Uncertainty aversion, multi utility representations and state independence of utility*

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January 22, 2015

Abstract

This paper proposes and characterises a model of uncertainty averse preferences that can simultaneously accommodate three divergences from subjective expected utility: imprecision of beliefs (or ambiguity), imprecision of tastes (or multi utility), and state dependence of utility. Moreover, it characterises, in this context, a notion of state independence of utility borrowed from the literature on incomplete preferences. This notion is then shown to be basically inconsistent with the standard state-independence axiom, monotonicity, whenever tastes are imprecise. A new notion of state independence in the context of imprecise tastes, which is characterised by monotonicity, is proposed.

Keywords: State independence of utility, imprecise tastes, uncertainty aversion, multi utility, multiple priors, state-dependent utility.

JEL classification: D81

*This paper is a preliminary version, and may be subject to subsequent revisions. The most recent version will always be downloadable from http://www.hec.fr/hill. The author gratefully acknowledges support from the ANR program DUSUCA (ANR-14-CE29-0003-01) and the Investissements d'Avenir program (ANR-11-IDEX-0003/Labex Ecodec/ANR-11-LABX-0047).

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1 Introduction

Consider a decision maker who has been diagnosed with a serious yet rare degenerative disease, and is faced with the choice between not doing anything or paying to receive a new, expensive but not very well understood treatment. If he does nothing, there is a 40% chance of an uncomfortable but largely normal life (and lifespan) and a 60% chance of a significantly impeded lifestyle for the rest of his days. The chances of success of the treatment are between 25% and 75%; in case of success, it will be life as usual (with no discomfort); in case of failure, he will have a seriously degraded quality of life. This decision has a marked resemblance with Ellsberg's (1961) examples, insofar as the probabilities of various outcomes are known for one of the options, and they are not known precisely for the other. Borrowing a term from statistics, we will say that there may be *imprecision in beliefs*.[1] It is well-known that in such cases, the decision maker may exhibit non-neutral attitudes to uncertainty, such as uncertainty aversion. Moreover, given that few people know what it's like to live in a severely handicapped situation, this decision may also involve what could be called *imprecision in tastes*: the decision maker may not be able to compare the impairment resulting from failed treatment with the impeded quality of life resulting from inaction. Finally, as is well-known (Drèze, 1987; Arrow, 1974; Cook and Graham, 1977; Karni, 1983), such health-related decisions often involve state-dependent utility: one's utility for the money saved by not undergoing the treatment may depend on one's state of health.

Examples such as this, which we take as representative of an important class of real-life cases that are relevant in health economics, the theory of insurance and beyond, involve several distinct violations of the standard axioms of subjective expected utility (Savage, 1954; Anscombe and Aumann, 1963).[2] These violations have generally been studied in isolation: the literature on ambiguity, for example, almost universally assumes precise tastes and state independence of utility. The first aim of this paper is to propose and characterise a theory of decision that displays uncertainty aversion and is sufficiently rich to capture all of these

[1]The term 'imprecise probabilities' is widely used in statistics and philosophy; see for example Walley (1991); Bradley (2014) and the references therein.

[2]Explicitly, in the Anscombe-Aumann framework and in the context of complete preferences: violations of the independence axiom for imprecision of beliefs, of the restriction of independence to risky prospects for imprecision of tastes, and of the monotonicity axiom for state independence of utilities.

violations simultaneously. Such a theory will be applicable in the sorts of situations just described.

Our basic result will involve the representation of preferences by the following functional form, for acts (that is, state-contingent consequences) f:

$$(1) \qquad \min_{U \in \mathcal{U}} \sum_{s \in S} U(s, f(s))$$

where $\mathcal{U}$ is a (closed convex) set of real-valued functions on states and consequences, which we call *evaluations*.[3] Evaluations occur in the literature on state-dependent utility, where they have been recognised to provide a natural representation of preferences in cases where state-independence axioms are violated, which is sufficient for many economic applications. For example, they can be used to determine the choice made in the decision discussed above, as well as to perform comparative statics on, for example, the cost of the treatment (Hill, 2010). We provide a representation theorem showing that, in the presence of some mild technical conditions, the standard order, continuity and uncertainty aversion axioms – without monotonicity or any form of independence – are necessary and sufficient for preferences to be representable according to (1). It follows that (1) also generalises the vast majority of existing models of uncertainty averse preferences.

Despite its uses, representation (1) does not allow discussion of beliefs and tastes. For this, a separation is required. We shall focus on state-independent separations, of which the most notable example comes from standard subjective expected utility: there is a separation of the single evaluation in the singleton version of (1) into a (suitably unique) state-independent utility function and a probability measure, which is ensured by the state-independence axioms (monotonicity, in the Anscombe-Aumann framework). The main finding of this paper is that the issue of separation, and with it the meaning of state independence of utility, is considerably more complex when there is both uncertainty aversion and imprecision of tastes.

[3] Some authors use the term 'state-dependent utilities' to refer to such functions. By contrast, we follow the literature on state-dependent utility (for example Karni et al. (1983); Karni (1993b,a); Karni and Schmeidler (1993); Karni and Mongin (2000); Karni (2011); Hill (2010)) in reserving the term 'state-dependent utility' for a representation involving a principled separation of beliefs and tastes: for example, a suitably unique probability and (state-dependent) utility.

Our first result is a characterisation of the following analogue, for uncertainty averse preferences, of a representation recently proposed by Galaabaatar and Karni (2013) in the context of incomplete preferences:

$$\min_{(u,p)\in\mathcal{U}^{si}} \sum_{s\in S} p(s)u(f(s)) \tag{2}$$

where $\mathcal{U}^{si}$ is a (closed convex) set of pairs of probabilities and state-independent utilities. Here there is separation evaluation-wise into a taste factor – the state-independent utility – and a belief factor – the probability. Since the state independence at issue holds at the level of each utility-probability pair, we shall refer to such representations as involving *multi state-independent utility*.

Surprisingly, this representation turns out to be in a sense inconsistent with the standard axiom for state independence, monotonicity. More precisely, multi state-independent utility and monotonicity taken together imply that, unless beliefs satisfy a particularly restrictive condition, preferences are represented by a single utility function. In other words, once there is imprecision in tastes, monotonicity – the standard axiom for state independence of utility – is incompatible with the notion of state independence embodied by (2) in all but a few special cases.

Investigating further, we show that, in the context of uncertainty averse preferences with imprecise tastes, monotonicity implies a particular representation of preferences, of which the following is a suggestive yet paradigmatic special case:

$$\min_{p\in\mathcal{C}} \sum_{s\in S} p(s) \min_{u\in\mathcal{U}^{ut}} u(f(s)) \tag{3}$$

where $\mathcal{C}$ is a (closed convex) set of probability measures on the state space and $\mathcal{U}^{ut}$ is a (closed convex) set of utility functions on the set of consequences. So monotonicity implies that the *set of utilities* involved is state independent, insofar as the same set is used in every state (in the second minimisation). We shall speak of *state-independent multi utility*, to emphasise that the state-independent separation applies at the level of sets, rather than individual utility-probability pairs.

These findings suggest that, once one leaves the realm of precise tastes, the single notion of state independence is replaced by two potentially relevant and basically incompatible notions. One, which is fast becoming the standard in studies of incomplete preferences, involves state independence at the level of each probability-utility pair involved in the representation. The other, which is implied by the most popular axiom for state independence, involves state independence at the level of the set of utilities. Of course, the difference between them may have consequences for behaviour, and in applications.

The paper is organised as follows. The framework is set out in Section 2. The basic result is stated and discussed in Section 3, which also contains a corollary for preferences over risky prospects. Section 4 contains the analysis of the issue of state independence: Section 4.1 focusses on multi state-independent utility and its incompatibility with monotonicity; Section 4.2 considers state-independent multi utility. Related literature is discussed in Section 5. Proofs are contained in an Appendix.

2 Framework

We use the standard Anscombe-Aumann (1963) framework, as adapted by Fishburn (1970). Let S be a finite set of states; Δ is the set of probability measures over S. Let X be a finite set of outcomes. *Consequences* are *lotteries* over X: that is, probability measures over X. $\Delta(X)$ is the set of consequences; of course $\Delta(X) \subset \Re^X$. An *act* is a function from states to consequences; let $\mathcal{A} = \Delta(X)^S$ be the set of acts. $\mathcal{A}$ is a mixture set with the mixture relation defined pointwise: for f, h in $\mathcal{A}$ and $\alpha \in [0,1]$, the mixture $\alpha f + (1-\alpha)h$ is defined by $(\alpha f + (1-\alpha)h)(s,x) = \alpha f(s,x) + (1-\alpha)h(s,x)$. We write $f_\alpha h$ as short for $\alpha f + (1-\alpha)h$. For state s and acts f and g, the act $f_s g_{s^c}$ is defined as follows: $f_s g_{s^c}(s) = f(s)$ and $f_s g_{s^c}(t) = g(t)$ for all $t \neq s$. With slight abuse of notation, a constant act taking consequence c for every state will be denoted c and the set of constant acts will be denoted $\Delta(X)$.

An *evaluation* is a function $U : S \times X \to \Re$. We assume $\Re^{S\times X}$ as well as $\mathcal{A}$ to be endowed with the Euclidean norm $|\;|$. For evaluation $U \in \Re^{S\times X}$, consequence $c \in \Delta(X)$ and state $s \in S$, we let $U(s,c) = \sum_{x\in X} U(s,x)c(x)$. Evaluations $U, U' \in \Re^{S\times X}$ are *cardinally equivalent* if there exists $b \in \Re^S$ with $\sum_{s\in S} b(s) = 0$ such that $U' = U + b$.[4] $U' \in \Re^{S\times X}$ is a *positive affine transformation* of U if there exists $a \in \Re_{>0}$ and $b \in \Re$ such that $U' = aU + b$. An evaluation representing preferences according to (1) with a singleton $\mathcal{U}$ is unique up to cardinal equivalence and positive affine transformation (see for

[4]Cardinally equivalent evaluations are such that $\sum_{s\in S} U(s, f(s)) = \sum_{s\in S} U'(s, f(s))$ for all $f \in \mathcal{A}$.

example Karni et al. (1983)). We extend these notions to sets of evaluations as follows. For $\mathcal{U}, \mathcal{U}' \subseteq \Re^{S \times X}$, $\mathcal{U}'$ is cardinally equivalent to $\mathcal{U}$ if, for each $U' \in \mathcal{U}'$, there exists $U \in \mathcal{U}$ such that U and U' are cardinally equivalent, and similarly for each $U \in \mathcal{U}$. $\mathcal{U}'$ is a positive affine transformation of $\mathcal{U}$ if there exist $a \in \Re_{>0}$ and $b \in \Re$, such that $\mathcal{U}' = a\mathcal{U} + b = \{aU + b \mid U \in \mathcal{U}\}$. We adopt analogous notation and terminology for state-independent utilities (ie. functions in $\Re^X$). Fix an arbitrary $\hat{x} \in X$, and let $\mathcal{N}^1 = \{U \in \Re^{S \times X} \mid \forall s \in S,\ U(s, \hat{x}) = 0,\ \ |U| = 1\}$. Up to cardinal equivalence and positive affine transformation, $\mathcal{N}^1$ contains a representative of each evaluation: it can thus be thought of as a set of appropriately normalised evaluations.

The binary relation $\preceq$ on $\mathcal{A}$ represents the decision maker's preferences over acts. The symmetric and asymmetric parts of $\preceq$, $\sim$ and $\prec$, are defined in the standard way. A functional $V : \mathcal{A} \to \Re$ *represents* $\preceq$ if and only if, for all $f, g \in \mathcal{A}$, $f \preceq g$ iff $V(f) \leqslant V(g)$.

An evaluation U is *constant* if $\sum_{s \in S} U(s, f(s)) = \sum_{s \in S} U(s, g(s))$ for all $f, g \in \mathcal{A}$. A set $\mathcal{U} \subseteq \Re^{S \times X}$ is *non-trivial* if it contains at least one non-constant evaluation. Let $\mathcal{F} \subseteq 2^{\Re^{S \times X}}$ be a family of closed convex sets of evaluations (for example, $\mathcal{F}$ could contain all closed convex sets). For any $\mathcal{U} \in \mathcal{F}$ representing $\preceq$ according to (1), let $\mathcal{U}(\mathcal{A}) = \{\min_{U \in \mathcal{U}} \sum_{s \in S} U(s, f(s)) \mid f \in \mathcal{A}\}$. Since $\mathcal{A}$ is closed, and the minimal functional and the U are continuous, this is a closed set. For any $\mathcal{U} \in \mathcal{F}$ representing $\preceq$ according to (1), we shall say that $\mathcal{U}$ is a *tight* closed convex representation of $\preceq$ from the family $\mathcal{F}$ if there exists no proper subset in the family, $\mathcal{U}' \subsetneq \mathcal{U}$ with $\mathcal{U}' \in \mathcal{F}$, representing $\preceq$ according to (1). A tight set is as small as a representation can be, in the sense that there are no extraneous evaluations. $\mathcal{U}$ is a *minimal* tight closed convex set from $\mathcal{F}$ representing $\preceq$ if for every tight closed convex set $\mathcal{U}' \in \mathcal{F}$ representing $\preceq$ with $\mathcal{U}'(\mathcal{A}) = \mathcal{U}(\mathcal{A})$, and for each $f \in \mathcal{A}$ and $U' \in \mathcal{U}'$ there exists $U \in \mathcal{U}$ such that $\sum_{s \in S} U'(s, f(s)) \geqslant \sum_{s \in S} U(s, f(s))$. A minimal representing set contains evaluations taking the minimal values possible, compared to other representing sets that are suitably normalised (so as to take the same range of values). Explicit mention of the family $\mathcal{F}$ will be omitted where it is evident from the context. Again, analogous notions are employed for sets of state-independent utilities (ie. functions in $\Re^X$).

3 The base result

3.1 Axioms

Consider the following axioms.

Axiom A1 (Weak order)**.** For all $f, g, h \in \mathcal{A}$: if $f \preceq g$ and $g \preceq h$, then $f \preceq h$; and $f \preceq g$ or $g \preceq f$.

Axiom A2 (Non-degeneracy)**.** There exists $f, g \in \mathcal{A}$ such that $f \succ g$.

Axiom A3 (Continuity)**.** For all sequences $f_n, g_n \in \mathcal{A}$, $f_n \to f$ and $g_n \to g$, if $f_n \succeq g_n$ for all n, then $f \succeq g$.

Axiom A4 (Uncertainty aversion)**.** For all $f, g \in \mathcal{A}$ and $\alpha \in (0, 1)$, if $f \sim g$ then $f_\alpha g \succeq g$.

Axiom A5 (Local non-satiation)**.** For every $f \in \mathcal{A}$, either $f \succeq g$ for all $g \in \mathcal{A}$ or there exists $g \in \mathcal{A}$ such that $g_\alpha f \succ f$ for all $\alpha \in (0, 1]$.

A1–A4 require little comment: A1 and A2 are totally standard, A3 is a standard topological continuity condition (equivalence with the more common mixture continuity axiom does not evidently hold here, given the absence of any independence axiom),[5] and A4 is the standard uncertainty aversion axiom from Schmeidler (1989); Gilboa and Schmeidler (1989). Local non-satiation, A5, is a version of a standard condition in consumer theory (for example, (Mas-Colell et al., 1995, Ch 3)), which ensures that all indifference curves are 'thin'. It is a consequence of monotonicity and independence, which are not generally assumed here.

One further axiom shall be of interest, for which the following notion will prove useful. For acts $f, g, h \in \mathcal{A}$ with $g \prec f \prec h$, f is a *potential fifty-fifty mixture* of g and h if there exist $g', h' \in \mathcal{A}$ with $g' \sim g$, $h' \sim h$ and $h'_{\frac{1}{2}} g' \sim f$. The potential fifty-fifty mixtures of g and h are those acts that are indifferent to a fifty-fifty mixture of – or if you will, to a fair coin toss yielding – two acts which are indifferent to g and h respectively. In the presence of independence, the set of potential fifty-fifty mixtures is the indifference class of the fifty-fifty mixture of g and h; to this extent, the notion of potential fifty-fifty mixture can be thought of as an analogue of the standard notion of fifty-fifty mixture for cases where the

[5] Cerreia-Vioglio (2009) shows that mixture continuity implies upper semi-continuity in our framework; however, the result used in the proof of Theorem 1 (by Kannai (1977)) requires full continuity.

axiom is not satisfied. Moreover, f is a *conservative potential fifty-fifty mixture* of g and h if it is a potential fifty-fifty mixture and for any other potential fifty-fifty mixture, f', of these acts $f' \succeq f$. Conservative potential fifty-fifty mixtures are the lowest acts, in the $\preceq$ order, that are potential fifty-fifty mixtures.

Now consider the following axiom.

Axiom A6 (Coherent calibration)**.** For any $f, g, h \in \mathcal{A}$ with $g \prec f \prec h$, there exists $\epsilon > 0$ such that, for every sequence $(f_i)_{0 \leqslant i \leqslant n}$ with $f_0 = g$, $f_n = h$, $f_i \succ f_{i-1}$ for all $1 \leqslant i \leqslant n$ and such that f_i is a conservative potential fifty-fifty mixture of f_{i-1} and f_{i+1} for all $1 \leqslant i \leqslant n-1$, $\frac{\max_{f_i \preceq f} i}{n} \leqslant 1 - \epsilon$.

A sequence of acts, each consecutive triple of which involves a conservative potential fifty-fifty mixture, can be thought of as forming a 'ruler' with equally-spaced gaps (in terms of preferences) between the elements of the sequence. (As such, there is conceptual analogy with the notion of standard sequence common in measurement theory (Krantz et al., 1971).) Coherent calibration, A6, demands that for any act strictly between two acts g and h, every such ruler gives a measurement of the position of f that is strictly bounded away from the distance between g and h. The intuition is obvious: if $f \prec h$, then f must be strictly 'closer' to g, according to any such ruler, than h. One could consider this axiom as guaranteeing that sequences of conservative potential fifty-fifty mixtures constitute reasonable 'rulers' to calibrate preferences. (Note that this is automatically the case in the presence of independence.) As will be noted below, the vast majority of uncertainty averse decision models proposed in the literature satisfy A6.

3.2 Representation Theorem

The following is the first basic result of the paper.

Theorem 1. *Let $\preceq$ be a preference relation on $\mathcal{A}$. The following are equivalent:*

(i) *$\preceq$ satisfies A1–A6,*

(ii) *there exists a non-trivial minimal tight closed convex set of evaluations $\mathcal{U} \subseteq \Re^{S \times X}$ such that $\preceq$ is represented by:*

$$V(f) = \min_{U \in \mathcal{U}} \sum_{s \in S} U(s, f(s)) \tag{1}$$

Moreover, for any minimal tight closed convex set $\mathcal{U}' \subseteq \Re^{S\times X}$ representing $\preceq$ according to (1) there exist $\mathcal{U}^\star \subseteq \Re^{S\times X}$, $a \in \Re_{>0}$ and $b \in \Re$ such that $\mathcal{U}$ is cardinally equivalent to $\mathcal{U}^\star$ and $\mathcal{U}' = a\mathcal{U}^\star + b$.

Hence the axioms given above – which, bar two largely technical assumptions, boil down to standard ordering, continuity and uncertainty aversion – characterise the multi evaluation representation of preferences mentioned in the Introduction. Since no monotonicity or independence assumptions are involved (not even independence on risky prospects), this representation is sufficiently rich to capture simultaneous violations of these axioms. In particular, it can accommodate state dependence of utility, imprecision of tastes and imprecision of beliefs, when the preferences exhibit aversion to uncertainty.

Indeed, representation (1) can be seen as a natural development of existing research focussing on some of these violations in isolation from the others. It is a natural extension of the single evaluation representation (where $\mathcal{U}$ is a singleton), which is obtained when one removes the state-independence axioms from the standard axiomatisation of subjective expected utility.[6] Moreover, it is a natural analogue, for uncertainty averse preferences, of recent representations of incomplete preferences obtained by Seidenfeld et al. (1995); Nau (2006); Galaabaatar and Karni (2013), which demand preference only when the expression in (1) is higher for all evaluations in $\mathcal{U}$. Such representations can accommodate imprecision in tastes and state dependence of utility, though not uncertainty aversion of preferences. Finally, representation (1) can also be thought of as a generalisation of the popular Maxmin Expected Utility model (Gilboa and Schmeidler, 1989) in the ambiguity literature, insofar as it involves minimisation over sets of evaluations as opposed to minimisation over sets of probability measures (with a single state-independent utility function). In fact, despite its prima facie resemblance to this model, the use of the minimum over a set of evaluations rather than something more refined is not restrictive: representation (1) generalises a large class of ambiguity models.

This is clear from the proof of the Theorem (see Appendix A.1), where it is shown that the axioms are equivalent to the representation of preferences by a continuous concave functional on $\mathcal{A}$. Since the vast majority of existing uncertainty averse decision the-

[6] This is a straightforward consequence of the von Neumann-Morgenstern theorem in the Anscombe-Aumann framework (Karni et al., 1983). Wakker and Zank (1999); Hill (2010) propose analogous results in the Savage framework.

ories involve continuous concave functionals – including the uncertainty averse Choquet preferences (Schmeidler, 1989), maxmin EU preferences (Gilboa and Schmeidler, 1989), variational preferences (Maccheroni et al., 2006), confidence preferences (Chateauneuf and Faro, 2009), and uncertainty averse smooth preferences (Klibanoff et al., 2005) – they are special cases of representation (1). This also indicates how mild the only new axiom, Coherent calibration (A6), is. To the knowledge of the author, the only uncertainty averse theory that does not fall under representation (1) is the general case of the uncertainty averse preference representation given in Cerreia-Vioglio et al. (2011b).

3.3 An Alternative Formulation

For future reference, it is useful to note that, by standard duality (Rockafellar, 1970), representation (1) is equivalent to the representation of $\preceq$ by the functional

$$V(f) = \min_{U \in \mathcal{N}^1, (a,b) \in \alpha(U)} \left(\sum_{s \in S} a.U(s, f(s)) + b(s) \right) \tag{4}$$

where $\alpha : \mathcal{N}^1 \to 2^{\Re_{>0} \times \Re}$ is a function with the following properties:[7]

- non-triviality: there exists non-constant $U \in \mathcal{N}^1$ such that $\alpha(U) \neq \emptyset$;
- convexity: if $(a, b) \in \alpha(U)$, $(a', b') \in \alpha(U')$ then for all $\lambda \in [0, 1]$, $(\lambda a + (1 - \lambda)a', \lambda b + (1 - \lambda)b') \in \alpha(\lambda U + (1 - \lambda)U')$;
- upper hemicontinuity: if $(a_n, b_n) \in \alpha(U_n)$ with $U_n \to U$ and $(a_n, b_n) \to (a, b)$, then $(a, b) \in \alpha(U)$;
- tightness: there exists no convex upper hemicontinuous α' representing $\preceq$ according to (4) with $\alpha'(U) \subseteq \alpha(U)$ for all $U \in \mathcal{N}^1$ where the inclusion is proper for at least one $U \in \mathcal{N}^1$;
- minimality: for any other tight convex upper hemicontinuous α' representing $\preceq$ according to (4) such that $\{\min_{U \in \mathcal{N}^1, (a,b) \in \alpha'(U)} (\sum_{s \in S} a.U(s, f(s)) + b(s)) \,|\, f \in \mathcal{A}\} = \{\min_{U \in \mathcal{N}^1, (a,b) \in \alpha(U)} (\sum_{s \in S} a.U(s, f(s)) + b(s)) \,|\, f \in \mathcal{A}\}$, and for all $f \in \mathcal{A}$, $U' \in \mathcal{N}^1$

[7] Note that α is not a correspondence: $\alpha(U)$ may be empty for some $U \in \mathcal{N}^1$. We adopt the convention that when $\alpha(U)$ is empty, U is not involved in the minimisation.

and $(a', b') \in \alpha'(U')$, there exists $U \in \mathcal{N}^1$ and $(a, b) \in \alpha(U)$ with $\sum_{s\in S} a.U(s, f(s))+b(s) \leqslant \sum_{s\in S} a'.U'(s, f(s)) + b'(s)$.

In fact, we have the following direct translation between representations (1) and (4):[8]

$$\alpha(U) = \{(a, b) \mid a > 0, b \in \Re,\ aU + b \in \mathcal{U}^\star\} \tag{5}$$

$$\mathcal{U}^\star = \left\{aU + b \mid U \in \mathcal{N}^1,\ (a, b) \in \alpha(U)\right\} \tag{6}$$

where $\mathcal{U}^\star$ is the set of evaluations cardinally equivalent to $\mathcal{U}$ such that $U(s, \hat{x}) = U(t, \hat{x})$ for all $s, t \in S$ and $U \in \mathcal{U}^\star$.[9]

Instead of involving a set of evaluations, representation (4) works with suitably normalised evaluations – elements of $\mathcal{N}^1$. As noted in Section 2, the elements of $\mathcal{N}^1$ suffice to represent preferences according to the version of (1) with a singleton set: any preference relation represented by an evaluation in this way can be represented by a member of $\mathcal{N}^1$. When several evaluations are involved in the representation, the function α determines how they are 'calibrated' for the decision maker (the a and b determine where the zero and unit are). Given the use of the maxmin rule, this in turn determines the 'weight' given to the different evaluations in the assessment of an act.[10]

3.4 Risk preferences

It is useful to remark that, applied in the special case of a state space containing a single state, Theorem 1 yields the following axiomatisation of preferences under risk.

Corollary 1. *Let $\preceq$ be a preference relation on $\Delta(X)$. $\preceq$ satisfies A1–A6 if and only if there exists a non-trivial minimal tight closed convex set of utility functions $\hat{\mathcal{U}} \subseteq \Re^X$ such that $\preceq$ is represented by:*

$$V(c) = \min_{u\in\hat{\mathcal{U}}} u(c) \tag{7}$$

[8]The equivalence of the representations and the properties of α follow directly from this translation.

[9]Recall that $\hat{x}$ is an arbitrary outcome used in the normalisation defining $\mathcal{N}^1$; see Section 2.

[10]This can be formulated alternatively using a set of evaluations-as-a-group analogy: one could consider the elements of $\mathcal{N}^1$ as representing the different possible preferences of members of a group and the α as setting the interpersonal comparison of evaluations among the group members.

Moreover, for any minimal tight closed convex $\hat{\mathcal{U}}' \subseteq \Re^X$ *representing* $\preceq$ *according to (7) there exists* $a \in \Re_{>0}$ *and* $b \in \Re$ *such that* $\hat{\mathcal{U}}' = a\hat{\mathcal{U}} + b$.

Moreover, it follows that, for each $\mathcal{U}$ representing a preference relation $\preceq$ over $\mathcal{A}$ (with a potentially non-singleton state space), one can define a unique 'restriction' of $\mathcal{U}$ which represents the restriction of $\preceq$ to constant acts $\Delta(X)$.

Proposition 1. *Let* $\preceq$ *be a preference relation on* $\mathcal{A}$*. Suppose that* $\preceq$ *satisfies A1–A6, and is represented by a non-trivial minimal tight closed convex* $\mathcal{U} \subseteq \Re^{S\times X}$ *according to (1). Then there exists a unique non-trivial tight closed convex* $\mathcal{U}_{\Delta(X)} \subseteq \Re^X$ *representing the restriction of* $\preceq$ *to constant acts* $\Delta(X)$ *according to (7), such that for each* $u \in \mathcal{U}_{\Delta(X)}$*, there exists* $U \in \mathcal{U}$ *with* $u(x) = \sum_{s\in S} U(s,x)$ *for all* $x \in X$.

Henceforth, for $\mathcal{U}$ representing $\preceq$, we call the set $\mathcal{U}_{\Delta(X)}$ identified in this Proposition the *restriction of* $\mathcal{U}$ *to constant acts* (and continue to use the notation $\mathcal{U}_{\Delta(X)}$).

4 A Tale of Two State Independences

Representation (1) does not provide the basis for consideration and discussion of beliefs and tastes. For this, a representation involving utilities (the standard representations of tastes) and probabilities (which are often interpreted as beliefs) is required. Representation (1) does however provide a springboard for studying such representations: the task is to provide a principled separation of the set of evaluations $\mathcal{U}$ into utilities and probabilities.[11] We shall focus here on representations involving state independence of utility.

Galaabaatar and Karni (2013) provide inspiration. Working with incomplete preferences, they consider a 'dominance' condition that, under their other assumptions, is stronger than the monotonicity axiom standard in the Anscombe-Aumann framework. They show that, in the context of a representation that may be thought of as an analogue of (1) for incomplete preferences, this condition is necessary and sufficient for preferences to be representable as follows: for $f, g \in \mathcal{A}$, $f \prec g$ if and only if

[11] As has been noted in the literature on state-dependent utility, any evaluation can be separated into a (state-dependent) utility and probability in many ways. The challenge is to provide a suitably unique separation. Demanding state-independence of the utility is a possible way of doing so, though it is not the only one.

$$(8) \qquad \sum_{s\in S} p(s)u(f(s)) < \sum_{s\in S} p(s)u(g(s)) \qquad \text{for all } (p,u) \in \mathcal{U}^{si}$$

where $\mathcal{U}^{si}$ is a set of pairs of probability measures over S and state-independent utilities over X. In other words, they study a notion of state independence that involves the separation of *each evaluation* into a state-independent utility and a probability, yielding a set of state-independent utility-probability pairs. This representation involves what we called multi state-independent utility.

In this section, we first characterise uncertainty averse preferences involving multi state-independent utility, and then go on to consider in depth the relationship with and consequences of monotonicity.

4.1 Multi state-independent utility

4.1.1 Basic state-independent representation

We begin by introducing a new axiom. To formulate it, we shall introduce the notion of δ-shift. Firstly, let $\mathcal{ZUR} = \{e \in \Re^X | \; \sum_{x\in X} e(x) = 0, \; |e| = 1\}$. For every $c \in \Delta(X)$, $\delta \in (0,1)$ and $e \in \mathcal{ZUR}$, the *δ-shift of c by e* is $c + \delta e$. If $c + \delta e \in \Delta(X)$ we shall say that that δ-shift exists. A δ-shift from c can be thought of as an act obtained by the addition of a small zero-sum risk, δe, to c. ($\mathcal{ZUR}$ is the set of zero-sum unit risks.) We denote the δ-shift of c by e, if it exists, by $c^{e,\delta}$.

A state $s \in S$ is *locally null* at $f \in \mathcal{A}$ if, for each $e \in \mathcal{ZUR}$, there exists $\epsilon > 0$ such that for all $\delta \in (0,\epsilon]$, $f(s)^{e,\delta}$ exists and $f(s)^{e,\delta}_s f_{s^c} \sim f$. For every $s \in S$, let $Null(s) = cl\left(\{f \in \mathcal{A} \mid s \text{ locally null at } f\}\right)$.[12] $s \in S$ is *locally nonnull* at $f \in \mathcal{A}$ if $f \notin Null(s)$.

Axiom A7 (Local state independence)**.** For all $f \in \mathcal{A}$, $s,t \in S$ locally nonnull at f, and all $e \in \mathcal{ZUR}$ such that $f(s)^{e,\delta}$ and $f(t)^{e,\delta}$ exist for some $\delta > 0$, there exists $\epsilon > 0$ such that $f(s)^{e,\delta}_s f_{s^c} \succeq f$ for all $\delta \leqslant \epsilon$ if and only if $f(t)^{e,\delta}_t f_{t^c} \succeq f$ for all $\delta \leqslant \epsilon$.

This axiom has the spirit of a state-independence condition, to the extent that it demands that adding a small zero-sum risk at a nonnull state of an act will have the same effect on preferences, irrespective of the nonnull state where it is added. It is local, because this

[12] For a set X, $cl(X)$ is the closure of X.

only holds for sufficiently small perturbations to the consequences obtained in the relevant states.

It turns out that this axiom is precisely what is required for a refinement of representation (1) where each evaluation is decomposed (uniquely) into a state-independent utility function and a probability measure.

Theorem 2. *Let $\preceq$ be a preference relation on $\mathcal{A}$. The following are equivalent:*

(i) *$\preceq$ satisfies A1–A6 and A7,*

(ii) *there exists a non-trivial minimal tight closed convex set of utility-probability pairs $\mathcal{U}^{si} \subseteq \Re^X \times \Delta$ such that $\preceq$ is represented by:*

$$V(f) = \min_{(u,p)\in\mathcal{U}^{si}} \sum_{s\in S} p(s)u(f(s)) \tag{2}$$

Moreover, for any minimal tight closed convex $\mathcal{U}^{si\prime} \subseteq \Re^X \times \Delta$ representing $\preceq$ according to (2), there exist $a \in \Re_{>0}$ and $b \in \Re$ such that $\mathcal{U}^{si\prime} = a\mathcal{U}^{si} + b$.[13]

This representation, as well as the uniqueness obtained, can be thought of as the analogue of Galaabaatar and Karni's representation (8) for uncertainty averse preferences. Like theirs, it involves a state-independent separation of beliefs and tastes (probabilities and utilities) evaluation-by-evaluation. The notion of state independence at issue here applies at the level of each utility-probability pair; for this reason, we call (2) a *multi state-independent utility (multi prior) representation*.

4.1.2 Set factorisation

Representation (8) involves a set of state-independent utility-probability pairs, but it does not guarantee that this set can be factorised into a set of state-independent utilities and a set of probabilities. Galaabaatar and Karni (2013) go on to provide a characterisation of what they dub a 'complete separation of beliefs and tastes'. It is possible to provide an analogous extension of Theorem 2 in the case of uncertainty averse preferences.

[13] It is understood in this result and those below that minimality and tightness are defined as stated in Section 2, with the obvious family $\mathcal{F}$; in this case $\mathcal{F}$ is the set of all closed convex subsets of $\Re^X \times \Delta$.

To this end, we formulate a strong version of the Belief consistency axiom proposed by Galaabaatar and Karni (2013). Like them, we shall make use of the following notion, introduced by Ok et al. (2012). For an act $f \in \mathcal{A}$ and a probability measure $p \in \Delta$, let $f^p \in \Delta(X)$ be defined by: $f^p(x) = \sum_{s \in S} p(s) f(s,x)$.[14] Let $\mathcal{P} = \{p \in \Delta \mid \forall f \in \mathcal{A},\ f^p \succeq f\}$.

Axiom A8 (Strong belief consistency)**.** For all $f, g \in \mathcal{A}$, $f \succeq g$ if and only if for all $p \in \mathcal{P}$, there exists $q \in \mathcal{P}$ such that $f^p \succeq g^q$.

Strong belief consistency can be interpreted in a similar way to the Belief consistency axiom proposed by Galaabaatar and Karni (2013). They point out that an act in the Anscombe-Aumann framework can be thought of as a tacit compound lottery, with as probabilities in the first stage the subjective probabilities that implicitly govern behaviour. They go on to interpret f^p as the reduction of this compound lottery (when the subjective probability is p). Applying this to sets of probability measures, one comes naturally to the idea that preferences over acts should correspond to preferences over appropriate reductions, where the reductions are taken with respect to the 'probability distributions that are consistent with preferences', as they call them (in the current case, the set $\mathcal{P}$).

This axiom ensures the desired factorisation into a set of state-independent utility functions and a set of probability measures, as the following result shows.

Theorem 3. *Let $\preceq$ be a preference relation on $\mathcal{A}$. The following are equivalent:*

(i) *$\preceq$ satisfies A1–A6, A7 and A8,*

(ii) *there exists a non-trivial minimal tight pair of closed convex sets of utility functions and probability measures, $\mathcal{U}^{ut} \subseteq \Re^X$ and $\mathcal{C} \subseteq \Delta$, such that $\preceq$ is represented by:*

$$V(f) = \min_{u \in \mathcal{U}^{ut},\ p \in \mathcal{C}} \sum_{s \in S} p(s) u(f(s)) \tag{9}$$

Moreover, for any minimal tight pair of closed convex sets $\mathcal{U}^{ut\prime} \subseteq \Re^X$ and $\mathcal{C}' \subseteq \Delta$ representing $\preceq$, $\mathcal{C}' = \mathcal{C}$ and there exist $a \in \Re_{>0}$ and $b \in \Re$ such that $\mathcal{U}^{ut\prime} = a\mathcal{U}^{ut} + b$.[15]

[14] The reader is referred to the cited papers for extended discussion of the interpretation of this notion.

[15] Minimality and tightness are defined as stated in Section 2, with the obvious family $\mathcal{F}$; in this case $\mathcal{F}$ is the set of all closed convex subsets of $\Re^X \times \Delta$ that can be factorised into the product of a subset of $\Re^X$ and a subset of Δ.

4.1.3 Multi state-independent utility and Monotonicity

Despite its adequacy for characterising representations involving multi state-independent utility, local state independence (A7) is far from standard. The standard state-independence condition in the Anscombe-Aumann framework is the following monotonicity axiom.[16]

Axiom A9 (Monotonicity)**.** For all $f, g \in \mathcal{A}$, if $f(s) \succeq g(s)$ for all $s \in S$, then $f \succeq g$.

The vast majority, if not all, ambiguity theories incorporate this axiom. Moreover, as noted in the Introduction, the vast majority also assume precise tastes. We shall say that a preference relation $\preceq$ represented by a minimal tight closed convex $\mathcal{U}$ according to (1) involves:

- *precise tastes* if $\mathcal{U}_{\Delta(X)}$ is a singleton;
- *rudimentary beliefs* if there exists $E \subseteq S$ and concave real-valued functions $\phi_s : \mathcal{U}(\mathcal{A}) \to \Re$ for all $s \in E$ such that $\preceq$ is represented by:

$$V(f) = \min_{s \in E} \phi_s \left(\min_{u \in \mathcal{U}_{\Delta(X)}} u((f(s)) \right) \tag{10}$$

Preferences involve precise tastes if there is a single utility function (recall from Section 3.4 that $\mathcal{U}_{\Delta(X)}$ is the restriction of $\mathcal{U}$ to constant acts). They involve rudimentary beliefs if there is no probabilistic dimension to beliefs: they are summed up by a set of states, over which the decision maker applies a maxmin rule.

For $c, c' \in \Delta(X)$, $c \prec c'$, let $\preceq_{[c,c']}$ be the restriction of $\preceq$ to $\{f \in \mathcal{A} \mid \forall s \in S,\ c' \preceq f(s) \preceq c\}$. We call $\preceq_{[c,c']}$ a *section* of the preferences. If $\preceq$ can represented according to (1), then any section can also be represented this way. A preference $\preceq$ representable according to (1) exhibits *section-wise precise tastes-rudimentary beliefs* if there exists a sequence $c_0, \ldots, c_n \in \Delta(X)$ with $c_0 \preceq f \preceq c_n$ for all $f \in \mathcal{A}$ and $c_i \prec c_{i+1}$ for each $i < n$, such that for each $i < n$, $\preceq_{[c_i, c_{i+1}]}$ involves precise tastes or rudimentary beliefs.

The following proposition sheds some light on the relationship between multi state-independent utility and monotonicity.

[16] In the presence of complete transitive preferences, monotonicity is implied by the following alternative condition for state independence: for every $f, g \in \mathcal{A}$, $c, d \in \Delta(X)$ and $s, t \in S$, $c_s f_{s^c} \preceq d_s f_{s^c}$ if and only if $c_t g_{t^c} \preceq d_t g_{t^c}$. The main results stated below thus continue to hold for this state-independence condition.

Proposition 2. *Let $\preceq$ be a preference relation on $\mathcal{A}$ represented according to (2). If $\preceq$ satisfies A9, then it exhibits section-wise precise tastes-rudimentary beliefs.*

This result says that whenever a multi state-independent utility representation satisfies monotonicity, it basically involves either a single utility function or non-probabilistic beliefs at every point in the preference order. It can be interpreted as an impossibility result: in the presence of all but the most simple beliefs, imprecision of tastes, multi state-independent utility and the monotonicity axiom are incompatible, when preferences are uncertainty averse.[17] If one takes the idea of imprecise tastes seriously, there is thus a tension between the prevalent notion of state independence for multi utility multi prior representations – according to which each utility in the representing set is state independent – and the standard axiom for state independence – namely monotonicity.

To gain some intuition for the result, consider the following simple example, to which we shall return below. Suppose that there are two states, s and t and two outcomes c and d. Let u_1, u_2 be state-independent utilities with $u_1(c) = 0$, $u_1(d) = 1$, $u_2(c) = 1$, $u_2(d) = 0$ and p be a probability with $p(s) = \frac{1}{4}$. Consider a multi state-independent utility representation (2) by the set $\mathcal{U}^{si}$, which is the convex closure of the following utility-probability pairs: (u_1, p), (u_2, p). It is evident that according to this representation $c \sim d$, but $c_s d_{s^c} \succ c$, contradicting monotonicity.

Proposition 2 suggests that, when tastes are imprecise, one can retain either multi state-independent utility or monotonicity, but not both. We have already considered one of the options; it is time to investigate the other.

4.2 State-independent multi utility

Does monotonicity deliver any form of state independence at all in the context of uncertainty averse preferences with imprecise tastes? The main result of this section provides an answer.

To formulate it, recall representation (4) (Section 3.3), which is equivalent to the basic multi evaluation representation (1) but involves a function from evaluations to sets of pairs, satisfying certain properties. We shall continue to use such functions (albeit on probabilities rather than evaluations), with properties as defined in Section 3.3,[18] and two new properties.

[17] For reasons suggested below, we conjecture that this incompatibility extends to weak ordered continuous preferences exhibiting a larger range of (non-neutral) uncertainty attitudes.

[18] To be precise, all properties are translated without change into the current context ($\mathcal{N}^1$ is replaced by

We shall say that a function $\alpha : \Delta \to 2^{\Re_{>0} \times \Re}$ is *grounded* if there exists $p \in \Delta$ such that $(1, 0) \in \alpha(p)$. It is *calibrated* with respect to a set of utilities $\mathcal{U}^{ut} \subseteq \Re^X$ if, for all $p \in \Delta$, $(a, b) \in \alpha(p)$, and $x \in \mathcal{U}^{ut}(\Delta(X))$,[19] $ax + b \geqslant x$. (Where it is obvious from the context, the set of utilities $\mathcal{U}^{ut}$ will not be mentioned.) Equipped with these notions, we now state the main theorem of this section.

Theorem 4. *Let $\preceq$ be a preference relation on $\mathcal{A}$. The following are equivalent:*

(i) *$\preceq$ satisfies A1–A6 and A9,*

(ii) *there exist a non-trivial minimal tight closed convex set of utility functions $\mathcal{U}^{ut} \subseteq \Re^X$ and a non-trivial, tight, grounded, calibrated, upper hemicontinuous, convex function $\alpha : \Delta \to 2^{\Re_{\geqslant 0} \times \Re}$ such that $\preceq$ is represented by:*

$$V(f) = \min_{p \in \Delta,\ (a,b) \in \alpha(p)} \left(a \sum_{s \in S} p(s) \min_{u \in \mathcal{U}^{ut}} u(f(s)) + b \right) \tag{11}$$

Moreover, for any minimal tight closed convex $\mathcal{U}^{ut\prime} \subseteq \Re^X$ and tight, grounded, calibrated upper hemicontinuous, convex function $\alpha' : \Delta \to 2^{\Re_{\geqslant 0} \times \Re}$ representing $\preceq$, there exists $a \in \Re_{\geqslant 0}$ and $b \in \Re$ such that $\mathcal{U}^{ut\prime} = a\mathcal{U}^{ut} + b$ and $\alpha' = \alpha$.

Monotonicity does imply a separation of tastes for outcomes and beliefs: the former are represented by a set of (state-independent) utilities and the latter are incorporated into a function that only operates at the level of states (and probabilities measures over them). Moreover, this separation does involve a form of state independence: the same set of utility functions is used to assess the consequence of every act in every state. It is this set – this multi utility – that is independent of the state. We shall thus refer to (11) as a *state-independent multi utility (multi prior) representation*.

In order to elucidate the relationship with the notion of multi state-independent utility considered in Section 4.1, it is useful to rewrite representation (11) in the same form as (1). To this end, for any $\mathcal{U}^{ut} \subseteq \Re^X$, let $(\mathcal{U}^{ut})^S = \{u : S \times X \to \Re \mid \forall s \in S,\ u(s, \bullet) \in \mathcal{U}\}$.

Δ, representation (4) by (11) and so on), except for non-triviality, where the requirement that the element is non-constant is removed.

[19] We continue to use the notation introduced in Section 2: $\mathcal{U}^{ut}(\Delta(X)) = \{\min_{u \in \mathcal{U}^{ut}} u(c) \mid c \in \Delta(X)\}$.

$(\mathcal{U}^{ut})^S$ is the set of state-dependent utilities whose restriction to any state coincides with a utility in $\mathcal{U}^{ut}$.[20] Let $\langle(\mathcal{U}^{ut})^S\rangle = \{a\mathcal{U}^S + b \,|\forall a \in \Re_{>0}, b \in \Re\}$.

A set $\mathcal{U} \subseteq \Re^{S\times X}$ is *rectangular* if i. there exist $\mathcal{U}^{ut} \subseteq \Re^X$, $\mathcal{C} \subseteq \Delta$ such that $\mathcal{U} \subseteq \langle(\mathcal{U}^{ut})^S\rangle \times \mathcal{C}$ and ii. for all $p \in \mathcal{C}$ and $U \in (\mathcal{U}^{ut})^S$, if $(p, au + b) \in \mathcal{U}'$ for some $a \in \Re_{>0}$, $b \in \Re$, then $(p, au' + b) \in \mathcal{U}$ for every $u' \in (\mathcal{U}^{ut})^S$. A rectangular set is, in an appropriately generalised sense, a product of a set of probabilities and the set of state-dependent utilities that are 'generated' from a set of utilities. In particular, rectangular sets consist of pairs of probability measures and state-dependent utility functions, with an added constraint that any probability that appears somewhere in the set is matched with every state-dependent utility that appears somewhere.

It is straightforward to show that any preference relation can be represented according to (11) if and only if it can be represented according to (1) by a minimal tight rectangular convex closed set $\mathcal{U}^{mon} \subseteq \Re^{S\times X}$. $\mathcal{U}^{mon}$ is a set of probability-utility pairs, but the utilities will not all be state independent in general. State independence at the level of sets of utilities and probabilities – as embodied in state-independent multi utility representations – thus yields state dependence at the level of probability-utility pairs, except in very special cases. Hence the incompatibility reported in Proposition 2.

The contrast may be illustrated on a continuation of the example given in Section 4.1.3. Using the same setup and notation, consider a state-independent multi utility representation (11) with as set of utilities $\mathcal{U}^{ut}$ the convex closure of $\{u_1, u_2\}$ and with sole probability p.[21] It is straightforward to check that $c \sim d$ and $c_s d_{s^c} \sim c$, as monotonicity demands. This is because, in the assessment of $c_s d_{s^c}$, the value assigned to the consequence c obtained in s is 0 as given by u_1, but the value assigned to the consequence d obtained in t is also 0, following u_2. So a state-dependent utility, which agrees with u_1 in state s and with u_2 in state t, is essentially involved in the assessment of this act under the state-independent multi utility representation. This is the sense in which this representation is state dependent.

On the other hand, the multi state-independent utility representation (2) with $\mathcal{U}^{si}$ (see Section 4.1.3) assigns a value higher than 0 to $c_s d_{s^c}$ because it uses the same state-independent utility to assess c in s and d in t (and when this utility gives a low value to one, it gives a high value to the other). This means in particular that whilst the utility u_1, giving value 0 to c, is used to assign a value to the constant act c, it is not used to assign a value to the

[20] Note that, unless $\mathcal{U}^{ut}$ is a singleton, $(\mathcal{U}^{ut})^S$ will contain at least one state-dependent utility.

[21] More precisely, the α is defined as follows: $\alpha(p) = \{(1, 0)\}$, and $\alpha(q) = \varnothing$ for $q \neq p$.

consequence c at state s in the context of the act $c_s d_{s^c}$. The *set* of utilities involved in the assessment of consequences under the multi state-independent utility representation may depend on the state and the act. In this sense, the representation is state dependent.

Remark 1. Like representation (1), representation (11) generalises many of the main theories of uncertainty averse preferences proposed in the literature (see references in Section 3.2). It does not assume a single utility function – characterised axiomatically by the restriction of independence to risky prospects[22] – whereas, as already mentioned, the vast majority of ambiguity theories do. Moreover, the function α generalises similar 'ambiguity indices' and 'confidence functions' in the variational and confidence models respectively (Maccheroni et al., 2006; Chateauneuf and Faro, 2009). It is relatively straightforward to show that, in the special case of a singleton $\mathcal{U}^{ut}$:

- preferences represented by (11) are variational (Maccheroni et al., 2006) whenever $a = 1$ for all $p \in \Delta$ and $(a, b) \in \alpha(p)$;
- preferences represented by (11) are confidence preferences (Chateauneuf and Faro, 2009) whenever $\min \mathcal{U}^{ut}(\Delta(X)) = 0$ and $b = 0$ for all $p \in \Delta$ and $(a, b) \in \alpha(p)$;[23]
- preferences represented by (11) are maxmin EU preferences (Gilboa and Schmeidler, 1989) when there exists a closed convex $\mathcal{C} \subseteq \Delta$ with $\alpha(p) = \{(1, 0)\}$ for all $p \in \mathcal{C}$ and $\alpha(p) = \varnothing$ for all $p \notin \mathcal{C}$.

Naturally, using the general form of (11) under any of these specifications yields multi utility versions of variational, confidence and maxmin EU preferences respectively. Axiomisation of these special cases involves relatively straightforward translations of the results in the aforementioned papers, combined with the insights in the proof of Theorem 4, and are omitted.

Finally, when there is a single utility function, representation (11) becomes a special case of the 'uncertainty averse preferences' characterised by Cerreia-Vioglio et al. (2011b), which are represented by a functional of the following form:

[22] This axiom is dubbed Risk Independence by Cerreia-Vioglio et al. (2011b,a).

[23] If one removes the condition that $\min \mathcal{U}^{ut}(\Delta(X)) = 0$, one obtains the more general class of homothetic preferences, as Cerreia-Vioglio et al. (2011b) call them. They do not involve Chateauneuf and Faro's assumption of a minimal element.

(12) $$V(f) = \inf_{p\in\Delta} G(\sum_{s\in S} p(s)u(f(s)), p)$$

It is straightforward to see that representation (11) with singleton $\mathcal{U}^{ut}$ corresponds to the special case where $G(t,p) = \min_{(a,b)\in\alpha(p)} at + b$. By duality, it is thus evident that representation (11) with singleton $\mathcal{U}^{ut}$ corresponds to the subclass of Cerreia-Vioglio et al's uncertainty averse preferences where G is concave in the first coordinate. As noted in Section 3.2, all major theories of uncertainty averse preferences, except for the general case of Cerreia-Vioglio et al. (2011b), belong to this class.

5 Related literature

As suggested previously, the results presented here tie into several different literatures. The relationship to some of these literatures has been discussed already: for example, the ambiguity literature in Sections 3.2 and 4.2, the state-dependent utility literature in the Introduction and Section 3.2. The literature on multi utility multi prior representations under incomplete preferences focusses on two of the three issues brought up in the Introduction (imprecise tastes and state dependence), and is discussed at various points in the paper. Beyond Galaabaatar and Karni (2013), whose relation to the work here has been explained in detail, Ok et al. (2012), who characterise a multi state-independent utility single prior representation, is also relevant.

Another related literature involves multi utility-style representations of preferences under risk. Maccheroni (2002) proposes a maxmin multi utility representation of preferences under risk, of which representation (7) is a generalisation. Moreover, Corollary 1 is comparable to the result for convex preferences under risk obtained by Cerreia-Vioglio (2009). Using a conditions reminiscent of our A1–A4, he obtains a representation by the infimum of a function of (suitably normalised) utility functions and the expected utility calculated with these functions. Rewriting (7) in a way similar to (4), it becomes clear that it is a special case of representation (2) in Cerreia-Vioglio (2009), with a specific function of utilities and expected utility values.

A final piece of related literature, at least technically, concerns the existence and uniqueness of representations of preference relations by concave functions. The proof of Theorem

1 draws on a characterisation result provided by Kannai (1977). This contribution and that of Debreu (1976) investigate uniqueness properties that play a role in our results.

6 Conclusion

There are apparently many important real-life decisions in which decision makers have difficulty forming precise probabilities, but also furnishing precise utilities. This paper proposes a theory of uncertainty aversion decision making that incorporates both of these imprecisions. Technical axioms aside, the basic model, which can also accommodate state dependence of utility, results from dropping the independence and monotonicity axioms from standard axiomatisations of subjective expected utility.

The central finding of the paper is the subtlety of the meaning of state independence of utility in this context. If tastes are imprecise, and beliefs sufficiently non-trivial, there turn out to be two *incompatible* notions of state independence of utility. One, which is popular in the burgeoning literature on incomplete preferences, involves sets of probability-utility pairs, and imposes state independence on each pair. The other, which corresponds to a standard state-independence axiom, requires state independence at the level of the sets of utilities used in the assessment of the various acts at the different states. The paper provides axiomatic characterisations of these different notions, thus identifying the behavioural differences between them.

Appendix A Proofs

Throughout the Appendix, $\leqslant$ on $\Re^n$ is the standard order, given by $a \leqslant b$ iff $a_i \leqslant b_i$ for all $1 \leqslant i \leqslant n$, for all $a, b \in \Re^n$. $\cdot$ is the standard scalar product of vectors.

A.1 Proofs of results in Section 3

Proof of Theorem 1. Consider firstly the (i) implies (ii) direction. A1, A2 A3, A4, along with (Cerreia-Vioglio et al., 2011b, Lemma 56) imply that $\preceq$ is a non-degenerate continuous convex complete preference order. A5 implies that, for each f that is not maximal under $\preceq$, there is no non-empty open subset of $\{g \in \mathcal{A} \mid g \sim f\}$.

For every $f_1, f_2, f_3 \in \mathcal{A}$ such that $f_1 \prec f_2 \prec f_3$, let

(13)
$$\alpha(f_1, f_2, f_3) = \sup\left\{\frac{|g_{1\beta}g_3 - g_1|}{|g_3 - g_{1\beta}g_3|} \mid g_i \in \mathcal{A}, \beta \in [0,1],\ g_1 \sim f_1, g_3 \sim f_3, g_{1\beta}g_3 \sim f_2\right\}$$

Note that $\mathcal{A}$ is a convex closed bounded subset of $\Re^{S\times X}$. For any $\hat{x} \in X$, $\mathcal{A}$ is isomorphic to a closed bounded subset of $\Re^{S\times X\backslash\{\hat{x}\}}$ which has non-empty interior.[24] By (Kannai, 1977, Theorem 2.4 and Remark 2.7), the following condition is necessary and sufficient for the existence of a concave continuous functional representing $\preceq$.

Condition 1 There exists a dense sequence of finite sequences of elements of $\mathcal{A}$, $\{f_{i,n}\}$, $0 \leqslant i \leqslant n$, where $f_{0,n} \prec f_{1,n} \prec \cdots \prec f_{n,n}$, $f_{0,n}$ is minimal with respect to $\preceq$ and $f_{n,n}$ is maximal, such that for every $f \in \mathcal{A}$ such that, when n is large enough, there exists $j = j(f,n)$ with $f_{j,n} = f$:[25]

$$\text{(14)} \qquad \lim_{n\to\infty}\left[\sum_{k=1}^{n}\prod_{i=k}^{n-1}\alpha\left(f_{i-1,n}, f_{i,n}, f_{i+1,n}\right)\right]^{-1}\sum_{k=1}^{j}\prod_{i=k}^{n-1}\alpha\left(f_{i-1,n}, f_{i,n}, f_{i+1,n}\right) < 1$$

Claim 1. *Condition 1 holds.*

Proof. Let g be a minimal element with respect to $\preceq$ and h be a maximal element (such elements exists because $\mathcal{A}$ is compact as a subset of $\Re^{S\times X}$ and $\preceq$ satisfies A3). For each $l \in \mathcal{A}$, $g \prec l \prec h$ with $\alpha(g,l,h) < 1$ (note that by definition $\alpha(g,l,h) \neq 0$), define the following sequence by induction.

- $f^l_0 = g$, $f^l_1 = l$
- suppose f^l_i has been defined. If $\alpha(f^l_{i-1}, f^l_i, h) \leqslant 1$, then, by the continuity of $\preceq$, there exists $h' \succ f^l_i$ such that $\alpha(f^l_{i-1}, f^l_i, h') = 1$; in this case, let $f^l_{i+1} = h'$. If $\alpha(f^l_{i-1}, f^l_i, h) > 1$, then stop the construction; f^l_i is the last element of the sequence.

[24] Explicitly: the image of $f \in \mathcal{A}$ is $\hat{f} \in \Re^{S\times X\backslash\{\hat{x}\}}$, where $\hat{f}(s,x) = f(s,x)$ for all $s \in S$, $x \in X\backslash\{\hat{x}\}$.

[25] Here we adopt the same conventions about sums and products as Kannai (1977): the summation over an empty set of indices is equal to 0, and the product is equal to 1. We also use his definition of a dense sequence of sequences: for every $g, h \in \mathcal{A}$ such that $g \prec h$, there exists N such that for every $n \geqslant N$, there exists i with $g \prec f_{i,n} \prec h$.

By construction, $\alpha(f^l_{i-1}, f^l_i, f^l_{i+1}) = 1$ for all i in this sequence. For a given l, let the length of the constructed sequence be n_l, and let $h_l = f^l_{n_l}$. It is straightforward to show that for $f_1 \prec f_2 \prec f'_2 \prec f_3$, $\alpha(f_1, f_2, f_3) \leqslant \alpha(f_1, f'_2, f_3)$ (it suffices to consider the triple where the supremum in (13) is reached; such a triple exists because $\mathcal{A}$ is compact). Moreover, for acts f_1, f_2 and sequences $(f_2)_j$ and $(f_3)_j$ with $(f_2)_j \to f_2$, if $f_1 \prec (f_2)_j, f_2 \prec (f_3)_j$ and $\alpha(f_1, (f_2)_j, (f_3)_j) = 1$ for all j, then by the continuity of $\preceq$ and the fact that $\mathcal{A}$ is compact, there exists f_3 and $(f_3)'_j \sim (f_3)_j$ for all j such that $(f_3)'_j \to f_3$ and $\alpha(f_1, f_2, f_3) = 1$. Hence it follows from the construction that, for any l_j, l_k, if $l_j \prec l_k$ then $n_{l_j} \geqslant n_{l_k}$ and if $l_j \prec l_k$ and $n_{l_j} = n_{l_k} = n$, then $f^{l_j}_n \prec f^{l_k}_n$. Moreover, if $l_j \to l$, then $f^{l_j}_i \to f^l_i$ for each i where this element exists.

Take any $n \geqslant 2$, and take any l such that $n_{l'} \geqslant n$ for all $l' \prec l$ and $n_{l'} < n$ for all $l' \succ l$; it is clear from the construction that such l exists. By the continuity property mentioned above, since l is a the limit of a sequence l_j with $l_j \prec l$ and $n_{l_j} \geqslant n$ for all j, $n_l \geqslant n$. By the continuity property again, for any sequence l_j such that $l_j \succ l$ and $n_{l_j} \leqslant n-1$ for all j, and $l_j \to l$, $\alpha(f^{l_j}_{n-2}, f^{l_j}_{n-1}, h) \to \alpha(f^l_{n-2}, f^l_{n-1}, h)$. So if $\alpha(f^l_{n-2}, f^l_{n-1}, h) < 1$, then there exists $l_j \succ l$ such that $\alpha(f^{l_j}_{n-2}, f^{l_j}_{n-1}, h) < 1$, and hence, by construction, there exists an element $f^{l_j}_n$, contradicting the assumption that $n_{l'} < n$ for all $l' \succ l$. Hence $\alpha(f^l_{n-2}, f^l_{n-1}, h) = 1$, and $h \sim f^l_n$. Hence, for any $n \geqslant 2$, there exists a sequence $f_{0,n}, \ldots f_{n,n}$ of length n with $f_{0,n} = g$, $f_{n,n} = h$ and $\alpha\left(f_{i-1,n}, f_{i,n}, f_{i+1,n}\right) = 1$ for all $1 \leqslant i \leqslant n-1$. Take a sequence of such sequences, $\{f_{i,n}\}$.

Now we show that this sequence is dense in $\preceq$. Without loss of generality, we may assume that $g \prec h_\alpha g \prec h$ for all $\alpha \in (0,1)$ (if not, replace g with the maximal $h_\beta g$ such that $h_\beta g \preceq g$, and similarly for h; such acts exist by the continuity of $\preceq$). For each sequence $(f_{i,n})_{0 \leqslant i \leqslant n}$ in the sequence of sequences, and for each $0 \leqslant i \leqslant n$, let $\alpha_{i,n} = \min\{\beta \in [0,1] \mid h_\beta g \succeq f_{i,n}\}$ (the minimum exists by the continuity of $\preceq$). $(\alpha_{i,n})_{0 \leqslant i \leqslant n}$ is a strictly increasing sequence with $\alpha_{0,n} = 0$ and $\alpha_{1n} = 1$. We claim that, for each $\delta > 0$, there exists $N > 0$ such that for all $n \geqslant N$, $\alpha_{i+1,n} - \alpha_{i,n} \leqslant \delta$ for all $0 \leqslant i \leqslant n-1$; it follows straightforwardly that the sequence is dense. (For any $f, f' \in \mathcal{A}$ such that $f \prec f'$, take $\alpha' = \min\{\beta \in [0,1] \mid h_\beta g \succeq f'\}$ and $\alpha = \max\{\beta \in [0,1] \mid \beta \leqslant \alpha',\ h_\beta g \preceq f\}$: it follows from the claim there exists $N > 0$ such that for all $n \geqslant N$, $\alpha_{i+1,n} - \alpha_{i,n} < \alpha' - \alpha$ for all $0 \leqslant i \leqslant n-1$, so there exists $\alpha < \alpha_{i,n} < \alpha'$ and thus $f \prec f_{i,n} \prec f'$.) To establish the claim, note firstly that, for all $1 \leqslant i \leqslant n-1$ and all n, since $\alpha\left(f_{i-1,n}, f_{i,n}, f_{i+1,n}\right) = 1$,

$\alpha_{i,n} - \alpha_{i-1,n} \leqslant \alpha_{i+1,n} - \alpha_{i,n}$. So it suffices to show that, for every $\delta > 0$ there exists $N > 0$ such that, for all $n \geqslant N$, $\alpha_{n,n} - \alpha_{n-1,n} \leqslant \delta$. Suppose that this is not the case, and that $\alpha_{n,n} - \alpha_{n-1,n} \geqslant \epsilon > 0$ for all sufficiently large n. Consider $h_{1-\epsilon}g$; since $\epsilon > 0$, $g \prec h_{1-\epsilon}g \prec h$. By the assumption, for every sequence $(f_{i,n})_{0\leqslant i\leqslant n}$, $\frac{\max_{f_{j,n}\preceq h_{1-\epsilon}g} j}{n} = \frac{n-1}{n}$. So $\frac{\max_{f_{j,n}\preceq h_{1-\epsilon}g} j}{n} \to 1$, contradicting A6. Hence for every $\delta > 0$ there exists $N > 0$ such that, for all $n \geqslant N$, $\alpha_{n,n} - \alpha_{n-1,n} \leqslant \delta$, and the sequence of sequences $\{f_{i,n}\}$ is dense as required.

For $f_1 \prec f_2 \prec f_3$, since $\mathcal{A}$ is compact, the supremum in (13) is obtained; let $f_1' \sim f_1$, $f_3' \sim f_3$ and $(f_1')_\beta f_3' \sim f_3$ be a triple where the supremum is attained. Since $\frac{|(f_1')_\beta f_3' - f_1'|}{|f_3' - (f_1')_\beta f_3'|} = \frac{\beta |f_3' - f_1'|}{(1-\beta)|f_3' - f_1'|} = \frac{\beta}{(1-\beta)}$, if $\alpha(f_1, f_2, f_3) = 1$, then $\beta = \frac{1}{2}$. It follows that, if $\alpha(f_1, f_2, f_3) = 1$, then f_2 is a conservative potential fifty-fifty mixture of f_1 and f_3. Hence, for any sequence in the sequence of sequences $\{f_{i,n}\}$ defined above, $f_{i,n}$ is a conservative potential fifty-fifty mixture of $f_{i-1,n}$ and $f_{i+1,n}$ for all $0 \leqslant i \leqslant n$. For each such sequence and each $f = f_{j,n}$, $\sum_{k=1}^{j} \prod_{i=k}^{n-1} \alpha\left(f_{i-1,n}, f_{i,n}, f_{i+1,n}\right) = j = \max_{f_{i,n}\preceq f} i$ and $\sum_{k=1}^{n} \prod_{i=k}^{n-1} \alpha\left(f_{i-1,n}, f_{i,n}, f_{i+1,n}\right) = n$. So, if f is such that there exists j for each n with $f = f_{j,n}$, the expression in the limit in (14) is $\frac{\max_{f_{i,n}\preceq f} i}{n}$. A6 implies that this expression is bounded away from 1, so the limit in (14) is less than 1, as required.

□

By Kannai (1977, Theorem 2.4 and Remark 2.7), there exists a concave continuous non-constant functional $V : \mathcal{A} \to \Re$ representing $\preceq$. In fact, as noted by (Kannai, 1977, p11), there exists a minimally concave functional (see also Debreu (1976)) in following sense: for any concave functional V' representing $\preceq$ such that $V'(h) = V(h)$ and $V'(g) = V(g)$ where g and h are minimal and maximal elements with respect to $\preceq$ respectively, $V'(f) \geqslant V(f)$ for all $f \in \mathcal{A}$. We henceforth assume that V is minimal in this sense.

Let $\hat{x}$ be a generic element of X. An evaluation $U \in \Re^{S\times X}$ is canonical if $U(s, \hat{x}) = 0$ for all $s \in S$. Let $\mathcal{CE}$ be the set of canonical evaluations. Note that for each evaluation, there exists a cardinally equivalent canonical evaluation.

Since $\mathcal{A}$ is a closed bounded convex subset of a finite-dimensional space, and V is concave and finite, for each $f \in ri(\mathcal{A})$,[26] there exists a canonical evaluation supporting V at f, ie. $U \in \mathcal{CE}$ such that $U \cdot g \geqslant V(g)$ for all $g \in \mathcal{A}$ and $U \cdot f = V(f)$. Let $\mathcal{U}$ be convex closure of the set of all canonical evaluations supporting V at some $f \in$

[26] For a set X, $ri(X)$ is the relative interior of X.

$ri(\mathcal{A})$. By construction, $V(f) = \min_{U \in \mathcal{U}} U \cdot f$ for all $f \in ri(\mathcal{A})$; by the continuity of V, this establishes representation (1). By construction $\mathcal{U}$ is closed and convex. Since there is a dense subset of $ri(\mathcal{A})$ on which V is differentiable – and hence at which it has unique supergradients – and the supergradients at all other points are contained in the convex closure of the supergradients at points where V is differentiable (Rockafellar, 1970, Theorems 25.5 and 25.6), $\mathcal{U}$ is tight. That $\mathcal{U}$ is minimal follows from the minimality of V and the following lemma.

Lemma 1. *Let V be a concave continuous function representing $\preceq$, and let $\mathcal{U} = cl(conv\{U \in \mathcal{CE} \mid \forall g \in \mathcal{A}\; U(g) \geqslant V(g),\; \exists f \in \mathcal{A},\; U(f) = V(f)\}))$.*[27] *Then $\mathcal{U}$ is a minimal (as a tight closed convex set representing $\preceq$) if and only if V is minimal (as a concave representation).*

Proof. Suppose that V is a minimally concave. Consider any tight closed convex $\mathcal{U}'$ representing $\preceq$ according to (1) such that $\mathcal{U}'(\mathcal{A}) = \mathcal{U}(\mathcal{A})$, and let $V'(f) = \min_{U \in \mathcal{U}'} U \cdot f$ for all $f \in \mathcal{A}$. Let g, h be minimal and maximal elements of $\mathcal{A}$ with respect to $\preceq$ (such elements exist by A3 and the fact that $\mathcal{A}$ is compact); it follows from the assumptions that $V'(g) = V(g)$ and $V'(h) = V(h)$. So, by the minimality of V, $V'(f) \geqslant V(f)$ for all $f \in \mathcal{A}$. Consider any $f \in \mathcal{A}$ and $U' \in \mathcal{U}'$. By the representation, $U' \cdot f \geqslant V'(f)$. However, by the representation, there exists $U \in \mathcal{U}$ such that $U \cdot f = V(f) \leqslant V'(f)$, so $U' \cdot f \geqslant U \cdot f$. Since this holds for every $f \in \mathcal{A}$ and $U' \in \mathcal{U}'$, $\mathcal{U}$ is minimal.

Now suppose that $\mathcal{U}$ is minimal. Let g, h be minimal and maximal elements of $\mathcal{A}$ with respect to $\mathcal{A}$. Let V' be any concave continuous functional representing $\preceq$ with $V'(g) = V(g)$ and $V'(h) = V(h)$, and let $\mathcal{U}' = cl\left(conv\left(\{U \in \mathcal{CE} \mid \forall g \in \mathcal{A}\; U(g) \geqslant V(g),\; \exists f \in \mathcal{A},\; U(f) = V'(f)\}\right)\right)$. Noting that $\mathcal{U}'(\mathcal{A}) = V'(\mathcal{A}) = V(\mathcal{A}) = \mathcal{U}(\mathcal{A})$, and that $\mathcal{U}'$ is a tight closed convex set representing $\preceq$, minimality of $\mathcal{U}$ implies that for each $U' \in \mathcal{U}'$ and $f \in \mathcal{A}$, there exists $U \in \mathcal{U}$ with $U' \cdot f \geqslant U \cdot f$. Hence $\min_{U' \in \mathcal{U}'} U' \cdot f \geqslant \min_{U \in \mathcal{U}} U \cdot f$ for all $f \in \mathcal{A}$, so $V'(f) \geqslant V(f)$ for all $f \in \mathcal{A}$. Hence V is minimally concave, as required.

□

Necessity The implication (ii) to (i) is standard for all axioms except A5 and A6. Suppose that $\preceq$ is represented by $\mathcal{U}$ according to (1) and let $V(f) = \min_{U \in \mathcal{U}} U \cdot f$ for all $f \in \mathcal{A}$.

For A5, consider any $f, g \in \mathcal{A}$ with $g \succ f$. By the representation, for all $U \in \mathcal{U}$, $U \cdot f \geqslant V(f)$ and $U \cdot g \geqslant V(g)$, so $U \cdot f_\alpha g \geqslant \alpha V(f) + (1-\alpha)V(g)$. Since $g \succ f$, it

[27] For a set X, $cl(X)$ is its closure and $conv(X)$ its convex hull.

follows that $g_\alpha f \succ f$ for all $\alpha \in (0, 1]$, as required.

As concerns A6, note that for any finite sequence $\{f_i\}$ of the sort described in the axiom, $\frac{\max_{f_i \preceq f} i}{n} < 1$, so it needs to be shown that there do not exist any sequence of such sequences $\{f_{i,n}\}$, each of length n, with $\frac{\max_{f_{i,n} \preceq f} i}{n} \to 1$ as $n \to \infty$. By Kannai (1977, Theorem 2.4 and Remark 2.7), if $\preceq$ is representable by a concave functional, then for all $g \prec f \prec h$ and for all dense sequences of sequences $\{f_{i,n}\}$ from g to h satisfying the properties described in Condition 1, (14) holds. By the argument in the proof of Claim 1, for any $g \prec h$, any sequence of sequences $\{f_{i,n}\}$ with $f_{i,n}$ being a conservative potential fifty-fifty mixture of $f_{i-1,n}$ and $f_{i+1,n}$ for all consecutive triples of elements is a dense sequence of sequences from g to h; hence (14) holds. Since, as noted in the proof above, the expression in (14) simplifies to $\frac{\max_{f_{i,n} \preceq f} i}{n}$ in the case of sequences involving conservative potential fifty-fifty mixes, it follows that $\frac{\max_{f_{i,n} \preceq f} i}{n} \nrightarrow 1$. So A6 holds.

Uniqueness Suppose that $\mathcal{U} \neq \mathcal{U}'$ are non-trivial minimal tight closed convex sets representing $\preceq$. Let g, h be minimal and maximal elements of $\preceq$ respectively. Let $V(f) = \min_{U \in \mathcal{U}} U \cdot f$ and $V'(f) = \min_{U' \in \mathcal{U}'} U' \cdot f$ for all $f \in \mathcal{A}$. Let $a \in \Re_{>0}$ and $b \in \Re$ be such that $aV'(g) + b = V(g)$ and $aV'(h) + b = V(h)$ (it is straightforward to show that such a and b exist), and let $\mathcal{U}'' = a\mathcal{U}' + b$ and $V'' = aV' + b$. If $V = V''$, then since $\mathcal{U}$ is the convex closure of the set of supergradients of V, for each member of $\mathcal{U}$, there exists a cardinally equivalent member of $\mathcal{U}''$. Since both sets are tight, it follows that $\mathcal{U}$ and $\mathcal{U}''$ are cardinally equivalent. It thus suffices to show that $V'' = V$. By the minimality of $\mathcal{U}$ and Lemma 1, $V''(f) \geqslant V(f)$ for all $f \in \mathcal{A}$. By the minimality of $\mathcal{U}'$ and Lemma 1, $V(f) \geqslant V''(f)$ for all $f \in \mathcal{A}$. Hence $V'' = V$, and $\mathcal{U} = a\mathcal{U}' + b$ as required.

□

Proof of Proposition 1. Let $\mathcal{U}$ represent $\preceq$, and let $V(f) = \min_{U \in \mathcal{U}} U \cdot f$ for all $f \in \mathcal{A}$. Let $\preceq_{\Delta(X)}$ be the restriction of $\preceq$ to $\Delta(X)$. For each $c \in \Delta(X)$, there exists a $U \in \mathcal{U}$ such that U supports V at c. Evidently, $\sum_{s \in S} U(s, \bullet)$ supports $V|_{\Delta(X)}$ at c. Let $\mathcal{U}_{\Delta(X)} = cl(conv(\{\sum_{s \in S} U(s, \bullet) \mid \exists c \in \Delta(X) \; s.t. \; U \text{ supports } V \text{ at } c\}))$. Note that, since $\mathcal{U}$ is closed and convex, if $u \in \mathcal{U}_{\Delta(X)}$, there exists $U \in \mathcal{U}$ such that $u = \sum_{s \in S} U(s, \bullet)$. $\mathcal{U}_{\Delta(X)}$ is closed and convex; by arguments similar to those used in the proof of Theorem 1, it is tight. Uniqueness follows by a similar argument to that used in the proof of Theorem 1.

□

A.2 Proofs of Results in Section 4

Proof of Theorem 2. First consider the (ii) to (i) direction. The necessity of all axioms apart from A7 follows from Theorem 1. As concerns this axiom, let $\mathcal{U}^{si}$ represent $\preceq$ according to (2) and let $V(f) = \min_{(u,p)\in\mathcal{U}^{si}}(u.p)\cdot f$ for all $f \in \mathcal{A}$.

First note that if $s \in S$ is locally null at $g \in \mathcal{A}$ and there is a unique $(u,p) \in \mathcal{U}^{si}$ supporting V at g, then since $(u.p)\cdot(g(s)+\delta e)_s g_{s^c} = \delta p(s)u\cdot e + (u.p)\cdot g = (u.p)\cdot g$ for all $e \in \mathcal{ZUR}$ and $\delta > 0$ sufficiently small, either $p(s) = 0$ or u is constant. By the closure properties of supergradients (Rockafellar, 1970, Theorem 25.6), $Null(s)$ contains all and only acts g where there exists a $(u,p) \in \mathcal{U}^{si}$ supporting V at g with $p(s) = 0$ or u constant. In particular, if $f \notin Null(s)$, then by the continuity and closure properties of supergradients (Rockafellar, 1970, Theorem 25.5 and 25.6), any $(u,p) \in \mathcal{U}^{si}$ supporting V at f is such that $p(s) \neq 0$ and u is not constant. So if s is locally nonnull at f then for every $(u,p) \in \mathcal{U}^{si}$ supporting V at f, $p(s) \neq 0$ and u is not constant.

Consider $f \in \mathcal{A}$ and $s,t \in S$ locally nonnull at f. By the previous observation, for every $(u,p) \in \mathcal{U}^{si}$ supporting V at f, $p(s) \neq 0$ and $p(t) \neq 0$. For $e \in \mathcal{ZUR}$ and $\epsilon > 0$ small enough, $f(s)^{e,\delta}_s f_{s^c} \succeq f$ for all $\delta \leqslant \epsilon$ if and only if $(u.p)\cdot(f(s)+\delta e)_s f_{s^c} = \delta p(s)u\cdot e + (u,p)\cdot f \geqslant (u.p)\cdot f$ for all $\delta \leqslant \epsilon$ and every (u,p) supporting f at V. This holds if and only if $u\cdot e \geqslant 0$ for every (u,p) supporting f at V. Similarly, $f(t)^{e,\delta}_t f_{t^c} \succeq f$ for all $\delta \leqslant \epsilon$ for some ϵ small enough, if and only if $u\cdot e \geqslant 0$ for every (u,p) supporting f at V. Hence there exists $\epsilon > 0$ such that $f(s)^{e,\delta}_s f_{s^c} \succeq f$ for all $\delta \leqslant \epsilon$ if and only if $f(t)^{e,\delta}_t f_{t^c} \succeq f$ for all $\delta \leqslant \epsilon$, and A7 is satisfied.

Now consider the (i) to (ii) implication. By Theorem 1, there is a representation of $\preceq$ by a non-trivial minimal tight closed convex $\mathcal{U} \subseteq \Re^{S\times X}$ according to (1). We will say that $U \in \Re^{S\times X}$ decomposes into $(u,p) \in \Re^X \times \Delta$ if, for all $g \in \mathcal{A}$, $\sum_{s\in S}\sum_{x\in X} U(s,x)g(s,x) = \sum_{s\in S}\sum_{x\in X} p(s)u(x)g(s,x)$. It suffices to show that every $U \in \mathcal{U}$ decomposes into a state-independent utility-probability pair.

Let $V(f) = \min_{U\in\mathcal{U}} U\cdot f$ for all $f \in \mathcal{A}$. For each $f \in \mathcal{A}$ and $s \in S$, let $\Delta(X)_{f,s} = \{c_s f_{s^c} \mid c \in \Delta(X)\}$. Let $UCS_{f,s} = \{g \in \Delta(X)_{f,s} \mid V(g) \geqslant V(f)\}$, the upper contour set of f in $\Delta(X)_{f,s}$. Finally, for any affine functional $\phi : \Re^X \to \Re$, we denote, with slight abuse of notation, the corresponding functional on $\Delta(X)_{f,s}$ – that is, the functional taking the value $\phi(c)$ on $c_s f_{s^c}$ – by ϕ.

We first establish the following lemma.

Lemma 2. *For every $f \in ri(\mathcal{A})$ and every $s, t \in S$ locally nonnull at f, if $\{g \mid \phi(g) = \Gamma\}$ is a hyperplane supporting $UCS_{f,s}$ in $\Delta(X)_{f,s}$ at f, then there exists Γ' such that $\{g \mid \phi(g) = \Gamma'\}$ supports $UCS_{f,t}$ in $\Delta(X)_{f,t}$ at f.*

Proof. Let $UCS^o_{f,s} = \{d - f(s) \in \Re^X \mid V(d_s f_{s^c}) \geqslant V(f)\}$ be the mapping of the upper contour set of f in $\Delta(X)_{f,s}$ into $\Re^X$ with the vertex shifted to 0 and let $cone(UCS^o_{f,s})$ be the cone generated by it. Note that by A4, if $a \in UCS^o_{f,s}$ then $\beta a \in UCS^o_{f,s}$ for every $\beta \in (0, 1)$.

Claim 2. *For every $f \in ri(\mathcal{A})$ and every $s, t \in S$ locally nonnull at f, $cone(UCS^o_{f,s}) = cone(UCS^o_{f,t})$.*

Proof. Suppose, for reductio, that $cone(UCS^o_{f,s}) \neq cone(UCS^o_{f,t})$ and, without loss of generality, that $a \in cone(UCS^o_{f,s}) \backslash cone(UCS^o_{f,t})$. Since $f \in ri(\mathcal{A})$, there exists $\epsilon > 0$ such that, for $s' = s, t$, $(\epsilon a + f(s'))_{s'} f_{s'^c} \in \mathcal{A}$. By the definition of $cone(UCS_{f,s})$, there exists $\epsilon' < \epsilon$ such that $(\delta a + f(s))_s f_{s^c} \succeq f$ for all $\delta \leqslant \epsilon'$. A7 implies that $(\delta a + f(t))_t f_{t^c} \succeq f$ for all $\delta \leqslant \epsilon''$, for some $\epsilon'' > 0$, so $a \in cone(UCS^o_{f,t})$, contradicting the assumption. So $cone(UCS^o_{f,s}) = cone(UCS^o_{f,t})$ as required. □

Since the cones coincide, a hyperplane will support $UCS^o_{f,s}$ if and only if it supports $UCS^o_{f,t}$. Since any hyperplane supporting $UCS_{f,s}$ is a translation of one supporting $UCS^o_{f,s}$, this is sufficient to establish the desired lemma. □

We now distinguish two cases.

Firstly, consider $f \in ri(\mathcal{A})$ such that V is differentiable at f, so there is a unique $U \in \mathcal{U}$ supporting V at f. If U is a constant evaluation, then it trivially decomposes; suppose henceforth that this is not the case. Consider $s \in S$. If $U(s, \bullet)$ is constant (ie. $U(s, x) = U(s, x')$ for all $x, x' \in X$), then $U \cdot (f(s) + \delta e)_s g_{s^c} = \delta U(s, e) + U \cdot f = U \cdot f$ for all $e \in \mathcal{ZUR}$ and δ sufficiently small. Since U is the unique support for V at f, this implies that s is locally null at f. If $U(s, \bullet)$ is non-constant, then by similar reasoning, the fact that U is the unique support at f and the continuity of the superdifferential mapping (Rockafellar, 1970, Theorem 24.4), s is locally nonnull at f. Note that, since U is a non-constant evaluation, there exists a cardinally equivalent U' such that $U'(s, x) = 0$ for all $x \in X$ and all s such that $U'(s, \bullet)$ is constant; we henceforth assume without loss of generality that $U(s, \bullet) = 0$ for all $s \in S$ such that $U'(s, \bullet)$ is constant. Now consider $s \in S$ such that $U(s, \bullet)$ is non-constant. Note that since V is differentiable at f, its restriction to $\Delta(X)_{f,s}$ is differentiable at f, so the hyperplane defined by U, $\{g \in \Delta(X)_{f,s} \mid U(g) = V(f)\}$ is the unique support for $UCS_{f,s}$. Note that this hyperplane can be written as $\{c_s f_{s^c} \mid c \in \Delta(X),\ U(s,c) = \Gamma\}$, for some constant Γ. For every other $t \in S$ such that $U(t, \bullet)$ is non-constant, s and t are locally nonnull, so by Lemma 2, $\{c_t f_{t^c} \mid c \in \Delta,\ U(s,c) = \Gamma'\}$ supports $UCS_{f,t}$ at f for some Γ'. Since, by the previous argument, $\{c_t f_{t^c} \mid c \in \Delta,\ U(t,c) = \Gamma''\}$ also supports $UCS_{f,t}$ for some Γ'', and $UCS_{f,t}$ has a unique supporting hyperplane at f, it follows that $U(t,x) = aU(s,x) + b$ for some $a \in \Re_{>0}$ and $b \in \Re$. This implies that, for each $s \in S$ such that $U(s, \bullet)$ non-constant, there exists $a^s \in \Re_{>0}$ and $b^s \in \Re$ such that $U(s,x) = a^s \sum_{s' \in S} U(s',x) + b^s$. Taking $u(x) = \sum_{s' \in S} U(s',x)$ for all $x \in X$ and $p(s) = \frac{a^s}{\sum_{s' \in S} a^{s'}}$ for s such that $U(s, \bullet)$ non-constant and $p(s) = 0$ otherwise, it is straightforward to show that, for all $g \in \mathcal{A}$, $\sum_{s \in S} \sum_{x \in X} U(s,x) g(s,x) = \sum_{s \in S} \sum_{x \in X} p(s) u(x) g(s,x)$, so U decomposes into (u,p).

Now consider any $f \in ri(\mathcal{A})$ that does not fall into the above case. Note that the set of points where V is differentiable is dense in $ri(\mathcal{A})$ (Rockafellar, 1970, Theorem 25.5), and at each of these points, by the previous case, the supporting U is decomposable into a state-independent utility and probability. Hence, there exists a sequence $(f_i)_{i \geqslant 1}$ with $f_i \to f$ such that, for each $i \geqslant 1$ there exist unique U_i supporting V at f_i, for which $U_i(s, \bullet)$ and $U_i(t, \bullet)$ are positive affine transformations of one another for all $s, t \in S$ for which they are not constant. Hence U, the limit of U_i, supports V at f (Rockafellar, 1970, Theorem 24.4) and is such that $U(s, \bullet)$ and $U(t, \bullet)$ are positive affine transformations of one another for all $s, t \in S$ for which they are not constant; so, by the previous argument, U decomposes into a state-independent-probability pair. Moreover, every support (or supergradient) of V at f is a convex combination of such limits of sequences (Rockafellar, 1970, Theorem 25.6); hence every U supporting V at f decomposes into a state-independent utility-probability pair.

Hence every $U \in \mathcal{U}$ supporting V at a point in $ri(\mathcal{A})$ decomposes into a state-independent utility-probabilty pair. By the tightness of $\mathcal{U}$, $\mathcal{U}$ is the convex closure of the set of such U. By the argument above, it follows that every element of $\mathcal{U}$ decomposes into a state-independent utility-probabilty pair, as required. Let $\mathcal{U}^{si}$ be the set of such pairs, for all $U \in \mathcal{U}$. By construction, this set represents $\preceq$ according to (2) and is minimal tight closed and convex.

Uniqueness follows immediately from the uniqueness clause in Theorem 1 and the decomposition into state-independent utility-probability pairs.

□

Proof of Theorem 3. The (ii) to (i) direction is straightforward; consider the (i) to (ii) implication. By Theorem 2, there is a representation of $\preceq$ by a minimal tight convex closed $\mathcal{U}^{si} \subseteq \Re^X \times \Delta$ according to (2). Let $V(f) = \min_{(u,p)\in\mathcal{U}^{si}}(u.p) \cdot f$, for all $f \in \mathcal{A}$. Let $\mathcal{U}^{ut} = cl\left(conv\{u \in R^X \mid (u,p) \in \mathcal{U}^{si},\ \exists c \in \Delta(X)\ s.t.\ u \text{ supports } V|_{\Delta(X)} \text{ at } c\}\right)$, and $\mathcal{C} = cl\left(conv\{p \in \Delta \mid \forall f \in \mathcal{A}\ f^p \succeq f\ ,\ \exists g \in ri(\mathcal{A})\ s.t.\ g^p \sim g\}\right)$. We will show that $\mathcal{U}^{ut}$, $\mathcal{C}$ represent $\preceq$ according to (9).

For any act $f \in \mathcal{A}$, let $c_f \in \Delta(X)$ be the minimal constant act (in the order $\preceq$) of the form f^p for some $p \in \mathcal{C}$. It follows from A8 that, for all $f, g \in \mathcal{A}$, $c_f \succeq c_g$ if and only if $f \succeq g$. Since V represents $\preceq$, it follows that $V(f) \geqslant V(g)$ if and only if $V(c_f) \geqslant V(c_g)$. By the definition of c_f and $\mathcal{U}^{ut}$, for every $f, g \in ri(\mathcal{A})$, $V(c_f) \geqslant V(c_g)$ if and only if $\min_{u\in\mathcal{U}^{ut}} \min_{p\in\mathcal{C}}(u.p) \cdot f \geqslant \min_{u\in\mathcal{U}^{ut}} \min_{p\in\mathcal{C}}(u.p) \cdot g$; by the continuity of V, this equivalence continues to hold for all $f, g \in \mathcal{A}$. This yields the desired representation.

By Proposition 1 and its proof, $\mathcal{U}^{ut}$ is a tight representation of the restriction of $\preceq$ to $\Delta(X)$. In particular, no proper closed convex subset of $\mathcal{U}^{ut}$ represents the restriction of $\preceq$ to $\Delta(X)$; so, to show that $\mathcal{U}^{ut}$ and $\mathcal{C}$ are tight, it suffices to show that there is no proper closed convex subset of $\mathcal{C}$ which, taken with $\mathcal{U}^{ut}$, represents $\preceq$. If $(u,p) \in \mathcal{U}^{ut} \times \Delta$ supports V at $f \in ri(\mathcal{A})$, then $g^p \succeq g$ for all $g \in A$ and $f^p \sim f$. If moreover V is differentiable at f, then (u,p) is the unique support for V at f. Hence, for all other $p' \in \Delta$ such that $g^{p'} \succeq g$ for all $g \in \mathcal{A}$, $f^{p'} \succ f$: if not, there would be another supergradient at f, contradicting the assumption that V is differentiable there. Let $\mathcal{D} = \{p \in \Delta \mid \exists f \in ri(\mathcal{A}), V \text{ differentiable at } f, \exists u \in \mathcal{U}^{ut}\ s.t.\ (u,p) \text{ supports } V \text{ at } f\}$; as just noted, $\mathcal{D} \subseteq \mathcal{C}$. If $(u,p) \in \mathcal{U}^{ut} \times \Delta$ supports V at $f \in ri(\mathcal{A})$ where V is not differentiable, $p \in cl\,(conv(\mathcal{D}))$ (Rockafellar, 1970, Theorem 25.6). Since, for any $p \in \Delta$ such that $g^p \succeq g$ for all $g \in A$ and $f^p \sim f$ for some $f \in ri(\mathcal{A})$, there exists $u \in \mathcal{U}^{ut}$ such that (u,p) supports V at f, it follows that $\mathcal{C} = cl\,(conv(\mathcal{D}))$. So there is no convex closed proper subset of $\mathcal{C}$ representing $\preceq$ in tandem with $\mathcal{U}^{ut}$ according to (9). So the pair $\mathcal{U}^{ut}$ and $\mathcal{C}$ is tight. Minimality follows from the minimality of $\mathcal{U}^{si}$.

As concerns the uniqueness clause, let $\mathcal{U}^{ut\prime}$ and $\mathcal{C}'$ be another pair representing $\preceq$ according to (9). Since $\mathcal{U}^{ut\prime}$ and $\mathcal{U}^{ut}$ both represent the restriction of $\preceq$ to $\Delta(X)$, and are minimal tight convex closed sets, it follows from the uniqueness clause in Theorem 1 (see also Corollary 1) that $\mathcal{U}^{ut\prime} = a\mathcal{U}^{ut} + b$ for some $a \in \Re_{>0}$, $b \in \Re$. We may thus assume without loss of generality that $\mathcal{U}^{ut\prime} = \mathcal{U}^{ut}$. Note that since, for each $f \in \mathcal{A}$, there exists $p \in \Delta$ such that $f^p \sim f$, $\mathcal{U}^{ut} \times \mathcal{C}'$ and $\mathcal{U}^{ut} \times \mathcal{C}$ generate the same functional $V : \mathcal{A} \to \Re$ representing $\preceq$ according to (9). Since both of these sets contain the convex closure of the set of supergradients of V, and $\mathcal{C}'$ and $\mathcal{C}$ are tight, it follows that $\mathcal{C}' = \mathcal{C}$, as required.

□

Proof of Proposition 2. Let $\mathcal{U}^{si} \subseteq \Re^X \times \Delta$ be a minimal tight closed convex set representing $\preceq$ according to (2), and let V be the corresponding functional. Suppose that A9 holds. As a point of notation, for every $s \in S$, let $\delta_s \in \Delta$ be the degenerate measure putting all weight on s: $\delta_s(s) = 1$, $\delta_s(t) = 0$ for $t \neq s$. If $p = \delta_s$ for some $s \in S$, then we shall say that p is *degenerate*. Let $\mathcal{ZR} = \{e \in \Re^X \mid \sum_{x \in X} e(x) = 0\}$. Finally, for $c, c' \in \Delta(X)$ with $c \prec c'$, let $(\mathcal{U}^{si})^{[c,c']} \subseteq \mathcal{U}^{si}$ be a minimal tight closed convex representation of $\preceq_{[c,c']}$ according to (2); by the uniqueness clause in Theorem 2 and the properties of minimality (Kannai, 1977), a unique such set exists.

For $c, c' \in \Delta(X)$ such that $c \prec c'$, let $(c, c') = \{d \in \Delta(X) | \ c \prec d \prec c'\}$; we call this the *interval* from c to c'. An interval (c, c') is *single-utilitied* if there exists non-constant $u \in \Re^X$ such that, for every $d \in (c, c')$, every support for $V|_{\Delta(X)}$ at d is a positive affine transformation of u. When writing sets of intervals $\{(\underline{c_1}, \overline{c_1}), \ldots, (\underline{c_n}, \overline{c_n})\}$, we shall henceforth adopt the convention that the intervals are pair-wise disjoint, and that $\overline{c_i} \preceq \underline{c_{i+1}}$ for all i. For any pair of sets of single-utilitied intervals, the appropriately defined union of them contains single-utilied intervals. There is thus a unique set $\mathcal{I}$ of pair-wise disjoint single-utilitied intervals that is maximal under containment: for any other set $\mathcal{I}'$ of single-utilitied intervals, $\bigcup_{(\underline{c_i}, \overline{c_i}) \in \mathcal{I}'} (\underline{c_i}, \overline{c_i}) \subseteq \bigcup_{(\underline{c_i}, \overline{c_i}) \in \mathcal{I}} (\underline{c_i}, \overline{c_i})$. We shall refer to this set henceforth as $\mathcal{I} = \{(\underline{c_1}, \overline{c_1}), \ldots, (\underline{c_n}, \overline{c_n})\}$.

By A9, there exist $\underline{c}, \overline{c} \in \Delta(X)$ such that $\underline{c} \preceq f \preceq \overline{c}$ for all $f \in \mathcal{A}$. Fix any such pair of constant acts $\underline{c}, \overline{c}$ for reference throughout the proof. Define the following sequence of elements of $\Delta(X)$: if $\underline{c_1} \sim \underline{c}$ and $\overline{c_n} \sim \overline{c}$, then take the sequence $\underline{c_1}, \overline{c_1}, \underline{c_2}, \ldots, \overline{c_{n-1}}, \underline{c_n}, \overline{c_n}$; if $\underline{c_1} \succ \underline{c}$ and $\overline{c_n} \prec \overline{c}$, then take the sequence $\underline{c} = \overline{c_0}, \underline{c_1}, \overline{c_1}, \underline{c_2}, \ldots, \overline{c_{n-1}}, \underline{c_n}, \overline{c_n}, \underline{c_{n+1}} = \overline{c}$; and similarly for the other cases.

The result will be a consequence of the following two claims.

Claim 3. *If an interval $(\underline{c_i}, \overline{c_i})$ is single-utilitied, then $(\mathcal{U}^{si})^{[\underline{c_i}, \overline{c_i}]}_{\Delta(X)}$ is a singleton.*

Claim 4. *For each i, there exists $f \in \mathcal{A}$ such that $\overline{c_i} \prec f(s) \prec \underline{c_{i+1}}$ for all $s \in S$. Moreover, for any such f, if V is differentiable at f with support (u, p), then p is degenerate.*

Proof of Claim 3. Since $(\underline{c_i}, \overline{c_i})$ is single-utilitied, there exists $\hat{u} \in \Re^X$ such that for each $u \in (\mathcal{U}^{si})^{[\underline{c_i}, \overline{c_i}]}_{\Delta(X)}$, $u = a\hat{u} + b$ for some $a \in \Re_{>0}$, $b \in \Re$. Note that $\hat{u}$ represents the restriction of $\preceq$ to $(\underline{c_i}, \overline{c_i})$: for if $\hat{u} \cdot c \geqslant \hat{u} \cdot d$, then for every $u \in (\mathcal{U}^{si})^{[\underline{c_i}, \overline{c_i}]}_{\Delta(X)}$, $u \cdot c \geqslant u \cdot d$, so by representation (2), $c \succeq d$, and similarly for strict inequalities.

Let $w : \Delta(X) \to \Re$ be given by: $w(c) = \min_{u \in (\mathcal{U}^{si})^{[\underline{c_i}, \overline{c_i}]}_{\Delta(X)}} u \cdot c = \min_{a\hat{u}+b \in (\mathcal{U}^{si})^{[\underline{c_i}, \overline{c_i}]}_{\Delta(X)}} a\hat{u} \cdot c + b$. w is obviously a concave functional, which represents the restriction of $\preceq$ to $(\underline{c_i}, \overline{c_i})$. We now show that w is linear: if this is the case, then all the (a, b) are equal, so since $(\mathcal{U}^{si})^{[\underline{c_i}, \overline{c_i}]}_{\Delta(X)} = cl(conv(\{u \mid \exists c \in (\underline{c_i}, \overline{c_i})\ s.t.\ (u, p) \in \mathcal{U}^{si}$ supports V at $c\}))$ (by the proofs of Theorem 1 and Proposition 1), it is a singleton. Suppose for reductio that w is not linear. Let $u' = A\hat{u} + B$ be such that $u'(\underline{c_i}) = \inf w((\underline{c_i}, \overline{c}_i))$ and $u'(\overline{c_i}) = \sup w((\underline{c_i}, \overline{c_i}))$. Note that u' represents the restriction of $\preceq$ to $(\underline{c_i}, \overline{c_i})$. Moreover, by the (strict) concavity of w, $u' \cdot d \leqslant w(d)$ for all $d \in (\underline{c_i}, \overline{c_i})$ with strict inequality for some $d \in (\underline{c_i}, \overline{c_i})$, and $u' \cdot d \geqslant w(d)$ for all $d \notin (\underline{c_i}, \overline{c_i})$. Let $v' : \Delta(X) \to \Re$ be defined by: $v'(c) = \min_{u \in \mathcal{U}^{si}_{\Delta(X)} \cup \{u'\}} u(c)$. It is a concave function representing the restriction of $\preceq$ to $\Delta(X)$. Moreover, $V|_{\Delta(X)}(c) \geqslant v'(c)$ for all $c \in \Delta(X)$ with strict inequality for some such $c \in (\underline{c_i}, \overline{c_i})$. Hence $V|_{\Delta(X)}$ is not a minimal concave representation of the restriction of $\preceq$ to $\Delta(X)$. This contradicts the fact, established by Lemma 6 in the proof of Theorem 4, that in the presence of A9, $V|_{\Delta(X)}$ is minimal. So w is linear and the claim is established.

□

Proof of Claim 4. To establish the claim, we require the following two Lemmas, which contain the basic insight at the heart of the whole result.

Lemma 3. *Suppose that V is differentiable at $f \in \mathcal{A}$, with support (u, p), where u is non-constant. Then for all $s \in S$, if $p(s) > 0$, then every support of $V|_{\Delta(X)}$ at $f(s)$ is a positive affine transformation of u.*

Lemma 4. *Let $f \in \mathcal{A}$ be such that V is differentiable at f, with support (u, p), where u is non-constant. Suppose that for $s \in S$, $\overline{c_i} \prec f(s) \prec \underline{c_{i+1}}$ for some i. If $p(s) > 0$, then $p = \delta_s$.*

Proof of Lemma 3. Let (u, p) be the unique support of V at some $f \in ri(\mathcal{A})$. Take $s \in S$ such that $p(s) > 0$. Since V is differentiable at f, its restriction to $\{c_s f_{s^c} \mid c \in \Delta(X)\}$ is differentiable at f; by representation (2), u is the unique support at this point. We show that every support for $V|_{\Delta(X)}$ at $f(s)$ is a positive affine transformation of u. Suppose that this is not the case, and that u', which is not a positive affine transformation of u, supports $V|_{\Delta(X)}$ at $f(s)$. Hence there exists $c', d' \in \Delta(X)$ such that $u(c') > u(d')$ and $u'(c') \leqslant u'(d')$. So, taking $e = c' - d'$ yields $e \in \mathcal{ZR}$ with $u \cdot e > 0$ and $u' \cdot e \leqslant 0$. Since $f \in ri(\mathcal{A})$, for sufficiently small $\delta > 0$, $f(s) + \delta e \in \Delta(X)$. The preceding facts, combined with the fact that u is the unique support to the restriction of V $\{c_s f_{s^c} \mid c \in \Delta(X)\}$ at f, imply that for sufficiently small $\delta > 0$, $(f(s) + \delta e)_s f_{s^c} \succ f$ whereas $f(s) + \delta e \preceq f(s)$, contradicting A9. Hence every support of $V|_{\Delta(X)}$ at $f(s)$ is a positive affine transformation of u. This establishes the claim for $f \in ri(\mathcal{A})$. By the continuity of superdifferentials (Rockafellar, 1970, Theorem 24.4), the claim holds for $f \in \mathcal{A}$ where V has a unique support. □

Proof of Lemma 4. Let f be such that V is differentiable at f, with support (u, p), where u is non-constant, and suppose that there exists $s_1 \in S$ such that $\overline{c_i} \prec f(s_1) \prec \underline{c_{i+1}}$. Suppose for reductio that $p(s_1) > 0$ and $p(s_2) > 0$, for some $s_2 \neq s_1$. Since V is differentiable at f with support (u, p), by Lemma 3 every support of $V|_{\Delta(X)}$ at $f(s_1)$ is a positive affine transformation of u, and similarly for $f(s_2)$.

Take any $d \in \Delta(X)$ such that $d \sim f(s_1)$ and let $g = d_{s_1} f_{s_1^c}$. By representation (2), there exists $e_1, e_2 \in \mathcal{ZR}$ and $\epsilon > 0$ such that, for all $\delta < \epsilon$, $(f(s_1) + \delta e_1)_{s_1} f_{s_1^c} \nsim f$, $(f(s_2) + \delta e_2)_{s_2} f_{s_2^c} \nsim f$, and $(f(s_1) + \delta e_1)_{s_1} (f(s_2) + \delta e_2)_{s_2} f_{\{s_1, s_2\}^c} \sim f$. By the continuity of $\preceq$, for each $\delta < \epsilon$, there exists $e_1^\delta, e_2^\delta \in \mathcal{ZR}$ such that $g(s_1) + e_1^\delta \sim f(s_1) + \delta e_1$ and $g(s_2) + e_2^\delta \sim f(s_2) + \delta e_2$. By A9, it follows that $(g(s_1) + e_1^\delta)_{s_1} g_{s_1^c} \nsim g$, $(g(s_2) + e_2^\delta)_{s_2} g_{s_2^c} \nsim g$ but $(g(s_1) + e_1^\delta)_{s_1} (g(s_2) + e_2')_{s_2} g_{\{s_1, s_2\}^c} \sim g$ for every δ (and associated e_1^δ). By representation (2), it follows that V is supported at g by (u', p') where $p'(s_1) > 0$ and $p'(s_2) > 0$, and that it is not supported by (u'', p'') with $p''(s_1) > 0$ and $p''(s_2) = 0$ or $p''(s_1) = 0$ and $p''(s_2) > 0$.

By the maximality of $\mathcal{I}$, the density of differentiable points of V and the determination of supergradients by supergradients at differentiable points (Rockafellar, 1970, Theorems 25.5 and 25.6), for every subinterval of $(\overline{c_i}, \underline{c_{i+1}})$ containing $f(s_1)$, there exists $d' \in (\overline{c_i}, \underline{c_{i+1}})$ with V differentiable at $g' = d'_{s_1} f_{s_1^c}$ and $V|_{\Delta(X)}$ not supported at d' by a positive affine transformation of u. Taking successively smaller intervals yields a sequence of d_i with these properties, with $d_i \to d$. Since, as shown above, V is supported at $g = d_{s_1} f_{s_1^c}$ by (u', p') where $p'(s_1) > 0$ and $p'(s_2) > 0$ and it is not supported by (u'', p'')

with $p''(s_1) > 0$ and $p''(s_2) = 0$ or $p''(s_1) = 0$ and $p''(s_2) > 0$, by the continuity of the superdifferential mapping (Rockafellar, 1970, Theorem 24.4), for i sufficiently large, V is supported at $g_i = (d_i)_{s_1} f_{s_1^c}$ by (u_i, p_i) with u_i non-constant, $p_i(s) \neq 0$ and $p_i(s') \neq 0$. Consider any such i: since $V|_{\Delta(X)}$ is supported at $f(s_2)$ only by positive affine transformations of u, whereas it is not supported at d_i by any positive affine transformation of u, we have a contradiction by Lemma 3. Hence there does not exist $s_2 \in S$ with $p(s_2) > 0$, so $p = \delta_{s_1}$, as required.

□

Now return to Claim 4. For any i, consider the interval $(\overline{c_i}, \underline{c_{i+1}})$. By the continuity of preferences, the indifference curve in $\Delta(X)$ containing $\overline{c_i}$ is determined by the unique utility function in $(\mathcal{U}^{si})^{[\underline{c_i},\overline{c_i}]}_{\Delta(X)}$ (this set is a singleton by Claim 3), and similarly for $\underline{c_{i+1}}$. So if $\overline{c_i} = \underline{c_{i+1}}$, these two indifference curves are identical, so the utility representing $(\mathcal{U}^{si})^{[\underline{c_i},\overline{c_i}]}_{\Delta(X)}$ is a positive affine transformation of that representing $(\mathcal{U}^{si})^{[\underline{c_{i+1}},\overline{c_{i+1}}]}_{\Delta(X)}$, contradicting the maximality of $\mathcal{I}$. $(\overline{c_i}, \underline{c_{i+1}})$ is thus non-empty, as required.

As concerns the other part of the claim, note that, by A9, for any $f \in \mathcal{A}$ such that $\overline{c_i} \prec f(s) \prec \underline{c_{i+1}}$ for all $s \in S$, $f \prec \underline{c_{i+1}}$, so it is not supported by (u, p) with u constant. So, for any such f with V is differentiable at f and having support (u, p) there, it follows directly from Lemma 4 that p is degenerate, as required.

□

Claim 3 establishes that each section $\preceq_{[\underline{c_i},\overline{c_i}]}$ exhibits precise tastes. We now show that each section $\preceq_{[\overline{c_i},\underline{c_{i+1}}]}$ has rudimentary beliefs. Consider $\preceq_{[\overline{c_i},\underline{c_{i+1}}]}$. By Claim 4, and the fact that supergradients of V are determined by supergradients at differentiable points, for every $f \in \mathcal{A}$ with $\overline{c_i} \preceq f(s) \preceq \underline{c_{i+1}}$ for all $s \in S$, V is supported at f by some (u, p) with degenerate p. We first show that for every $s \in S$ such that there exists $(u, \delta_s) \in \mathcal{U}^{si}$ supporting V at some f with $\overline{c_i} \prec f(s') \prec \underline{c_{i+1}}$ for all $s' \in S$, $\{u \mid (u, \delta_s) \in \mathcal{U}^{si}\}$ represents the restriction of $\preceq$ to $\{d \in (\underline{c_i}, \overline{c_i}) \mid d_s \overline{c}_{s^c} \prec \overline{c}\}$.

Consider any such $s \in S$. By representation (2), there exists $d \in (\overline{c_i}, \underline{c_{i+1}})$ with $d_s \overline{c}_{s^c} \prec \overline{c}$. We show that, for every $d, d' \in (\overline{c_i}, \underline{c_{i+1}})$ with either $d_s \overline{c}_{s^c} \prec \overline{c}$ or $d'_s \overline{c}_{s^c} \prec \overline{c}$, if $d' \succ d$ then $d'_s \overline{c}_{s^c} \succ d_s \overline{c}_{s^c}$. Suppose that this is not the case: by A9 it follows that there exist $d, d' \in (\overline{c_i}, \underline{c_{i+1}})$ with $d_s \overline{c}_{s^c} \prec \overline{c}$ and $d' \succ d$ but $d'_s \overline{c}_{s^c} \sim d_s \overline{c}_{s^c}$. By A9, for every $c \in \Delta(X)$ such that $d' \succeq c \succeq d$, $c_s \overline{c}_{s^c} \sim d_s \overline{c}_{s^c}$. For every $g \in \mathcal{A}$, if $g(s) \succeq d$, then $d \preceq g(s)_\alpha d \preceq d'$

for sufficiently small $\alpha \in (0,1)$. Since, by the aforementioned properties and Lemma 4, there exists $(u', \delta_s) \in \mathcal{U}^{si}$ such that $(u'.\delta_s) \cdot g_\alpha(d_s\overline{c}_{s^c}) = (u'.\delta_s) \cdot (g(s)_\alpha d)_s\overline{c}_{s^c} = V(d_s\overline{c}_{s^c})$, it follows that $g_\alpha(d_s\overline{c}_{s^c}) \preceq d_s\overline{c}_{s^c}$ for sufficiently small $\alpha \in (0,1)$. By A9, the same holds for $g \in \mathcal{A}$ with $g(s) \preceq d$. So, for every $g \in \mathcal{A}$, for sufficiently small $\alpha \in (0,1)$, $g_\alpha(d_s\overline{c}_{s^c}) \preceq d_s\overline{c}_{s^c}$, contradicting A5. Hence the restriction of $\preceq$ to $\{d_s\overline{c}_{s^c} \mid d \in (\overline{c_i}, \underline{c_{i+1}}), d_s\overline{c}_{s^c} \prec \overline{c}\}$ coincides with the restriction of $\preceq$ to $\{d \in (\underline{c_i}, \overline{c_i}) \mid d_s\overline{c}_{s^c} \prec \overline{c}\}$. Since, by Lemma 4 and representation (2), $\{u \mid (u, \delta_s) \in \mathcal{U}^{si}\}$ represents the former relation, it also represents the latter.

By Lemma 6 in the proof of Theorem 4, $V|_{\Delta(X)}$ is a minimal concave representation of the restriction of $\preceq$ to $\Delta(X)$. So, for each $s \in S$ such that there exists $(u, \delta_s) \in \mathcal{U}^{si}$ supporting V at some f with $\overline{c_i} \prec f(s) \prec \underline{c_{i+1}}$ for all $s \in S$, since $\{u \mid (u, \delta_s) \in \mathcal{U}^{si}\}$ represents the restriction of $\preceq$ to $\{d \in (\underline{c_i}, \overline{c_i}) \mid d_s\overline{c}_{s^c} \prec \overline{c}\}$, there exists a concave function $\psi_s : (V|_{\Delta(X)}(\overline{c_i}), V|_{\Delta(X)}(\underline{c_{i+1}})) \to \Re$ such that $\min_{(u,\delta_s)\in\mathcal{U}^{si}} u(c) = \psi_s \circ V|_{\Delta(X)}(c)$ for all $c \in \Delta(X)$. (ψ_s is constant on $V|_{\Delta(X)}(d \in (\underline{c_i}, \overline{c_i}) \mid d_s\overline{c}_{s^c} \sim \overline{c}\})$.) Let $\phi_s : V|_{\Delta(X)}(\Delta(X)) \to \Re$ be any concave function extending ψ_S. Hence, for every act f with $\overline{c_i} \prec f(s) \prec \underline{c_{i+1}}$ for all $s \in S$, $V(f) = \min_{s\in E} \phi_s(\min_{u\in\mathcal{U}^{si}_{\Delta(X)}} u(f(s)))$, where E is the set of $s \in S$ such that there exists $(u, \delta_s) \in \mathcal{U}^{si}$ supporting V at some f with $\overline{c_i} \prec f(s) \prec \underline{c_{i+1}}$ for all $s \in S$. By the continuity of $\preceq$, this representation continues to hold for f with $\overline{c_i} \preceq f(s) \preceq \underline{c_{i+1}}$ for all $s \in S$; hence $\preceq_{[\overline{c_i}, \underline{c_{i+1}}]}$ involves rudimentary beliefs, as required. □

Proof of Theorem 4. The (ii) to (i) direction is straightforward; consider the (i) to (ii) implication. Theorem 1 implies that there is a representation of $\preceq$ by a non-trivial minimal tight closed convex $\mathcal{U} \subseteq \Re^{S\times X}$ according to (1). Let $V(f) = \min_{U\in\mathcal{U}} U \cdot f$ for all $f \in \mathcal{A}$; as noted in the proof of Theorem 1 (see Lemma 1), V is a minimally concave representation, in the sense of (Kannai, 1977, p11). Let $\mathcal{U}^{ut} = \mathcal{U}_{\Delta(X)} = \{\sum_{s\in S} U(s, \bullet) \mid \exists c \in \Delta(X)\ s.t.\ U$ supports V at $c\}$. By Proposition 1 and its proof, $\mathcal{U}^{ut}$ is a tight closed convex set of utilities representing the restriction of $\preceq$ to $\Delta(X)$ according to (7). Let $V^{ut} : \Delta(X) \to \Re$ be defined by $V^{ut}(c) = \min_{u\in\mathcal{U}^{ut}} u \cdot c$; by definition, it is the restriction of V to $\Delta(X)$. Let $K = V^{ut}(\Delta(X))$, and $B(K)$ be the set of functions from $S \to K$. With slight abuse of notation, for $f \in \mathcal{A}$, we use $V^{ut}(f)$ to denote the element of $B(K)$ with $V^{ut}(f)(s) = V^{ut}(f(s))$ for all $s \in S$. By A9, the function $I : B(K) \to \Re$ defined by $I(a) = V(f)$ for any f such that $V^{ut}(f) = a$ is well-defined. By definition, $V(f) = I(V^{ut}(f))$.

Lemma 5. *I is concave, continuous, monotonic, and normalised (ie. for all $x \in K$, $I(x^*) = x$, where x^* is the constant function in $B(K)$ taking value x).*

Proof. Normalisation follows from the definition of I. Monotonicity of I is immediate from A9. Continuity follows from the continuity of V and the fact that V^{ut} is a quotient map. To establish concavity, we first introduce the following notion. For any $f, g \in \mathcal{A}$, a *state-wise calibrated sequence* from f to g is a sequence $(f_i)_{0 \leqslant i \leqslant n}$ with $f_0 = f$, $f_n = g$ such that for every $1 \leqslant i \leqslant n-1$ and every $s \in S$, $\ell(f_{i-1}(s), f_i(s), f_{i+1}(s)) = \frac{1}{2}$, where:

(15)
$$\ell(g,f,h) = \begin{cases} \sup\{\alpha \mid \exists g', h' \in \mathcal{A} \; s.t \; g' \sim g, h' \sim h, h'_\alpha g' \sim f\} & \text{if } g \prec f \prec h \\ \sup\{1-\alpha \mid \exists g', h' \in \mathcal{A} \; s.t \; g' \sim g, h' \sim h, h'_\alpha g' \sim f\} & \text{if } g \succ f \succ h \\ \frac{1}{2} & \text{if } g \sim f \sim h \\ 0 & \text{otherwise} \end{cases}$$

We begin with the following claim.

Claim 5. *For every $f, g \in \mathcal{A}$, for every state-wise calibrated sequence $(f_i)_{0 \leqslant i \leqslant n}$ from f to g and every $0 \leqslant i \leqslant n$, $V(f_i) \geqslant \frac{n-i}{n} V(f) + \frac{i}{n} V(g)$.*

Proof. Let $(f_i)_{0 \leqslant i \leqslant n}$ be a state-wise calibrated sequence, and consider three successive elements f_{i-1}, f_i, f_{i+1}. We first show that $V(f_i) \geqslant \frac{1}{2} V(f_{i-1}) + \frac{1}{2} V(f_{i+1})$. Since the sequence is state-wise calibrated, for each $s \in S$, $f_i(s) \sim c^s_{\frac{1}{2}} d^s$ for some $c^s, d^s \in \Delta(X)$ with $c^s \sim f_{i-1}(s)$ and $d^s \sim f_{i+1}(s)$. (Notice that the suprema in the definition of ℓ are obtained because $\mathcal{A}$ is compact.) Define $f'_{i-1}, f'_{i+1} \in \mathcal{A}$ by $f'_{i-1}(s) = c^s$ and $f'_{i+1}(s) = d^s$ for all $s \in S$. By A9, $f_i \sim f'_{i-1 \frac{1}{2}} f'_{i+1}$. Since V represents $\preceq$ and is concave, $V(f_i) \geqslant \frac{1}{2} V(f'_{i-1}) + \frac{1}{2} V(f'_{i+1}) = \frac{1}{2} V(f_{i-1}) + \frac{1}{2} V(f_{i+1})$, as required.

Applying this fact inductively yields that $V(f_i) \geqslant \frac{1}{i+1} V(f_0) + \frac{i}{i+1} V(f_{i+1})$, and $V(f_i) \geqslant \frac{n-i}{n-i+1} V(f_{i-1}) + \frac{1}{n-i+1} V(f_n)$. It follows (plugging the second inequality into the first) that $V(f_i) \geqslant \frac{n-i}{n} V(f_0) + \frac{i}{n} V(f_n)$, as required. ☐

To establish the concavity of I, it suffices to show that, for every $a, b \in B(K)$, $I(\frac{1}{2}a + \frac{1}{2}b) \geqslant \frac{1}{2} I(a) + \frac{1}{2} I(b)$. Consider any such a, b with $a \neq b$, and let $f, g \in \mathcal{A}$ be such that $a = V^{ut}(f)$ and $b = V^{ut}(g)$.

Take a sequence of state-wise calibrated sequences $f = f_{0,n}, \ldots, f_{i,n}, \ldots, f_{2n,n} = g$ from f to g; such a sequence can be constructed as in the proof of Theorem 1. Since

$\mathcal{A}$ is compact, there is a subsequence of $f_{n,n}$ that converges as $n \to \infty$. Consider this subsequence of the sequences $\{f_{i,n}\}$ and let $f_{n,n} \to h$. Since $a \neq b$, there exists $s \in S$ such that $f(s) \nsim g(s)$. It follows from the definition of state-wise calibrated sequences and of the function α introduced in the proof of Theorem 1 (equation (13)), that, for every $s \in S$ such that $f(s) \nsim g(s)$, and every n and $0 < i < 2n$:

$$\begin{aligned}\alpha(f_{i-1,n}(s), f_{i,n}(s), f_{i+1,n}(s)) =& \frac{\ell(f_{i-1,n}(s), f_{i,n}(s), f_{i+1,n}(s))}{1 - \ell(f_{i-1,n}(s), f_{i,n}(s), f_{i+1}(s))} \\ =& 1\end{aligned}$$

Consider any $s \in S$ such that $f(s) \nsim g(s)$ and suppose without loss of generality that $f(s) \prec g(s)$. It follows from the construction used by Kannai (1977, Theorem 2.4) that the restriction of V to $\{f' \in \mathcal{A} \mid f(s) \preceq f' \preceq g(s)\}$ is the limit of a sequence of concave functions V_n, each of which takes value $\frac{\sum_{k=1}^{n} \prod_{i=k}^{2n-1} \alpha(f_{i-1,n}(s), f_{i,n}(s), f_{i+1,n}(s))}{\sum_{k=1}^{2n} \prod_{i=k}^{2n-1} \alpha(f_{i-1,n}(s), f_{i,n}(s), f_{i+1,n}(s))} V(f(s)) + \frac{\sum_{k=n+1}^{2n} \prod_{i=k}^{2n-1} \alpha(f_{i-1,n}(s), f_{i,n}(s), f_{i+1,n}(s))}{\sum_{k=1}^{2n} \prod_{i=k}^{2n-1} \alpha(f_{i-1,n}(s), f_{i,n}(s), f_{i+1,n}(s))} V(g(s))$ on $f_{n,n}(s)$. Since this tends to $\frac{1}{2}V(f(s)) + \frac{1}{2}V(g(s))$ as $n \to \infty$, $V(h(s)) = \frac{1}{2}V(f(s)) + \frac{1}{2}V(g(s))$. Since this holds for every s such that $f(s) \nsim g(s)$, $V^{ut}(h) = \frac{1}{2}a + \frac{1}{2}b$. By Claim 5, for each sequence n, $V(f_{n,n}) \geqslant \frac{1}{2}V(f) + \frac{1}{2}V(g)$, so by the continuity of V, $V(h) \geqslant \frac{1}{2}V(f) + \frac{1}{2}V(g)$. Hence $I(\frac{1}{2}a + \frac{1}{2}b) \geqslant \frac{1}{2}I(a) + \frac{1}{2}I(b)$, as required. So I is concave.

□

Since $B(K)$ is closed and bounded and I is concave and continuous, for each $a \in B(K)$, there exists an affine functional $\phi : \Re^S \to \Re$ supporting I at a: i.e. such that $\phi(b) \geqslant I(b)$ for all $b \in B(K)$ and $\phi(a) = I(a)$. Each such ϕ can be written as $\phi(b) = x \cdot b + \mu$ for some $x \in \Re^S$, $\mu \in \Re$. We first show that, if such a ϕ supports I at an interior point, then $x(s) \geqslant 0$ for every $s \in S$. Take any $a \in ri(B(K))$ such that ϕ supports I at a, and consider $a_s \in B(K)$ defined by $a_s(s) = a(s) + \epsilon$, $a_s(s') = a(s)$ for $s' \neq s$, where $\epsilon > 0$ such that $a(s) + \epsilon \in K$. By the monotonicity of I, $I(a_s) \geqslant I(a)$; since ϕ supports I at a, we have that $\phi(a_s) = x \cdot a_s + \mu \geqslant I(a_s) \geqslant I(a) = x \cdot a + \mu$, so $x(s).\epsilon \geqslant 0$. Since this holds for every $s \in S$, we have that $x(s) \geqslant 0$ for every $s \in S$. Moreover, since I is non-constant, $x \neq 0$. So each ϕ supporting I at an interior point can be written in the form $\phi(a) = (\bar{a}.p) \cdot a + \bar{b}$ for some $p \in \Delta \subseteq \Re^S$, $\bar{a} \in \Re_{>0}$ and $\bar{b} \in \Re$. Let $\mathcal{P} = cl\left(conv\{(p, \bar{a}, \bar{b}) \in \Delta \times \Re_{>0} \times \Re \mid \exists a \in ri(B(K))\ s.t. (\bar{a}.p) + \bar{b} \text{ supports } I \text{ at } a\}\right)$. By the fact that I is differentiable on a dense subset of $ri(B(K))$, the continuity of the superdifferential mapping, and the continuity of I, $I(a) = \min_{(p,\bar{a},\bar{b})\in\mathcal{P}}(\bar{a}.p) \cdot a + \bar{b}$ for all $a \in B(K)$. By the fact that supergradients at differentiable points determine the supergradients elsewhere (Rockafellar, 1970, 25.6), $\mathcal{P}$ is tight: no proper closed convex subset of it represents I in this way. For all $x \in K$, since $I(x^*) = x$, $\bar{a}x + b \geqslant x$ for all $(p, \bar{a}, \bar{b}) \in \mathcal{P}$. Moreover, we argue that there exists $p \in \Delta$ such that $(p, 1, 0) \in \mathcal{P}$. To show this, note firstly that for every $x > \min K$, there exists no $(p, a, b) \in \mathcal{P}$ such that $a > 1$ and $a.x + b = x$: if there were such a, b, then for any $y \in K$ with $y < x$, $a.y + b = x - a.(x - y) < y$, contradicting the fact that $\mathcal{P}$ represents I and $I(y^*) = y$. Similarly, for every $x < \max K$, there exists no $(p, a, b) \in \mathcal{P}$ such that $a < 1$ and $a.x + b = x$. Since I is normalised, it follows from the representation of I by $\mathcal{P}$ that there exists $p \in \Delta$ with $(p, 1, 0) \in \mathcal{P}$.

Define $\alpha : \Delta \to \Re_{>0} \times \Re$ by $\alpha(p) = \{(a, b) \in \Re_{>0} \times \Re \mid (p, a, b) \in \mathcal{P}\}$. By the definition of $\mathcal{P}$, and in particular the fact that it is non-empty, closed, convex and tight, α is non-trivial, upper hemicontinuous, convex and tight. By the last two properties of $\mathcal{P}$ mentioned above, α is calibrated (with respect to $\mathcal{U}^{ut}$) and grounded. So α is a non-trivial, tight, grounded, calibrated, upper hemicontinuous, convex function representing $\preceq$ according to (11), as required.

It remains to show that $\mathcal{U}^{ut}$ is minimal. This is established by the following lemma.

Lemma 6. *$\mathcal{U}_{\Delta(X)}$ is a minimal tight closed convex representation of the restriction of $\preceq$ to $\Delta(X)$.*

Proof. By Lemma 1, it suffices to show that $V|_{\Delta(X)}$ is a minimal concave representation of the restriction of $\preceq$ to $\Delta(X)$. Suppose that this is not the case, so there exists a concave functional on $\Delta(X)$, v', representing the restriction of $\preceq$ to $\Delta(X)$ and such that $v'(\Delta(X)) = V|_{\Delta(X)}(\Delta(X))$ and $V|_{\Delta(X)}(c) \geqslant v'(c)$ for all $c \in \Delta(X)$ with strict inequality for some c (Kannai, 1977; Debreu, 1976). By Lemma 5, there exists a normalised, monotone, concave functional I such that $V = I \circ V|_{\Delta(X)}$. Hence $V' = I \circ v'$ represents $\preceq$, is concave and is such that $V(f) \geqslant V'(f)$ for all $f \in \mathcal{A}$ with strict inequality for some f, contradicting the minimality of V. So $\mathcal{U}_{\Delta(X)}$ is minimal, as required.

□

Consider finally the uniqueness clause. The uniqueness of $\mathcal{U}^{ut}$ follows from the arguments used to establish the uniqueness clause in Theorem 1, applied to the restriction to constant acts. Suppose that $\mathcal{U}^{ut}$ and α, and $\mathcal{U}^{ut}$ and α' represent $\preceq$. Then α and α' both represent the same functional $I : B(K) \to \Re$: $I(a) \leqslant I(b)$ iff $\min_{p\in\Delta,\ (\bar{a},\bar{b})\in\alpha(p)} \bar{a}(a \cdot p) + \bar{b} \leqslant \min_{p\in\Delta,\ (\bar{a},\bar{b})\in\alpha(p)} \bar{a}(b \cdot p) + \bar{b}$, for all $a, b \in B(K)$, and similarly for α'. Since α and α' are grounded and calibrated, $I(a) = \min_{p\in\Delta,\ (\bar{a},\bar{b})\in\alpha(p)} \bar{a}(a\cdot p)+\bar{b} = \min_{p\in\Delta,\ (\bar{a},\bar{b})\in\alpha'(p)} \bar{a}(a\cdot p)+\bar{b}$ for all $a \in B(K)$. By the arguments in the proof of Theorem 1, $\alpha = \alpha'$, as required.

□

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