$, $\bar{J}(f) \leq \bar{J}(g)$, whence, by unanimity, $A(\bar{J})(f) \leq A(\bar{J})(g)$. It follows that $\phi(a) \leq \phi(b)$, as required.

Now consider $a, b \in \Re^n$ with $a \leq b$, not satisfying the conditions in the previous case. There exists $x \in \Re$ such that $a + \overrightarrow{x}, b + \overrightarrow{x} \geq \overrightarrow{0}$; moreover, there exists $0 < \alpha \leq 1$ such that $\alpha(a + b + 2\overrightarrow{x}) \leq \overrightarrow{1}$. Since $\alpha(a + \overrightarrow{x}) \leq \alpha(b + \overrightarrow{x})$, by the previous case, $\phi(\alpha(a + \overrightarrow{x})) \leq \phi(\alpha(b + \overrightarrow{x}))$, whence, thanks to Lemma 5, $\phi(a) \leq \phi(b)$ as required.

□

Lemma 7. *ϕ is concave.*

Proof. Consider first $a, b \in \Re^n$ such that $\overrightarrow{0} \leq a, b$ and $\overrightarrow{1} \geq a + b$ and $\phi(a) = \phi(b)$. We will show that $\phi(\frac{1}{2}a + \frac{1}{2}b) \geq \frac{1}{2}\phi(a) + \frac{1}{2}\phi(b) = \phi(a)$. By Lemma 4, there exists $\bar{J} \in \mathcal{EU}_u^n$ and $f, g \in F$ with $\bar{J}(f) = a$ and $\bar{J}(g) = b$. Since $\bar{J} \in \mathcal{EU}_u^n$, $\bar{J}(\frac{1}{2}f + \frac{1}{2}g) = \frac{1}{2}\bar{J}(f) + \frac{1}{2}\bar{J}(g) = \frac{1}{2}a + \frac{1}{2}b$. Since $\phi(a) = \phi(b)$, $A(\bar{J})(f) = A(\bar{J})(g)$, and since the functional $A(\bar{J}) \in \mathcal{J}_u$ and hence is concave, $A(\bar{J})(\frac{1}{2}f + \frac{1}{2}g) \geq A(\bar{J})(f)$. It follows that $\phi(\frac{1}{2}a + \frac{1}{2}b) = \phi(\bar{J}(\frac{1}{2}f + \frac{1}{2}g)) = A(\bar{J})(\frac{1}{2}f + \frac{1}{2}g) \geq A(\bar{J})(f) = \phi(a)$, as required.

By the homogeneity and constant additivity of ϕ, it follows that, for any $a, b \in \Re^n$ with $\phi(a) = \phi(b)$, we have that $\phi(\frac{1}{2}a + \frac{1}{2}b) \geq \frac{1}{2}\phi(a) + \frac{1}{2}\phi(b) = \phi(a)$. Finally, consider a and b with $\phi(a) \neq \phi(b)$ and suppose without loss of generality that $\phi(a) < \phi(b)$ and that $x = \phi(b) - \phi(a)$. By the previous result and constant additivity of ϕ, we have that $\phi(\frac{1}{2}a + \frac{1}{2}b) + \frac{1}{2}x = \phi(\frac{1}{2}(a + \overrightarrow{x}) + \frac{1}{2}b) \geq \frac{1}{2}\phi(a + \overrightarrow{x}) + \frac{1}{2}\phi(b) = \frac{1}{2}\phi(a) + \frac{1}{2}\phi(b) + \frac{1}{2}x$, hence the desired result.

□

ϕ is thus a monotonic, homogeneous, concave, constant additive function on $\Re^n$, and the Gilboa and Schmeidler (1989) reasoning can be applied, precisely as in Crès et al. (2011, Lemma 8), to obtain representation (3). Moreover, since u

is non-degenerate and $cone(B) = \Re^n$, the uniqueness clause of the Gilboa and Schmeidler (1989) result applies, and implies that Λ is the unique convex closed set satisfying (3).

□

B Proofs of other results

Proof of Proposition 1. Evidently, strong independence is equivalent to the existence of a function $\psi : B \to \Re$ such that, for every $J = (J_1, \ldots, J_n) \in \mathcal{J}_u^n$ and $f \in F$,

$$A(J)(f) = \psi(J(f))$$

By Lemma 3, weak independence and unanimity imply strong independence. It is evident that strong independence implies weak independence. Now we show that strong independence implies unanimity.

Note firstly that the arguments in Lemma 5 go through using only the properties if the maxmin EU functional. Hence ψ is homogeneous and constant additive.

Now consider acts $f, g \in F$ and $J = (J_1, \ldots, J_n) \in \mathcal{J}_u^n$ such that $J_i(f) \leq J_i(g)$ for all $1 \leq i \leq n$. Let $a = J(f)$ and $b = J(g)$ respectively; by hypothesis, $a \leq b$. We need to show that $A(J)(f) \leq A(J)(g)$, or, in other terms, $\phi(a) \leq \phi(b)$. It suffices to consider the case where $a, b \geq \overrightarrow{0}$, and $b \leq \overrightarrow{1}$; other cases are derived by the homogeneity and constant additivity of ψ. Consider the vector $b - a \in \Re^n$; since $b \geq a$, $b - a \geq \overrightarrow{0}$. Take three disjoint non-empty events E_1, E_2, E_3 whose union is S, and define the n-tuple of probability measures $(p_1, \ldots, p_n)$ on S as follows:

$$p_i(E_1) = b_i - a_i$$
$$p_i(E_2) = a_i$$
$$p_i(E_3) = 1 - b_i$$

By the definition of the case, these are all well-defined (the values are between 0 and 1 and sum to 1). Take $\bar{J}_i$ to be the expected utility functional in $\mathcal{EU}_u$ with probability p_i. Let $x, z \in L$ be such that $u(x) = 1$, $u(z) = 0$, and define $h, h' \in F$ as follows: $h(s) = x$ for $s \in E_1 \cup E_2$ and z otherwise, and $h'(s) = x$ for $s \in E_2$ and z otherwise. It is clear by construction that $\bar{J}(h) = b$ and $\bar{J}(h') = a$. Moreover, since $h(s) \succeq h'(s)$ for every $s \in S$, by the monotonicity of the maxmin EU functional, $A(\bar{J})(h) \geq A(\bar{J})(h')$. Hence $\phi(b) \geq \phi(a)$, as required.

□

Proof of Theorem 2. That (ii) implies (i) is immediate. The proof that (i) implies (ii) is of a similar structure to the proof of Theorem 1. (We use the notation introduced in that proof.) First of all, since u is unbounded, Lemma 2 is not required to establish Lemma 3: the arguments (notably in cases (ii) and (iii)) go through with no need to scale the functions in $B(S)$ by some $\alpha \in (0, 1)$.[4] Moreover, since the maxmin EU model is a special case of the variational preference models, these arguments imply that $A^{var}(J)(f) = \phi(J(f))$ for some $\phi : B \to \Re$. To establish (6), we show that ϕ is monotonic, concave, normalized (that is, $\phi(\overrightarrow{c}) = c$ for any $c \in u(X)$) and vertically invariant (that is, $\phi(\alpha a + (1-\alpha)\overrightarrow{c}) = \phi(\alpha a) + (1-\alpha)c$

[4]The proof given here is for a u that is unbounded above and below. However the arguments continue to apply if u is only unbounded above or unbounded below, because, given the constant additivity of the variational preference functional (4), $\widehat{\phi}$ and ϕ are constant additive (by an argument similar to that establishing Lemma 5), and constant additivity can play the role played by homogeneity and constant additivity in Theorem 1 (in particular in Lemmas 3, 6 and 7).

for all $c \in u(X)$ and $\alpha \in (0,1)$). To this end, the following lemma will play a similar role to Lemma 4.

Lemma 8. *For each $a, b \in \Re^n$, there exists $\bar{J} = (\bar{J}_1, \ldots, \bar{J}_n) \in \mathcal{EU}_u^n$ and $f, g \in F$ with $\bar{J}(f) = a$ and $\bar{J}(g) = b$.*

Proof. Take any $r \in \Re_+$ with $|a_i| + |b_i| \leq r$ for all $1 \leq i \leq n$. Take any three disjoint non-empty events E_1, E_2, E_3 with $\bigcup E_j = S$, and, for each $1 \leq i \leq n$, take $\bar{J}_i$ to be the expected utility functional with probability measure p_i defined as follows:

$$
\begin{aligned}
p_i(E_1) &= \frac{|a_i|}{r} \\
p_i(E_2) &= \frac{|b_i|}{r} \\
p_i(E_3) &= 1 - \frac{|a_i|}{r} - \frac{|b_i|}{r}
\end{aligned}
$$

Such measures exist by the choice of r. Let $x, y, z \in L$ be such that $u(x) = r$, $u(y) = -r$ and $u(z) = 0$ (such elements exist since u is unbounded), and define $f, g \in F$ as follows: if $a_i \geq 0$, $f(s) = x$ for $s \in E_1$ and z for $s \notin E_1$, and if $a_i < 0$, $f(s) = y$ for $s \in E_1$ and z for $s \notin E_1$; similarly, if $b_i \geq 0$, $g(s) = x$ for $s \in E_2$ and z for $s \notin E_2$, and if $b_i < 0$, $g(s) = y$ for $s \in E_2$ and z for $s \notin E_2$. It is clear by construction that $\bar{J}(f) = a$ and $\bar{J}(g) = b$. □

Given Lemma 8, $B = \Re^n$, and, moreover, the arguments in the proofs of Lemmas 6 and 7 go through without any need for homogeneity or constant additivity of ϕ, establishing that ϕ is monotonic and concave. The fact that $A(J)$ and the J_i are represented by the same utility function implies that ϕ is normalized. Vertical invariance is established in the following lemma.

Lemma 9. *ϕ is vertically invariant: for any $a \in \Re^n$, $c \in \Re$ and $\alpha \in (0,1)$, $\phi(\alpha a + (1-\alpha)\overrightarrow{c}) = \phi(\alpha a) + (1-\alpha)c$.*

Proof. By Lemma 8, there exists $J \in \mathcal{EU}_u^n$ with $J(f) = a$. Let $x, z \in F_c$ with $u(x(s)) = c$, and $u(z(s)) = 0$ for all $s \in S$, and consider the acts $g = \alpha f + (1-\alpha)x$ and $h = \alpha f + (1-\alpha)z$. Since u is affine, $u(g(s)) = \alpha u(f(s)) + (1-\alpha)c$ for all $s \in S$ and $u(h(s)) = \alpha u(f(s))$ for all $s \in S$, and since the functionals J_i are expected utility, $J(h) = \alpha J(f) = \alpha a$. Moreover, since $A(J) \in \mathcal{J}_u^{var}$, $A(J)(g) = A(J)(h) + (1-\alpha)c$. Hence $\phi(\alpha a + (1-\alpha)\overrightarrow{c}) = A(J)(g) = A(J)(h) + (1-\alpha)c = \phi(\alpha a) + (1-\alpha)c$, as required.

□

So ϕ is a concave normalized niveloid (Maccheroni et al., 2006, Lemma 25) and, by Maccheroni et al. (2006, Lemma 26), we obtain the representation (6) as required. The uniqueness clause is an immediate consequence of the aforementioned lemma and the assumption that u is unbounded.

It remains to show that (ii) and (iii) are equivalent. Let J be the functional defined by (5) (ie. $J(f) = \min_{p\in\Delta(\Sigma)}(\int u(f)dp + c_{A^{var}(\preceq)}(p))$ with $c_{A^{var}(\preceq)}$ as in (5)) and let J' be the functional defined by (6) (ie. $J'(f) = \min_{\lambda\in\Delta(\{1,\ldots,n\})}(\sum_{i=1}^n \lambda_i J_i(f) + e(\lambda))$). Choose λ to minimise the expression in (6), choose, for each $1 \leq i \leq n$, p_i to minimise $\int u(f(s))dp_i(s) + c_i(p_i)$, and let $p = \sum_{i=1}^n \lambda_i p_i$. It follows immediately that $J(f) \leq \int u(f)dp + \sum_{i=1}^n \lambda_i c(p_i) + e(\lambda) = \sum_{i=1}^n \lambda_i(\int u(f)dp_i + c(p_i)) + e(\lambda) = J'(f)$. Conversely, let $p \in \Delta(\Sigma)$ be such that $J(f) = \int u(f)dp + c_{A^{var}(\preceq)}(p)$. By (5), there exists $(\lambda, p_1, \ldots, p_n) \in \Delta(\{1,\ldots,n\}) \times \Delta(\Sigma)^n$ such that $p = \sum_{i=1}^n \lambda_i p_i$. So $J'(f) \leq \sum_{i=1}^n \lambda_i J_i(f) + e(\lambda) \leq \sum_{i=1}^n \lambda_i(\int u(f)dp_i + c(p_i)) + e(\lambda) = \sum_{i=1}^n \lambda_i \int u(f)dp_i + \sum_{i=1}^n \lambda_i c(p_i) + e(\lambda) = J(f)$. Hence $J(f) = J'(f)$. The required equivalence is an immediate consequence of the uniqueness clause in the Maccheroni et al. (2006) result (in particular Proposition 6).

□

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