A Non-Bayesian Theory of State-Dependent Utility*

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Abstract

Many decision situations involve two or more of the following divergences from subjective expected utility: imprecision of beliefs (or ambiguity), imprecision of tastes (or multi-utility), and state dependence of utility. This paper proposes and characterises a model of uncertainty averse preferences that can simultaneously incorporate all three phenomena. The representation supports a principled separation of (imprecise) beliefs and (potentially state-dependent, imprecise) tastes. Moreover, the representation permits comparative statics separating the roles of beliefs and tastes, and is modular: it easily delivers special cases involving various combinations of the phenomena, as well as state-dependent multi-utility generalisations covering popular ambiguity models.

Keywords: State-dependent utility, uncertainty aversion, multiple priors, ambiguity, imprecise tastes, multi-utility.

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1 Introduction

Decision makers sometimes do not know precisely the probabilities of the relevant outcomes given their actions; they may thus exhibit aversion to this uncertainty or '*imprecision in beliefs*' (Ellsberg, 1961).[1] They sometimes may have trouble comparing the relevant outcomes; this *imprecision in tastes* has long been studied in economics (Aumann, 1962; Dubra et al., 2004) and has been connected to Allais-style violations of subjective expected utility (Levi, 1986; Cerreia-Vioglio et al., 2015). They sometimes have state-dependent utility (Arrow, 1974; Karni, 1983b). And sometimes, they may exhibit all three of these effects in a single decision situation. For instance, although classic analyses of health insurance—which are essentially monetary bets on one's future state of health—focus uniquely on state-dependent utilities (Cook and Graham, 1977), in many cases individuals have limited information about the probabilities of future health states—or imprecise beliefs—and, given the lack of familiarity with some of the states, may have trouble evaluating their utility in them—or have imprecise tastes. Or, to take another example, whilst some have proposed explanations of the equity premium puzzle in terms of belief imprecision and uncertainty aversion (e.g. Ju and Miao, 2012), others have claimed to explain it using state-dependent utility (e.g. Melino and Yang, 2003).

Applications typically focus on (at most) one of these factors, ignoring the others. However, their simultaneous presence poses the question of the *robustness* of the conclusions drawn from single-factor analyses to the presence of other factors. Moreover, establishing a potential role for several factors in explaining a given phenomenon naturally leads to the question of their *relative importance*: is, say, state-dependence or belief imprecision a bigger driver of a given asset pricing pattern? And if different factors impact an economic variable in diverse ways, how do they trade off? Systematic study of such questions would naturally be grounded in a formal model accommodating all of these phenomena. The present paper provides such a model.

From a decision-theoretic perspective, these effects correspond to distinct violations of the standard axioms of subjective expected utility (Savage, 1954),[2] so the required model

[1]The term 'imprecise probabilities' is widely used in statistics and philosophy; see for example Walley (1991); Bradley (2014) and the references therein.

[2]More precisely, in the Anscombe and Aumann (1963) framework and in the context of complete preferences: violations of the Independence axiom for imprecision of beliefs, of the restriction of Independence to

should be able handle all of them simultaneously, whilst ideally preserving the separation of beliefs and tastes. Such a model is still lacking in the theoretical literature, which almost exclusively focusses on these violations taken in isolation: work on ambiguity, for example, generally assumes precise tastes and state-independent utility, and that on state-dependent utility mainly works in the context of precise beliefs and tastes. In fact, simultaneous violations pose specific technical and conceptual challenges. Technically, the bulk of the literature on state-dependent utility relies strongly on the expected utility framework (Karni et al., 1983; Karni, 1993a), whereas many of the principal axioms in the ambiguity literature explicitly use constant acts (Gilboa and Schmeidler, 1989; Maccheroni et al., 2006), which lose their meaning as soon as utilities are state dependent. Conceptually, it is unclear what functional form to use when modelling situations with imprecise tastes, imprecise beliefs and state-dependent utility. We are aware of no proposed model accommodating all three phenomena under uncertainty aversion; the closest related literature, on incomplete preferences (Galaabaatar and Karni, 2013), suggests a functional form that violates one of the central axioms we defend below (Section 4.1).

We propose and axiomatise the following state-dependent non-expected utility representation:

$$\min_{p \in \mathcal{C}} \sum_{s \in S} p(s) \min_{u \in \upsilon(s)} u(f(s)) \tag{1}$$

where $\mathcal{C}$ is a (closed convex) set of probability measures over states and υ is a function assigning to each state s a (closed convex) set of utility functions over consequences. This representation can accommodate all the aforementioned phenomena. Indeed, it displays: (i) imprecise beliefs, in the use of multiple priors $\mathcal{C}$; (ii) imprecise tastes, in the multiple utility functions $\upsilon(s)$; and (iii) state dependence of utility, in the possible dependence of the set of utility functions used to evaluate consequences on the state.

On the axiomatic front, given the absence of expected utility both over states and consequences, no form of the Independence axiom appears in our most general result. In its stead is an uncertainty aversion condition that retains the same spirit as, though strengthens the classic Uncertainty Aversion axiom due to Schmeidler (1989). Similarly, given the state dependence of utility, there is no Monotonicity axiom, but instead State Consistency, which basically says that one's preferences regarding the consequence obtained in a given state is

risky prospects for imprecision of tastes, and of the Monotonicity axiom for state dependence of utility.

independent of what one would get in the other states. This is a minimal axiom ensuring a coherent notion of preferences conditional on a state; our results suggest moreover that it is a basic axiom for state-dependent utility in the context of imprecise beliefs or tastes.

Whilst our benchmark model can accommodate all of the aforementioned violations of subjective expected utility, it provides a springboard for the study of models incorporating only some violations. As an illustration, we axiomatise special cases in which each of the factors is 'shut down'. Moreover, we provide a simple extension, permitting an extremely general treatment of uncertainty that covers virtually all existing uncertainty averse models in the literature. As such, our approach straightforwardly yields state-dependent multi-utility generalisations of standard ambiguity models, beyond the Gilboa and Schmeidler (1989) multiple prior representation used in (1) above.

To corroborate the interpretations of the different elements of the model, we provide comparative statics separating their impacts on choice. In particular, our model clearly separates the impact of belief imprecision from taste imprecision and state-dependence. It thus can smoothly incorporate changes in or 'addition' of different factors in a given economic application, hence lending itself naturally to the study of the previously mentioned robustness or relative importance questions.

The paper is organised as follows. The framework is set out in Section 2. The benchmark model is stated and characterised in Section 3. Section 4 maps out special cases and extensions, and clarifies the relationship to the existing literature on the various violations, whilst Section 5 provides a comparative statics analysis. Proofs and technical material are contained in the Appendix.

2 Preliminaries

We use a version of the standard Anscombe-Aumann (1963) framework. Let S be a finite set of states; Δ is the set of probability measures over S. Let X be the set of *consequences*. An *act* is a function from states to consequences; $\mathcal{A} = X^S$ is the set of acts. For state s and acts f and g, the act $f_s g$ is defined as follows: $f_s g(s) = f(s)$ and $f_s g(t) = g(t)$ for all $t \neq s$. With slight abuse of notation, a constant act taking consequence x in every state will be denoted x and the set of constant acts will be denoted X.

We assume that X is the set of lotteries (Borel probability measures) over a compact metric space of prizes Z, with the topology of weak convergence. Special cases of this setup include the set of (standard) lotteries over a closed interval of monetary prizes, or over a finite set of prizes. Our results also hold for X any compact polyhedral convex subset of a finite-dimensional vector space,[3] so they apply in cases where consequences are commodity bundles or allocations (e.g. as in social choice under uncertainty) taking values in bounded intervals. X admits the standard mixing operation. Moreover, $\mathcal{A}$ is also a compact convex subset of a vector space with the inherited mixture relation, defined pointwise as standard. For $f, h \in \mathcal{A}$ and $\alpha \in [0,1]$, we write $f_\alpha h$ for the mixture of f and h; similarly, we write $x_\alpha y$ for the mixture of $x, y \in X$. $\mathcal{A}$ is endowed with the product topology. For $h, h' \in \mathcal{A}$, $\mathcal{A}^{h,h'} = \{f \in \mathcal{A} : \forall s \in S, \exists \beta \in [0,1] \; s.t. \; f(s) = h(s)_\beta h'(s)\}$.

The binary relation $\succeq$ on $\mathcal{A}$ depicts the decision maker's preferences over acts. The symmetric and asymmetric parts of $\succeq$, $\sim$ and $\succ$, are defined in the standard way. A state $s \in S$ is said to be *null* if $f_s h \sim h$ for all $f, h \in \mathcal{A}$; otherwise it is non-null. A functional $V : \mathcal{A} \to \Re$ *represents* $\succeq$ if, for all $f, g \in \mathcal{A}$, $f \succeq g$ if and only if $V(f) \geqslant V(g)$.

A utility function is a continuous function $u : Z \to \Re$. We endow $\mathbb{U}$, the set of utility functions, with the supnorm. With slight abuse of notation, for any $x \in X$ and $u \in \mathbb{U}$, we set $u(x) = \int u dx$. A set of utility functions $\mathcal{U} \subseteq \mathbb{U}$ is *non-trivial* if, for every constant function $u' \in \mathcal{U}$, there exists $x \in X$ and a non-constant $u \in \mathcal{U}$ with $u(x) < u'(x)$. For any $c, c' \in X$, $\mathcal{U} \subseteq \mathbb{U}$ is *c, c'-precise* if there exists $u \in \mathcal{U}$ such that $u(c') = \inf_{u' \in \mathcal{U}} u'(c') \leqslant \inf_{u' \in \mathcal{U}} u'(d) \leqslant \inf_{u' \in \mathcal{U}} u'(c) = u(c)$ for all $d \in X$. We define the following order on sets of utility functions: $\mathcal{U}^1 \leqslant \mathcal{U}^2$ if and only if, for every $x \in X$ and $u \in \mathcal{U}^2$, there exists $u' \in \mathcal{U}^1$ with $u'(x) \leqslant u(x)$. Positive affine transformations of sets of utility functions are defined pointwise: for any $\mathcal{U} \subseteq \mathbb{U}$, $\kappa \in \Re_{>0}$ and $\lambda \in \Re$, $\kappa\mathcal{U} + \lambda = \{\kappa u + \lambda | \; u \in \mathcal{U}\}$.

A function $\upsilon : S \to 2^{\mathbb{U}} \backslash \varnothing$ is non-trivial, closed, convex and h, h'-precise, for $h, h' \in \mathcal{A}$, if $\upsilon(s)$ is non-trivial, closed, convex and $h(s), h'(s)$-precise for every $s \in S$. It is *h, h'-constant* if, for every $\alpha \in [0,1]$ and $s, t \in S$, $\min_{u \in \upsilon(s)} u(h(s)_\alpha h'(s)) = \min_{u \in \upsilon(t)} u(h(t)_\alpha h'(t))$. Containment, unions and positive affine transformations are defined statewise: for every pair of $\upsilon^1, \upsilon^2 : S \to 2^{\mathbb{U}} \backslash \varnothing$, $\upsilon^1 \subseteq \upsilon^2$ if and only if $\upsilon^1(s) \subseteq \upsilon^2(s)$ for all $s \in S$, $\upsilon^1 \cup \upsilon^2 : S \to 2^{\mathbb{U}} \backslash \varnothing$ is defined by $\upsilon^1 \cup \upsilon^2(s) = \upsilon^1(s) \cup \upsilon^2(s)$ for all $s \in S$, and υ^2 is a positive affine transformation of υ^1 if there exists $\kappa \in \Re_{>0}$ and

[3] A polyhedral convex set is a set of the points satisfying a finite collection of linear inequalities (Rockafellar, 1970). A polytope is an example of such a set.

$\lambda \in \Re$ such that $\upsilon^2(s) = \kappa\upsilon^1(s) + \lambda$ for all non-null $s \in S$. For any such υ, we let $\upsilon(X) = \bigcup_{s\in S}\{\min_{u\in\upsilon(s)} u(x) \mid x \in X\}$.[4]

Finally, a closed convex set of priors $\mathcal{C} \subseteq \Delta$ is *null-consistent* if, for every $p, q \in \mathcal{C}$, and every $s \in S$, $p(s) = 0$ if and only if $q(s) = 0$.

3 Benchmark model

3.1 Axioms

First consider the following three Basic Axioms on preferences, which are standard for our consequence space.[5]

Axiom A1 (Weak Order)**.** $\geq$ is complete and transitive.

Axiom A2 (Non-degeneracy)**.** There exists $f, g \in \mathcal{A}$ such that $f > g$.

Axiom A3 (Continuity)**.** For all $f \in \mathcal{A}$, the sets $\{g \in \mathcal{A} | \ f \leq g\}$ and $\{g \in \mathcal{A} | \ f \geq g\}$ are closed.

Each of the following axioms has been used in standard treatments to impose state independence of utility.

Axiom A4 (Monotonicity)**.** For all $f, g \in \mathcal{A}$, if $f(s) \geq g(s)$ for all $s \in S$, then $f \geq g$.

Axiom A5 (State Independence)**.** For all $h, h' \in \mathcal{A}$, $x, y \in X$ and non-null $s, t \in S$, $x_s h \geq y_s h$ if and only if $x_t h' \geq y_t h'$.

In the presence of the standard Independence axiom[6] (and Weak Order), these two axioms are equivalent; this is no longer the case in the context of imprecise beliefs and tastes (Section 4.1). Since the aim is to go beyond state-independent utility, neither of them will be imposed here. In their place, consider the following axiom.

Axiom A6 (State Consistency)**.** For every $h, h' \in \mathcal{A}$, $x, y \in X$ and $s \in S$, if $x_s h \geq y_s h$, then $x_s h' \geq y_s h'$.

[4] All terminology and notation extends immediately to notions such as a single set of utilities or a state-dependent utility, considering them to be special cases where υ is constant or singleton-valued respectively.

[5] In the special case of a finite-dimensional consequence space, the topological Continuity axiom can be weakened to mixture continuity.

[6] Independence states that, for all $f, g, h \in \mathcal{A}$ and $\alpha \in (0, 1)$, $f \geq g$ if and only if $f_\alpha h \geq g_\alpha h$.

This is a considerable weakening of State Independence, concerning only cases involving a single state. As such, it says nothing about the relationship between preferences conditional on different states, and so does not imply state independence of utility. On the other hand, it does ensure that the standard notion of preference conditional on a state can be coherently defined (see Proposition 6 in Appendix A). Note that State Consistency is implied by the standard Independence axiom, and so holds automatically in the expected utility context adopted by most of the state-dependent utility literature.

A crucial role shall be played by certain maximal and minimal elements under $\succeq$. To be able to refer to such elements, we introduce the following definition.

Definition 1. $\overline{h}, \underline{h} \in \mathcal{A}$ are *$\succeq$-best-and-worst* if, for every non-null $s \in S$ and $h \in \mathcal{A}$, $\underline{h}(s)_s h \preceq x_s h \preceq \overline{h}(s)_s h$ for all $x \in X$ and $(\overline{h}(s))_\beta(\underline{h}(s))_s h \prec \overline{h}(s)_s h$ for all $\beta < 1$.

Note firstly that in the presence of the previous axioms, such acts always exist.

Proposition 1. *If $\succeq$ satisfies the Basic Axioms and State Consistency, then there exist $\succeq$-best-and-worst $\overline{h}, \underline{h} \in \mathcal{A}$.*

In many cases where our framework typically applies, it is fairly straightforward to identify $\succeq$-best-and-worst $\overline{h}, \underline{h}$. For instance, if X is the set of lotteries over a closed interval (or finite set) of monetary prizes, in most cases (e.g. under first-order stochastic dominance), $\overline{h}$ (respectively $\underline{h}$) can be taken to be lottery yielding the top (resp. bottom) prize for sure. Or, if X is a set of commodity bundles or allocations, in most cases (e.g. under monotonicity or Pareto), $\overline{h}$ can be taken to be the bundle or allocation yielding the maximal amount for each commodity or individual, and similarly for $\underline{h}$.

In the absence of Monotonicity or State Independence, constant acts—acts taking the same consequence in every state—cease to have any special status. This poses a significant challenge, given their central role both as concerns state independence of utility and uncertainty aversion. For the former, they are the acts having the same utility in all states, and hence are key to 'tying' together the utilities assigned to consequences in different states. For the latter, they constitute the 'safe options', of sure precise value, that play a role in the axiomatisations of many popular ambiguity models (Schmeidler, 1989; Gilboa and Schmeidler, 1989; Ghirardato et al., 2004; Maccheroni et al., 2006), as well as in the definition of notions such as relative ambiguity aversion (Ghirardato and Marinacci, 2002). In the face of this challenge, we mobilise an insight from the literature on state-dependent utility (Drèze, 1987; Karni, 1993a,b; Hill, 2009), namely to use *essentially constant acts*: acts that, though they yield different consequences in different states, yield the same precise utility in all states. In our model, the set of mixtures of $\geq$-best-and-worst acts $\overline{h}, \underline{h}$ (ie. $\{\overline{h}_\alpha \underline{h} : \alpha \in [0,1]\}$) will be a set of essentially constant acts; this comes out in the formulation of the uncertainty aversion axiom.

Recall firstly the classic uncertainty aversion axiom from Schmeidler (1989).

Axiom A7 (Uncertainty Aversion)**.** For all $f, g \in \mathcal{A}$ and $\alpha \in [0,1]$, if $f \sim g$ then $f_\alpha g \geq g$.

Now consider, for any $\overline{h}, \underline{h} \in \mathcal{A}$, the following axiom.

Axiom A8 (Strong Uncertainty Aversion with respect to $\overline{h}, \underline{h}$)**.** For all $f, g \in \mathcal{A}$ and $\alpha, \beta, \beta' \in [0,1]$, if $f \geq \overline{h}_\beta \underline{h}$ and $g \geq \overline{h}_{\beta'} \underline{h}$ then $f_\alpha g \geq (\overline{h}_\beta \underline{h})_\alpha (\overline{h}_{\beta'} \underline{h})$.

The only instance of this axiom used here will involve $\geq$-best-and-worst acts $\overline{h}, \underline{h}$; in this case, it is a strengthening of Uncertainty Aversion. To understand it, consider first the case where consequences are lotteries over monetary prizes, preferences over lotteries are represented by a single state-independent utility function, and $\overline{h}, \underline{h}$ are best and worst prizes respectively. It then implies[7] that if the decision maker values the act f at over $\$x$, and the act g at over $\$y$, then he prefers the 50-50 mixture of the two acts, $f_{\frac{1}{2}}g$, to the 50-50 lottery over the two monetary payments $\$x$ and $\$y$. This is just a strong version of the preference-for-hedging motive behind classical Uncertainty Aversion: mixing over acts may bring a hedging advantage with respect to (the mixing of) sure monetary payments.

Strong Uncertainty Aversion with respect to $\geq$-best-and-worst $\overline{h}, \underline{h}$ is the formulation of precisely this condition when nothing can be assumed about the precision, form and state independence of utility. In this context, mixtures $\overline{h}_\beta \underline{h}$ are a proxy for 'sure' acts—monetary payments in the previous example. The axiom, when formulated for $\geq$-best-and-worst acts, says that this hedging motive holds on this 'scale' of essentially constant acts. Since, as remarked previously, it is fairly straightforward in many cases to identify $\geq$-best-and-worst acts, Strong Uncertainty Aversion with respect to such acts is in practice as easy to test as other typical decision-theory axioms.

Finally consider, for any $\overline{h}, \underline{h} \in \mathcal{A}$, the following axiom.

Axiom A9 (EC-Independence with respect to $\overline{h}, \underline{h}$)**.** For all $f, g \in \mathcal{A}^{\overline{h},\underline{h}}$ and $\alpha \in [0,1], \beta \in (0,1)$, $f \geq g$ if and only if $f_\beta(\overline{h}_\alpha \underline{h}) \geq g_\beta(\overline{h}_\alpha \underline{h})$.

[7]The implication is immediate noting that, in this case, Independence holds over constant acts.

This is Gilboa and Schmeidler's C-Independence,[8] but formulated in terms of mixtures of $\overline{h}, \underline{h}$ and acts yielding such mixtures as consequences. As noted, in the context of state dependence and imprecision of tastes, constant acts lose their special status; so C-independence no longer has its original sense. By contrast, mixtures of $\succeq$-best-and-worst $\overline{h}, \underline{h}$ are essentially constant—they yield the same precise utility in all states. So, in using these in the place of constant acts, EC-Independence (for Essentially Constant-Independence) with respect to $\succeq$-best-and-worst $\overline{h}, \underline{h}$ retains the essence of the original axiom even in our more general setting.

3.2 Representation Theorem

The previously discussed axioms yield the following representation.

Theorem 1. *Let $\succeq$ be a preference relation on $\mathcal{A}$, and $\overline{h}, \underline{h} \in \mathcal{A}$. The following are equivalent:*

(i) *$\succeq$ satisfies the Basic Axioms, State Consistency, Strong Uncertainty Aversion and EC-Independence with respect to $\overline{h}, \underline{h}$, and $\overline{h}, \underline{h}$ are $\succeq$-best-and-worst acts*

(ii) *there exists a non-trivial, closed, convex, $\overline{h}, \underline{h}$-constant, $\overline{h}, \underline{h}$-precise function $\upsilon : S \to 2^{\mathbb{U}} \backslash \varnothing$ and a null-consistent, closed, convex set of priors $\mathcal{C} \subseteq \Delta$ such that $\succeq$ is represented by a continuous $V : \mathcal{A} \to \mathbb{R}$ with:*

$$V(f) = \min_{p \in \mathcal{C}} \sum_{s \in S} p(s) \min_{u \in \upsilon(s)} u(f(s)) \tag{1}$$

Note that, by Proposition 1, the first two axioms in part (i) imply the existence of $\succeq$-best-and-worst acts, so the final clause of (i) really only identifies $\overline{h}, \underline{h}$ as two such acts. So the second half of part (i) just says that Strong Uncertainty Aversion and EC-Independence hold with respect to $\succeq$-best-and-worst acts $\overline{h}, \underline{h} \in \mathcal{A}$; we use this shorter formulation below.

Theorem 1 tells us that the axioms yield a general state-dependent utility representation, incorporating imprecision of both beliefs and tastes. On the one hand, tastes for consequences are represented by a function υ assigning a set of utility functions to each state. To the extent that sets of utilities are involved, this captures imprecision of tastes; to the

[8] C-Independence states that, for every $f, g \in \mathcal{A}$, every constant act $c \in \mathcal{A}$ and every $\alpha \in (0,1)$, $f \succeq g$ if and only if $f_\alpha c \succeq g_\alpha c$.

extent that the set may depend on the state, state-dependence is also accommodated. The non-triviality, closure and convexity of v are familiar in the literature. The other properties translate the (double) role of $\overline{h}, \underline{h}$-mixtures as essentially constant acts: $\overline{h}, \underline{h}$-constancy says that they receive the same evaluation in all states, and $\overline{h}, \underline{h}$-precision ensures that they can be seen as 'sure options', insofar as hedging among them provides no particular advantage.[9] Since the set of utilities depends on the state, we call the non-trivial, closed, convex, $\overline{h}, \underline{h}$-constant, $\overline{h}, \underline{h}$-precise function v a $\overline{h}, \underline{h}$*-state-dependent multi-utility*, or simply a *state-dependent multi-utility* when $\overline{h}, \underline{h}$ is clear from the context.

On the belief side, the representation involves a set of priors $\mathcal{C}$; as such, it is a straightforward extension of the maxmin EU model (Gilboa and Schmeidler, 1989) to incorporate state-dependence of utility and imprecision of tastes. Closure and convexity of $\mathcal{C}$ are standard; null-consistency is a non-nullness condition, guaranteeing that if a state is non-null according to one probability measure in $\mathcal{C}$, then it is non-null according to all of them. Mimicking the terminology used for utilities, we refer to $\mathcal{C}$ as a *multi-prior*.

3.3 Uniqueness

To discuss the uniqueness of the representation, we require some terminology. A state-dependent multi-utility v representing $\succeq$ in tandem with multi-prior $\mathcal{C}$ according to (1) is said to be *tight* if there exists no state-dependent multi-utility v' representing $\succeq$ in tandem with $\mathcal{C}$ according to (1) such that $v'(s) \subseteq v(s)$ for all $s \in S$, with strict containment for some s. A tight state-dependent multi-utility is as small as a representation can be, in the sense that there are no extraneous members of the relevant sets.

Proposition 2. *Let $\succeq$ satisfy the Basic Axioms, State Consistency, Strong Uncertainty Aversion and EC-Independence with respect to $\succeq$-best-and-worst $\overline{h}, \underline{h} \in \mathcal{A}$. Then there exists a tight v and $\mathcal{C}$ representing $\succeq$ according to (1). Moreover, $\mathcal{C}$ is unique and v is unique up to positive affine transformation.*

A central challenge in the state-dependent utility literature (under expected utility) is to provide a suitably unique representation, separating in particular the (state-dependent) utility part from the belief side. This result shows that our representation has the desired uniqueness: the state-dependent multi-utility is unique up to positive affine transformation, and the multi-prior is unique.

[9] $\overline{h}, \underline{h}$-precision implies that the restriction of V to the set of $\overline{h}, \underline{h}$-mixtures is affine.

4 Special cases and Extensions

The three phenomena—state-dependence of utility, imprecision of tastes and imprecision of beliefs—can be straightforwardly separated in representation (1); indeed, any combination of them can be 'shut down', yielding potentially useful special cases. Moreover, the treatment of uncertainty can be extended beyond the multi-prior approach adopted in representation (1). The relevant special cases and extensions are summarized in Table 1, which involves the following two axioms.

Axiom A10 (State-wise Independence)**.** For every $x, y, z \in X$, $h \in \mathcal{A}$, $\alpha \in (0,1)$ and $s \in S$, $x_s h \geq y_s h$ if and only if $(x_\alpha z)_s h \geq (y_\alpha z)_s h$.

Axiom A11 (Restricted Independence with respect to $\overline{h}, \underline{h}$)**.** For all $f, g, h \in \mathcal{A}^{\overline{h},\underline{h}}$ and $\alpha \in (0,1)$, $f \geq g$ if and only if $f_\alpha h \geq g_\alpha h$.

The table is to be read in the context of the following result.

Proposition 3. *Let $\geq$ be a preference relation on $\mathcal{A}$. Then, for each row of Table 1, the following are equivalent:*

(i) $\geq$ satisfies the axioms in Theorem 1 augmented by (for the 1st three rows) or with the exception of (last row) the axiom in second column of Table 1

(ii) there exists a pair as stated in the third column of Table 1 such that $\geq$ is represented by a continuous $V : \mathcal{A} \to \mathbb{R}$ as specified in that column.

Moreover, any combination of the axiom additions or removals in the second column of Table 1 characterises the corresponding combination of the representations in the third column.

Finally, in each case, the uniqueness of the representation is the specification or natural generalisation of that in Proposition 2.[10]

We now discuss these characterisations in turn.

[10]More precisely: $\mathcal{C}$ and p are unique, tight $\mathcal{U}$, u and v are unique up to positive affine transformation, and tight α is unique up to the corresponding transformation (see Theorem 2 in Appendix A for details).

Table 1: Special cases and Extensions

Shut down	Add Axiom	Representation
State-dependence	Monotonicity	$$V(f) = \min_{p \in \mathcal{C}} \sum_{s \in S} p(s) \min_{u \in \mathcal{U}} u(f(s)) \quad (2)$$ $\mathcal{U} \subseteq \mathbb{U}$: a non-trivial, $\overline{h}, \underline{h}$-precise, closed, convex set of utility functions; $\mathcal{C}$ as in Theorem 1
Taste imprecision	State-wise Independence	$$V(f) = \min_{p \in \mathcal{C}} \sum_{s \in S} p(s) u(s, f(s)) \quad (3)$$ $u : S \times Z \to \Re$: a non-trivial, $\overline{h}, \underline{h}$-constant function that is continuous in its second coordinate; $\mathcal{C}$ as in Theorem 1
Belief imprecision	Restricted Independence with respect to $\geq$-best-and-worst $\overline{h}, \underline{h}$	$$V(f) = \sum_{s \in S} p(s) \min_{u \in \upsilon(s)} u(f(s)) \quad (4)$$ $p \in \Delta$: probability measure on S; υ as in Theorem 1

Extend	Remove Axiom	Representation
Treatment of uncertainty	EC-Independence with respect to $\geq$-best-and-worst $\overline{h}, \underline{h}$	$$V(f) = \min_{p \in \Delta,\ (a,b) \in \alpha(p)} \left(a \sum_{s \in S} p(s) \min_{u \in \upsilon(s)} u(f(s)) + b \right) \quad (5)$$ $\alpha : \Delta \to 2^{\Re_{>0} \times \Re}$: a non-trivial, null-consistent, grounded, calibrated, closed, convex function; ν as in Theorem 1[11]

[11] $\alpha : \Delta \to 2^{\Re_{>0} \times \Re}$ is: *non-trivial* if there exists $p \in \Delta$ such that $\alpha(p) \neq \varnothing$; *convex* if, whenever $(a, b) \in \alpha(p)$ and $(a', b') \in \alpha(p')$, then for all $\lambda \in [0, 1]$, $(\lambda a + (1 - \lambda)a', \lambda b + (1 - \lambda)b') \in \alpha\left(\frac{\lambda a}{\lambda a + (1-\lambda)a'} p + \frac{(1-\lambda)a'}{\lambda a + (1-\lambda)a'} p'\right)$; *closed* if, whenever $(a_n, b_n) \in \alpha(p_n)$ with $p_n \to p$ and $(a_n, b_n) \to (a, b) \in \Re_{>0} \times \Re$, then $(a, b) \in \alpha(p)$; *grounded* if there exists $p \in \Delta$ such that $(1, 0) \in \alpha(p)$; *calibrated* (with respect to υ) if, for all $p \in \Delta$, $(a, b) \in \alpha(p)$ and $z \in \upsilon(X)$, $az + b \geqslant z$; *null-consistent* (with respect to υ) if, for every p, q such that $\alpha(p), \alpha(q) \neq \varnothing$, $p(s) = 0$ and $q(s) \neq 0$ for some $s \in S$ if and only if, for each $r \in (int(\upsilon(X)))^S$, there exists $p' \in \Delta$ and $(a', b') \in \alpha(p')$ such that $a' \sum_{s \in S} p'(s) r_s + b' < a \sum_{s \in S} p(s) r_s + b$ for all $(a, b) \in \alpha(p)$. We adopt the convention that, when $\alpha(p)$ is empty, p is not involved in the minimisation.

4.1 'Shutting down' factors

The first row of Table 1 gives a general state-independent representation with imprecise beliefs and tastes. The set of utilities $\mathcal{U}$ is state-independent; hence the state-dependence of utility is 'shut down' by adding the standard Monotonicity axiom (Section 3.1).[12] The same representation is obtained by replacing State Consistency with the State Independence axiom (see Proposition 7 in Appendix A). In particular, unlike in the expected utility case, these two axioms are not equivalent in the presence of imprecise beliefs and tastes. The maxmin multi-utility representation over consequences in (2) is similar to that obtained by Maccheroni (2002) in the context of decision under risk, though his characterisation uses a weakened version of the Independence axiom in the place of Strong Uncertainty Aversion.

The second row involves what, to our knowledge, is the first precise state-dependent utility uncertainty averse representation. The function u is a standard (precise) state-dependent utility function as in the state-dependent utility literature; its uniqueness is comparable to that obtained in this literature (Karni, 1993a,b, 2011).[13] Hence taste imprecision is 'shut down' by the addition of a weakened independence axiom, applying only to preferences conditional on states.

The third row provides a state-dependent multi-utility representation with a single-prior belief. Hence, imprecision in beliefs (and thus uncertainty aversion) is 'shut down' by the restriction of the standard independence axiom to acts yielding mixtures of $\succeq$-best-and-worst $\overline{h}, \underline{h}$ as consequences in every state (i.e. acts in $\mathcal{A}^{\overline{h},\underline{h}}$; see Section 2).

The second clause of Proposition 3 implies that concurrent shut downs of several factors are obtained by combining the axioms in the table, yielding representations that can be read off from those provided. For instance, the combination of the axioms for the first and third rows characterises a single prior and state-independent multi-utility representation. (Riella, 2015, Thms 5 & 6) proposes a representation for precisely this case; though, unlike the representations involved here, it uses the certainty equivalents of consequences rather than

[12]The fact that Monotonicity yields the expected effects—namely state-independence of the multi-utility—is a non-trivial property of the representation. For instance, the representation $V(f) = \min_{p\in\mathcal{C}} \min_{u\in\mathcal{U}} \sum_{s\in S} p(s)u(f(s))$, which is the uncertainty averse counterpart of the incomplete preference imprecise belief and taste model due to Galaabaatar and Karni (2013), is basically incompatible with Monotonicity (Hill, 2017). In fact, that representation violates State Consistency.

[13]Other approaches (Karni et al., 1983; Karni and Schmeidler, 2016, 1993; Karni and Mongin, 2000) yield a weaker uniqueness: up to cardinal unit comparable transformation (Karni et al., 1983).

their utility values, in the style of Cerreia-Vioglio et al. (2015).

4.2 Extensions

The final row of Table 1 shows that dropping EC-Independence yields a representation with a more general treatment of uncertainty. It encompasses many of the main theories of uncertainty averse preferences proposed in the literature. One way to see this is to compare such models, which generally assume precise state-independent utility, with the corresponding special case of (5). By Proposition 3, removing EC-Independence and adding Monotonicity and State-wise Independence yields this case:

$$V(f) = \min_{p \in \Delta,\ (a,b) \in \alpha(p)} \left(a \sum_{s \in S} p(s) u(f(s)) + b \right) \tag{6}$$

where u is a precise state-independent utility, and α is as in the final row of Table 1.

This representation clearly contains not only maxmin EU preferences, but also other families such as variational preferences (Maccheroni et al., 2006) and confidence preferences (Chateauneuf and Faro, 2009) as special cases.[14] The α generalises similar sets of priors, 'ambiguity indices' or 'confidence functions' in these models. In fact, representation (6) is essentially the class of uncertainty averse preferences (Cerreia-Vioglio et al., 2011) that can be represented by a concave functional on $\mathcal{A}$ (Appendix A.1). Since the vast majority of existing uncertainty averse models involve concavifiable functionals[15]—including uncertainty averse Choquet (Schmeidler, 1989) and smooth ambiguity preferences (Klibanoff et al., 2005)—they belong to the family of preferences characterised by representation (6). They can thus be extended to incorporate state dependence and imprecision of tastes by mobilising the insight behind the EC-Independence axiom in Theorem 1: use essentially constant acts—mixtures of $\overline{h}, \underline{h}$—in the roles played by constant acts in standard axiomatisations.

Note that further extensions of (5) can be obtained by weakening Strong Uncertainty Aversion to standard Uncertainty Aversion (and adding some technical assumptions); in

[14]Precisely: variational preferences correspond to the case where $a = 1$ for all $p \in \Delta$ and $(a, b) \in \alpha(p)$, confidence preferences to the case where $u(\underline{h}(s)) = 0$ and $b = 0$ for all $p \in \Delta$ and $(a, b) \in \alpha(p)$, and maxmin EU to the case $a = 1$ and $b = 0$ for all $p \in \Delta$ and $(a, b) \in \alpha(p)$.

[15]A functional I is concavifiable if there exists a strictly monotone transformation of it which is concave. For instance, the smooth ambiguity functional under uncertainty aversion—often presented as $I(\varphi) = \phi^{-1}\left(\int_\Delta \phi\left(\int_S \varphi dp\right) du\right)$ for concave, strictly increasing ϕ—is concavifiable because $\phi \circ I$ is concave.

doing so, however, the simple multi-utility representation of tastes in (1) and (5) is lost. See Hill (2018, Appendix A) for details.

5 Comparative Statics

We now explore the comparative statics of our benchmark representation (1). (Similar results hold for the representations in Section 4; see Appendix A.4.) A popular notion of relative uncertainty aversion is formulated in terms of the propensity of decision makers to prefer an act over a constant act. Such comparisons rely on the constant act having a fixed, precise value across states for the decision maker, and so lose their intuition when utility may be state dependent or imprecise. Fortunately, in the context of (5), there is a suitable proxy for constant acts, namely mixtures of $\succeq$-best-and-worst acts $\overline{h}, \underline{h}$. Substituting them into the standard definition of comparative uncertainty aversion yields the following notion.

Definition 2. The (decision maker with) preference $\succeq^1$ with $\succeq^1$-best-and-worst acts $\overline{h}^1, \underline{h}^1$ is *more imprecision averse* than (one with) preference $\succeq^2$ with $\succeq^2$-best-and-worst acts $\overline{h}^2, \underline{h}^2$ if and only if, for each $f \in \mathcal{A}$ and $\alpha \in [0, 1]$,

$$f \succeq^1 \overline{h}^1_\alpha \underline{h}^1 \quad \text{implies} \quad f \succeq^2 \overline{h}^2_\alpha \underline{h}^2 \tag{7}$$

Definition 2 does not assume that the decision makers are using the same best and worst acts and hence it does not assume that the decision makers share the same essentially constant acts. For a given imprecision aversion comparison, this makes it difficult to disentangle aspects pertaining to the decision makers' tastes from those concerning their beliefs: what is a comparison of essentially constant acts for one decision maker—and hence one involving only tastes—may not be for the other. Indeed, to conduct comparative statics in the context of state-dependent utility, it is not uncommon to invoke some assumption of comparability between the decision makers' preferences (Karni, 1979, 1983a,b; Drèze and Rustichini, 2004). For our main result, we only require the mild assumption that the less imprecision-averse decision maker's best act $\overline{h}^2$ is considered a maximal act by the more imprecision-averse decision maker.

Proposition 4. *Let $\succeq^1$ and $\succeq^2$ be represented according to (1) by pairs of $\overline{h}^1, \underline{h}^1$- (respectively $\overline{h}^2, \underline{h}^2$-)state-dependent multi-utilities and multi-priors $(\upsilon^1, \mathcal{C}^1)$ and $(\upsilon^2, \mathcal{C}^2)$, and* *suppose that they are normalised so that* $\upsilon^1(X) = \upsilon^2(X)$. *Suppose that* $\overline{h}^2$ *is a maximal element of* $\succeq^1$*. Then the following are equivalent.*

(i) $\succeq^1$ *is more imprecision averse than* $\succeq^2$

(ii) $\mathcal{C}^2 \subseteq \mathcal{C}^1$ *and* $\upsilon^1(s) \leqslant \upsilon^2(s)$ *for all* $\succeq^1$*-non-null states* $s \in S$.

Imprecision aversion corresponds to simultaneous but separate comparisons of the belief and taste elements. Less imprecision-averse preferences have higher state-dependent multi-utilities and smaller multi-priors than more imprecision-averse ones. In particular, this means that the utilities involved in the representation of less imprecision-averse preferences are 'redundant' with respect to more imprecision-averse preferences: adding them does not change the fact of representing $\succeq^1$ (see also Proposition 8, Appendix A.4).

These effects on tastes and beliefs can be characterised separately by using more refined notions of comparative imprecision aversion.

Definition 3. Preference $\succeq^1$ with $\succeq^1$-best-and-worst acts $\overline{h}^1, \underline{h}^1$ is *more imprecision averse on consequences* than $\succeq^2$ with $\succeq^2$-best-and-worst acts $\overline{h}^2, \underline{h}^2$ if and only if, for every $\succeq^1$-non-null $s \in S$, $h \in \mathcal{A}$, $x \in X$ and $\alpha \in [0,1]$,

$$x_s h \succeq^1 (\overline{h}\,^1_\alpha\, \underline{h}\,^1)_s h \quad \text{implies} \quad x_s h \succeq^2 (\overline{h}\,^2_\alpha\, \underline{h}\,^2)_s h \tag{8}$$

Moreover, preference $\succeq^1$ is *more imprecision averse on states* than $\succeq^2$ if and only if, for every $\alpha \in [0,1]$, every $f \in \mathcal{A}^{\overline{h}^1, \underline{h}^1}$, and for $\hat{f} \in \mathcal{A}^{\overline{h}^2, \underline{h}^2}$ with $\hat{f}(s) = \overline{h}^2_\beta\, \underline{h}^2(s)$ if and only if $f(s) = \overline{h}^1_\beta\, \underline{h}^1(s)$ for all $s \in S$,

$$f \succeq^1 \overline{h}\,^1_\alpha\, \underline{h}\,^1 \quad \text{implies} \quad \hat{f} \succeq^2 \overline{h}\,^2_\alpha\, \underline{h}\,^2 \tag{9}$$

These notions are as to be expected. Imprecision aversion on consequences compares consequences with the 'essentially constant' mixtures of best and worst acts on each non-null state, but eschews comparisons of acts differing on several states. Imprecision aversion on states considers only acts f yielding 'essentially constant' consequences, and asks that if f is preferred to an essentially constant act by decision maker 1, then 2 retains the same preference. A complication with the latter condition is that the decision makers may have different essentially constant acts: because of this, it uses the 'equivalent' act to f but formulated in terms of decision maker 2's essentially constant acts ($\hat{f}$).

Proposition 5. *Let $\succeq^1$ and $\succeq^2$ be represented according to (1) by pairs of $\overline{h}^1, \underline{h}^1$- (respectively $\overline{h}^2, \underline{h}^2$-)state-dependent multi-utilities and multi-priors $(\upsilon^1, \mathcal{C}^1)$ and $(\upsilon^2, \mathcal{C}^2)$, and suppose that they are normalised so that $\upsilon^1(X) = \upsilon^2(X)$. Then:*

(i) *$\succeq^1$ is more imprecision averse on consequences than $\succeq^2$ if and only if $\upsilon^1(s) \leqslant \upsilon^2(s)$ for all $\succeq^1$-non-null $s \in S$.*

(ii) *$\succeq^1$ is more imprecision averse on states than $\succeq^2$ if and only if $\mathcal{C}^2 \subseteq \mathcal{C}^1$.*

These results are consistent with known characterisations of uncertainty attitudes. A standard notion of relative uncertainty aversion (Ghirardato and Marinacci, 2002) coincides with Definition 2 in the special case of precise state-independent utility—when constant acts are essentially constant in our sense. Proposition 4 then yields the known characterisation of uncertainty aversion for the maxmin-EU model (Ghirardato and Marinacci, 2002): $\succeq^1$ is more uncertainty averse than $\succeq^2$ if and only the latter's representing multi-prior is contained in the former's and the decision makers share the same (normalised) utilities. Hence the essential change in the notion of relative uncertainty aversion required under taste imprecision and state dependence of utility is the switch from standard constant acts to essentially constant ones (mixtures of $\overline{h}, \underline{h}$). Interestingly, imprecision aversion on states separates out the containment of multi-priors from the identity of (normalised) utilities, only implying the former (Proposition 5 (ii)).

Similarly, in the presence of state-independent utility, the notion of imprecision aversion on consequences can be reformulated with preferences over constant acts in the place of preferences conditional on states. It is a straightforward corollary of Proposition 5 (i) that, in this case, $\succeq^1$ is more imprecision averse on consequences than $\succeq^2$ if and only if there is the appropriate ordering of the representing multi-utilities in (2): $\mathcal{U}^1 \leqslant \mathcal{U}^2$. Whilst this is, to our knowledge, the first comparative static result for the maxmin multi-utility representation featuring in (1), it echoes results obtained for other multi-utility-style representations in the case of monetary lotteries (e.g. Cerreia-Vioglio et al., 2015). It also corroborates the interpretation of υ as state-dependent multi-utility.

In summary, the proposed model permits comparative statics, relating a state-dependent-utility extension of standard relative uncertainty aversion to concordant changes in the two primitives of model—the multi-prior and the state-dependent multi-utility. Moreover, the changes in the two primitives can be separately characterised, in terms of more refined imprecision aversions on consequences and on states.[16] This attests to the possibility of separately studying changes in the belief- and taste-factors under this model. Importantly, it allows a grasp on cases where beliefs and tastes *do not* move in the same direction. Consider a hurricane in a region that has never known natural disasters. Given the novelty of the experience, the inhabitants' tastes concerning the consequences of a hurricane would typically become more precise after the event. Moreover, since it may be unclear whether it was a one-off or a new trend, their ex ante (quasi-)certainty of the absence of hurricanes would naturally give way to a wider range of probabilities of future hurricanes ex post—betraying an increase in belief imprecision. In such cases, tastes and beliefs may move in opposite directions. Whereas the general notion of imprecision aversion does not apply, the more refined notions provide the foundations for the study of such changes. By allowing clearly understood and separate interventions on the belief and taste parameters, representation (1) provides the modeling tools required to investigate the consequences of such simultaneous yet diametric movements on, say, insurance purchasing.

[16]Indeed, Propositions 4 and 5 show that the general notion of imprecision aversion can be 'factorised' into the conjunction of more refined imprecision aversions on consequences and on states.

Appendix A Proofs

Throughout the Appendices, $\leqslant$ on $\mathbb{R}^n$ is the standard order, given by $a \leqslant b$ iff $a_i \leqslant b_i$ for all $1 \leqslant i \leqslant n$, for all $a, b \in \mathbb{R}^n$. Note that X is a closed convex subset of the Banach space $ca(Z)$ of signed Borel measures of bounded variation over Z, under the total variation norm (denoted here $\|\cdot\|$). Moreover, since Z is compact, $ca(Z)$ (under the total variation norm) is isometrically isomorphic to the topological dual of $\mathbb{U}$ (which, recall, is the set of continuous real functions on Z), under the duality $< f, x >= \int udx$. This duality generates the weak* topology on $ca(Z)$; unless specified, we adopt this topology throughout the Appendix. $\cdot$ is the standard scalar product of vectors in finite-dimensional vector spaces, and the duality for $(\mathbb{U}, ca(Z))$ (in particular, $u \cdot x = \int udx$ for $x \in X$, $u \in \mathbb{U}$).

A.1 General result

We begin with a preparatory proposition.

Proposition 6. *For every non-null $s \in S$, the relation $\succeq_s$ defined by, for all $x, y \in X$, $x \succeq_s y$ if and only if $x_s h \succeq y_s h$ for some $h \in \mathcal{A}$ is a continuous weak order.*

Proof. For all $x, y \in X$, if $x \not\succeq_s y$, then there exists no $h \in \mathcal{A}$ with $x_s h \not\succeq y_s h$, so $x_s h \prec y_s h$ for all $h \in \mathcal{A}$, so $x \preceq_s y$. Hence $\succeq_s$ is complete. Moreover, for all $x, y, z \in X$ if $x \succeq_s y$ and $y \succeq_s z$, then for some $h, h' \in \mathcal{A}$ $x_s h \succeq y_s h$ and $y_s h' \succeq z_s h'$. It follows by A6 that $y_s h \succeq z_s h$, so $x_s h \succeq z_s h$ by A1 and hence $x \succeq_s z$; so $\succeq_s$ is transitive. Finally, for any $y \in X$, $\{x \in \mathcal{A} : x \succeq_s y\} = \{x \in \mathcal{A} : \exists h \in \mathcal{A},\ x_s h \succeq y_s h\} = \{x \in \mathcal{A} : x_s h \succeq y_s h,\ \forall h \in \mathcal{A}\} = \bigcap_{h \in \mathcal{A}} \{x \in \mathcal{A} : x_s h \succeq y_s h\}$ (where the middle equality is due to A6), which is closed since, by A3, each of the sets $\{x \in \mathcal{A} : x_s h \succeq y_s h\}$ is. A similar argument holds for $\{x \in X : x \preceq_s y\}$, so $\succeq_s$ is continuous. □

The following is the fundamental technical result, which underpins the others. A pair (υ, α) representing $\succeq$ according to (5) below is *tight* if there exists no (υ', α') representing $\succeq$ according to (5) such that $\upsilon'(s) \subseteq \upsilon(s)$ and $\alpha'(p) \subseteq \alpha(p)$ for all $s \in S$ and $p \in \Delta$, with strict containment for some s or p.

Theorem 2. *Let $\succeq$ be a preference relation on $\mathcal{A}$, and $\overline{h}, \underline{h} \in \mathcal{A}$. The following are equivalent:*

(i) $\succeq$ *satisfies the Basic Axioms, State Consistency and Strong Uncertainty Aversion with respect to* $\overline{h}, \underline{h}$*, and* $\overline{h}, \underline{h}$ *are* $\succeq$*-best-and-worst acts*

(ii) there exists a tight pair consisting of a non-trivial, closed, convex, $\overline{h}, \underline{h}$*-constant,* $\overline{h}, \underline{h}$*-precise function* $\upsilon : S \to 2^{\mathbb{U}} \backslash \varnothing$ *and a non-trivial, null-consistent, grounded, calibrated, closed, convex function* $\alpha : \Delta \to 2^{\Re_{>0} \times \Re}$ *such that* $\succeq$ *is represented by a continuous* $V : \mathcal{A} \to \mathbb{R}$ *with:*

$$V(f) = \min_{p \in \Delta,\ (a,b) \in \alpha(p)} \left(a \sum_{s \in S} p(s) \min_{u \in \upsilon(s)} u(f(s)) + b \right) \tag{5}$$

Moreover, for any other tight pair (υ', α') *representing* $\succeq$ *according to (5), there exists* $\kappa \in \Re_{>0}$ *and* $\lambda \in \Re$ *such that* $\upsilon'(s) = \kappa \upsilon(s) + \lambda$ *for all non-null* $s \in S$*, and* $\alpha'(p) = \{(a, \kappa b + \lambda(1-a)) \mid \forall (a,b) \in \alpha(p)\}$ *for all* $p \in \Delta$*.*

We shall refer to a function $\alpha : \Delta \to 2^{\Re_{>0} \times \Re}$ with the properties specified in this Theorem as an ambiguity index.

Proof of Theorem 2. Consider firstly the (i) implies (ii) direction.

By State Consistency and the fact that $\underline{h}, \overline{h}$ are $\succeq$-best-and-worst, $\overline{h}(s) \succ_s \overline{h}(s)_\alpha\, \underline{h}(s)$ for all $\alpha \in (0,1)$, whence it follows from A1 and A6 that $\overline{h} \succ \overline{h}_\alpha\, \underline{h}$ for all $\alpha \in (0,1)$. We now show the following stochastic dominance property for $\overline{h}, \underline{h}$: for every $\alpha, \beta \in [0,1]$, $\alpha \geqslant \beta$ iff $\overline{h}_\alpha\, \underline{h} \succeq \overline{h}_\beta\, \underline{h}$. If $\alpha \geqslant \beta$, then $\overline{h}_\alpha\, \underline{h} = \overline{h}_{\frac{\alpha-\beta}{1-\beta}}(\overline{h}_\beta\, \underline{h})$. Since $\overline{h}_\beta\, \underline{h} \succeq \overline{h}_\beta\, \underline{h}$ and $\overline{h} \succeq \overline{h}_\beta\, \underline{h}$, it follows from A8 that $\overline{h}_\alpha\, \underline{h} \succeq \overline{h}_\beta\, \underline{h}$, as required. For the other direction, suppose for reductio that there exist $\alpha, \beta \in [0,1]$ with $\alpha < \beta$, and $\overline{h}_\alpha\, \underline{h} \succeq \overline{h}_\beta\, \underline{h}$. It follows from the previous argument that $\overline{h}_\alpha\, \underline{h} \sim \overline{h}_\beta\, \underline{h}$. Without loss of generality, we can assume that $[\alpha, \beta]$ is a maximal interval with this property, in the following sense: for every $\alpha' < \alpha$ and every $\beta' > \beta$, $\overline{h}_{\alpha'}\, \underline{h} \prec \overline{h}_\alpha\, \underline{h} \sim \overline{h}_\beta\, \underline{h} \prec \overline{h}_{\beta'}\, \underline{h}$. Since $\overline{h}_{\alpha'}\, \underline{h} \prec \overline{h}$ for all $\alpha' \in (0,1)$, $\beta < 1$. Take any $\gamma \in \left(0, \frac{\beta-\alpha}{1-\alpha}\right)$ (this set is non-empty since $\beta > \alpha$). So $\overline{h}_\gamma(\overline{h}_\alpha\, \underline{h}) = \overline{h}_\delta\, \underline{h}$ for some $\alpha < \delta < \beta$. By the previous result, $\overline{h}_\delta\, \underline{h} \sim \overline{h}_\alpha\, \underline{h} \sim \overline{h}_\beta\, \underline{h}$. However, since $\overline{h}_\beta\, \underline{h} \preceq \overline{h}_\alpha\, \underline{h}$, it follows from A8 that $\overline{h}_{\gamma+\beta(1-\gamma)}\, \underline{h} = \overline{h}_\gamma(\overline{h}_\beta\, \underline{h}) \preceq \overline{h}_\gamma(\overline{h}_\alpha\, \underline{h}) = \overline{h}_\delta\, \underline{h} \sim \overline{h}_\beta\, \underline{h}$, whence, by the previous result $\overline{h}_{\gamma+\beta(1-\gamma)}\, \underline{h} \sim \overline{h}_\beta\, \underline{h}$, contradicting the maximality of $[\alpha, \beta]$. So $\overline{h}_\alpha\, \underline{h} \prec \overline{h}_\beta\, \underline{h}$ whenever $\alpha < \beta$, as required.

Relying on this result, we apply standard arguments to obtain a real-valued functional representing $\succeq$. It follows from the fact that $\underline{h}, \overline{h}$ are $\succeq$-best-and-worst, A1 and A6 that $\overline{h} \succeq f \succeq \underline{h}$ for all $f \in \mathcal{A}$. It follows from this observation, the previous one and A3 that, for each $f \in \mathcal{A}$, there exists a unique $\alpha_f \in [0,1]$ with $f \sim \overline{h}_{\alpha_f}\, \underline{h}$. Define $V : \mathcal{A} \to [0,1]$ by $V(f) = \alpha_f$ for each $f \in \mathcal{A}$. By definition and A1, V represents $\succeq$. By A2, V is non-constant. It follows from A8 that, for all $f, g \in \mathcal{A}$, $\alpha \in [0,1]$, $V(f_\alpha g) \geqslant \alpha V(f) + (1-\alpha)V(g)$. So V is concave. By A3, V is continuous.

Now fix a non-null $s \in S$. We proceed as above, but now for $\succeq_s$. We first show that for every $\alpha, \beta \in [0,1]$, $\alpha \geqslant \beta$ iff $\overline{h}(s)_\alpha\, \underline{h}(s) \succeq_s \overline{h}(s)_\beta\, \underline{h}(s)$. If $\alpha \geqslant \beta$, then $\overline{h}_\alpha\, \underline{h} = \overline{h}_{\frac{\alpha-\beta}{1-\beta}}(\overline{h}_\beta\, \underline{h})$. Since $\overline{h}_\beta\, \underline{h} \succeq \overline{h}_\beta\, \underline{h}$ and $\overline{h}_s(\overline{h}_\beta\, \underline{h}) \succeq \overline{h}_\beta\, \underline{h}$, it follows from A8 for $\overline{h}, \underline{h}$ that $(\overline{h}_\alpha\, \underline{h})_s(\overline{h}_\beta\, \underline{h}) \succeq \overline{h}_\beta\, \underline{h}$, so, by A6, $\overline{h}(s)_\alpha\, \underline{h}(s) \succeq_s \overline{h}(s)_\beta\, \underline{h}(s)$ as required. For the other direction, suppose for reductio that there exist $\alpha, \beta \in [0,1]$ with $\alpha < \beta$, and $\overline{h}(s)_\alpha\, \underline{h}(s) \succeq_s \overline{h}(s)_\beta\, \underline{h}(s)$. It follows from the previous argument that $\overline{h}(s)_\alpha\, \underline{h}(s) \sim_s \overline{h}(s)_\beta\, \underline{h}(s)$. Without loss of generality, we can assume that $[\alpha, \beta]$ is a maximal interval with this property, in the same sense as above. By State Consistency and the fact that $\overline{h}, \underline{h}$ are $\succeq$-best-and-worst, $\overline{h}(s)_{\alpha'}\, \underline{h}(s) \prec_s \overline{h}(s)$ for all $\alpha' \in (0,1)$, so $\beta < 1$. Take any $\gamma \in \left(0, \frac{\beta-\alpha}{1-\alpha}\right)$ (this set is non-empty since $\beta > \alpha$). So $\overline{h}(s)_\gamma(\overline{h}(s)_\alpha\, \underline{h}(s)) = \overline{h}(s)_\delta\, \underline{h}(s)$ for some $\alpha < \delta < \beta$. By the previous result, $\overline{h}(s)_\delta\, \underline{h}(s) \sim_s \overline{h}(s)_\alpha\, \underline{h}(s) \sim_s \overline{h}(s)_\beta\, \underline{h}(s)$. However, since, by A6, $\overline{h}_\beta\, \underline{h} \preceq (\overline{h}_\alpha\, \underline{h})_s(\overline{h}_\beta\, \underline{h})$, it follows from A8 that $\overline{h}_{\gamma+\beta(1-\gamma)}\, \underline{h} = \overline{h}_\gamma(\overline{h}_\beta\, \underline{h}) \preceq \overline{h}_\gamma((\overline{h}_\alpha\, \underline{h})_s(\overline{h}_\beta\, \underline{h})) = (\overline{h}_\delta\, \underline{h})_s(\overline{h}_{\gamma+\beta(1-\gamma)}\, \underline{h}) \sim (\overline{h}_\beta\, \underline{h})_s(\overline{h}_{\gamma+\beta(1-\gamma)}\, \underline{h})$, whence, by the previous result $\overline{h}_{\gamma+\beta(1-\gamma)}\, \underline{h} \sim_s \overline{h}_\beta\, \underline{h}$, contradicting the maximality of $[\alpha, \beta]$. So $\overline{h}(s)_\alpha\, \underline{h}(s) \prec_s \overline{h}(s)_\beta\, \underline{h}(s)$ whenever $\alpha < \beta$, as required.

By this observation, Proposition 6 and the fact that $\underline{h}, \overline{h}$ are $\succeq$-best-and-worst, for every $x \in X$, there is a unique α_s^x such that $x \sim_s \overline{h}(s)_{\alpha_s^x}\, \underline{h}(s)$. Define the function $V_s : X \to [0,1]$ by $V_s(x) = \alpha_s^x$. By Proposition 6, V_s represents $\succeq_s$. Moreover, for any $x, y \in X$, by A6, $x_s(\overline{h}_{\alpha_s^x}\, \underline{h})_{s^c} \sim \overline{h}_{\alpha_s^x}\, \underline{h}$ and $y_s(\overline{h}_{\alpha_s^y}\, \underline{h})_{s^c} \sim \overline{h}_{\alpha_s^y}\, \underline{h}$, whence it follows from A8 that $(x_\gamma y)_s(\overline{h}_{\gamma\alpha_s^x+(1-\gamma)\alpha_s^y}\, \underline{h})_{s^c} \succeq \overline{h}_{\gamma\alpha_s^x+(1-\gamma)\alpha_s^y}\, \underline{h}$ for all $\gamma \in [0,1]$, so $V_s(x_\gamma y) \geqslant \gamma V_s(x) + (1-\gamma)V_s(y)$ and V_s is concave. By Proposition 6, V_s is (weak*-)continuous.

We have the following consequence.

Lemma A.1. *For each non-null $s \in S$, there exists a non-trivial $\overline{h}(s), \underline{h}(s)$-precise closed convex set of utility functions $\mathcal{U}_s$ such that $V_s(x) = \min_{u \in \mathcal{U}_s} u \cdot x$ for all $x \in X$. Moreover there exists such a set which is such that every other non-trivial $\overline{h}(s), \underline{h}(s)$-precise closed* *convex set of utility functions with these properties is a superset.*

Proof. Fix a non-null $s \in S$. Let $\mathbf{1} : Z \to \mathbb{R}$ be the constant function yielding value 1. We first extend V_s to the set of non-negative measures $ca^+(Z) \subset ca(Z)$: $\bar{V}_s : ca(Z) \to \Re$ is defined by $\bar{V}_s(\mu) = (\mathbf{1} \cdot \mu)V_s(\frac{\mu}{\mathbf{1}\cdot\mu})$. For $\mu, \nu \in ca^+(Z)$ and $\lambda \in [0, 1]$, we have:

$$\begin{aligned}
\bar{V}_s(\lambda\mu + (1-\lambda)\nu) =& \mathbf{1} \cdot (\lambda\mu + (1-\lambda)\nu)V_s\left(\frac{\lambda\mu + (1-\lambda)\nu}{\mathbf{1} \cdot (\lambda\mu + (1-\lambda)\nu)}\right) \\
=& \mathbf{1} \cdot (\lambda\mu + (1-\lambda)\nu)V_s\left(\frac{\lambda\mathbf{1} \cdot \mu}{\mathbf{1} \cdot (\lambda\mu + (1-\lambda)\nu)}.\frac{\mu}{\mathbf{1} \cdot \mu} + \frac{(1-\lambda)\mathbf{1} \cdot \nu}{\mathbf{1} \cdot (\lambda\mu + (1-\lambda)\nu).}\frac{\nu}{\mathbf{1} \cdot \nu}\right) \\
\geqslant& \mathbf{1} \cdot (\lambda\mu + (1-\lambda)\nu)\left(\frac{\lambda\mathbf{1} \cdot \mu}{\mathbf{1} \cdot (\lambda\mu + (1-\lambda)\nu)}V_s(\frac{\mu}{\mathbf{1} \cdot \mu}) + \frac{(1-\lambda)\mathbf{1} \cdot \nu}{\mathbf{1} \cdot (\lambda\mu + (1-\lambda)\nu)}V_s(\frac{\nu}{\mathbf{1} \cdot \nu})\right) \\
=& \lambda\bar{V}_s(\mu) + (1-\lambda)V_s(\nu)
\end{aligned}$$

where the inequality follows from the concavity of V_s. Hence $\bar{V}_s$ is concave. Since V_s is weak*-continuous (and the duality is weak*-continuous), it follows that $\bar{V}_s$ is weak*-continuous. It follows that $\bar{V}_s$ is superdifferentiable at every $\mu \in int(ca^+(Z))$ (Aliprantis and Border, 2007, Thm 7.12)—that is, for each such μ, there exists $u \in \mathbb{U}$ with $u \cdot \nu + \bar{V}_s(\mu) - u \cdot \mu \geqslant \bar{V}_s(\nu)$ for all $\nu \in ca^+(Z)$. Note that if this holds for $u \in \mathbb{U}$ at $\mu \in ca^+(Z)$ with $\mu \neq 0$, then we have, for $u' \in \mathbb{U}$ defined by $u'(z) = u(z) + \bar{V}_s(\mu) - u \cdot \mu$ for all $z \in Z$, that $u' \cdot \nu \geqslant \bar{V}_s(\nu)$ for all $\nu \in ca^+(Z)$ and $u' \cdot \mu = V_s(\mu)$. With slight abuse of terminology, we will refer to u' satisfying these conditions for $\mu \in ca^+(Z)$ with $\mu \neq 0$ as a subgradient at μ. It follows from the definition of $\bar{V}_s$ that u is a subgradient of $\mu \in X$ if and only if it is a subgradient of $\lambda\mu$, for all $\lambda > 0$.

X is weak*-compact because Z is compact (Aliprantis and Border, 2007, Thm 15.11), and since it linearly spans $ca(Z)$, $ca(Z)$ is weakly compactly generated in the sense of Phelps (1993, Defn 2.41). Since $\bar{V}_s$ is weak*-continuous (and the norm topology is stronger than the weak*-topology), it is norm-continuous. It follows from Phelps (1993, Thm 2.45) that the set of points in $int(ca^+(Z))$ where $\bar{V}_s$ is Gâteaux differentiable—and hence at which it has unique supergradients—is dense in $int(ca^+(Z))$. Let $\mathcal{U}_s$ be convex closure of the union of the subgradients at all $\mu \in int(ca^+(Z))$ where $\bar{V}_s$ is Gâteaux differentiable. It follows from the upper hemi-continuity of the superdifferential mapping (Phelps, 1993, Prop 2.5), that, for each point in $int(ca^+(Z))$, some supergradient at this point is contained in $\mathcal{U}_s$. It follows that $\bar{V}_s(\mu) = \min_{u \in \mathcal{U}_s} u \cdot \mu$ for all $\mu \in int(ca^+(Z))$, and, by the continuity of $\bar{V}_s$, this holds for all $\mu \in ca^+(Z)$. By construction $\mathcal{U}_s$ is closed and convex, and we have that $V_s(x) = \min_{u \in \mathcal{U}_s} u \cdot x$ for all $x \in X$. Since V_s is not a constant function, $\mathcal{U}_s$ is non-trivial. Since, by construction, V_s is linear on $\{\overline{h}_\alpha \underline{h} \mid \alpha \in [0,1]\}$, there exists $u \in \mathcal{U}_s$ supporting V_s at every point in $\{\overline{h}_\alpha \underline{h} \mid \alpha \in [0,1]\}$; it follows from this, and the fact that $\underline{h}, \overline{h}$ are $\succeq$-best-and-worst that $\mathcal{U}_s$ is $\overline{h}(s), \underline{h}(s)$-precise. Finally, any other $\mathcal{U}'_s$ representing V_s as stated must contain supergradients at points where $\bar{V}_s$ is Gâteaux differentiable, and hence be a superset of $\mathcal{U}_s$, as required.

□

Take any non-null $s \in S$, and for all null $s' \in S$, define $V_{s'} = V_s$ and $\mathcal{U}_{s'} = \mathcal{U}_s$. By construction, $V_s(\overline{h}(s)_\alpha \underline{h}(s)) = \alpha = V_{s'}(\overline{h}(s')_\alpha \underline{h}(s'))$ for every $\alpha \in [0,1]$ and $s, s' \in S$, so $\upsilon : S \to 2^{\mathbb{U}} \backslash \varnothing$ defined by $\upsilon(s) = \mathcal{U}_s$ for all $s \in S$, is $\overline{h}, \underline{h}$-constant. Moreover, by Lemma A.1, it is non-trivial, closed, convex and $\overline{h}, \underline{h}$-precise.

Let $B = [0,1]^S$. For every $f \in \mathcal{A}$, define $\hat{V}(f) \in B$ by $\hat{V}(f)(s) = V_s(f(s))$ for all $s \in S$. By A1 and A6, for every $f, g \in \mathcal{A}$, if $f(s) \sim_s g(s)$ for every $s \in S$, then $f \sim g$, so the functional $I : B \to \Re$ defined by $I(a) = V(f)$ for any f such that $\hat{V}(f) = a$ is well-defined. By definition $V(f) = I(\hat{V}(f))$.

Lemma A.2. *I is concave, continuous, monotonic, and normalised (ie. for all $z \in [0,1]$, $I(z^*) = z$, where z^* is the constant function in B taking value z). Moreover, for every $z, w \in [0,1]$ with $z \neq w$, $a \in B$ and $s \in S$, $I(z_s a) = I(w_s a)$ if and only if $s \in S$ is null.*[17]

Proof. The Lemma follows from standard arguments. Normalisation follows from the definition of I, V_s and A6. Monotonicity of I is immediate from A6. Continuity of I follows from the continuity of V and the fact that $\hat{V}$ is a quotient map. As concerns concavity, for every $a \in B$, let $h^a \in \mathcal{A}$ be such that $h^a(s) = \overline{h}_{a(s)} \underline{h}$ for all $s \in S$. By construction, $\hat{V}(h^a) = a$. Note moreover that, for every $a, b \in B$ and $\gamma \in [0,1]$, $\hat{V}(h^a_\gamma h^b) = \gamma a + (1-\gamma)b$. By the concavity of V, $I(\gamma a + (1-\gamma)b) = V(h^a_\gamma h^b) \geqslant \gamma V(h^a) + (1-\gamma)V(h^b) = \gamma I(a) + (1-\gamma)I(b)$, so I is concave. As concerns the final property, if $s \in S$ is null, then $V(x_s f) = V(y_s f)$ for all $x, y \in X$, $f \in \mathcal{A}$ by definition, and the corresponding property for I follows immediately. If $s \in S$ is non-null, then, since, as shown above $\overline{h}(s)_\alpha \underline{h}(s) \nsim_s \overline{h}(s)_\beta \underline{h}(s)$ for $\alpha \neq \beta$, it follows from A6 that

[17] $z_s a$ is defined in an analogous way to $x_s f \in \mathcal{A}$.

$V((\overline{h}(s)_\alpha\, \underline{h}(s)))_s f) \neq V((\overline{h}(s)_\beta\, \underline{h}(s))_s f)$ for all $f \in \mathcal{A}$; the corresponding property for I follows immediately.

□

Since B is closed, bounded and convex and I is concave, for each $a \in ri(B)$, there exists an affine functional $\phi : \Re^S \to \Re$ supporting I at a: i.e. such that $\phi(b) \geqslant I(b)$ for all $b \in B$ and $\phi(a) = I(a)$. Each such ϕ can be written as $\phi(b) = \eta \cdot b + \mu$ for some $\eta \in \Re^S$, $\mu \in \Re$. We first show that $\eta_s \geqslant 0$ for every $s \in S$. Take any $a \in ri(B)$ such that ϕ supports I at a, and consider $a^s \in B$ defined by $a^s(s) = a(s) + \epsilon$, $a^s(s') = a(s)$ for $s' \neq s$, where $\epsilon > 0$ such that $a(s) + \epsilon \in [0, 1]$. By the monotonicity of I, $I(a^s) \geqslant I(a)$; since ϕ supports I at a, we have that $\phi(a^s) = \eta \cdot a^s + \mu \geqslant I(a^s) \geqslant I(a) = \eta \cdot a + \mu$, so $\eta_s.\epsilon \geqslant 0$. Since this holds for every $s \in S$, we have that $\eta_s \geqslant 0$ for every $s \in S$. We now show that $\eta_s = 0$ if and only if s is null. Take any $a \in ri(B)$ such that ϕ supports I at a, and consider a^s as defined previously and $a^{-s} \in B$ defined by $a^{-s}(s) = a(s) - \delta$, $a^{-s}(s') = a(s)$ for $s' \neq s$, where $\delta > 0$ such that $a(s) - \delta \in [0, 1]$. If $\eta_s = 0$, then $\phi(a^s) = \phi(a) = I(a)$, whence it follows, since $\phi(a^s) \geqslant I(a^s)$, that $I(a^s) = I(a)$, so s is null, by the final clause in Lemma A.2. Conversely, by the final clause in Lemma A.2, if s is null, then $I(a^{-s}) = I(a)$; since $\phi(a^{-s}) = \eta \cdot a^{-s} + \mu = \phi(a) - \eta_s.\epsilon$, it follows from the fact that ϕ supports I at a that $\eta_s \leqslant 0$, and thus, in the light of the previous result, that $\eta_s = 0$, as required.

Let $\Phi = cl\,(conv\{(\eta, \mu) \mid \exists a \in ri(B) \text{ s.t. } \phi(b) = \eta \cdot b + \mu \text{ supports } I \text{ at } a\})$. By the continuity of the superdifferential mapping and of I, $I(a) = \min_{(\eta,\mu)\in\Phi} \eta \cdot a + \mu$ for all $a \in B$. Since I is differentiable on a dense subset of B and supergradients at differentiable points determine the supergradients elsewhere (Rockafellar, 1970, Theorem 25.6), this does not hold for any proper closed convex subset of Φ. Moreover, since I is normalised, for all $\mu' < 1$, $(0, \mu') \notin \Phi$. (It is clear from the construction that $(0, \mu') \notin \Phi$ for all $\mu' > 1$.) Hence, for every $(\eta, \mu) \in \Phi \backslash \{(0, 1)\}$, there is a unique $p \in \Delta \subseteq \Re^S$ and $\bar{a} \in \Re_{>0}$ such that $\eta = \bar{a}.p$. Let $\mathcal{P} = \{(p, \bar{a}, \bar{b}) \in \Delta \times \Re_{>0} \times \Re \mid (\bar{a}.p, \bar{b}) \in \Phi\backslash\{(0, 1)\}\}$.

By the previous observations, for all $a \in B$, $I(a) \leqslant \min_{(p,\bar{a},\bar{b})\in\mathcal{P}} (\bar{a}.p) \cdot a + \bar{b}$ with equality whenever $a \notin I^{-1}(1)$. Since I is normalised, it follows that $\bar{a}r + \bar{b} \geqslant r$ for all $(p, \bar{a}, \bar{b}) \in \mathcal{P}$ and $r \in [0, 1]$. We now show that there exists $p \in \Delta$ such that $(p, 1, 0) \in \mathcal{P}$. To establish this, note firstly that for every $r > 0$, there exists no $(p, a, b) \in \mathcal{P}$ such that $a > 1$ and $a.r + b = r$: if there were such a, b, then for any $r' \in [0, 1]$ with $r' < r$, $a.r' + b = r - a.(r - r') < r'$, contradicting the fact that $\mathcal{P}$ represents I and that I is normalised. Similarly, for every $r < 1$, there exists no $(p, a, b) \in \mathcal{P}$ such that $a < 1$ and $a.r + b = r$. Since I is normalised, it follows from the previous observation about the representation of I by $\mathcal{P}$ that there exists $p \in \Delta$ with $(p, 1, 0) \in \mathcal{P}$. Since, for any $p \in \Delta$ and $a \in B$, $p \cdot a \leqslant 1$, it follows from the fact that $\mathcal{P}$ represents I on $(I^{-1}(1))^c$ that, for every $(p, 1, 0) \in \mathcal{P}$ and $a \in I^{-1}(1)$, $p \cdot a = 1$. Hence $I(a) = \min_{(p,\bar{a},\bar{b})\in\mathcal{P}}(\bar{a}.p) \cdot a + \bar{b}$ for all $a \in B$.

Define $\alpha : \Delta \to \Re_{>0} \times \Re$ by $\alpha(p) = \{(a, b) \in \Re_{>0} \times \Re \mid (p, a, b) \in \mathcal{P}\}$. By the definition of $\mathcal{P}$, the properties of Φ (in particular non-emptiness, closure, convexity, the fact about null states) and the continuity of the superdifferential mapping, α is non-trivial, null-consistent, closed and convex. By the last two properties of $\mathcal{P}$ mentioned above, α is calibrated (with respect to υ) and grounded. So α is a non-trivial, null-consistent, grounded, calibrated, closed, convex function representing $\succeq$ along with υ according to (5), as required.

Since no proper closed convex subset of Φ represents I in the specified way, there is no closed, convex $\alpha' : \Delta \to \Re_{>0} \times \Re$ with $\alpha'(p) \subseteq \alpha(p)$ for all $p \in \Delta$ where the inclusion is proper for at least one $p \in \Delta$ that represents $\succeq$ along with υ according to (5). It follows from this and Lemma A.1 that the pair (υ, α) is tight.

Now consider the (ii) to (i) implication. It is standard for the Basic Axioms: Weak Order is immediate, Non-degeneracy follows from the non-triviality of υ and α and Continuity follows from the continuity of V. State Consistency follows immediately from the form of the representation, and the fact that α is null-consistent. By the null-consistency of α, for every non-null state $s \in S$, $x, y \in X$ and $h \in \mathcal{A}$, $x_s h \succeq y_s h$ if and only if $\min_{u\in\upsilon(s)} u(x) \geqslant \min_{u\in\upsilon(s)} u(y)$. Since υ is $\overline{h}, \underline{h}$-precise, it follows that, for every non-null $s \in S$, $\underline{h}(s) \prec_s (\overline{h}(s))_\beta(\underline{h}(s)) \prec_s \overline{h}(s)$ for all $\beta \in (0, 1)$ and $\underline{h}(s) \preceq_s x \preceq_s \overline{h}(s)$ for all $x \in X$. Hence $\overline{h}, \underline{h}$ are $\succeq$-best-and-worst acts. Since υ is $\overline{h}, \underline{h}$-constant and $\overline{h}, \underline{h}$-precise, for each $\beta \in [0, 1]$, $V(\overline{h}_\beta\, \underline{h}) = \beta V(\overline{h}) + (1 - \beta)V(\underline{h})$; and hence, by the representation, $f \succeq \overline{h}_\beta\, \underline{h}$ iff $a \sum_{s\in S} p(s)u_s(f(s)) + b \geqslant \beta V(\overline{h}) + (1 - \beta)V(\underline{h})$ for all $p \in \Delta$, $(a, b) \in \alpha(p)$ and $u_s \in \upsilon(s)$. Strong Uncertainty Aversion with respect to $\overline{h}, \underline{h}$ follows immediately from the form of the representation.

Finally, consider the uniqueness clause, and suppose that (υ, α), and (υ', α') are tight pairs of state-dependent multi-utilities and ambiguity indices representing $\succeq$. Let $V_{\upsilon,\alpha}$ and $V_{\upsilon',\alpha'}$ be the functionals defined from them according to (5). By Lemma A.3 below, we can assume without loss of generality that υ and υ' are calibrated and constant with respect to the same $\overline{h}, \underline{h}$. Let $\kappa \in \Re_{>0}$ and $\lambda \in \Re$ be such that $\kappa V_{\upsilon',\alpha'}(\underline{h}) + \lambda = V_{\upsilon,\alpha}(\underline{h})$ and

$\kappa V_{\upsilon',\alpha'}(\overline{h}) + \lambda = V_{\upsilon,\alpha}(\overline{h})$ (it is straightforward to show that such κ and λ exist). Define $V'' = \kappa V_{\upsilon',\alpha'} + \lambda$, $\upsilon'' = \kappa\upsilon' + \lambda$ and α'' by $\alpha''(p) = \{(a, \kappa b + \lambda(1-a)) \mid \forall(a,b) \in \alpha'(p)\}$ for all $p \in \Delta$; it is clear that V'', υ'' and α'' are related according to (5) and that V'' represents $\geq$. υ'' is $\overline{h}, \underline{h}$-precise and $\overline{h}, \underline{h}$-constant since (υ', α') is, and because υ is $\overline{h}, \underline{h}$-precise and $\overline{h}, \underline{h}$-constant, $V''(\overline{h}_\alpha\, \underline{h}) = V(\overline{h}_\alpha\, \underline{h})$ for all $\alpha \in [0,1]$. It follows by the fact that for each $f \in \mathcal{A}$ there exists an unique $\alpha \in [0,1]$ with $f \sim \overline{h}_\alpha\, \underline{h}$ and the fact that they both represent $\geq$ that $V'' = V$. Since υ and υ'' are $\overline{h}, \underline{h}$-constant, it follows that, for all $s, t \in S$, and $\beta \in [0,1]$, $V_s(\overline{h}(s)_\beta\, \underline{h}(s)) = V_t(\overline{h}(s)_\beta\, \underline{h}(s)) = V(\overline{h}_\beta\, \underline{h}) = V''(\overline{h}_\beta\, \underline{h}) = V''_t(\overline{h}(s)_\beta\, \underline{h}(s)) = V''_s(\overline{h}(s)_\beta\, \underline{h}(s))$, where $V_s(x) = \min_{u \in \upsilon(s)} u \cdot x$ and similarly for V''_s. Since, for each non-null $s \in S$ and $x \in X$, there exists a unique β with $x \sim_s \overline{h}(s)_\beta\, \underline{h}(s)$, it follows that $V''_s = V_s$ for each non-null $s \in S$. Since $\upsilon(s)$ is tight, by the reasoning in the proof of Lemma A.1, it is the convex closure of the set of supergradients of V_s; however, since $\upsilon''(s)$ is tight, the same holds for it, and so the two sets are identical. So $\upsilon(s) = \upsilon''(s) = \kappa\upsilon'(s) + \lambda$ for all non-null $s \in S$ as required.

Let $K = V(X) = V''(X)$; since υ and υ'' are $\overline{h}, \underline{h}$-constant, $K = \{\min_{u \in \upsilon(s)} u \cdot x \mid x \in X\} = \{\min_{u \in \upsilon''(s)} u \cdot x \mid x \in X\}$ for all non-null $s \in S$. Let I be the functional on K^S defined from V as in the proof of Theorem 2, and similarly for I'' and V''. Since $V = V''$, $I = I''$. By the construction in the proof of Theorem 2, since α is tight, it is generated by the set of supergradients of I; however, since α'' is tight, the same holds for it, so $\alpha = \alpha''$. So $\alpha = \{(a, \kappa b + \lambda(1-a)) \mid \forall(a,b) \in \alpha'(p)\}$ for all $p \in \Delta$, as required.

□

Lemma A.3. *Let (υ, α) and (υ', α') be pairs of $\overline{h}, \underline{h}$- (respectively $\overline{h}', \underline{h}'$-)state dependent multi-utilities and ambiguity indices representing $\geq$ according to* (5)*. Then $\overline{h}_\alpha\, \underline{h} \sim \overline{h}'_\alpha\, \underline{h}'$ for all $\alpha \in [0,1]$. It follows in particular that υ and υ' are both $\overline{h}, \underline{h}$-constant, $\overline{h}, \underline{h}$-precise and $\overline{h}', \underline{h}'$-constant, $\overline{h}', \underline{h}'$-precise.*

Proof. By Theorem 2 and its proof, the Strong Uncertainty Aversion holds with respect both to $\overline{h}, \underline{h}$ and $\overline{h}', \underline{h}'$. Since (υ', α') represents $\geq$, we have that $\overline{h}'$ and $\underline{h}'$ are maximal and minimal elements of $\geq$, respectively, and similarly for $\overline{h}$ and $\underline{h}$. So $\overline{h} \sim \overline{h}'$ and $\underline{h} \sim \underline{h}'$. It follows that, for any $\alpha \in [0,1]$, by Strong Uncertainty Aversion with respect to $\overline{h}, \underline{h}$, that $\overline{h}'_\alpha\, \underline{h}' \geq \overline{h}_\alpha\, \underline{h}$, and by Strong Uncertainty Aversion with respect to $\overline{h}', \underline{h}'$, that $\overline{h}_\alpha\, \underline{h} \geq \overline{h}'_\alpha\, \underline{h}'$, whence $\overline{h}_\alpha\, \underline{h} \sim \overline{h}'_\alpha\, \underline{h}'$, as required. The remaining clauses in the lemma follow immediately. □

A.2 Proofs of results in Section 3

Proof of Proposition 1. For every non-null $s \in S$, it follows from Proposition 6 and the compactness of X that there exists a minimal and maximal element of $\geq_s$; for each such s, let $\underline{c}_s$ and $\overline{c}_s$ respectively be such elements. Without loss of generality, we can assume that $\underline{c}_s$ and $\overline{c}_s$ are such that $\underline{c}_s <_s (\overline{c}_s)_\beta(\underline{c}_s) <_s \overline{c}_s$ for all $\beta \in (0,1)$. If this is not the case for some $\overline{c}_s$ and $\underline{c}_s$, take the maximum and minimum $(\overline{c}_s)_\beta(\underline{c}_s)$ for which it is; they exist by the continuity of $\geq_s$. Pick any non-null $s' \in S$, and define the acts $\underline{h}, \overline{h}$ by: $\underline{h}(s) = \underline{c}_s$ (respectively $\overline{h}(s) = \overline{c}_s$) for every non-null $s \in S$; and $\underline{h}(t) = \underline{c}_{s'}$ (resp. $\overline{h}(t) = \overline{c}_{s'}$) otherwise. By construction, $\overline{h}, \underline{h}$ are $\geq$-best-and-worst acts. □

Proof of Theorem 1. We first consider the (i) to (ii) implication. By Theorem 2, $\geq$ is represented according to (5) by a tight pair (υ, α). Let V and I be as in the proof of Theorem 2, so $I(c) = \min_{p\in\Delta,(a,b)\in\alpha(p)} (a.c \cdot p + b)$ for all $c \in V(\mathcal{A})^S$. By standard arguments (see for instance Gilboa and Schmeidler (1989)), it follows from EC-Independence that I is positive homogeneous ($I(\lambda c) = \lambda I(c)$ for all $c \in V(\mathcal{A})^S$, $\lambda \in \Re_{>0}$) and constant additive ($I(c + r^*) = I(c) + r$, where r^* is the constant function taking value $r \in \Re$ everywhere). Consider any p, a, b with $(a,b) \in \alpha(p)$ such that $ap + b$ is the unique support to I at some $c \in ri(V(\mathcal{A})^S)$; such points exist by the arguments in the proof of Theorem 2. So $ap \cdot c + b = I(c)$. Now consider $c + \epsilon^*$ for some $\epsilon \in \mathbb{B}$ with $|\epsilon|$ sufficiently small, so that $c + \epsilon^* \in V(\mathcal{A})^S$. By the representation $I(c + \epsilon^*) \leqslant ap \cdot (c + \epsilon^*) + b = ap \cdot c + b + a.\epsilon = I(c) + a.\epsilon$. But by the constant additivity of I, $I(c + \epsilon^*) = I(c) + \epsilon$. It follows, taking $\epsilon > 0$, that $a \geqslant 1$; however, taking the case of $\epsilon < 0$ implies that $a \leqslant 1$. So $a = 1$. Similarly, considering λc for $\lambda > 0$, $\lambda \neq 1$ such that $\lambda c \in V(\mathcal{A})^S$, we have $I(\lambda c) \leqslant ap \cdot (\lambda c) + b = \lambda(ap \cdot c + b) + (1-\lambda)b = \lambda I(c) + (1-\lambda)b$. Positive homogeneity implies that $I(\lambda c) = \lambda I(c)$. Again, taking $\lambda < 1$ implies that $b \geqslant 0$ whereas the case with $\lambda > 1$ implies that $b \leqslant 0$; so $b = 0$. It thus follows that at all points where I has a unique support, $a = 1$ and $b = 0$ for the supporting $ap + b$. It follows from the arguments in the proof of Theorem 2, and in particular the fact that α is determined by the closure of such points, that $a = 1$ and $b = 0$ for every (a,b) and $p \in \Delta$ such that $(a,b) \in \alpha(p)$. By the closure and convexity of α, it reduces to a closed convex set $\mathcal{C} \subseteq \Delta$, yielding representation (1). Since α is null-consistent, for every $p, q \in \mathcal{C}$ and $s \in S$, if $p(s) = 0$ and $q(s) \neq 0$, then for all $r \in int(\nu(X))^S$, $I(r) < \sum_{s\in S} p(s) r_s$, whence, by the constant additivity and positive homogeneity of I, this holds for all $r \in \nu(X)^S$, contradicting the fact that $p \in \mathcal{C}$

(and the construction of α). Hence, $\mathcal{C}$ is null-consistent, as required.

The only new case in the (ii) to (i) implication with respect to Theorem 2—concerning EC-Independence—is straightforward to show, once one notes that the (multi-)utility representation is affine on mixtures of $\overline{h}, \underline{h}$.

□

Proof of Proposition 2. Follows from the uniqueness clause of Theorem 2. □

A.3 Proofs of Results in Section 4

Proof of Proposition 3. Theorem 2 is the final row of Table 1. The lemmas below (all except the first of which rely on standard results) establish that adding the relevant axiom to those in Theorem 2 imposes specific properties on υ or α. The main clause of the Proposition follows as an immediate Corollary of this and Theorem 1, as does the second clause (about performing several additions or removals). The uniqueness clause follows from Theorem 2.

□

Lemma A.4. *Let $\geq$ be a preference relation satisfying the axioms in Theorem 2. $\geq$ satisfies Monotonicity if and only if it can be represented according to* (5) *with constant υ (i.e. $\upsilon(s) = \upsilon(s') = \mathcal{U}$ for all $s, s' \in \mathcal{S}$).*

Proof. The necessity of the axiom is straightforward to check; we consider sufficiency. Let $\geq_s$ be defined as in the proof of Theorem 2. We first show that $\geq_s = \geq_{s'}$ for all non-null $s, s' \in S$. Take any such s, s' and suppose, for $x, y \in X$, $x >_s y$. By A4, it follows from the fact that $\overline{h}(s')$ and $\underline{h}(s')$ are $\geq_{s'}$-maximal and $\geq_{s'}$-minimal elements in X that they are $\geq$-maximal and $\geq$-minimal elements of X, and hence of $\geq_s$-maximal and $\geq_s$-minimal elements, respectively. By A3, it follows that there exist $\alpha, \beta \in (0, 1)$, $\alpha > \beta$, such that $x >_s \overline{h}(s')_\alpha \underline{h}(s'), \overline{h}(s')_\beta \underline{h}(s') >_s y$. By A4, it follows that $x > \overline{h}(s')_\alpha \underline{h}(s'), \overline{h}(s')_\beta \underline{h}(s') > y$, and hence that $x \geq_{s'} \overline{h}(s')_\alpha \underline{h}(s'), \overline{h}(s')_\beta \underline{h}(s') \geq_{s'} y$. It follows from the stochastic dominance property for $\geq_s$ established in the proof of Theorem 2 that $x \geq_{s'} \overline{h}(s')_\alpha \underline{h}(s') >_{s'} \overline{h}(s')_\beta \underline{h}(s') \geq_{s'} y$, as required.

By the reasoning in the proof of Theorem 2, there exists a non-constant, continuous, concave functional $V : \mathcal{A} \to \Re$, linear on $\{\overline{h}_\beta \underline{h} \mid \beta \in [0, 1]\}$, representing $\geq$, and continuous concave functionals V_s, linear on $\{\overline{h}(s)_\beta \underline{h}(s) \mid \beta \in [0, 1]\}$, representing $\geq_s$ for each non-null $s \in S$. Take any such V_s. Since $\geq_s = \geq_{s'}$ for all non-null $s, s' \in S$, V_s represents $\geq_{s'}$. Moreover, for similar reasons, it represents the restriction of $\geq$ to X. Note that since V_s is linear on $\{\overline{h}(s)_\beta\, \underline{h}(s) \mid \beta \in [0,1]\}$, with $\overline{h}(s)$ and $\underline{h}(s)$ maximal and minimal elements of $\geq_{s'}$ respectively, it is a minimal concave representation of $\geq_{s'}$, in the sense of Debreu (1976) (see also Kannai (1977)): every other concave representation V' of $\geq_{s'}$ with $V'(X) = V(X)$ is such that $V'(x) \geqslant V(x)$ for all $x \in X$. Since the same holds for $V_{s'}$, and since minimal representations are unique (Kannai, 1977, pp11-13), it follows that $V_s = V_{s'}$, and more generally that $V_s = V_t$ for every non-null $s, t \in S$. In particular, it follows that, for every non-null $s, s' \in S$, $\overline{h}(s)_\beta\, \underline{h}(s) \sim_s \overline{h}(s')_\beta\, \underline{h}(s')$ for all $\beta \in [0,1]$. Since v in representation (5) can be taken to be $\overline{h}, \underline{h}$-constant and tight, it follows that $v(s) = v(t)$ for all non-null $s, t \in \mathcal{S}$. Setting $v(t) = \mathcal{U} = v(s)$ for all $t \in S$ and any non-null $s \in S$ yields the desired representation.

□

Lemma A.5. *Let $\geq$ be a preference relation satisfying the axioms in Theorem 2. $\geq$ satisfies State-wise Independence if and only if it can be represented according to* (5) *with singleton-valued v (i.e. $v(s)$ is a singleton, for all $s \in \mathcal{S}$).*

Proof. The necessity of the axiom is straightforward. As concerns sufficiency, by State-wise Independence, $\geq_s$ satisfies the standard Independence axiom for each state $s \in S$. It follows by standard arguments that the V_s defined in the proof of Theorem 2 is affine for every $s \in S$, so $v(s)$, being tight, is a singleton for all non-null $s \in S$. $v(s)$ can thus be taken as a singleton for all $s \in S$ (for instance, by setting v on null states as in the proof of Theorem 2).

□

Lemma A.6. *Let $\geq$ be a preference relation satisfying the axioms in Theorem 2. $\geq$ satisfies Restricted Independence with respect to $\geq$-best-and-worst $\overline{h}, \underline{h}$ if and only if it can be represented according to* (5) *with singleton $\mathcal{C}$.*

Proof. The necessity of the axiom is straightforward. Their sufficiency is a direct extension of the proof of Theorem 1, noting that Restricted Independence implies that I is affine, and hence, by standard arguments, is generated by a (single) probability measure p.

□

Proposition 7. *In the presence of the Basic Axioms and Strong Uncertainty Aversion with respect to $\succeq$-best-and-worst $\overline{h}, \underline{h}$, Monotonicity and State Consistency are equivalent to State Independence.*

Proof of Proposition 7. State Independence clearly implies State Consistency (taking $t = s$). Moreover, in the presence of Weak Order, State Independence implies that, if $x \succeq y$, then $x_s f \succeq y_s f$ for every non-null $s \in S$ and $f \in \mathcal{A}$: if this were not the case, then $x_s f \prec y_s f$ for some and hence every non-null $s \in S$ (by State Independence), whence $x \prec x_{s_1} y \prec x_{\{s_1,s_2\}} y \prec \cdots \prec y$ (with indifferences for null states), contradicting the fact that $x \succeq y$. So, for $f, g \in \mathcal{A}$ satisfying the conditions in the Monotonicity axiom, $f \succeq g_{s_1} f \succeq g_{\{s_1,s_2\}} f \succeq \cdots \succeq g$, where each step is an application of the previous fact; hence State Independence implies Monotonicity in the presence of Weak Order.

The other direction was established by the reasoning at the beginning of the proof of Lemma A.4. □

A.4 Proofs of Results in Section 5

Proposition 4 follows directly from Proposition 8 (and the proof of Theorem 1).

Proposition 8. *Let $\succeq^1$ and $\succeq^2$ be represented according to (5) by pairs of tight $\overline{h}^1, \underline{h}^1$- (respectively $\overline{h}^2, \underline{h}^2$-)state-dependent multi-utilities and ambiguity indices (υ^1, α^1) and (υ^2, α^2), and suppose that they are normalised so that $\upsilon^1(X) = \upsilon^2(X)$. Suppose that $\overline{h}^2$ is a maximal element of $\succeq^1$. Then the following are equivalent.*

- *(i)* $\succeq^1$ *is more imprecision averse than* $\succeq^2$
- *(ii)* $(\upsilon^1 \cup \upsilon^2, \alpha^1 \cup \alpha^2)$ *represents* $\succeq^1$ *according to (5)*
- *(iii)* $\alpha^1 \leqslant \alpha^2$ *and* $\upsilon^1(s) \leqslant \upsilon^2(s)$ *for all* $\succeq^1$*-non-null states* $s \in S$.

where, $\alpha^1 \leqslant \alpha^2$ if and only if, for every $c \in (\upsilon^1(X))^S$ and $(a, b) \in \alpha^2(p)$ for $p \in \Delta$, there exists $(a', b') \in \alpha^1(p')$ for $p' \in \Delta$ such that $a' \sum_{s\in S} p'(s).c_s + b' \leqslant a \sum_{s\in S} p(s).c_s + b$.

Furthermore, (υ^1, α^1) is the unique tight subset of $(\upsilon^1 \cup \upsilon^2, \alpha^1 \cup \alpha^2)$ representing $\succeq^1$.

Proof of Proposition 8. It is clear that (iii) implies (ii), once one notes that the values of $\upsilon^1 \cup \upsilon^2$ are immaterial on $\succeq^1$-null states. We now show that (ii) implies (i). Define the representing functionals V^1 and V^2 from (υ^1, α^1) and (υ^2, α^2) according to (5); since, as shown in the proof of Theorem 2, $\upsilon^1(X) = V^1(\mathcal{A})$, it follows from the normalisation assumption that $V^1(\mathcal{A}) = V^2(\mathcal{A})$. We now show, by a relatively standard argument, that $\succeq^1$ is more imprecision averse than $\succeq^2$ iff $V^1(f) \leqslant V^2(f)$ for all $f \in \mathcal{A}$. By the representation and the $\overline{h}^{\,i}, \underline{h}^{\,i}$-precision of the (υ^i, α^i) (for $i = 1, 2$), for every $f \in \mathcal{A}$, $f \sim^1 \overline{h}^{\,1}_{V^1(f)}\, \underline{h}^{\,1}$ and $f \sim^2 \overline{h}^{\,2}_{V^2(f)}\, \underline{h}^{\,2}$. If $\succeq^1$ is more imprecision averse, then $f \sim^1 \overline{h}^{\,1}_{V^1(f)}\, \underline{h}^{\,1}$ implies that $f \succeq^2 \overline{h}^{\,2}_{V^1(f)}\, \underline{h}^{\,2}$, from which it follows, by the stochastic dominance property for $\overline{h}^{\,2}, \underline{h}^{\,2}$ (see proof of Theorem 2), that $V^2(f) \geqslant V^1(f)$. Conversely, if $V^2(f) \geqslant V^1(f)$, then by the same stochastic dominance property, $f \succeq^1 \overline{h}^{\,1}_{\alpha}\, \underline{h}^{\,1}$ iff $V^1(f) \geqslant \alpha$, and this implies that $V^2(f) \geqslant \alpha$ and hence $f \succeq^2 \overline{h}^{\,2}_{\alpha}\, \underline{h}^{\,2}$, establishing the claim. It follows that (i) holds iff $V^1(f) \leqslant V^2(f)$ for all $f \in \mathcal{A}$, which is the case iff $\min\{V^1(f), V^2(f)\} = V^1(f)$ for all $f \in \mathcal{A}$. Since (ii) implies that $\min\{V^1(f), V^2(f)\} = V^1(f)$ for all $f \in \mathcal{A}$, it implies (i).

We now show that (i) implies (iii); henceforth, assume (i) to hold. Let V^1 and V^2 be the representing functionals from (υ^1, α^1) and (υ^2, α^2) defined above, and let $V^1_s(x) = \min_{u \in \upsilon^1(s)} u(x)$ for all $x \in X$ and $s \in S$, and similarly for V^2_s. We first show that $\overline{h}^1_\alpha\, \underline{h}^1 \sim^1 \overline{h}^2_\alpha\, \underline{h}^2$ for all $\alpha \in [0, 1]$. Recall firstly that $\overline{h}^2 \succeq^1 f$ for all $f \in \mathcal{A}$ (it is a maximal element of $\succeq^1$), and note that $\underline{h}^2$ is a minimal element of $\succeq^1$: for if not, there would exist $\alpha > 0$ such that $\underline{h}^2 \succeq^1 \overline{h}^1_\alpha\, \underline{h}^1$, whence $\underline{h}^2 \succeq^2 \overline{h}^2_\alpha\, \underline{h}^2$ by the fact that $\succeq_1$ is more imprecision averse than $\succeq_2$, contradicting the stochastic dominance property of $\succeq^2$ with respect to $\overline{h}^2_\alpha\, \underline{h}^2$ (see proof of Theorem 2). By the stochastic dominance property of $\succeq^1$ with respect to $\overline{h}^1_\alpha\, \underline{h}^1$ (see proof of Theorem 2), for every $\alpha \in [0, 1]$, there exists a unique $\beta_\alpha \in [0, 1]$ with $\overline{h}^2_\alpha\, \underline{h}^2 \sim^1 \overline{h}^1_{\beta_\alpha}\, \underline{h}^1$; let $\mu(\alpha) = \beta_\alpha$ for all $\alpha \in [0, 1]$. Since V_1 is concave and continuous, and $V_1(\overline{h}^1_\beta\, \underline{h}^1) = \beta$ for all $\beta \in [0, 1]$, μ is a concave continuous function. Since $\overline{h}^2$ and $\underline{h}^2$ are maximal and minimal elements of $\succeq^1$ respectively, $\mu(0) = 0$ and $\mu(1) = 1$. Finally, if $\mu(\alpha) > \alpha$, then, by the stochastic dominance property of $\succeq^2$ with respect to $\overline{h}^2_\alpha\, \underline{h}^2$, $\overline{h}^2_{\mu(\alpha)}\, \underline{h}^2 \succ^2 \overline{h}^2_\alpha\, \underline{h}^2$, whence it follows, by the fact that $\succeq^1$ is more imprecision averse than $\succeq^2$, that $\overline{h}^1_{\mu(\alpha)}\, \underline{h}^1 \succ^1 \overline{h}^2_\alpha\, \underline{h}^2$, contradicting the definition of μ. So $\mu(\alpha) \leqslant \alpha$ for all $\alpha \in [0, 1]$. Since μ is concave, it follows that is the identity function, as required.

It follows that $\succeq_1$ satisfies Strong Uncertainty Aversion with respect to $\overline{h}^2, \underline{h}^2$.

Observe furthermore that for every $s \in S$, if s is $\succeq^1$-non-null, then it is $\succeq^2$-non-null. For if s is $\succeq^1$-non-null, then by A6, $\overline{h}^2_s\, \underline{h}^2 \succ^1 \underline{h}^1$, so $\overline{h}^2_s\, \underline{h}^2 \succeq^1 \overline{h}^1_\alpha\, \underline{h}^1$ for some $\alpha > 0$. Since $\succeq^1$ is more imprecision averse, $\overline{h}^2_s\, \underline{h}^2 \succeq^2 \overline{h}^2_\alpha\, \underline{h}^2 \succ \underline{h}^2$, and hence s is $\succeq^2$-non-null. Now we have the following claim.

Claim 1. *For every $\succeq^1$-non-null $s \in S$, $\overline{h}^1_\alpha\,\underline{h}^1 \sim^1_s \overline{h}^2_\alpha\,\underline{h}^2$.*

Proof. Fix a $\succeq^1$-non-null $s \in S$, and let $\succeq^1_s$ be as defined in Proposition 6. Since v^1 is $\overline{h}^1, \underline{h}^1$-constant and $\succeq^1_s$ is represented by V^1_s, $\succeq^1_s$ satisfies the stochastic dominance property with respect to $\overline{h}^1_\alpha\,\underline{h}^1$: for every $\alpha, \beta \in [0,1]$, $\alpha \geqslant \beta$ iff $\overline{h}^1_\alpha\,\underline{h}^1 \succeq^1_s \overline{h}^1_\beta\,\underline{h}^1$. Since $\overline{h}^2$ is a maximal element of $\succeq^1$, it follows that $\overline{h}^2(s)$ is a maximal element of $\succeq^1_s$; similarly $\underline{h}^2(s)$ is a minimal element of $\succeq^1_s$. We now show that the stochastic dominance property holds for $\succeq^1_s$ with respect to $\{\overline{h}^2_\alpha\,\underline{h}^2\}$. Let $\alpha > \beta$; by the stochastic dominance property for $\succeq^2_s$ (see proof of Theorem 2), $\overline{h}^2_\alpha\,\underline{h}^2 \succ^2_s \overline{h}^2_\beta\,\underline{h}^2$, and so $\overline{h}^2_\alpha\,\underline{h}^2 \succ^2 (\overline{h}^2_\beta\,\underline{h}^2)_s(\overline{h}^2_\alpha\,\underline{h}^2)$, by A6. It follows from the fact that $\succeq^1$ is more imprecision averse and the observation above that $\overline{h}^2_\alpha\,\underline{h}^2 \sim^1 \overline{h}^1_\alpha\,\underline{h}^1 \succ^1 (\overline{h}^2_\beta\,\underline{h}^2)_s(\overline{h}^2_\alpha\,\underline{h}^2)$. Hence $\overline{h}^2_\alpha\,\underline{h}^2 \succ^1_s \overline{h}^2_\beta\,\underline{h}^2$. For the other direction, suppose that $\overline{h}^2_\beta\,\underline{h}^2 \succeq^1_s \overline{h}^2_\alpha\,\underline{h}^2$. So $(\overline{h}^2_\beta\,\underline{h}^2)_s(\overline{h}^2_\alpha\,\underline{h}^2) \succeq^1 \overline{h}^2_\alpha\,\underline{h}^2 \sim^1 \overline{h}^1_\alpha\,\underline{h}^1$, whence, by the fact that $\succeq^1$ is more imprecision averse, $(\overline{h}^2_\beta\,\underline{h}^2)_s(\overline{h}^2_\alpha\,\underline{h}^2) \succeq^2 \overline{h}^2_\alpha\,\underline{h}^2$ and hence $\overline{h}^2_\beta\,\underline{h}^2 \succeq^2_s \overline{h}^2_\alpha\,\underline{h}^2$. It follows from the stochastic dominance property for $\succeq^2_s$ that $\beta \geqslant \alpha$. So $\alpha > \beta$ iff $\overline{h}^2_\alpha\,\underline{h}^2 \succ^1_s \overline{h}^2_\beta\,\underline{h}^2$ as required.

It follows from this property, and the fact that $\succeq^1$ satisfies Strong Uncertainty Aversion with respect to $\overline{h}^2_\alpha\,\underline{h}^2$ (as well as State Consistency) that, by the reasoning in the proof of Theorem 2, there is a continuous concave functional $\bar{V}_s : X \rightarrow [0,1]$ representing $\succeq^1_s$ which is linear on $\{\overline{h}^2_\alpha\,\underline{h}^2 : \alpha \in [0,1]\}$. Since this function is linear on a set on which it takes values ranging over its whole co-domain, it is minimal in the sense of Debreu (1976); Kannai (1977). However, the same holds for V^1_s, which is (by the proof of Theorem 2) a continuous concave functional repxresenting $\succeq^1_s$ which is linear on $\{\overline{h}^1_\alpha\,\underline{h}^1 : \alpha \in [0,1]\}$. Since minimal concave representations are unique up to positive affine transformation (Kannai, 1977), and $\bar{V}_s(X) = V^1_s(X)$, we have that $\bar{V}_s = V^1_s$. It follows in particular that V^1_s is linear on both $\{\overline{h}^1_\alpha\,\underline{h}^1 : \alpha \in [0,1]\}$ and $\{\overline{h}^2_\alpha\,\underline{h}^2 : \alpha \in [0,1]\}$, taking the value 1 and $\alpha = 1$ and 0 at $\alpha = 0$ in both cases, so $V^1_s(\overline{h}^1_\alpha\,\underline{h}^1) = V^1_s(\overline{h}^2_\alpha\,\underline{h}^2)$ for all $\alpha \in [0,1]$. Since V^1_s represents $\succeq^1_s$, it follows that $\overline{h}^1_\alpha\,\underline{h}^1 \sim^1_s \overline{h}^2_\alpha\,\underline{h}^2$ for all $\alpha \in [0,1]$, as required.

□

We now show that $\succeq^1$ is more imprecision averse on consequences than $\succeq^2$, in the sense of Definition 3. For every $x \in X$ and $\alpha \in [0,1]$, by Claim 1 (and Weak Order applied repeatedly), $x_s(\overline{h}^2_\alpha\,\underline{h}^2) \sim^1 x_s(\overline{h}^1_\alpha\,\underline{h}^1)$. So the fact that $\succeq^1$ is more imprecision averse implies that, for any $\succeq_1$-non-null $s \in S$, whenever $x_s h \succeq^1 (\overline{h}^1_\alpha\,\underline{h}^1)_s h$, then $x_s(\overline{h}^2_\alpha\,\underline{h}^2) \sim^1 x_s(\overline{h}^1_\alpha\,\underline{h}^1) \succeq^1 \overline{h}^1_\alpha\,\underline{h}^1$ (by A6), and hence $x_s(\overline{h}^2_\alpha\,\underline{h}^2) \succeq^2 \overline{h}^2_\alpha\,\underline{h}^2$ (by Imprecision Aversion), and so $x_s h \succeq^2 (\overline{h}^2_\alpha \underline{h}^2)_s h$. So $\succeq^1$ is more imprecision averse on consequences than $\succeq^2$; by Proposition 5, $\upsilon_1(s) \leqslant \upsilon_2(s)$ for all $\succeq^1$-non-null states, as required.

Finally, we show that $\succeq_1$ is more imprecision averse for states than $\succeq_2$. Consider any $f \in \mathcal{A}^{\overline{h}^1, \underline{h}^1}$. By Claim 1, $f \sim^1 \hat{f}$, with the latter act defined as in Definition 3. So the fact that $\succeq^1$ is more imprecision averse implies that, whenever $\hat{f} \sim^1 f \succeq^1 \overline{h}^1_\alpha \underline{h}^1$, then $\hat{f} \succeq^2 \overline{h}^2_\alpha \underline{h}^2$. Hence $\succeq_1$ is more imprecision averse for states than $\succeq_2$. By Proposition 5, $\alpha_1 \leqslant \alpha_2$ as required.

The uniqueness clause follows from Theorem 2 and the fact that the representations by (υ^1, α^1) and $(\upsilon^1 \cup \upsilon^2, \alpha^1 \cup \alpha^2)$ give the same range of values.

□

Proposition 5 follows from the following proposition.

Proposition 9. *Let $\succeq^1$ and $\succeq^2$ be represented according to (5) by pairs of $\overline{h}^1, \underline{h}^1$- (respectively $\overline{h}^2, \underline{h}^2$-)state-dependent multi-utilities and ambiguity indices (υ^1, α^1) and (υ^2, α^2), and suppose that they are normalised so that $\upsilon^1(X) = \upsilon^2(X)$. Then:*

(i) $\succeq^1$ is more imprecision averse on consequences than $\succeq^2$ if and only if $\upsilon^1(s) \leqslant \upsilon^2(s)$ for all $\succeq^1$-non-null $s \in S$.

(ii) $\succeq^1$ is more imprecision averse on states than $\succeq^2$ if and only if $\alpha^1 \leqslant \alpha^2$.

Proof of Proposition 9. Part (i) For each $s \in S$, define the functionals $V^1_s : X \to \Re$ and $V^2_s : X \to \Re$ by $V^1_s(x) = \min_{u \in \upsilon^1(s)} u(x)$ and similarly for V^2_s and υ^2. By the assumption and Proposition 2, we can assume without loss of generality that $V^1_s(\overline{h}_1) = V^2_s(\overline{h}_2) = 1$ and $V^1_s(\underline{h}_1) = V^2_s(\underline{h}_2) = 0$ for all $s \in S$. We first show that $\succeq^1$ is more imprecision averse on consequences than $\succeq^2$ iff $V^1_s(x) \leqslant V^2_s(x)$ for all $x \in X$ and $\succeq^1$-non-null $s \in S$. By the representation, the previous normalisation and the $\overline{h}^i, \underline{h}^i$-precision and -constancy of the υ^i, for every $x \in \mathcal{A}$, $x \sim^1_s \overline{h}^1_{V^1_s(x)} \underline{h}^1$ and $x \sim^2_s \overline{h}^2_{V^2_s(x)} \underline{h}^2$. If $\succeq^1$ is more imprecision averse on consequences, then $x \sim^1_s \overline{h}^1_\alpha \underline{h}^1$ implies that $x \succeq^2_s \overline{h}^2_\alpha \underline{h}^2$, from which it follows, by the stochastic dominance property for $\succeq^2_s$ with respect to $\overline{h}^2, \underline{h}^2$ (see proof of Theorem 2), that $V^2_s(x) \geqslant V^1_s(x)$. Conversely, if $V^2_s(x) \geqslant V^1_s(x)$ for all $x \in X$, then by the same stochastic dominance property, $x \succeq^1_s \overline{h}^1_\alpha \underline{h}^1$ iff $V^1_s(x) \geqslant \alpha$, and this implies that $V^2_s(x) \geqslant \alpha$ and hence $x \succeq^2_s \overline{h}^2_\alpha \underline{h}^2$; since this holds for every $\succeq^1$-non-null $s \in S$, it establishes the claim. We have thus established that $\succeq^1$ is more imprecision averse on consequences iff $V_s^1(x) \leqslant V_s^2(x)$ for all $x \in X$ and $\succeq^1$-non-null $s \in S$. This is the case iff $\min\{V_s^1(x), V_s^2(x)\} = V_s^1(x)$ for all $x \in X$ and $\succeq^1$-non-null $s \in S$. By the definition of V_s^i, we have that $\min\{V_s^1(x), V_s^2(x)\} = \min_{u \in \upsilon^1(s) \cup \upsilon^2(s)} u(x)$ for all $x \in X$, so $\succeq^1$ is more imprecision averse on consequences iff $\min_{u \in \upsilon^1(s)} u(x) = \min_{u \in \upsilon^1(s) \cup \upsilon^2(s)} u(x)$ for all $x \in X$ and $\succeq^1$-non-null $s \in S$, and this is the case if and only if $\upsilon^1(s) \leqslant \upsilon^2(s)$ for all $\succeq^1$-non-null $s \in S$, as required.

Part (ii) By Proposition 2 and the normalisation of the υ^i, we can assume without loss of generality that $\upsilon^i(X) = [0,1]$ for $i \in \{1,2\}$. Define $\hat{V}^i : \mathcal{A} \to [0,1]^S$ by $\hat{V}^i(f)(s) = \min_{u \in \upsilon^i(s)} u(f(s))$ for all $f \in \mathcal{A}$ and $i \in \{1,2\}$. Note that, by the $\overline{h}^i, \underline{h}^i$-precision and the fact that the υ^i are $\overline{h}^i, \underline{h}^i$-constant, $\hat{V}^1(f)(s) = \alpha$ whenever $f(s) = \overline{h}^1_\alpha \, \underline{h}^1$, and similarly for $\hat{V}^2$. Now define $I^1 : [0,1]^S \to \Re$ by $I^1(c) = \min_{p \in \Delta,\ (a,b) \in \alpha^1(p)} (ap \cdot c + b)$, and similarly for I^2. By definition, $I^i \circ \hat{V}^i$ represents $\succeq^i$.

We now show that $\succeq^1$ is more imprecision averse on states than $\succeq^2$ iff $I^1(c) \leqslant I^2(c)$ for all $c \in [0,1]^S$. Take any $f \in \mathcal{A}^{\overline{h}^1, \underline{h}^1}$ with $\hat{f}$ as in Definition 3. For each $s \in S$, $f(s) = \overline{h}^1_\beta \, \underline{h}^1$ and $\hat{f} = \overline{h}^2_\beta \, \underline{h}^2$ for some $\beta \in [0,1]$, so by the previously observed property of $\hat{V}^i$, $\hat{V}^1(f)(s) = \beta = \hat{V}^2(f)(s)$. So $\hat{V}^1(f) = \hat{V}^2(\hat{f})$. Moreover, by the representation and the $\overline{h}^i, \underline{h}^i$-precision of the (υ^i, α^i), for each $g \in \mathcal{A}$, $g \sim^1 \overline{h}^1_{I^1(\hat{V}^1(g))} \, \underline{h}^1$, and similarly for $\succeq^2$. It follows that, for every $f \in \mathcal{A}^{\overline{h}^1, \underline{h}^1}$, $f \sim^1 \overline{h}_{I^1(\hat{V}^1(f))} \, \underline{h}$ and $\hat{f} \sim^2 \overline{h}_{I^2(\hat{V}^1(f))} \, \underline{h}$. Condition (9) for a given $f \in \mathcal{A}^{\overline{h}^1, \underline{h}^1}$ implies, given the stochastic dominance property for $\succeq^2$ with respect to $\overline{h}^2, \underline{h}^2$, that $I^1(\hat{V}^1(f)) \leqslant I^2(\hat{V}^1(f))$. Conversely, if $I^1(\hat{V}^1(f)) \leqslant I^2(\hat{V}^1(f))$, then, by the same stochastic dominance property, $f \succeq^1 \overline{h}^1_\beta \, \underline{h}^1$ iff $I^1(\hat{V}^1(f)) \geqslant \beta$, and this implies that $I^2(\hat{V}^1(f)) \geqslant \beta$ and hence $\hat{f} \succeq^2 \overline{h}^1_\beta \, \underline{h}^2$; so Condition (9) holds for this f. So $\succeq^1$ is more imprecision averse on states than $\succeq^2$ iff $I^1(\hat{V}^1(f)) \leqslant I^2(\hat{V}^1(f))$ for all $f \in \mathcal{A}^{\overline{h}^1, \underline{h}^1}$. Since, for each $c \in [0,1]^S$, there exists $f \in \mathcal{A}^{\overline{h}^1, \underline{h}^1}$ with $\hat{V}^1(f) = c$—namely $f(s) = \overline{h}^1_{c(s)} \, \underline{h}^1$ for all $s \in S$—this holds if and only if $I^1(c) \leqslant I^2(c)$ for all $c \in [0,1]^S$. This is the case iff $\min\{I^1(c), I^2(c)\} = I^1(c)$ for all $c \in [0,1]^S$. By the definition of I^i, we have that $\min\{I^1(c), I^2(c)\} = \min_{p \in \Delta,\ (a,b) \in \alpha^1(p) \cup \alpha^2(p)} (a \sum_{s \in S} p(s)c(s) + b)$. So $\succeq^1$ is more imprecision averse on states iff $\min_{p \in \Delta,\ (a,b) \in \alpha^1(p)} (a \sum_{s \in S} p(s)c(s) + b) = \min_{p \in \Delta,\ (a,b) \in \alpha^1(p) \cup \alpha^2(p)} (a \sum_{s \in S} p(s)c(s) + b)$, and this is the case if and only if $\alpha^1 \leqslant \alpha^2$, as required.

□

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