Updating Confidence in Beliefs*

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September 12, 2019

Abstract

This paper develops a belief update rule under ambiguity, motivated by the maxim: in the face of new information, retain those conditional beliefs in which you are more confident, and relinquish only those in which you have less confidence. We provide a preference-based axiomatisation, drawing on the account of confidence in beliefs developed in Hill (2013). The proposed rule constitutes a general framework of which several existing rules for multiple priors (Full Bayesian, Maximum Likelihood) are special cases, but avoids the problems that these rules have with updating on complete ignorance. Moreover, it can handle surprising and null events, such as crises or reasoning in games, recovering traditional approaches, such as conditional probability systems, as special cases.

Key words: Belief Update, Ambiguity, Multiple Priors, Confidence, Complete ignorance, Update on Surprising or Null Events

JEL Codes: D81, D83.

*The author gratefully acknowledges support from the ANR project DUSUCA (ANR-14-CE29-0003-01).
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1 Introduction

1.1 Motivation and intuition

Reasons for going beyond the Bayesian representation of beliefs by probability measures abound. Whether it be decision makers' observed non-neutrality to ambiguity (Ellsberg, 1961), the purported injustifiability of the Bayesian requirement of belief precision (Gilboa et al., 2009, 2011; Bradley, 2014) or the difficulty of forming warranted beliefs satisfying the Bayesian tenets in real decisions (Cox, 2012; Gilboa and Marinacci, 2013), many have argued for non-probabilistic representations of belief. But how should such non-Bayesian beliefs be updated?

Non-Bayesian beliefs are particularly relevant in two sorts of sitations: under complete ignorance—as when an investor is considering a revolutionary technology—and after paradigm-shattering events—such as a financial crisis. In terms of the popular multiple prior model (Gilboa and Schmeidler, 1989), complete ignorance is naturally characterised by the set of all (relevant) priors, and the 'shock' of a crisis seems to lead an initial 'small' set of priors to expand to an ex post 'large' set of priors. Any normatively reasonable account of belief update for non-Bayesian beliefs should provide reasonable guidance for learning in these two sorts of cases. However, the update models proposed to date struggle. On the one hand, existing update rules for multiple priors in the literature tend to deal with complete ignorance in an 'extreme' way (Gilboa and Marinacci, 2013), for instance by not learning at all on the receipt of new information, or by immediately jumping to a perfectly precise posterior belief, no matter how scant the information (see Section 4.1). On the other hand, the literature on updating non-Bayesian beliefs has not tackled the issue of updating on surprise events—and *a fortiori* on null events—which plagues the standard Bayesian account (e.g. De Bondt and Thaler, 1985, 1987). This issue is particularly relevant in the context of non-Bayesian beliefs, as attested by a growing body of research indicating that ambiguity increases in a financial crisis, with significant economic consequences (e.g. Caballero and Krishnamurthy, 2008; Ilut and Schneider, 2014).

These two challenges are related by the issue of *conditional beliefs*. Bayesian conditionnalisation assumes that ex ante conditional beliefs fully determine ex post conditional beliefs—and in fact, that they coincide. Under basic behavioural axioms, this invariance of conditional beliefs is a direct implication of the famous Dynamic Consistency condition (see Section 4.2.1 and Ghirardato 2002). It renders the Bayesian account silent about learning null events and, as just noted, is known to be problematic when the learnt event is surprising (i.e. assigned low probability ex ante). Moreover, in cases of complete ignorance, conditional belief may be completely indeterminate ex ante, but it is unreasonable to insist that it should remain so after learning. After all, if some of the ex ante interdeterminacy was related to admitting priors that gave a very low probability to the observed event—according to them, it was a surprise—this might be a reason to disregard them ex post, and hence adopt more determinate conditional beliefs. For a theory of rational update, the key to meeting these challenges is thus how it relates ex ante conditional beliefs with ex post ones.

Drawing on new insights about this relationship, this paper proposes and behaviourally characterises a novel account of the update of non-Bayesian beliefs. Conceptually, it taps into an intuition as to *why* beliefs may be non-Bayesian: decision makers may be more or less *confident* in different beliefs. This 'second-order' aspect—confidence in beliefs—is something that the Bayesian model has trouble rendering properly, whilst it can be captured, and related to preferences, in some non-Bayesian models (e.g. Hill, 2019). Our account recognises that it *also* has a role to play in update. Put succinctly: in updating beliefs, *retain* those conditional beliefs in which you are *more* confident, and relinquish only those in which you have less confidence.

To see this intuition, note that acquiring new information may give one cause to withdraw some previously held beliefs. Moreover, there are often several prior beliefs that could be withdrawn. To take a simple example, suppose that you are to observe a sequence of tosses of a (single) coin. Ex ante, you believe that the coin will remain unchanged throughout (the process is IID), and that it is unbiased (so that the probability of heads on, say, the 101th toss is 0.5). Then you observe 100 tosses, 75 of which are heads. Such an observation is very unlikely given your previous beliefs—just like in the financial crisis example above. This effectively leaves you with three ways of forming new beliefs: (a) retaining both prior beliefs and accepting that you have observed something extremely unlikely; (b) accepting that the observation is not that unlikely, but revising your belief about the bias of the coin, retaining the belief that it remains constant; (c) accepting that the observation is not so unlikely, retaining the belief that the coin is *now* unbiased, but withdrawing the belief that it did not change during sampling. Which should you adopt?

Bayesian conditionnalisation (and Dynamic Consistency) obliges you to proceed as in (a). By contrast, the answer proposed by our account looks at the *ex ante confidence* in the various beliefs. If you are more confident that the coin does not change and that you have not seen an extremely unlikely sequence of events, but less confident in your belief that the coin is unbiased, then you should retain the former beliefs and revise the latter one (as in (b)). If, on the other hand, you are very confident that the coin is unbiased on the 101th toss, but not very confident that its bias is constant, then you should revise the latter belief (c). This seems a natural, and indeed rational, account of the role of confidence in belief change: a decision maker's confidence in a belief reflects how sure he is of it, so it makes sense that it determines how tenaciously he will hold onto that belief in the light of new, perhaps conflicting information. As this example indicates, the proposed approach will be able to deal comfortably with surprising events; indeed, it encompasses a popular approach to updating on null events as a special case (Section 4.2).

This intuition naturally has implications for the understanding of the 'information' purveyed by an observation. Learning that there were 100 tosses, 75 of which came up heads, lends more weight to some hypotheses about the bias of the coin than to others. Accordingly, it indicates something about which beliefs are more or less reasonable to hold in the light of this observation. These *relative* judgements can be formulated in the language of confidence. The observation warrants a large amount of confidence that, if the process was IID, the probability of heads on the next toss is 0.25 or higher, though less confidence that the probability is 0.80 or lower on the next toss. This suggests that learning an event effectively warrants specific amounts of confidence in certain beliefs or probability judgements: confidence which can be compared to the decision maker's ex ante confidence in relevant beliefs, according to the maxim mooted above. So, for instance, deciding how to update the initial belief that the coin is unbiased in the light of the observation of 75 heads out of 100 tosses basically involves comparing the confidence that the coin is unbiased with the confidence that, given these observations, the bias of the coin is greater than 0.5. If one has greater confidence in the latter judgement, then one will relinquish (or revise) the initial belief that the coin was unbiased. By contrast, if one has greater confidence in one's initial belief, it is retained on update. Of course, some observations (e.g. 75 heads out of 100 tosses) will warrant sufficient confidence to force revision of the initial belief, whereas others (e.g. 55 heads out of 100 tosses) may not.

This insight allows the account to cope naturally with learning in situations of complete ignorance. Since the observation itself warrants differing degrees of confidence in various beliefs, even in cases where one begins with a 'clean slate'—no confidence in any belief—one will end up holding relevant beliefs to appropriate levels of confidence. Indeed, it will turn out that several standard update rules for multi-prior beliefs can be recovered as special cases of the account proposed here, corresponding to restrictions to extreme levels of confidence. Moreover, some recently suggested rules, such as that used by Epstein and Schneider (2007), can be recovered as other, more reasonable, special cases; to that extent, we provide an axiomatic analysis of them.

1.2 Outline of the proposal

To develop our account, we employ a representation of beliefs specifically developped to capture confidence, namely the notion of *confidence ranking* proposed by Hill (2013). A confidence ranking is a nested family of sets of probability measures, where different sets are understood as representing the beliefs, or probability judgements, held at different levels of confidence. As explained in the cited paper, larger sets in the family involve fewer beliefs, and accordingly correspond to higher levels of confidence. Structures of this sort have long been employed in econometrics (e.g. Manski and Nagin, 1998; Manski, 2013).

The previous discussion motivates an update rule for the confidence ranking Ξ on learning E of the following form (see Section 2.4 for details):

$$\Xi_E = \left\{ \{p \in \mathcal{C} : p(E) \geq \rho_E(\mathcal{C})\}_E : \mathcal{C} \in \Xi,\ \{p \in \mathcal{C} : p(E) \geq \rho_E(\mathcal{C})\} \neq \emptyset \right\} \tag{1}$$

where $\rho_E : \Xi \to [0, 1]$ is a decreasing function and, for every set of probability measures $\mathcal{C}$ and event E, $\mathcal{C}_E$ is the well-known Full Bayesian update defined as follows:

$$\mathcal{C}_E = \{p(\bullet/E) : \ p \in \mathcal{C}, \ p(E) > 0\} \tag{2}$$

The *probability-threshold* function ρ_E assigns a probability value to every set in the confidence ranking, and hence implicitly to every confidence level. In so doing, it effectively specifies a set of probability measures, namely those which assign ex ante probability to E greater than the ρ_E-value for that confidence level. These can be thought of as representing the conclusions the decision maker is warranted to deduce from the observation of E with that much confidence: any probability measure giving a value to E that is less than this threshold 'gets it too wrong' to be considered plausible at the confidence level. Since ρ_E is decreasing, this set is larger for larger confidence levels, corresponding to the fact, noted above, that weaker conclusions can be drawn from the data at higher levels of confidence. Probability thresholds are reminiscent of significance levels in hypothesis testing, and indeed, at one level, the proposed update rule retains the spirit of classical statistical reasoning; see Section 5 for further discussion.

So, for every confidence level, the prior beliefs held at that level are represented by the appropriate set of probability measures in Ξ, whereas the conclusions that can be drawn from the data with that level of confidence are summarized by the set of probability measures singled out as 'reasonable' by ρ_E. If these are compatible—if the two sets of probability measures overlap—the update rule (1) retains all of these as posterior beliefs at that confidence level—it takes the intersection. This corresponds to the maxim that conditional beliefs held or conclusions drawn with high confidence are, as far as possible, retained. By contrast, at lower confidence levels, where the (more precise) initial beliefs may contradict the (stronger) conclusions drawn from the data with that much confidence, neither are retained. As discussed above, beliefs held with low confidence may be withdrawn in cases of conflict with observation. Finally, the update rule conditions the structure obtained on the learnt event E.

Let us illustrate the workings of this rule on the previous example. For each $x \in [0,1]$, let p_x^{IID} be the probability measure according to which the sampling process is IID and the coin has bias x, i.e. the probability of heads is x on every toss (see Section 4 for technical details). Let $p_{x,y}^{non-IID}$ be a probability measure according to which the probability of heads is x for each of the first 100 tosses, and y thereafter. Consider a decision maker who believes that the process is IID and the coin is unbiased, but who is more confident in the former belief than the latter. His confidence in beliefs can be modelled by a three-level confidence ranking: $\left\{\left\{p_{0.5}^{IID}\right\}, \left\{p_x^{IID} : \ x \in [0,1]\right\}, \left\{p_{x,y}^{non-IID} : \ x,y \in [0,1]\right\}\right\}$. According to the bottom element, the process is IID with an unbiased coin: this captures the stated ex ante beliefs about the process and the bias. At the next level up, all probability meaasures in the set agree on the character of the process, but not on the bias. This captures a judgement that he is more confident that the process is IID than that the coin is unbiased. Finally, neither the belief about the bias nor that concerning the process are retained at the highest confidence level. This confidence

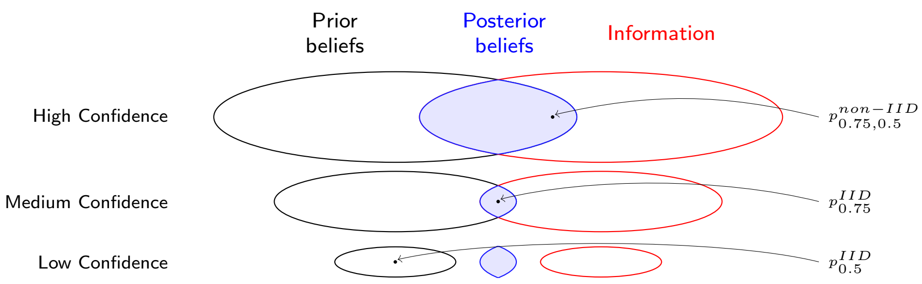

Figure 1: Confidence Update

ranking is drawn in black in Figure 1.

On learning that 75 tosses out of 100 came up heads, one can assign to each confidence level a probability threshold, determining which measures can be ruled out as 'having got the prediction for the 100 tosses too wrong' with that much confidence. For example, one could apply a threshold of 0.1 for the lowest level in the confidence ranking, 0.05 for the next level up, and 0.01 for the final level. The sets of probability measures giving a probability to the observation higher than the threshold are shown in red in Figure 1. The update rule proposed here yields the confidence ranking given by the blue sets in Figure 1.[1] As is clear from the Figure, at the top two confidence levels, the intersection of the sets is non-empty, and these yield the posterior beliefs. It follows that all the prior conditional beliefs are retained—for instance, at the intermediate confidence level, the belief that the process is IID is retained. Perhaps new beliefs are added—in the example, the posterior beliefs about the bias are more precise than they were ex ante at that confidence level. The prior beliefs specific to the bottom confidence level—and in particular that concerning the bias of the coin—are dropped on learning: the only beliefs held at that level are those inherited from higher confidence levels. This is in accordance with the intuition suggested previously: since this is a decision maker who has more confidence in the process being IID, this belief is retained—and the belief that the coin is unbiased is dropped—on update. Whilst this example focusses on one of the possible cases mentioned above (case (b)), by considering a different confidence ranking (capturing a decision maker who is more confident in the judgement about the bias than in that about the process) or different probability thresholds (which are more permissive at lower confidence levels), one can account for the other cases.

In this paper, we provide a behavioural characterisation of a generalisation of (1), which we call *(general) confidence update*. All the parameters, and in particular that playing the role of ρ_E, are revealed from preferences. Special cases, including a version of the rule (1) discussed above, are also characterised.

The paper is organised as follows. Section 2 sets out the framework, the confidence model and update rules. Section 3 contains the main results of the paper, characterising general

[1] More specifically, this is the confidence ranking $\{\{p^{IID}_x : x \in [0,1],\ p^{IID}_x(s_{75}) \geq 0.05\}, \{p^{non-IID}_{x,y} : x, y \in [0,1],\ p^{IID}_{x,y}(s_{75}) \geq 0.01\}\}$, where s_{75} is the event where 75 tosses out of 100 come up heads.

and specific versions of confidence update, and considering its comparative statics. Section 4 brings out the contributions of the proposed approach with respect to the issues of update for non-Bayesian beliefs and update on surprising or null events, including null events in game-theoretical reasoning. Section 5 discusses extensions and *inter alia* the relationship to Bayesian and classical statistical reasoning. Proofs and other material are to be found in the Appendix.

2 Preliminaries

2.1 Setup

Let $\mathcal{S}$ be a non-empty set of states, with a σ-algebra Σ of subsets of $\mathcal{S}$, called *events*. $\Delta(\Sigma)$ is the set of finitely-additive probability measures over $(\mathcal{S}, \Sigma)$ endowed with the weak* topology. For every subset $\mathcal{C} \subseteq \Delta(\Sigma)$, $\overline{\mathcal{C}}$ denotes the closure of $\mathcal{C}$. Let $\mathcal{X}$, the set of *consequences*, be a convex subset of a vector space; for instance it could be the set of lotteries over a set of prizes, as in the Anscombe and Aumann (1963) setting. $\mathcal{A}$ is the set of *(simple) acts*: finite-valued Σ-measurable functions from states to consequences. $\mathcal{A}^c$ is the set of constant acts (acts taking a constant value). Mixtures of acts are defined pointwise as standard: for any $f, g \in \mathcal{A}$ and $\alpha \in [0, 1]$, the α-mixture of f and g, which we denote by $f_\alpha g$, is defined by $f_\alpha g(s) = \alpha f(s) + (1 - \alpha)g(s)$ for all $s \in \mathcal{S}$. For every $f, g \in \mathcal{A}$ and $E \in \Sigma$, $f_E g \in \mathcal{A}$ is defined by $f_E g(s) = f(s)$ if $s \in E$, $f_E g(s) = g(s)$ otherwise.

We use $\succeq$ (perhaps with subscripts) to denote a preference relation on $\mathcal{A}$. The symmetric and asymmetric parts of $\succeq$, $\sim$ and $\succ$, are defined as standard. We say that $\succeq$ is *degenerate* if $f \sim g$ for all $f, g \in \mathcal{A}$. A functional $V : \mathcal{A} \to \mathbb{R}$ is said to represent preferences $\succeq$ if $f \succeq g$ if and only if $V(f) \geq V(g)$.

Henceforth, $\succeq$ (with no subscript) will denote the decision maker's ex ante preferences. Ex post preferences will be denoted with subscripts, depending on the information received; there is a class of preferences $\{\succeq_E\}_{E\in\Sigma}$. For each event E, $\succeq_E$ is the decision maker's preference after having learnt (only) that E obtains. Finally, an event $E \in \Sigma$ will be said to be null if it is null with respect to $\succeq$: for any $f, g, h, h' \in \mathcal{A}$, $f_{E^c}h \succeq g_{E^c}h'$ if and only if $f \succeq g$.

2.2 Confidence model

We adopt the confidence framework set out and developped in Hill (2013, 2016, 2019). Beliefs are represented by a *confidence ranking* on $\mathcal{S}$: a nested family of non-empty subsets of $\Delta(\Sigma)$. (More formally, a confidence ranking Ξ is a subset of $2^{\Delta(\Sigma)} \setminus \emptyset$ such that, for all $\mathcal{C}, \mathcal{C}' \in \Xi$, $\mathcal{C} \subseteq \mathcal{C}'$ or $\mathcal{C}' \subseteq \mathcal{C}$.) Different sets in the confidence ranking represent beliefs held with different levels of confidence. Note that a single set of probability measures à la Gilboa and Schmeidler (1989) is a degenerate special case of a confidence ranking; it can be interpreted as the case where the same beliefs are held at all levels of confidence.

A confidence ranking Ξ is said to be *closed* (resp. *convex*) if each set in the family is. We let $\min \Xi = \bigcap_{\mathcal{C}' \in \Xi} \mathcal{C}'$ and $\max \Xi = \bigcup_{\mathcal{C}' \in \Xi} \mathcal{C}'$; these can be loosely thought of as the bottom and top elements of Ξ. For a confidence ranking Ξ, its min-closure, $\Xi^{\min} = \Xi \cup \{\min \Xi\}$.[2] Ξ is *min-closed* if $\Xi = \Xi^{\min}$. Throughout the axiomatic treatment (Section 3), we shall only be concerned with closed, convex and min-closed confidence rankings.

As discussed in the aforementioned papers, there are several decision models in the confidence family. Here we use the maximin-EU version, according to which preferences are represented by:

$$V(f) = \min_{p \in D(f)} \mathbb{E}_p u(f(s)) \tag{3}$$

where u is a non-constant affine utility function, Ξ is a closed, convex, min-closed confidence ranking Ξ and D is a function from $\mathcal{A}$ to Ξ, satisfying the following *richness* condition: for every $f \in \mathcal{A} \backslash \mathcal{A}^c$ and $\mathcal{C} \in \Xi \backslash \{\min \Xi\}$, there exist $d \in \mathcal{A}^c$ and $\alpha \in (0, 1]$ such that $D(f_\alpha d) = \mathcal{C}$. This function, called the *cautiousness coefficient* for Ξ, captures the decision maker's ambiguity attitudes—or attitudes to choosing on the basis of limited confidence. We refer to the cited papers for discussion, details and other models in the confidence family, to which the approach developed here can be extended.

When (3) holds for preferences $\succeq$, we say that the triple (Ξ, D, u) *represents* $\succeq$. Whenever $\succeq$ is non-degenerate, there is a unique triple (up to positive affine transformation of the utility function) representing $\succeq$ (Hill, 2013), which we refer to as *the representation of* $\succeq$. We adopt the convention that $\succeq$ is degenerate if and only if it is represented by $\Xi = \{\emptyset\}$ and D the only function from $\mathcal{A}$ to $\{\emptyset\}$. For mere technical convenience, we will suppose throughout that utility is unbounded: $u(\mathcal{X}) = \mathbb{R}$.

We assume that all preferences, ex ante and ex post, are represented according the confidence model (3), and focus on non-degenerate ex ante preferences.

Assumption 1. *$\succeq$ and $\succeq_E$ are represented according to (3) for all $E \in \Sigma$, and $\succeq$ is non-degenerate.*

Behavioural foundations for this version of the confidence model have been provided in Hill (2013). They can be used to provide a reformulation of this assumption in terms of preferences.

2.3 Tastes and stakes

A central idea behind the confidence model is that the beliefs one relies on to decide are held to a level of confidence that is appropriate given the importance of the decision (Hill, 2013, 2019). In the light of this, when fewer beliefs are invoked—i.e. when a larger set of priors is used, say $D(f) \supset D(g)$—then this is an indication that the decision maker considers the choice of f to be more important than the choice of g: it involves higher stakes. The converse is not necessarily true: a decision maker may use the same beliefs—the same $\mathcal{C} \in \Xi$—for decisions

[2] By convention, if Ξ is empty or the family consisting of the empty set, then $\Xi^{\min}$ is taken to be $\{\emptyset\}$.

of differing importance. Indeed, the standard maximin-EU decision rule with a single set of priors (Gilboa and Schmeidler, 1989) is a special case of (3) of just this sort: the same set of probability measures are used for all acts, no matter the stakes involved.

Throughout, we assume that only beliefs—in the context of this model, the confidence ranking Ξ—change on learning, and in particular that there is no change in the utility function or in the level of stakes that a decision is considered to involve.

Assumption 2. *For representations* (Ξ, D, u) *and* $\{(\Xi_E, D_E, u_E)\}_{E\in\Sigma}$ *of* $\succeq$ *and* $\{\succeq_E\}_{E\in\Sigma}$ *respectively:*

1. u *and* u_E *are identical up to positive affine transformation for every* $E \in \Sigma$*;*
2. *there exists a complete transitive relation* $\geqq$ *on* $\mathcal{A}$ *such that for all* $E \in \Sigma$ *and all* $f, g \in \mathcal{A}$*,* $f \geqq g$ *implies* $D(f) \supseteq D(g)$ *and* $D_E(f) \supseteq D_E(g)$ *.*

The first part of this assumption is standard. The second clause states that there is a single notion of stakes (captured by $\geqq$) that all preferences, ex ante and ex post, can be thought of as respecting. It reflects the assumption that the decision maker's view of the relative importance of decisions remains constant under learning.

Whilst stated on the models for ease, Assumption 2 can be reformulated in behavioural terms. The first clause corresponds to the standard axiom that preferences over constant acts are unaffected by learning. The latter is built into axiomatisations of the confidence model assuming an exogenously given notion of stakes (Hill, 2013); framework-specific axioms characterise it in setups where stakes are endogenous (Hill, 2015).

Given Assumption 2, $\mathcal{A}$ can be partitioned into *stakes levels* according to $\geqq$. We use σ_f to denote the stakes level of f: that is, the set of acts having the same stakes as f, $\sigma_f = \{g \in \mathcal{A} : g \geqq f \ \& \ g \leqq f\}$. We use σ, σ' as notation for stakes levels. With this notation, $f \in \sigma$ if f involves stakes of level σ. The obvious order on stakes levels is defined as standard: for stakes levels σ, σ', $\sigma \geq \sigma'$ if and only if, for all $f \in \sigma$, $f' \in \sigma'$, $f \geqq f'$.

Finally, given a preference relation $\succeq$ represented according to (3) and a stakes level σ, we define the derived relation $\succeq^\sigma$ as follows: for all $f, g \in \mathcal{A}$, $f \succeq^\sigma g$ if and only if there exists $c, d, d' \in \mathcal{A}^c$ and $\alpha, \alpha' \in (0, 1]$ such that $D(f_\alpha d) = D(g_{\alpha'} d') = D(h)$ for all $h \in \sigma$, $f_\alpha d \succeq c_\alpha d$ and $c_{\alpha'} d' \succeq g_{\alpha'} d'$.[3] As discussed in Hill (2013), $f \succeq^\sigma g$ essentially says that, if the acts were evaluated 'as if' they were both of stakes level σ, then f would be preferred. For example, consider a bet f on the Democrat candidate winning the 2020 US President election, yielding $\$1$ million if you win and a loss of $\$1$ million if not, and a similar bet g on the 2024 election, with stakes (winnings and losses) 100000 times less in utility terms. An agent with beliefs that are more precise and slightly more favorable for the 2020 election might nevertheless prefer g to f because of the difference in stakes: with lower stakes, he can rely on low-confidence beliefs in evaluating g, but not for f. However, if both options were evaluated at the same

[3] This is well-defined because of the richness of D.

stakes level—for instance, if bets on the 2020 and 2024 elections with stakes of $\$1$ million were compared—then f would typically be preferred: i.e. $f \succeq^{\sigma} g$, where σ is the appropriate stakes level. When $f \succeq^{\sigma} g$, we say that f *is preferred to* g *at stakes level* σ.

2.4 Update

We now formally present the updates rules that we will consider. We shall say that a correspondence $\gamma : X \rightrightarrows Y$ between two ordered sets $(X, \geq_X)$, $(Y, \geq_Y)$ is *increasing (resp. decreasing)* if, for every $y, y' \in Y$, $x, x' \in X$, if $y \in \gamma(x)$, $y' \in \gamma(x')$ and $x \geq_X x'$, then $y \geq_Y y'$ (resp. $y \leq_Y y'$).[4] We use the natural order, given by containment, on confidence rankings.

Our benchmark update rule is as the following.

Definition 1. For confidence rankings Ξ and Ξ_E and an event $E \in \Sigma$, Ξ_E is a *(general) confidence update of* Ξ *by* E if there exists a confidence ranking Ξ_{UpdE} and an increasing correspondence $c_{UpdE} : \Xi \rightrightarrows \Xi_{UpdE}$ such that

$$\Xi_E = \left\{ \overline{(\mathcal{C} \cap \mathcal{C}')_E} : \mathcal{C} \in \Xi,\ \mathcal{C}' \in \Xi_{UpdE}\ s.t.\ \mathcal{C}' \in c_{UpdE}(\mathcal{C}),\ \mathcal{C} \cap \mathcal{C}' \neq \emptyset \right\}^{\min} \tag{4}$$

where, for $\mathcal{C} \subseteq \Delta(\Sigma)$ and $E \in \Sigma$, $\mathcal{C}_E$ is the Full Bayesian update defined in (2). The *calibration correspondence* $c_E : \Xi \rightrightarrows \Xi_E$ is defined by $c_E(\mathcal{C}) = \left\{ \overline{(\mathcal{C} \cap \mathcal{C}')_E} : \mathcal{C}' \in c_{UpdE}(\mathcal{C}),\ \mathcal{C} \cap \mathcal{C}' \neq \emptyset \right\}$ if there exists $\mathcal{C}' \in c_{UpdE}(\mathcal{C})$ with $\mathcal{C}' \cap \mathcal{C}$ non-empty and $c_E(\mathcal{C}) = \{\min \Xi_E\}$ otherwise.

General confidence update generalises the update logic discussed in the Introduction. The information that E is taken to indicate something about how reasonable (prior) probability measures are; however, unlike (1), it not assumed to amount to a probability threshold at each confidence level. Rather, a set of 'reasonable' probability measures in the light of the fact that E has been learnt is specified for each confidence level, representing the conclusions that can be drawn with various levels of confidence. The conclusions are weaker (and the sets are larger) for higher confidence levels, so the information can be represented as confidence ranking, Ξ_{UpdE}. The correspondence c_{UpdE} picks out the appropriate sets in Ξ_{UpdE} for the various confidence levels. Beyond this difference, the rule operates as discussed previously. Whenever the initial beliefs and conclusions drawn from the data at a confidence level are compatible, they are both retained—by taking the intersection of the sets. Whenever they aren't, neither is retained and the posterior beliefs are inherited from higher confidence levels. Note that if Ξ and Ξ_{UpdE} are closed, convex and min-closed confidence rankings, then Ξ_E defined according to (4) is as well.[5]

[4] A correspondence $\gamma : X \rightrightarrows Y$ is a function from X to $2^Y \setminus \emptyset$. If Y is a lattice, γ is increasing in the defined sense if and only if it is increasing in the Strong Set Order.

[5] Since the Full Bayesian update of a closed set of priors is not necessarily closed (see Section 4.1), the closure is required in (4) to ensure that the ex post confidence ranking is closed. We work with closed confidence rankings for mere convenience (the confidence ranking revealed from preferences is only unique up to closure); the results can be extended to versions of the rule that do not impose closure.

General confidence update is permissive in how conclusions are drawn from the learnt event E; we also consider more restrictive special cases.

Definition 2. For confidence rankings Ξ and Ξ_E and an event $E \in \Sigma$, Ξ_E is a *probability-threshold confidence update of* Ξ *by* E if there exists a decreasing correspondence $\rho_E : \Xi \rightrightarrows [0,1]$ such that

$$\Xi_E = \left\{ \overline{\{p \in \mathcal{C} : p(E) \geq r\}_E} : \mathcal{C} \in \Xi,\ r \in \rho_E(\mathcal{C}),\ \{p \in \mathcal{C} : p(E) \geq r\} \neq \emptyset \right\}^{\min} \tag{5}$$

ρ_E is called the *probability-threshold correspondence.*

Probability-threshold confidence update—or *PT-confidence update*, for short—is the special case where the updating confidence ranking Ξ_{UpdE} consists of sets of probability measures satisfying probability thresholds.[6] It is built on the same intuition that learning E indicates something about how reasonable probability measures are: here, those that give too low a probability to E ex ante 'got it more wrong' than others, and hence may not be retained at certain confidence levels. At every confidence level, the correspondence ρ_E can be interpreted as providing a threshold that picks out the probability measures retained at that level. If the decision maker has a different set of beliefs at each confidence level, then ρ_E is a function; indeed, modulo some technicalities, the update rule (1) in the Introduction corresponds precisely to this case. However, to accommodate cases where the decision maker holds the same beliefs at different confidence levels—as in the special case of a single set of priors (Section 2.2)—we allow ρ_E to be a correspondence. This allows him to have the same initial beliefs at two different confidence levels, but to consider that observation warrants the use of different probability thresholds.

Given preference relations representable by the confidence model, $\succeq$ and $\succeq_E$ for an event $E \in \Sigma$, we say that $\succeq_E$ is a *general confidence update* of $\succeq$ *by* E if, for any representations (Ξ, D, u) and (Ξ_E, D_E, u) of $\succeq$ and $\succeq_E$ respectively, Ξ_E is a general confidence update of Ξ and $D_E(f) \in c_E(D(f))$ for all $f \in \mathcal{A}$. PT-confidence update of preferences is defined analogously.

3 Characterising Confidence Update

We now provide behavioural characterisations of the general and special cases of confidence update.

[6] More formally, Ξ_E is a probability-threshold confidence update of Ξ by E if and only if Ξ_E is a general confidence update of Ξ by E, with $\Xi_{UpdE} = \{\{p \in \Delta(\Sigma) : p(E) \geq x\} : x \in [0,1]\}$ and $c_{UpdE}(\mathcal{C}) = \{\{p \in \Delta(\Sigma) : p(E) \geq r\} : r \in \rho_E(\mathcal{C})\}$ for all $\mathcal{C} \in \Xi$.

3.1 General Confidence Update

A specific comparison of acts with constant acts will play a special role in the axioms below. Recall that the preference of an act f over a constant act c betrays that the decision maker values f at least as highly as c. Whilst the value assigned to f may change on learning, the assumption of constant tastes (Assumption 2) ensures that the value of c will not: to that extent, it provides a constant 'benchmark'. The acts $f_E c$ and c coincide whenever E is not the case, in which case the constant benchmark c obtains. A preference for $f_E c$ over c thus indicates that, *conditional on* E, f is evaluated as better than the constant benchmark c. This is a special case of the standard definition of conditional preferences under expected utility, which compares $f_E h$ and $c_E h$. (For expected utility, unlike for ambiguity models, this comparison is independent of h.) The specific case used here—where $h = c$—is the only one where one of the acts is guaranteed to be constant, and hence has a value that is independent of beliefs. To the extent that it ties in with the use of constant acts as a benchmark for evaluating others, preference comparisons of $f_E c$ and c thus provide a natural conception of conditional preferences. As noted in the Introduction, conditional preferences are important, because the central issue in update concerns what happens to conditional beliefs.

We introduce the following terminology. For an event $E \in \Sigma$ and a stakes level σ, we say that σ is E-*resilient* if, for all $f \in \mathcal{A}$, $c \in \mathcal{A}^c$, if $f_E c \succeq^\sigma c$, then $f_E c \succeq^\sigma_E c$. E-resilient stakes levels are those for which all relevant ex ante conditional preferences are retained on learning E: if f is evaluated as better than c conditional on E prior to learning, then it continues to enjoy this evaluation afterwards.

These concepts are familiar in the literature on (Bayesian and non-Bayesian) updating. For instance, Pires's (2002) axiomatisation of Full Bayesian update under the maxmin-EU model involves a related notion of conditional preferences, and his main axiom is a strengthening of the condition that all stakes levels are E-resilient, for every non-null E. And, as discussed in Section 4.2.1, Dynamic Consistency also imposes a stronger conditional-preference preservation property than E-resilience, for all stakes levels.

3.1.1 Main Axiom

The following is the central behavioural axiom behind confidence update.

Axiom A1 (Confidence Consistency)**.** *For all stakes levels σ, σ' with $\sigma' > \sigma$ and every non-null $E \in \Sigma$, if σ is E-resilient, then so is σ'.*

Confidence Consistency translates the maxim mooted in the Introduction: retain those conditional beliefs in which you are more confident, and relinquish those in which you have less confidence. If σ is E-resilient, then all ex ante conditional evaluations of acts relative to 'benchmark' constant acts are retained ex post. This indicates that the beliefs underlying these preferences are retained on update. Confidence consistency implies that if all such conditional preferences are retained at some stakes level, then the conditional preferences at any higher stakes level are also retained. If the decision maker is confident enough in the beliefs underlying the former preferences to hold onto them in the face of the information E, then he will also hold onto the beliefs underlying the latter preferences. This is precisely as the maxim demands: if he retains beliefs held at a given level of confidence, then he certainly cannot relinquish beliefs held with higher confidence, for he should have relinquished the former beliefs first!

3.1.2 Other Axioms

Now consider the following axioms.

Axiom A2 (Consequentialism). *For every non-null $E \in \Sigma$, if $f(s) = g(s)$ for all $s \in E$, then $f \sim_E g$.*

Axiom A3 (Non-degeneracy). *For every non-null $E \in \Sigma$, $\succeq_E$ is non-degenerate.*

Axiom A4 (Information-based Learning). *For every $f \in \mathcal{A}$, $c \in \mathcal{A}^c$ and $E \in \Sigma$, if $f \not\succeq_E^\sigma c$ for every E-resilient stakes level σ, then $f \not\succeq_E^{\sigma'} c$ for every σ'.*

Consequentialism is a well-known and relatively standard axiom in the dynamic context; see e.g. Epstein and Le Breton (1993); Ghirardato (2002) for further discussion of it. Non-degeneracy is the standard property that update by non-null events yields non-degenerate preferences.

Confidence Consistency concerns what happens when learning E does not shake beliefs held with a certain level of confidence. By contrast, Information-based Learning constrains what happens when learning E *does* shake beliefs at a particular confidence level—that is, at stakes levels which are not E-resilient, and hence where some ex ante conditional preferences are not retained on learning. The condition basically implies that preferences at these stakes levels are determined by preferences at higher, E-resilient stakes levels, where the information can be incorporated without relinquishing ex ante beliefs. Hence it demands that learning is entirely driven by the new information E. If learning E undermines beliefs held only to a low level of confidence, they will not be replaced with anything specific. The information is only understood as saying that such low-confidence beliefs are inappropriate, but not as specifying other beliefs to replace them, except those beliefs inherited from higher confidence levels.

We shall call these three axioms the *Basic Axioms*.

3.1.3 Representation

Confidence Consistency and the Basic Axioms yield our most general update rule.

Theorem 1. *Let $\succeq$ and $\{\succeq_E\}_{E\in\Sigma}$ satisfy Assumptions 1 and 2. They satisfy Confidence Consistency and the Basic Axioms if and only if, for every non-null $E \in \Sigma$, $\succeq_E$ is a general confidence update of $\succeq$.*

So, in the presence of the Basic Axioms, Confidence Consistency characterises the heart of the proposed approach, namely the general confidence update rule. In fact, the central behavioural properties of the approach essentially boil down to Confidence Consistency, Consequentialism and Non-degeneracy. Information-based Learning merely controls what happens at the bottom of the confidence ranking, where there is incompatibility with prior beliefs. It can be shown that in its absence, the essence of confidence update is retained, except at confidence levels at the bottom of the ranking.

3.2 Probability-Threshold Confidence Update

3.2.1 Axioms

To obtain the specification of general confidence update involving probability thresholds in Definition 2, consider the following axioms.

Axiom A5 (Probability Consistency)**.** *Consider any non-null $E \in \Sigma$ and E-resilient stakes levels $\sigma \leq \sigma'$. For every $\lambda \in (0,1]$ and $f, g \in \mathcal{A}$, if, for every $c, \underline{c} \in \mathcal{A}^c$ with $c \succ \underline{c}$, $f_E c \sim^{\sigma'} \underline{c}_\lambda c$ implies $(f_E c)_{\frac{1}{2}}(c_E \underline{c}) \succeq_E^{\sigma'} c_{\frac{1}{2}}\underline{c}$, then, for every $d, \underline{d} \in \mathcal{A}^c$ with $d \succ \underline{d}$, $g_E d \sim^{\sigma} \underline{d}_\lambda d$ implies $(g_E d)_{\frac{1}{2}}(d_E \underline{d}) \succeq_E^{\sigma} d_{\frac{1}{2}}\underline{d}$.*

Axiom A6 (Null consistency)**.** *For every non-null $E \in \Sigma$, E-resilient stakes level σ, $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$, if $f_E e \preceq^{\sigma} c$ for all $e \in \mathcal{A}^c$, then $f_E c \preceq_E^{\sigma} c$.*

To interpret Probability Consistency, note that $(f_E c)_{\frac{1}{2}}(c_E \underline{c})$ is a 50-50 mixture of $f_E c$ with a bet on the event E—the act $c_E \underline{c}$—whereas $c_{\frac{1}{2}}\underline{c}$ is a 50-50 mixture of $\underline{c}_\lambda c$ with a bet yielding the winning option c with probability λ—that is, $c_\lambda \underline{c}$. So if a decision maker weakly prefers the first bets ($f_E c$; the bet on E) over the second ($\underline{c}_\lambda c$; the bet with probability λ of winning) in each case, then she would weakly prefer the first mixture ($(f_E c)_{\frac{1}{2}}(c_E \underline{c})$) over the second ($c_{\frac{1}{2}}\underline{c}$).[7] This can be thought of as an 'implication' of the previous two preferences. But a weak preference for the bet on E over that with probability λ (i.e. $c_E \underline{c} \succeq c_\lambda \underline{c}$) betrays a judgement that the probability of E is λ or greater. So a weak preference for $(f_E c)_{\frac{1}{2}}(c_E \underline{c})$ over $c_{\frac{1}{2}}\underline{c}$ would be an 'implication' of a prior weak preference for $f_E c$ over $\underline{c}_\lambda c$ *and* a judgement that the probability of E is λ or greater. In the light of this, the axiom says that if the decision maker's ex post preferences at some stakes level are consistent with all such 'implications' of the judgement that the probability of E is λ or greater (i.e. she weakly prefers each relevant $(f_E c)_{\frac{1}{2}}(c_E \underline{c})$ ex post), then they remain consistent with all such 'implications' of the judgement at any lower E-resilient stakes level. In other words, if the decision maker's preferences are consistent with her incorporating the opinion that E is more probable than λ at some stakes level, then they are consistent with her incorporating that opinion at any lower stakes level. This is the sort

[7]Since the second bets are both constant acts, this is a consequence of the uncertainty aversion of the maximin-EU model.

of pattern one would expect given the intuitions about the information purveyed on learning E: if at some confidence level, the decision maker considers the observation that E to warrant a judgement that its probability was greater than λ, then she still considers it to warrant that judgement at any lower confidence level.

Null consistency concerns the case where the ex ante evaluation of an act $f_E e$ remains bounded by a constant act c, no matter how attractive e is. This indicates that E^c involves a certain form of nullness—certainly, if its probability were bounded away from 0 across the relevant set of priors, then such preferences would not occur. The axiom says that, in such cases of ex ante nullness, the ex post evaluation concerning f remains bounded by c. This is reasonable: if the event E was already treated as if had probability 1 in that region ex ante, then on learning E, the decision maker's evaluation of f cannot rise very much.

3.2.2 Representation

Theorem 2. *Let $\succeq$, $\{\succeq_E\}_{E\in\Sigma}$ satisfy Assumptions 1 and 2. Then they satisfy Confidence Consistency, the Basic Axioms, Probability Consistency and Null consistency if and only if, for every non-null $E \in \Sigma$, $\succeq_E$ is a probability-threshold confidence update of $\succeq$.*

In the presence of Null consistency, Probability Consistency thus characterises the probability-threshold specification of general confidence update discussed in the Introduction and Section 2.4. In other words, it guarantees the existence of a probability-threshold correspondence that characterises update according to (5). The following result characterises the uniqueness of this correspondence.

Proposition 1. *Let $\succeq$, $\{\succeq_E\}_{E\in\Sigma}$ satisfy the conditions in Theorem 2, with the former represented by (Ξ, D, u). There exists a unique maximal probability-threshold correspondence $\rho_E : \Xi \rightrightarrows [0,1]$ representing the update of Ξ by E, in the following sense: for every other ρ'_E representing the update by E and for every $\mathcal{C} \in \Xi$, if $y \in \rho'_E(\mathcal{C})$, then there exists $x \in \rho_E(\mathcal{C})$ with $x \geq y$. Moreover, if for every stakes level σ, there exists $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$ such that $f_E c \succeq^\sigma_E c$ but $f_E c \not\succeq^\sigma c$, then there exists $\mathcal{C} \in \Xi$ such that the correspondence ρ_E representing the update by E is unique on all $\mathcal{C}' \in \Xi$ with $\mathcal{C}' \supset \mathcal{C}$.*

There is a unique maximal probability-threshold correspondence, in the sense that it yields values higher than those given by any other correspondence representing the update. Moreover, whenever something is learnt (i.e. preferences change) at every stakes level, then on all sufficiently large confidence levels, the probability-threshold correspondence is uniquely revealed from preferences.

3.3 Further special cases

PT-confidence update (5) involves a probability-threshold correspondence ρ_E for each event E, without assuming any relationship between them. However, the approach can easily accommodate richer structures, involving closer relationships between the probability-threshold correspondences for different events. These may be useful for applications, or in connecting the approach to existing work in statistics (Section 5). By way of illustration, we provide an axiomatisation of the simplest such special case: where a single probability-threshold correspondence represents update for all (non-null) events.[8] To this end, consider the following strengthening of Probability Consistency.

Axiom A7 (Strong Probability Consistency)**.** *Consider any non-null $E, F \in \Sigma$ and E- and F-resilient stakes levels $\sigma \leq \sigma'$. For every $\lambda \in (0,1]$ and $f, g \in \mathcal{A}$, if, for every $c, \underline{c} \in \mathcal{A}^c$ with $c \succ \underline{c}$, $f_E c \sim^{\sigma'} \underline{c}_\lambda c$ implies $(f_E c)_{\frac{1}{2}}(c_E \underline{c}) \succeq_E^{\sigma'} c_{\frac{1}{2}} \underline{c}$, then, for every $d, \underline{d} \in \mathcal{A}^c$ with $d \succ \underline{d}$, $g_F d \sim^\sigma \underline{d}_\lambda d$ implies $(g_E d)_{\frac{1}{2}}(d_E \underline{d}) \succeq_F^\sigma d_{\frac{1}{2}} \underline{d}$.*

The central difference in this axiom with respect to Probability Consistency is that it compares across different events; apart from that, the interpretation in terms of the lower stakes levels retaining the judgements whose 'implications' are respected at higher stakes levels remains the same. This strengthening yields the desired special case.

Proposition 2. *Let $\succeq$, $\{\succeq_E\}_{E\in\Sigma}$ satisfy Assumptions 1 and 2. Then they satisfy Confidence Consistency, the Basic Axioms, Null consistency and Strong Probability Consistency if and only if there exists a probability-threshold correspondence $\rho : \Xi \rightrightarrows [0,1]$ such that, for every non-null $E \in \Sigma$, $\succeq_E$ is a confidence update of $\succeq$ by E represented by ρ. The uniqueness of ρ is as in Proposition 1.*

3.4 Comparative statics

In this section, we take a brief look at the comparative statics of the PT-confidence update rule, as concerns ex post ambiguity aversion. We adopt a standard definition of comparative ambiguity aversion (Ghirardato and Marinacci, 2002), according to which agent $\succeq'$ is more ambiguity averse than $\succeq$ if and only if, for all $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$, if $f \succeq' c$, then $f \succeq c$.

Proposition 3. *Let $\succeq$, $\{\succeq_E\}_{E\in\Sigma}$ and $\succeq'$, $\{\succeq'_E\}_{E\in\Sigma}$ be two families satisfying Assumptions 1 and 2, and the conditions in Theorem 2, and suppose that $\succeq = \succeq'$. Let (Ξ, D, u) be the representation of $\succeq$, and let $\{\rho_E\}$ and $\{\rho'_E\}$ be the families of maximal correspondences as specified in Proposition 1 representing updates yielding $\{\succeq_E\}_{E\in\Sigma}$ and $\{\succeq'_E\}_{E\in\Sigma}$ respectively. Then the following are equivalent, for each non-null event E:*

(i) for every E-resilient stakes level σ according to $\succeq$, $\succeq'^{\sigma}_E$ is more ambiguity averse than $\succeq_E^\sigma$;

[8] The aim of this exercise is to illustrate the strength of the approach; we by no means wish to suggest that equality of probability-threshold correspondences is reasonable or desirable. For instance, it seems more reasonable to look at equal likelihood ratio-thresholds across events; however, note that in our extremely general framework (where the space of probability measures is the whole of $\Delta(\Sigma)$), the likelihood ratio coincides with the likelihood. Restricting the space of measures considered and using an adapted version of the techniques presented here can provide a likelihood ratio-version of the result below; details go beyond the scope of the current paper.

(ii) for every $\mathcal{C} \in \Xi$ *and* $y \in \rho'_E(\mathcal{C})$*, there exists* $x \in \rho_E(\mathcal{C})$ *with* $x \geq y$*.*

This result sheds light on the role of the probability-threshold correspondence. For decision makers with identical ex ante preferences, differences in ex post ambiguity attitude at stakes levels where the relevant conditional preferences are retained on learning (E-resilient stakes levels) correspond to differences in the probability-threshold correspondence. The latter essentially reflects the strength of the conclusions a decision maker is willing to draw from a given observation for each confidence level: as discussed in the Introduction, it reflects how 'wrong' a probability measure has to be ex ante about the new information for it to be ruled out as plausible. The higher the probability-threshold at a given confidence level, the stricter this constraint, and hence the stronger the implicit conclusion the decision maker is drawing from the data. So, if one decision maker always uses a higher probability threshold than another, the former can be thought of as more daring, or less cautious, in the conclusions he is prepared to draw from the same data. This translates to him being less ambiguity averse ex post.

As discussed below (Section 5), the probability-threshold correspondence plays a similar role to significance levels in statistics, with the difference that it assigns a significance level to each level of confidence. Decision makers which differ in the probability-threshold correspondence (or, equivalently, *ceteris paribus*, ex post ambiguity aversion) can thus be thought of, roughly, as differing in the significance level they deem appropriate for a given level of confidence.

4 Situating Confidence Update

As noted in the Introduction, update of ambiguous beliefs presents a certain number of challenges, concerning in particular complete ignorance and surprising events. We shall now consider how the approach fairs with respect to these challenges, and compares to existing update rules in the literature.

4.1 Complete ignorance, and other updating rules for ambiguous beliefs

The most notable generally-applicable consequentialist update rules that have been proposed and axiomatised for multi-prior models, and in particular the maximin-EU model (Gilboa and Schmeidler, 1989) are Full Bayesian (Pires, 2002; Walley, 1991) and Maximum Likelihood update (Gilboa and Schmeidler, 1993; Dempster, 1967).[9] As Gilboa and Marinacci (2013) point out, both are extreme, which of course sheds doubt on their descriptive adequacy as

[9]Other approaches to update in the literature drop consequentialism (Hanany and Klibanoff, 2007), restrict to ex ante beliefs satisfying a particular property with respect to a given filtration of events representing the potential new information (Epstein and Schneider, 2003), or define update in a multi-stage context with a given information structure, with update depending on the first-stage partition (Gul and Pesendorfer, 2018). Note that in the sort of challenging cases mentioned above—complete ignorance, learning on surprise or null events—a decision maker would not typically have a full and correct conception of the information structure (filtration) he is facing.

well as their normative validity. Some other, apparenty milder, rules have been proposed, for instance in Epstein and Schneider (2007). As we shall now show, all of these rules come out as special instances of confidence update.

Let us assume that the initial confidence ranking is a singleton containing the closed convex set of probability measures $\mathcal{P} \subseteq \Delta(\Sigma)$, so initial preferences are maximin-EU (Gilboa and Schmeidler, 1989). For update by an event E, applying the PT-confidence update rule to these preferences yields preferences represented according to (3), with confidence ranking

$$\Xi_E = \left\{ \overline{\{p \in \mathcal{P} : p(E) \geq r\}_E} : r \in R_E \; s.t. \; \{p \in P : p(E) \geq r\} \neq \emptyset \right\}^{min} \tag{6}$$

where $\rho_E(\mathcal{P}) = R_E \subseteq [0,1]$. As discussed previously, the confidence rule allows one to distinguish on update according to how reasonable probability meaures are in the light of the information. Hence it can yield, even for a degenerate initial confidence ranking (i.e. a single set of probability measures) a richer posterior confidence ranking; indeed, this will typically be the case whenever R_E is not a singleton. We shall say that a PT-confidence update by E is *maximally refined* whenever $R_E = [0,1]$, in which case (6) simply becomes:

$$\Xi_E^{mr} = \left\{ \overline{\{p \in \mathcal{P} : p(E) \geq r\}_E} : r \in [0,1] \; s.t. \; \{p \in P : p(E) \geq r\} \neq \emptyset \right\} \tag{7}$$

The previously mentioned update rules from the literature correspond to particular confidence levels in the confidence ranking resulting from the maximally refined application of the PT-confidence update rule. Consider, for instance, the largest set in Ξ_E^{mr}, capturing the beliefs held with the highest level of confidence. This is (the closure of) $\mathcal{P}_E$—the Full Bayesian update of the initial set of priors $\mathcal{P}$ (see (2)). On the other hand, the smallest set in Ξ_E^{mr}—capturing all beliefs held, no matter how little confidence there is in them—is clearly (the closure of) $\{p \in \mathcal{P} : \; p(E) = \max_{q \in \mathcal{P}} q(E)\}_E$: the Maximum-Likelihood update of $\mathcal{P}$. Moreover, for every 'significance level' $\alpha \in [0,1]$, the 'classical-style' update which retains all probability measures giving a probability greater than α to E—ie. $\{p \in \mathcal{P} : \; p(E) \geq \alpha\}_E$—coincides (up to closure) with some non-extremal set in Ξ_E^{mr}, corresponding to some intermediate confidence level. Whilst axiomatisations of the two previous rules are well-known in the economics literature (Gilboa and Schmeidler, 1993; Pires, 2002), the results in Section 3 also yield a behavioural characterisation of this last rule.[10] Morover, they provide what to our knowledge is the first unified behavioural characterisation of this whole family of rules.

We illustrate these points, and the consequences for the 'complete ignorance' cases that have been argued to be problematic for existing rules, on an example.

Example 1 (Complete Ignorance)**.** Consider a coin with a bias about which you know absolutely nothing, except that it is fixed. This can be cast in a statistical decision-style framework as follows. The period state space $S_t = S = \{h, t\}$ (h for heads, t for tails), and the full state space $\mathcal{S} = S^\infty$, with

[10] After completing the paper, our attention was drawn to Kovach (2015), which develops a different, specific axiomatisation of the last rule.

the product σ-algebra Σ. The standard statistical decision framework assumes in addition a parameter space Θ. Under the assumption that the process is IID, we can take $\Theta = [0,1]$, with each $\theta \in \Theta$ associated to the probability distribution over the period state space (ie. an element of $\Delta(S)$)[11] $\ell(\bullet/\theta)$, where $\ell(h/\theta) = \theta$. Each $\theta \in \Theta$ thus generates the distribution $\ell^\infty(\bullet/\theta)$ over $\mathcal{S}$. Just as a distribution over the parameter space $\mu \in \Delta(\Theta)$ generates a *predictive* distribution of the full state space $\overline{\mu} = \int_\Theta \ell^\infty(\bullet/\theta)d\mu(\theta) \in \Delta(\mathcal{S})$, a set of probability measures $\mathcal{M} \subseteq \Delta(\Theta)$ generates a set of probability measures $\overline{\mathcal{M}} \subseteq \Delta(\mathcal{S})$, defined as follows:

$$\overline{\mathcal{M}} = \left\{ \int_\Theta \ell^\infty(\bullet/\theta)d\mu(\theta) : \mu \in \mathcal{M} \right\} \tag{8}$$

We adopt a multiple prior representation, $\mathcal{M}$, of ex ante beliefs; following the standard way of representing a complete lack of prior knowledge about the bias of the coin in this context, we set $\mathcal{M} = \Delta(\Theta)$. In particular, $\mathcal{M}$ contains every Dirac measure; we denote by μ_θ the Dirac measure putting all weight on θ.

Suppose that you observe 100 tosses, 75 of which were heads—call this event s_{100}—and consider your posterior belief concerning h_{101}—getting a head on the next toss. Applying the maximally refined confidence update, as in (7), to $\overline{\mathcal{M}}$ yields:

$$\Xi_{s_{100}} = \left\{ \overline{\left\{ \int_\Theta \ell^\infty(\bullet/\theta)d\mu_{s_{100}}(\theta) : \mu \in \Delta(\Theta),\ \int_\Theta \ell^\infty(s_{100}/\theta)d\mu(\theta) \geq r,\ \int_\Theta \ell^\infty(s_{100}/\theta)d\mu(\theta) > 0 \right\}} \right. \tag{9}$$
$$\left. : r \in [0, \max_{\mu\in\Delta(\Theta)} \int_\Theta \ell^\infty(s_{100}/\theta)d\mu(\theta)] \right\}$$

where, as standard

$$\mu_{s_{100}}(A) = \frac{\int_A \ell^\infty(s_{100}/\theta)d\mu(\theta)}{\int_\Theta \ell^\infty(s_{100}/\theta)d\mu(\theta)} \tag{10}$$

for any (measurable) $A \subseteq \Theta$.

The Full Bayesian update of $\overline{\mathcal{M}}$ on s_{100} is $\left\{\int_\Theta \ell^\infty(\bullet/\theta)d\mu_{s_{100}}(\theta) : \mu \in \Delta(\Theta),\ \int_\Theta \ell^\infty(s_{100}/\theta)d\mu(\theta) > 0\right\}$. Up to closure, this coincides with the set of probability measures corresponding to the highest confidence level in $\Xi_{s_{100}}$, $\max \Xi_{s_{100}}$. In particular, since the only $\mu \in \Delta(\Theta)$ with $\int_\Theta \ell^\infty(s_{100}/\theta)d\mu(\theta) = 0$ are the Dirac measures μ_0 and μ_1, this set is $\overline{\mathcal{M}}$ itself. Since, under the maximin-EU rule, a set of priors is behaviourally indistinguishable from its closure, this means that, under Full Bayesian update, the decision maker's preferences do not change on learning. In particular, his posterior probability interval for h_{101} is behaviourally indistinguishable from the prior interval $[0,1]$. Full Bayesianism allows for no learning in such cases of ex ante complete ignorance. This problem, known as the issue of 'complete ignorance' or 'vacuous priors' (Walley, 1991, §6.6.1, 9.3), has been the topic of intense debate in some circles, where it is considered a major challenge for non-Bayesian accounts (e.g. Joyce, 2010; Bradley, 2017; Vallinder, 2017).

[11] We use $\Delta(S)$ in the context of this example to denote the set of probability distributions over S, and similary for $\Delta(\mathcal{S})$ and $\Delta(\Theta)$.

On the other hand, Maximum Likelihood update yields $\{\int_\Theta \ell^\infty(\bullet/\theta)d\mu_{s_{100}}(\theta) : \int_\Theta \ell^\infty(s_{100}/\theta)d\mu(\theta) = \max_{\mu\in\Delta(\Theta)} \int_\Theta \ell^\infty(s_{100}/\theta)d\mu(\theta)\}$, which coincides up to closure with the set of probability measures corresponding to the lowest confidence level in $\Xi_{s_{100}}$, namely $\min \Xi_{s_{100}}$. Since $\ell^\infty(s_{100}/0.75) > \ell^\infty(s_{100}/\theta)$ for every $\theta \neq 0.75$, this set is the singleton containing $\ell^\infty(\bullet/0.75)$, giving all weight to the bias being 0.75 in favour of heads. After learning s_{100}, the decision maker using this rule assigns a precise probability of 0.75 to h_{101}. So the Maximum Likelihood update rule goes to the opposite extreme: the decision maker settles on a precise opinion about the bias after a finite number of observations; indeed, it does so even if the number of observations is very small.

More reasonable than these extremes are the update rules one gets when restricting to intermediate confidence levels. Up to closure, these yield posterior sets of probability measures such as $\mathcal{C}^\alpha = \{\int_\Theta \ell^\infty(\bullet/\theta)d\mu_{s_{100}}(\theta) : \int_\Theta \ell^\infty(s_{100}/\theta)d\mu(\theta) \geq \alpha\}$, for $\alpha \in [0, \max_{\mu\in\Delta(\Theta)} \int_\Theta \ell^\infty(s_{100}/\theta)d\mu(\theta)]$. For non-extreme α, these sets are neither as imprecise as $\overline{\mathcal{M}}$, nor as specific as a singleton. As a simple illustration, suppose the initial set of priors is the set of Dirac measures, $\mathcal{M}^D = \{\mu_\theta : \theta \in \Theta\}$. In this case, the maximally refined confidence update of $\overline{\mathcal{M}^D}$ yields:[12]

$$\Xi_{s_{100}} = \left\{\overline{\{\ell^\infty(\bullet/\theta) : \theta \in \Theta,\ \ell^\infty(s_{100}/\theta) \geq r,\ \ell^\infty(s_{100}/\theta) > 0\}} : r \in [0,1]\right\} \tag{11}$$

which, at intermediate confidence levels, involves sets of the form $\{\ell^\infty(\bullet/\theta) : \theta \in \Theta,\ \ell^\infty(s_{100}/\theta) \geq \alpha\}$, up to closure. (Full Bayesian and Maximum Likelihood update on this set yields analogous results to those above.) It is clear that these sets are smaller for larger values of α, which correspond in turn to lower confidence levels. Furthermore, they will typically be non-extremal.

Moreover, setting $\beta = \frac{\alpha}{\max_{\mu\in\Delta(\Theta)} \int_\Theta \ell^\infty(s_{100}/\theta)d\mu(\theta)}$, we can rewrite $\mathcal{C}^\alpha = \{\int_\Theta \ell^\infty(\bullet/\theta)d\mu_{s_{100}}(\theta) : \int_\Theta \ell^\infty(s_{100}/\theta)d\mu(\theta) \geq \beta \max_{\mu\in\mathcal{M}} \int_\Theta \ell^\infty(s_{100}/\theta)d\mu(\theta)\}$ for $\beta \in [0,1]$. This is the essence of the update rule proposed by Epstein and Schneider (2007, Eqn (6)), albeit in a recursive setup involving incomplete learning.[13] As noted previously, it falls out as a consequence of confidence update.

Confidence update thus offers a new perspective on existing update rules for ambiguous beliefs, and with it a new resolution of the 'complete ignorance' problem. Full Bayesian update is what you get if you use the maximally refined version of the confidence update rule, but then only restrict to beliefs in which you have most confidence. It thus basically retains only conclusions that can be gleaned from observation with maximal confidence, ignoring the rest. As such, it comes to appear as a particularly cautious update rule. Maximum-likelihood update, on the other hand, is what get if you apply confidence update and then allow yourself to rely on all beliefs, even those held with little confidence. It thus admits any conclusion that can be gleaned from observation, no matter the confidence with which it can be deduced: it makes the boldest use possible of observations. The third sort of rule described above corresponds to

[12]Recall (Section 2.2) that we are not restricting to convex confidence rankings in this section. See also Section 5.

[13]Although we have illustrated the relationship with their update rule in a standard IID context, it is possible to write their incomplete learning model in our general framework and to recover their version of the rule.

taking beliefs held to an intermediate level of confidence, and hence embodies an intermediate level of caution. Note that this latter rule is perhaps the closest to much practice: taking the set corresponding to a probability threshold of 0.01, for instance, would be consistent with the classical practice in statistics of taking a 1% significance value (see also Section 5).

These 'standard' rules thus turn out to differ not in the underlying update mechanism—they are all retreivable from a single PT-confidence update—but rather in the confidence level that they embrace in posterior beliefs. However, under the confidence approach, the appropriate confidence level for an ex post decision depends on its importance and the cautiousness coefficient which, recall, reflects ambiguity attitude (Section 2.2; see Hill, 2013, 2019 for extended discussion). This new perspective thus suggests that the aforementioned rules are in fact confounding update with ex post ambiguity attitude. Full Bayesian update, for instance, is so cautious because it implicitly recommends, even for the most trivial decision, demanding maximal confidence in the beliefs used; and this is at the heart of its problems with complete ignorance cases. By contrast, the confidence approach does not require one to settle on a *single* confidence level for *all* updates and decisions. Different ex post decisions and different ambiguity attitudes will call for different confidence levels. Whilst in very high stakes decisions, a confidence decision maker may behave as if he is using Full Bayesian update, at less severe stakes levels, his ex post choices will be characterised by less extreme interim rules. So whilst, in the troublesome 'complete ignorance' cases, the confidence approach recognises that decision makers may behave as if there was no learning when the stakes are extremely high, in moderate-stakes decisions behaviour will be consistent with non-trivial learning.[14] Confidence update thus provides a generalisation of standard approaches that can situate and resolve the tension between them, and resolve the 'complete ignorance' problem mentioned in the Introduction.

4.2 Conditional beliefs and surprising or null events

It was claimed in the Introduction that a specificity of the confidence update rule was the way it deals with conditional belief. We now elaborate on this point, first via a comparison with Bayesian conditionnalisation, and then with a consideration of consequences for updating on surprising or null events.

4.2.1 Bayesian Conditionnalisation and Dynamic Consistency

It is well-known that Bayesian conditionnalisation relies on the assumption that conditional probabilities on a non-null event E are unchanged after learning E (e.g. Jeffrey, 1992; Bradley, 2005; Dietrich et al., 2016). Translated into confidence terms, the assumption that the conditional probability of an event F given E is unaffected by learning E essentially boils down to

[14] Moreover, such a decision maker will behave as if the ex ante 'complete ignorance' set of priors contracts more for less important ex post decisions, which require less confidence and hence admit drawing bolder conclusions from observation.

the assumption that the decision maker has sufficient confidence in his judgement about the conditional probability of F given E to retain it in the face of the new information. However, his confidence is a fact about his ex ante beliefs, as encapsulated in his confidence ranking. Indeed, it is a straightforward consequence of confidence update that whenever he is maximally confident in his judgement about the conditional probability of F given E, his conditional beliefs with respect to these events will be invariant. We summarize this in the following fact.

Fact 1. *Let Ξ be a confidence ranking with $\min \Xi = \{p\}$ and $E, F \in \Sigma$ with E non-null. If $q(F/E) = p(F/E)$ for all $q \in \max \Xi$, then for any Ξ_E resulting from a (general) confidence update of Ξ by E, $q'(F/E) = p(F/E)$ for all $q' \in \max \Xi_E$. In particular, if $\min \Xi_E$ is a singleton containing p_E, then $p_E(F/E) = p(F/E)$.*

Confidence rankings containing a singleton set are discussed and characterised in Hill (2013), where they were called *centred* confidence rankings. Decision makers represented by such confidence rankings are Bayesians with confidence: they can assign a precise probability value to any event, but may have limited confidence in some of these assignments. They are thus a natural context for exploring the relationship with standard Bayesian techniques.

The previous observation suggests that the essence of Bayesian update boils down to a property of the decision maker's ex ante beliefs: namely, a large amount of confidence in conditional probabilities. Indeed, it is straightforward to check that if the decision maker is maximally confident in all conditional probabilities, then we return to the Bayesian special case: the confidence ranking contains only one set, which is a singleton. So the proposal here diverges from Bayesianism insofar as it acknowledges that decision makers might, not unreasonably, have limited confidence in some of their conditional probability judgements. In the face of certain information, they may thus relinquish some conditional probability judgements in order to retain others—and hence violate the central tenet behind Bayesian conditionnalisation.

Dynamic Consistency represents the behavioural counterpart of the aforemtioned invariance property: in the presence of other basic axioms, it is equivalent to the statement that preferences conditional on E are invariant on learning E (Ghirardato, 2002, Lemma 1). On the behavioural front, confidence update leads to an analogous weakening of Dynamic Consistency: the Confidence Consistency axiom allows relinquishing some conditional preferences, whereas Dynamic Consistency preserves them all.[15]

Whilst this is not the place for an extended discussion of Dynamic Consistency's normative credentials, note that its defense is strongest in cases where the decision makers have full and correct ex ante knowledge of the information structure they are faced with (Hill, 2018). In such cases, decision makers will typically be very confident in their conditional beliefs, and hence confidence update will coincide with Bayesian conditionnalisation, and satisfy Dynamic Consistency (Fact 1). By contrast, learning surprising or null events often give decision

[15]More precisely, using the terminology introduced in Section 2: Dynamic Consistency implies that $f_E c \succeq^\sigma c$ if and only if $f_E c \succeq^\sigma_E c$ for all f, c and stakes levels σ (and is equivalent to this condition in our setup whenever ex ante preferences are expected utility), whereas Confidence Consistency clearly weakens this condition.

makers reason to question their prior conceptions; the assumption of full and correct ex ante awareness underpinning Dynamic Consistency is ill-adapted to such cases. Indeed, as noted in the Introduction, updates on surprising or null events are particularly challenging for Bayesian update. By eschewing Bayesianism's insistence on the invariance of conditional beliefs and preferences, confidence update provides a more constructive treatment of such cases.

4.2.2 Scientific Discovery, Crises and Surprises

Consider the following example.

Example 2. Prior to a sequence of successive coin tosses, a decision maker is fairly sure that the coin will be unchanged throughout the sequence—so the process is IID—though he is unsure of the bias of the coin. Suppose (to avoid issues with improper priors) that he uses the standard statistical setup for IID processes (Example 1), with as parameter space $\Theta = \{0, 0.1, 0.2, \ldots, 0.9, 1\}$ where $\ell(h/\theta) = \theta$ for each $\theta \in \Theta$. He is Bayesian, and takes a uniform prior μ on Θ, which generates the predictive $\overline{\mu} \in \Delta(\mathcal{S})$. He then observes 10 000 tosses, which turn out as follows: $(h, t, h, t, \ldots, h, t, h, t)$. Let us call this history of observed tosses s^{10000}.

Under Bayesian conditionnalisation, his posterior probability for the next coin being a head would coincide with his prior conditional probability, taking the value 0.5 ($p_{s^{10000}}(h_{10001}) = p_{s^{10000}}(h_{10001}/s^{10000}) = \overline{\mu}(h_{10001}/s^{10000}) = 0.5$). Similarly, Dynamic Consistency insists that his ex post evaluation of bets on the 10 001th toss should coincide with his ex ante conditional evaluation: so he is indifferent between betting on heads or tails in both cases. However, whilst under the IID assumption the sequence s^{10000} is as probable as every other sequence involving 5 000 heads out of 10 000 tosses, the particular pattern in fact seems rather surprising, and may give the decision maker reasonable grounds to question the IID nature of the process. If so, he would tend to think that a head on the next toss is more likely, given the alternating nature of the sequence: $p_{s^{10000}}(h_{10001}) = p_{s^{10000}}(h_{10001}/s^{10000}) > 0.5 = \overline{\mu}(h_{10001}/s^{10000})$. Accordingly, he would have a strict preference for betting on heads ex post. His conditional probabilities would thus change on learning, and he would violate Dynamic Consistency.

Such a decision maker can be straightforwardly modelled by the update rule and framework developed here. Consider the confidence ranking on $\mathcal{S}$

$$\Xi = \left\{ \{\overline{\mu}\}, \left\{ \pi_{\lambda}^{Markov} : \lambda \in [0,1] \right\} \cup \{\overline{\mu}\} \right\}$$

where π_{λ}^{Markov} is (the Markov hypothesis) defined by $\pi_{\lambda}^{Markov}(h_{t+1}/h_t) = \pi_{\lambda}^{Markov}(t_{t+1}/t_t) = \lambda$ for all t. This confidence ranking reflects the fact that the parameter space Θ and prior μ over it captures what the decision maker thinks about the sequence of tosses he is about to observe: they (or rather the generated predictive) naturally characterise the centre of his confidence ranking. He may be fairly confident in this judgement, and hence use this prior at medium stakes levels (or loss values, for a statistician). However, according to Ξ he is not maximally confident that the process is IID, so there will be confidence levels at which he does not hold this belief. At such levels, the corresponding set of probability measures contains measures that do not correspond to IID processes but, for instance, to Markov processes. For an appropriate $\rho_{s^{10000}}$, setting reasonable probability thresholds,[16] the PT-confidence update of Ξ by s^{10000} is such that $\pi_0^{Markov}(\bullet/s^{10000}) \in \min \Xi_{s^{10000}}$: the decision maker will not have a posterior precise probability of 0.5 for heads on the next toss. Indeed, the minimum probability of heads over $\min \Xi_{s^{10000}}$ will be greater than 0.5, and will generally depend on the probability threshold set by $\rho_{s^{10000}}$. On this analysis, the confidence approach seems to agree with pre-theoretical intuition. On the one hand, the decision maker sticks with his best-estimate (Bayesian) belief as long as the observations are not very surprising: in the absence of this peculiar pattern, PT-confidence update typically recommends applying Bayesian conditionnalisation on $\overline{\mu}$. On the other hand, in the presence of a surprising event (or pattern), he retains only those beliefs held with higher confidence, and moves to the most reasonable conjecture according to those beliefs: in this case, that the process is not IID.

This example can be thought of as a parable of (some) scientific discovery. Prominent discoveries—Fleming's discovery of penicillin, for instance[17]—involve noting surprising patterns where one was not expecting them. One certainly would not like to qualify such cases as irrational, and it can be taken as an advantage of our approach that it can capture them comfortably, as the example illustrates. Indeed, it can account for such updates whilst retaining ex ante preferences that are consistent with the initial assumption of an IID process at medium stakes levels; preferences only diverge at high stakes, where lots of confidence is required. By contrast, any Bayesian approach capable of accounting for these sorts of belief change would require the decision maker to place a small probability ex ante on the process not being IID, so as to guarantee that the conditional beliefs remain invariant on learning (Section 4.2.1). This would complicate calculations; behaviourally, it would lead to ex ante preferences which contravene the basic assumption of an IID process. The Bayesian framework cannot accommodate *both* ex ante preferences at medium stakes that are fully compatible with the IID assumption *and* updates which deviate from this assumption in surprising cases. Indeed, a long tradition of experimental and empirical evidence suggests that people do not employ Bayesian updating, especially in the face of ex ante surprising (low probability) events.[18]

Similar points hold for economic crises. Consider how people should update on events such as the collapse of Lehman Brothers, or the a priori surprising economic circumstances since. These are cases where the appropriateness of the Bayesian approach has been questioned (Gilboa et al., 2017; Giacomini et al., 2019). By contrast, assimilating the IID assumption in Example 2 with 'previously accepted' economic thinking, it is clear that the confidence approach once again provides guidance. In such situations, it recommends relinquishing those beliefs or hypotheses conflicting with the observed events in which confidence is most limited;

[16]For instance, take $\rho_{s^{10000}}$ with $\rho_{s^{10000}}(\{\overline{\mu}\}) = 0.05$, $\rho_{s^{10000}}(\{\pi_\lambda^{Markov} : \lambda \in [0,1]\} \cup \{\overline{\mu}\}) = 0.01$. In this case the conditionnalisation of the IID prior $\overline{\mu}$ is no longer held at the bottom of the updated confidence ranking: $\overline{\mu}(\bullet/s^{10000}) \notin \min \Xi_{s^{10000}} = \{\pi_\lambda^{Markov}(\bullet/s^{10000}) : \lambda \in [0, \sqrt[9999]{0.01}]\}$.

[17]Fleming noticed a petri dish containing Staphylococci bacteria that had been mistakenly left open was contaminated by blue-green mould from an open window, and that, surprisingly, there was a halo of inhibited bacterial growth around the mould.

[18]See for example Kahneman et al. (1982); Grether (1980, 1992); Griffin and Tversky (1992); Camerer et al. (2011) and De Bondt and Thaler (1985, 1987); Gallagher (2014) for experimental and empirical evidence respectively. Note that the points made here hold for both first- and second-order Bayesian approaches.

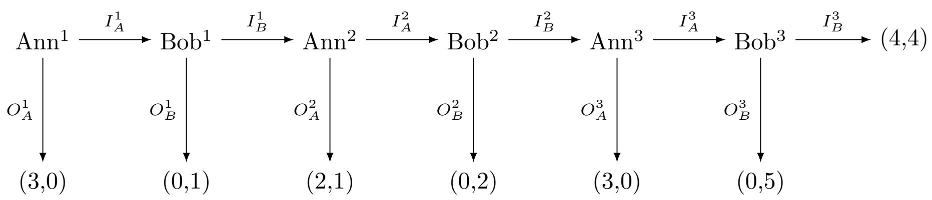

Figure 2: A Game. (The first number in each pair is Ann's payoff, the second is Bob's.)

only beliefs and hypotheses in which one has relatively high confidence are retained. So, to continue the analogy, if one is sure that there are only two viable relevant economic theories (that the process is IID or Markov), and one is initially fairly sure that one is right (IID, in the example), then in the face of observations which are very unlikely on the basis of that theory, one relinquishes one's adherence to that theory, falling back on one's (more solid) belief that at least one of the theories is right.

4.2.3 Reasoning in games

Another situation where conditional beliefs may change on update involves surprising or null events in games.

Example 3. Consider the game in Figure 2 and suppose that Bob thinks that Ann will adopt the Backwards Induction strategy, though admits a small probability ϵ of her making a 'trembling hand' mistake at each node. He acts as a Bayesian, and thus places probability $1-\epsilon$ on her going Out at every node.[19] Suppose now that Ann plays In at the first node, then Bob plays In, and then Ann plays In again. By standard Bayesian conditionnalisation on the (unexpected, but non-null) event of Ann going in twice, Bob continues to believe that Ann will play the Backwards Induction strategy, and hence go Out at node Ann^3. However, given the very small probability of her making two successive mistakes, he might come to reconsider his assumption that she is trying to play the Backwards Induction strategy. He might wonder whether the deviations from Backwards Induction play are intentional: Ann could be aiming for the gain she would get if Bob went In at every node. In other words, he might switch to Forward Induction-style reasoning (Pearce, 1984; Reny, 1992; Stalnaker, 1998; Battigalli and Siniscalchi, 2002). Under this assumption, he would expect Ann to move In at node Ann^3. That is, his belief about what Ann would do at node Ann^3 after having observed her play In twice differs from his prior conditional belief about what Ann will do if she gets to node Ann^3. This is another case where conditional beliefs change on learning (and hence, where Dynamic Consistency is violated).

Given the obvious analogy to Example 2—the assumption that Ann is playing the Backwards Induction strategy plays the role of the IID assumption; the strategy in which she is aiming for both playing In at all nodes plays the role of the alternative Markov hypotheses—it shoud be no surprise that the confidence approach can comfortably capture the reasoning in this example. Bob's initial beliefs can be characterised by the centred confidence ranking

[19]It is simple to check that Ann's Backward Induction strategy is to play Out at each node.

$$\Xi = \{\{\mu_{BI,\epsilon}\}, \{\mu_{BI,\epsilon}, \mu_{FI,\epsilon}\}\}$$

where $\mu_{BI,\epsilon}$ is the probability measure over Ann's play corresponding to the Backwards Induction assumption with 'trembling hand' errors—$\mu_{BI,\epsilon}(Out) = 1 - \epsilon > 0.5$ at every node—and $\mu_{FI,\epsilon}$ is the probability measure corresponding to the thesis that Ann is aiming for Bob going In at every node, with 'trembling hand' errors—$\mu_{FI,\epsilon}(In) = 1 - \epsilon > 0.5$ at every node. This represents Bob as a Bayesian with confidence: at the centre of the confidence ranking is the Bayesian belief $\mu_{BI,\epsilon}$, capturing the fact that at intermediate confidence levels, he acts and reasons as a Bayesian accepting that Ann will play the Backward Induction strategy with errors. However, at high levels of confidence, he is not sure of this prediction, entertaining alternative conjectures, and in particular the possibility that Ann is 'aiming' for everyone playing In at every node. For appropriate ρ_E, corresponding to appropriate probability thresholds about (the reasoning behind) Ann's play, the confidence update will shift to $\mu_{FI,\epsilon}$ if she makes 'too many' mistakes. For instance, if $\rho_{In}(\{\mu_{BI,\epsilon}\}), \rho_{In,In}(\{\mu_{BI,\epsilon}\}), \rho_{In}(\{\mu_{BI,\epsilon}, \mu_{FI,\epsilon}\}), \rho_{In,In}(\{\mu_{BI,\epsilon}, \mu_{FI,\epsilon}\}) \in (\frac{\epsilon}{2}, \epsilon)$,[20] then $\min \Xi_{In} = \{\mu_{BI,\epsilon}\}$[21], whereas $\min \Xi_{In,In} = \{\mu_{FI,\epsilon}\}$. If Ann goes In once, this can be seen as a mistake, so Bob reasons as a Bayesian and, updating by conditionnalisation, sticks with his Backwards Induction assumption. However, if she goes In again, then this is too surprising, and Bob looks to the most reasonable alternative conjecture that he admits as possible, which interprets Ann's play as intentional. As in the previous example, confidence update can comfortably capture such learning patterns.

Under the confidence analysis, Bob's update (and reasoning) varies as one would expect with ϵ. For a fixed probability-threshold correspondence ρ_E, as ϵ increases, there may be a value such that, after Ann plays In twice, Bob retains his Backwards Induction assumption: $\min \Xi_{In,In} = \{\mu_{BI,\epsilon}\}$.[22] This is as to be expected: if the probability of error is high enough, he need not consider two successives plays of In to be sufficiently surprising, and hence has less reason to doubt his initial beliefs about her strategy. On the other hand, as ϵ decreases with ρ_E fixed, there will be a value below which Bob will interpret Ann's play as intentional after she plays In just once: $\min \Xi_{In} = \{\mu_{FI,\epsilon}\}$.[23] If he considers a 'trembling hand' mistake to be sufficiently unlikely, then seeing just one deviation from the expected Backwards Induction play will be enough to trigger alternative reasoning. This is how Bob would update under this specification in the limit case of no 'trembling hand' errors ($\epsilon = 0$). There is thus a 'continuity' in reasoning between very small and zero ex ante probabilities of 'trembling hand' errors—that is, between update on surprising and null events.

By contrast, standard approaches retain Bayesian conditionnalisation whenever the observed event is non-null: so in the example, whenever $\epsilon > 0$, Bob will hold onto his Backwards Induction assumption no matter how many times Ann plays In. Bayes rule need only be

[20] In is the event that Ann plays In at the first node; In, In is the event that she plays In at the first two nodes, and so on.

[21] Note that $\mu_{BI,\epsilon} = \mu_{BI,\epsilon}(\bullet/In)$ and similarly for $\mu_{FI,\epsilon}$ as defined.

[22] This occurs whenever $\epsilon^2 \geq \rho_{In}(\{\mu_{BI,\epsilon}\}), \rho_{In,In}(\{\mu_{BI,\epsilon}\})$.

[23] This occurs whenever $\epsilon < \rho_{In}(\{\mu_{BI,\epsilon}\})$.

supplemented for cases of update on null events—when $\epsilon = 0$—and generalisations of probabilities (and Bayesian update) such as conditional probability systems (CPS) or lexicographic conditional probability systems (LCPS) have been proposed for such cases (Rényi, 1955; Myerson, 1986; Blume et al., 1991a,b; Dekel and Siniscalchi, 2015). Since they coincide with Bayesian conditionnalisation on non-null events, there is a discontinuity at $\epsilon = 0$: although under the smallest positive probability of error, Bob continues to hold onto the assumption of future Backwards Induction play after several deviations, as soon as the probability hits zero he can change his assumption on update after a single deviation. The continuity supported by the confidence-based approach may seem a more desirable property of reasoning in games.[24] Whether or not this is so, the example indicates that the confidence framework can cope with update by null events: indeed, the aforementioned generalisations of Bayesian conditionnalisation to null events can be recovered as special cases of confidence update, as we now illustrate on CPS's.[25]

For simplicity, let us assume that the state space $\mathcal{S}$ is finite (and retain all other terminology). A conditional probability system on $\mathcal{S}$ is a map $p^{CPS} : \Sigma \times (\Sigma/\emptyset) \to [0,1]$ such that $p^{CPS}(\bullet/E) \in \Delta(\Sigma)$, $p^{CPS}(E/E) = 1$, and $p^{CPS}(E/G) = p^{CPS}(E/F).p^{CPS}(F/G)$ for all $E, F, G \in \Sigma$ with $E \subseteq F \subseteq G$ and $F \neq \emptyset$. $p^{CPS}(E/\mathcal{S})$ can be thought of as representing prior beliefs. If $p^{CPS}(E/\mathcal{S}) > 0$, then $p^{CPS}(\bullet/E)$ is the standard Bayesian conditionnalisation of p^{CPS}; however, $p^{CPS}(\bullet/E)$ is well-defined and non-trivial even when $p^{CPS}(E/\mathcal{S}) = 0$. Recall (Section 4.2.1) that a confidence ranking Ξ is centred if it contains a singleton set; in this case, we use p_Ξ to denote the member of the singleton set, and call it the *centre* of Ξ.

Proposition 4. *Let p^{CPS} be a conditional probability system on a finite space $\mathcal{S}$. Then there exists a centred confidence ranking Ξ and a family of functions $(\rho_E)_{E\in\Sigma}$, $\rho_E : \Xi \to [0,1]$ such that: i. the centre of Ξ, p_Ξ=$p^{CPS}(\bullet/\mathcal{S})$; and ii. for each non-empty event E, Ξ_E, the confidence update of Ξ by E according to ρ_E is a centred confidence ranking whose centre, p_{Ξ_E} satisfies $p_{\Xi_E}(F) = p(F/E)$ for all $F \in \Sigma$.*

So any decision maker that can be modelled using a CPS can alternatively be modelled using confidence update. Focussing on decisions where the stakes are limited, the decision maker's ex ante and ex post preferences would be precisely as according to the CPS model: in particular confidence update picks out his ex post beliefs properly, even for update on events that are null according to the centre of his confidence ranking. By contrast, his lack of full confidence about his best-guess probability measure (and his relative degree of confidence in the alternatives) does come out in his ex ante preferences under the confidence approach—though not under the CPS approach—in decisions with high or extremely high stakes. On such decisions, his preferences may be non-Bayesian.

[24] We hasten to add that this discussion concerns the reasoning (and update) of one player in a game; evaluating potential consequences for equilibria would require further concepts (e.g. Dekel and Siniscalchi, 2015), and goes beyond the scope of this paper.

[25] See for instance Hammond (1994) on the relation with LCPS.

This suggests that confidence update, in addition to dealing with update under ambiguity, can comfortably and fruitfully deal with issues arising from update on surprising or null events. Indeed, unlike standard approaches, it offers a uniform treatment of both sorts of update.

5 Discussion

We now briefly consider the relationships with other learning paradigms, as well as potential extensions.

Bayesian and Classical Statistical Reasoning Confidence update subsumes elements of both Classical and Bayesian statistical reasoning. The way it deals with confidence, and in particular the use of probability thresholds over the ex ante probability (or likelihood) of the learnt event under different probability measures, is classical in spirit. The recognition that on learning an event, one ultimately has to use (some) probabilities conditional on that event is Bayesian. This can be illustrated on Example 1 (Section 4.1).

On the one hand, the penultimate case in the example (involving Dirac measures) reveals a strong analogy to the reasoning in classical statistics: there is a set of parameters (the ex ante set of Dirac measures), and on observation, one can rule out those according to which the observation was too unlikely. The probability threshold in the confidence approach plays a role analogous to the significance level in classical hypothesis testing. However, the confidence approach does not demand a fixed significance level. Rather, the update encompasses all relevant significance levels. The level to be used in an ensuing decision is determined on the basis of its importance and the decision maker's attitude to choosing on the basis of limited confidence, as represented by his cautiousness coefficient. In other words, the approach sheds light on how the appropriate significance level should be fixed, revealing the value judgement or taste it corresponds to.

On the other hand, since initial beliefs representable by a Bayesian probability generate a special type of confidence ranking, the confidence update rule can be applied, yielding as posterior beliefs the conditional probability measure (or, more precisely, the confidence ranking whose only element is the singleton containing it). So confidence update coincides with standard Bayesian statistical practice whenever the decision maker holds single-prior beliefs with maximal confidence.

Belief Revision Confidence update is also reminiscent of a substantial literature in Artificial Intelligence, logic and philosophy on 'belief revision' (e.g. Gardenfors, 1988), which focuses on belief change in cases where incoming information contradicts initial beliefs. In such cases, there is usually a choice of which among several ex ante beliefs to give up. A popular approach employs the notion of the 'entrenchment' of a belief, and is guided by a maxim similar to ours: hold on to the beliefs that are more 'entrenched', relinquishing those that are less 'entrenched'.

This affinity is doubtless related to some of the points made in the preceding sections; indeed, the relevance of belief revision for scientific theory change (Alchourron et al., 1985) and reasoning in games (Stalnaker, 1998) has long been recognised.

However, given the focus on categorical rather than probabilistic beliefs in that literature, it contains, to the best of our knowledge, no rule corresponding to the one proposed here. Moreover, and crucially, they typically do not consider decision. As such, one could consider this paper as developing a decision-theoretic approach to learning that was lacking from the belief revision literature.

Choice and Learning An important characteristic of the Bayesian paradigm is the connection between ex ante preferences and update: under it, ex ante and ex post conditional preferences coincide (Section 4.2.1). The current proposal involves a strong, albeit different connection, modulated by the double role of confidence in choice (according to (3)) and learning (via (4)): a decision maker's confidence in a belief regulates both how willing he is is to choose on the basis of it and how tenaciously he is ready to hold onto it in the face of conflicting information.[26] This guarantees that ex post preferences are partially determined by ex ante ones (in particular those held at sufficiently high stakes levels).

This connection is a central plank of our approach. It draws normative support from the aforementioned intuitions. The relationships it implies between ex ante and ex post preferences enhance testability, hence lending descriptive clout. And it sets our approach apart from others dealing with null or surprising events. For instance, under the CPS model (Section 4.2.3), ex ante preferences impose very few constraints on ex post preferences after updating on a null event.

Ortoleva (2012) proposes a 'Hypothesis Testing' update rule of Bayesian beliefs which is similar in spirit to the CPS and LCPS models, except that it 'moves to' another Bayesian probability when the learnt event is surprising enough (i.e. its ex ante probability falls below a threshold), rather than when it is null.[27] The rule is motivated by classical hypothesis-testing reasoning, of the sort mentioned above. However, unlike the confidence-based approach, ex ante preferences in Ortoleva's model impose virtually no constraints on the ex post preferences an agent will adopt on learning surprising information. In fact, given some underlying technical similarilities,[28] it may be possible to retrieve the 'Hypothesis Testing' rule as a special case of confidence update, via a result similar to Proposition 4 for the CPS model. This may be a way of linking the update to ex ante behaviour.

Gilboa et al. (2017) propose a model of choice which combines case-based and expected-

[26]Note that in the Bayesian paradigm, no single concept plays such a double role: the strength of a Bayesian probability in particular is quite distinct from how tenaciously it is retained on update (e.g. Leitgeb, 2017).

[27]Ortoleva (2014) extends the approach to multiple prior beliefs.

[28]Specifically: the proof of our Proposition 4 relies on the fact that CPS's are equivalent to certain orders on the space of probability measures, as are confidence rankings, and Ortoleva's update is also determined by an order on the probability space (Ortoleva, 2012, Prop 2).

utility reasoning, claiming that the former is more appropriate and widespread in the aftermath of surprising events. Since the model is static, it does not draw any link between preferences prior to a (surprising) event and posterior preferences, whereas, as noted, confidence will play a role in relating the two under the approach proposed here.

Extensions and future research Most of the technical assumptions on confidence rankings adopted in Section 3—notably closure and convexity—are inessential to the workings of the update rule. Similar results can be obtained in their absence, albeit with added technicalities to deal, for instance, with the fact that non-convexities do not show up in preferences. Moreover, whilst we have focussed on the standard case of update on events, the general logic of the update rule—and in particular the intersection of the sets of probability measures reflecting the information with the ex ante confidence ranking—applies for other 'input formats', such as information representable by a subset of the probability space (as in Gajdos et al., 2008) or a probability assignment for certain events (as in Jeffrey, 1972; Dietrich et al., 2016). Future work could set out the consequences of confidence update in such cases. A final important extension would be to sequential learning situations, as commonly found in statistical decision theory. This would be essential for understanding the long-run implications of the approach, and its consequences in a range of economic applications.

6 Conclusion

This paper proposes a novel update rule under ambiguity. Starting from the intuition that one's confidence in beliefs has a central role to play in learning, we formulate a model of update of confidence in beliefs, drawing on an existing model of confidence and decision (Hill, 2013). It is based on a simple, but reasonable intuition: when updating in the face of information that conflicts with prior beliefs, *retain* as far as possible those conditional beliefs in which you are *more* confident, and relinquish only those in which you have less confidence. A simple and intuitive axiom—Confidence Consistency—characterises a general confidence update rule that conforms to this maxim.

We also characterise a more refined version: probability-threshold confidence update. In a way reminiscent of classical statistical reasoning, it uses a confidence level-dependent threshold to eliminate probability measures that were too 'wrong' about the learnt event ex ante.

Confidence update can comfortably handle update on complete ignorance, on which standard multi-prior update rules struggle. It provides a general framework that can recover prominent existing update rules as special cases, providing a new perspective on their credentials and relationship. It can also fruitfully deal with update on surprising events, such as crises, and on null events, encompassing the standard game-theoretical tools for the latter as special cases.

A Proofs

A.1 Proofs of Results in Section 3

Proof of Theorems 1 and 2 . We prove Theorem 2. The proof of Theorem 1 is similar. We first show sufficiency of the axioms.

Fix non-null $E \in \Sigma$; since $\succeq$ is non-degenerate by Assumption 1, such events exist.. By Assumption 1, there exists a triple (Ξ, D, u) representing $\succeq$ according to (3). For every stakes level σ, let $\mathcal{C}^\sigma = D(f)$ for some $f \in \sigma$. It follows from the confidence representation (Hill, 2013) that $\mathcal{C}^\sigma$ represents $\succeq^\sigma$ (in tandem with u) according to standard maximin EU representation; ie. $\succeq^\sigma$ is represented by:

$$V(f) = \min_{p \in \mathcal{C}^\sigma} \mathbb{E}_p u(f(s)) \tag{12}$$

As a point of notation, for any $x \in [0,1]$, we use $[E, x]$ to denote $\{p \in \Delta(\Sigma) : \ p(E) \geq x\}$.

By Non-degeneracy, $\succeq_E$ is non-degenerate. Moreover, there exists an E-resilient stakes-level σ: if not, by Information-Based Learning, $f \not\preceq_E^{\sigma'} c$ for every $f \in \mathcal{A}$, $c \in \mathcal{A}^c$ and stakes level σ', contradicting the monotonicity of the confidence representation (3).

Lemma 1. *For any E-resilient stakes level σ, there exists $x_E^\sigma \in [0,1]$ such that $\succeq_E^\sigma$ is represented by:*

$$V_E^\sigma(f) = \underline{\min_{p \in \left(C^\sigma \cap [E, x_E^\sigma]\right)_E}} \mathbb{E}_p u(f) \tag{13}$$

where $(\mathcal{C}^\sigma \cap [E, x_E^\sigma])_E$ is as defined in (2). Moreover:

1. *if there exists $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$ such that $f_E c \sim_E^\sigma c$ but $f_E c \nsim^\sigma c$, then there is a unique $x_E^\sigma \in [0,1]$ having this property;*
2. *if for all $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$, whenever $f_E c \sim_E^\sigma c$, then $f_E c \sim^\sigma c$, and* there exists no $e, d \in \mathcal{A}^c$ with $e \succ d \succ c$ and $f_E e \succeq^\sigma d$, *then every $x_E^\sigma \in [0,1]$ has this property;*
3. *if for all $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$, whenever $f_E c \sim_E^\sigma c$, then $f_E c \sim^\sigma c$, and for some such $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$,* there exists $e, d \in \mathcal{A}^c$ with $e \succ d \succ c$ and $f_E e \succeq^\sigma d$, then there exists $\overline{x_E^\sigma} \in [0,1]$ such that every $x_E^\sigma \in [0, \overline{x_E^\sigma}]$ *has this property.*

Proof. Fix an E-resilient stakes level σ. For every $f \in \mathcal{A}$, by the representation (Assumption 1), there exists a unique $c \in \mathcal{A}^c$, up to indifference, such that $f_E c \sim_E^\sigma c$; consider any such f and c. For any $e, d \in \mathcal{A}^c$ with $e \succ d \succeq c$ and $f_E e \succeq^\sigma d$, let $\lambda_{e,d;f}$ be the (unique) $\lambda \in [0,1]$ such that $f_E e \sim^\sigma e_{1-\lambda} d$. (By the E-resilience of σ, $f_E c \preceq^\sigma c$, whence, by the representation, $f_E e \preceq^\sigma e$, so such a λ exists; by the representation, it is unique.) Note that, by definition, for any $p \in \Delta$ such that $\mathbb{E}_p u(f_E e) \geq \mathbb{E}_p u(e_{1-\lambda} d)$ and $\mathbb{E}_p u(e_E d) \geq \mathbb{E}_p u(e_\lambda d)$, we have that $\mathbb{E}_p u((f_E e)_{\frac{1}{2}}(e_E d)) = \mathbb{E}_p u(e_{\frac{1}{2}}(f_E d)) \geq \mathbb{E}_p u((e_{1-\lambda} d)_{\frac{1}{2}}(e_\lambda d)) =$

$\mathbb{E}_p u(e_{\frac{1}{2}}d)$. Let $\Lambda_f^\sigma = \{\lambda_{e,d;f} : e, d \in \mathcal{A}^c, \beta \in (0,1], f_E e \succeq^\sigma d, e \succ d \succ c\}$, and $\overline{\Lambda_f^\sigma} = \{\lambda_{e,d;f} : e, d \in \mathcal{A}^c, \beta \in (0,1], f_E e \succeq^\sigma d, e \succ d \succeq c\}$.

Claim 1.

$$\Lambda_f^\sigma = \left\{\hat{\lambda} \in [0,1] : \exists \hat{e}, \hat{d} \in \mathcal{A}^c \; s.t. \; \hat{e} \succ \hat{d}, f_E\hat{e} \sim^\sigma \hat{d}_{\hat{\lambda}}\hat{e}, \hat{\lambda} > \lambda_{\hat{e},c;f}\right\}$$
$$= \left\{\hat{\lambda} \in [0,1] : \exists \hat{e}, \hat{d} \in \mathcal{A}^c \; s.t. \; \hat{e} \succ \hat{d}, f_E\hat{e} \sim^\sigma \hat{d}_{\hat{\lambda}}\hat{e}, (f_E\hat{e})_{\frac{1}{2}}(\hat{e_E d}) \prec_E^\sigma \hat{e}_{\frac{1}{2}}\hat{d}\right\}$$

and

$$\overline{\Lambda_f^\sigma} = \left\{\hat{\lambda} \in [0,1] : \exists \hat{e}, \hat{d} \in \mathcal{A}^c \; s.t. \; \hat{e} \succ \hat{d}, f_E\hat{e} \sim^\sigma \hat{d}_{\hat{\lambda}}\hat{e}, \hat{\lambda} \geq \lambda_{\hat{e},c;f}\right\}$$
$$= \left\{\hat{\lambda} \in [0,1] : \exists \hat{e}, \hat{d} \in \mathcal{A}^c \; s.t. \; \hat{e} \succ \hat{d}, f_E\hat{e} \sim^\sigma \hat{d}_{\hat{\lambda}}\hat{e}, (f_E\hat{e})_{\frac{1}{2}}(\hat{e_E d}) \preceq_E^\sigma \hat{e}_{\frac{1}{2}}\hat{d}\right\}.$$

Proof. Note firstly that, by the representation, for any $e \succ d, d'$, $\lambda_{e,d;f} > \lambda_{e,d';f}$ if and only if $d \succ d'$. For any $\hat{\lambda} \in [0,1]$ and $\hat{e}, \hat{d} \in \mathcal{A}^c$ with $\hat{e} \succ \hat{d}$ and $f_E\hat{e} \sim^\sigma \hat{d}_{\hat{\lambda}}\hat{e}$, if $\hat{\lambda} > \lambda_{\hat{e},c;f}$, then $\hat{d} \succ c$ by the previous observation. So $\Lambda_f^\sigma \supseteq \left\{\hat{\lambda} \in [0,1] : \exists \hat{e}, \hat{d} \in \mathcal{A}^c \; s.t. \; f_E\hat{e} \sim^\sigma \hat{d}_{\hat{\lambda}}\hat{e} \succ^\sigma c, \hat{\lambda} > \lambda_{\hat{e},c;f}\right\}$, and similarly for $\overline{\Lambda_f^\sigma}$. Moreover, for such $\hat{\lambda}$, $\hat{e}, \hat{d}$, it follows from the representation that $(f_E\hat{e})_{\frac{1}{2}}(\hat{e_E d}) \prec_E^\sigma \hat{e}_{\frac{1}{2}}\hat{d}$ if and only if $f_E\hat{d} \prec_E^\sigma \hat{d}$, and since $f_E c \sim_E^\sigma c$, this can only be the case if $\hat{d \succ c}$. So $\Lambda_f^\sigma \supseteq \left\{\hat{\lambda} \in [0,1] : \exists \hat{e}, \hat{d} \in \mathcal{A}^c \; s.t. \; \hat{e} \succ \hat{d}, f_E\hat{e} \sim^\sigma \hat{d}_{\hat{\lambda}}\hat{e}, (f_E\hat{e})_{\frac{1}{2}}(\hat{e_E d}) \prec_E^\sigma \hat{e}_{\frac{1}{2}}\hat{d}\right\}$, and similarly for $\overline{\Lambda_f^\sigma}$. As for the other direction, for any $\hat{e}, \hat{d} \in \mathcal{A}^c$ with $\hat{e} \succ \hat{d} \succ c$ and $f_E\hat{e} \succeq \hat{d}$, if $f_E\hat{e} \sim \hat{e}_{1-\hat{\lambda}}\hat{d}$, then by the previous remark about the ordering of $\lambda_{e,d;f}, \lambda_{e,d';f}$, $\hat{\lambda} > \lambda_{e,c;f}$; it follows that $\Lambda_f^\sigma \subseteq \left\{\hat{\lambda} \in [0,1] : \exists \hat{e}, \hat{d} \in \mathcal{A}^c \; s.t. \; f_E\hat{e} \sim^\sigma \hat{d}_{\hat{\lambda}}\hat{e} \succ^\sigma c, \hat{\lambda} > \lambda_{\hat{e},c;f}\right\}$, and similarly for $\overline{\Lambda_f^\sigma}$. Finally, for any such $\hat{e}, \hat{d} \in \mathcal{A}^c$, by the representation (and in particular C-Independence at a given stakes level) and the fact that $f_E c \sim_E c$, it follows from the $\hat{d} \succ c$ that $f_E\hat{d} \prec_E \hat{d}$, so $(f_E\hat{e})_{\frac{1}{2}}(\hat{e_E d}) \prec_E^\sigma \hat{e}_{\frac{1}{2}}\hat{d}$, and hence $\lambda_{\hat{e},\hat{d};f} \in \left\{\hat{\lambda} \in [0,1] : \exists \hat{e}, \hat{d} \in \mathcal{A}^c \; s.t. \; \hat{e} \succ \hat{d}, f_E\hat{e} \sim^\sigma \hat{d}_{\hat{\lambda}}\hat{e}, (f_E\hat{e})_{\frac{1}{2}}(\hat{e_E d}) \prec_E^\sigma \hat{e}_{\frac{1}{2}}\hat{d}\right\}$, and similarly for the case of $\hat{d} \succeq c$. This establishes the claim.

□

If, for all $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$ such that $f_E c \sim_E^\sigma c$, $f_E c \sim^\sigma c$, then the result immediately holds with $x_E^\sigma = 0$, so assume henceforth that this is not the case. For clarity, we divide the remainder of the proof into cases.

Case 1. We first consider the case in which there exists $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$ with $f_E c \sim_E^\sigma c$ but $f_E c \nsim^\sigma c$ such that there exists $e, d \in \mathcal{A}^c$ with $e \succ d \succ c$ and $f_E e \succeq^\sigma d$. So Λ_f^σ and $\overline{\Lambda_f^\sigma}$ are non-empty. Since $f_E c \nsim^\sigma c$, and σ is E-resilient, it follows that $f_E c \prec^\sigma c$; this, in combination with the fact that $f_E c \sim_E^\sigma c$ implies that $f \notin \mathcal{A}^c$. Since, for any $e \in \mathcal{A}^c$ with $f_E e \sim^\sigma e$, $f_E e \succ_E^\sigma e$ by the representation, $0 \notin \Lambda_f^\sigma$. Let $\lambda_f = \inf \Lambda_f^\sigma$. Since, for any $d \succ c$, $\lambda_{\hat{e},d;f} > \lambda_{\hat{e},c;f}$ for all $\hat{e} \in \mathcal{A}^c$, $\lambda_f \notin \Lambda_f^\sigma$, and hence, for every $\overline{c}, \underline{c} \in \mathcal{A}^c$ with $\overline{c} \succ \underline{c}$ and $f_E\overline{c} \sim^\sigma \underline{c}_{\lambda_f}\overline{c}$, it holds that $(f_{\frac{1}{2}}\overline{c})_E(\overline{c}_{\frac{1}{2}}\underline{c}) \succeq_E^\sigma \overline{c}_{\frac{1}{2}}\underline{c}$. Since

$\lambda_{e,d;f}$ is continuous in d for every $e \succ c$, for every such e, $\lambda_{e,c;f} \geq \lambda_f$.

We now show that, for every $p \in \mathcal{C}^\sigma \cap [E, \lambda_f]$, $\mathbb{E}_p u(f_E c) \geq \mathbb{E}_p u(c)$. First consider any $q \in \mathcal{C}^\sigma \cap \{p \in \Delta(\Sigma) : \ p(E) > \lambda_f\}$; by the definition of λ_f, there exist $e, d \in \mathcal{A}^c$, with $e \succ d \succ c$, $f_E c \succeq^\sigma d$ and $q(E) \geq \lambda_{e,d;f}$. By the previous remark, since $\mathbb{E}_q u(f_E e) \geq \mathbb{E}_q u(e_{1-\lambda_{e,d;f}} d)$ and $\mathbb{E}_q u(e_E d) \geq \mathbb{E}_q u(e_{\lambda_{e,d;f}} d)$, it follows that $\mathbb{E}_q u((f_E e)_{\frac{1}{2}}(e_E d)) \geq \mathbb{E}_q u(e_{\frac{1}{2}} d)$, and hence, by the linearity of the EU functional, $\mathbb{E}_q u(f_E d) \geq \mathbb{E}_q u(d)$. It follows from the properties of the EU functional that $\mathbb{E}_q u(f_E c) \geq \mathbb{E}_q u(c)$. Since this holds for all $q \in \mathcal{C}^\sigma \cap \{p \in \Delta(\Sigma) : \ p(E) > \lambda_f\}$, by the continuity of the EU functional, it holds for the closure $\mathcal{C}^\sigma \cap [E, \lambda_f]$, as required.

Now we show that, for each $d \succ c$, there exists $p \in \mathcal{C}^\sigma \cap [E, \lambda_f]$ with $\mathbb{E}_p u(f_E d) < u(d)$. For reductio, suppose that there exists $d \succ c$ such that $\mathbb{E}_p u(f_E d) \geq \mathbb{E}_p u(d)$ for all $p \in \mathcal{C}^\sigma \cap [E, \lambda_f]$. It follows that $\mathbb{E}_p u(f_E c) > \mathbb{E}_p u(c)$ for all $p \in \mathcal{C}^\sigma \cap [E, \lambda_f]$. For each $e \succ c$, consider $I_{e,\lambda_f} = \left\{p \in \Delta(\Sigma) : \mathbb{E}_p u(f_E e) = \mathbb{E}_p u(e_{1-\lambda_{e,c;f}} c)\right\} \cap \{p \in \Delta(\Sigma) : p(E) = \lambda_f\}$; since $\mathbb{E}_p u(f_E c) = \mathbb{E}_p u(c)$ for all p in this set (by the previous observation), it follows that $I_{e,\lambda_f} \cap (\mathcal{C}^\sigma \cap [E, \lambda_f]) = \emptyset$ for all such e. Let $\lambda' = \inf\{x \in [0,1] : \mathbb{E}_p u(f_E c) \geq \mathbb{E}_p u(c), \ \forall p \in \mathcal{C}^\sigma \cap [E, x]\}$. By the previous observations $\lambda' < \lambda_f$. Moreover, by continuity of the EU functional, there exists $p \in \mathcal{C}^\sigma \cap [E, \lambda']$ such that $\mathbb{E}_p u(f_E c) = \mathbb{E}_p u(c)$. It follows that $I_{e,\lambda'} \cap (\mathcal{C}^\sigma \cap [E, \lambda']) \neq \emptyset$ for at least one $e \succ c$, where $I_{e,\lambda'} = \left\{p \in \Delta(\Sigma) : \mathbb{E}_p u(f_E e) = \mathbb{E}_p u(e_{1-\lambda_{e,c;f}} c)\right\} \cap \{p \in \Delta(\Sigma) : p(E) = \lambda'\}$. Since, for any $p \in I_{e,\lambda'}$, $\mathbb{E}_p u((f_E e)_{\frac{1}{2}}(e_E c)) = \mathbb{E}_p u((e_{1-\lambda_{e,c;f}} c)_{\frac{1}{2}}(e_{\lambda'} c) = u(c_{\frac{1}{2}}(e_{1-(\lambda_{e,c;f}-\lambda')} c))$, and since, for any $p \in I_{e,\lambda'} \cap (\mathcal{C}^\sigma \cap [E, \lambda'])$, $\mathbb{E}_p u(f_E c) \succeq \mathbb{E}_p u(c)$, it follows that $\lambda_{e,c;f} = \lambda' < \lambda_f$ for any such e, contradicting the definition of λ_f. So for each $d \succ c$, there exists $p \in \mathcal{C}^\sigma \cap [E, \lambda_f]$ with $\mathbb{E}_p u(f_E d) < u(d)$, as required.

Now consider any $f' \in \mathcal{A}$ with $f'_E c' \sim^\sigma_E c'$. We consider two cases separately.

Case i. First consider the case where $f'_E c' \nsim^\sigma c'$. We first treat the case in which there exists $e \in \mathcal{A}^c$ with $e \succ c$ and $f'_E e \succ^\sigma c'$, so, as above, $\Lambda^\sigma_{f'}$ and $\overline{\Lambda^\sigma_{f'}}$ are non-empty. By Probability Consistency, Claim 1 and the previous observations, $\lambda_f \notin \Lambda^\sigma_{f'}$. Applying the same axiom again yields that $\inf \Lambda^\sigma_{f'} \notin \Lambda^\sigma_f$, so $\lambda_f = \inf \Lambda^\sigma_{f'}$. By the arguments used above, $\mathbb{E}_p u(f'_E c') \geq \mathbb{E}_p u(c')$ for all $p \in \mathcal{C}^\sigma \cap [E, \lambda_f]$, and, for each $d' \succ c'$, there exists $p \in \mathcal{C}^\sigma \cap [E, \lambda_f]$ with $\mathbb{E}_p u(f_E d') < u(d')$. Now consider the case where, for all $e \in \mathcal{A}^c$, $f'_E e \preceq^\sigma c'$. So $\Lambda^\sigma_{f'} = \emptyset$, which by Claim 1, contradicts A5 and the fact that $\Lambda^\sigma_f \neq \emptyset$, so this case cannot occur.

Case ii. Now consider the case where $f'_E c' \sim^\sigma c'$. So $\mathbb{E}_p u(f'_E c') \geq \mathbb{E}_p u(c')$ for all $p \in \mathcal{C}^\sigma \cap [E, \lambda_f]$. If there exists $e, d \in \mathcal{A}^c$ with $e \succ d \succ c'$ and $f'_E e \succeq^\sigma d$, then $\Lambda^\sigma_{f'} \neq \emptyset$. By Probability Consistency and the last characterisation of Λ^σ_f in Claim 1, $\lambda_f < \lambda$ for all $\lambda \in \Lambda^\sigma_{f'}$. By an argument similar to that used above that, for each $d' \succ c'$, there exists $p \in \mathcal{C}^\sigma \cap [E, \lambda_f]$ with $\mathbb{E}_p u(f'_E d') < u(d')$. If there exists no $e, d \in \mathcal{A}^c$ with $e \succ d \succ c'$ and $f'_E e \succeq^\sigma d$, then $f'_E e \sim f'_E c' \sim^\sigma c'$ for all $e \in \mathcal{A}^c$ with $e \succ c'$. It follows from the representation that there exists $p \in \mathcal{C}^\sigma$ with $\mathbb{E}_p u(f_E d') = u(c') < u(d')$ for all $d' \succ c'$ and $p(E) = 1$; since $p \in \mathcal{C}^\sigma \cap [E, \lambda_f]$, for every $d' \succ c'$, there exists $p \in \mathcal{C}^\sigma \cap [E, \lambda_f]$ with $\mathbb{E}_p u(f_E d') < u(d')$.

Case 2. Now we consider the case in which there exists $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$ such that $f_E c \sim^\sigma_E c$ but $f_E c \nsim^\sigma c$, and for all such f, c, $f_E e \preceq^\sigma c$ for all $e \in \mathcal{A}^c$. By Null consistency, for each such f, c, there exists $e \in \mathcal{A}^c$ with $f_E e \sim^\sigma c$. Since $f_E e' \sim^\sigma f_E e$ for any $e' \succ e$ and any such f, c, it follows from the representation that there exists $p \in \mathcal{C}^\sigma$ with $\mathbb{E}_p u(f_E e) = u(c)$ and $p(E) = 1$ and that, for any other $q \in \mathcal{C}^\sigma$ with $q(E) = 1$, $\mathbb{E}_q u(f_E e) \geq u(c)$. It thus follows that for all $p \in \mathcal{C}^\sigma \cap [E, 1]$, $\mathbb{E}_p u(f_E c) \geq \mathbb{E}_p u(c)$. Moreover, for every $d \succ c$, if $\mathbb{E}_p u(f_E d) \geq \mathbb{E}_p u(d)$ for all $p \in \mathcal{C}^\sigma \cap [E, 1]$, then $f_E d \succ^\sigma c$, contradicting the definition of the case; so for each $d \succ c$, there exists $p \in \mathcal{C}^\sigma \cap [E, 1]$ with $\mathbb{E}_p u(f_E d) < u(d)$. Now consider any $f' \in \mathcal{A}$ with $f'_E c' \sim^\sigma_E c'$ and $f'_E c' \sim^\sigma c'$. If there exists $e, d \in \mathcal{A}^c$ with $e \succ d \succ c'$ and $f'_E e \succeq^\sigma d$, then $\Lambda^\sigma_{f'} \cap [0, 1) \neq \emptyset$. By Probability Consistency and the last characterisation of Λ_f in Claim 1, it follows that, for every $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$ such that $f_E c \sim^\sigma_E c$ but $f_E c \nsim^\sigma c$, and $f_E e \preceq^\sigma c$ for all $e \in \mathcal{A}^c$, there exists $e' \succ d' \succ c$ with $f_E e' \succeq^\sigma d' \succ c$, which is a contradiction. So for every $f' \in \mathcal{A}$ with $f'_E c' \sim^\sigma_E c'$ and $f'_E c' \sim^\sigma c'$, $f'_E e \sim^\sigma c'$ for all $e \in \mathcal{A}^c$ with $e \succ c'$. It follows from the representation that there exists $p \in \mathcal{C}^\sigma$ with $\mathbb{E}_p u(f_E d') = u(c') < u(d')$ for all $d' \succ c'$ and $p(E) = 1$; since $p \in \mathcal{C}^\sigma \cap [E, 1]$, for every $d' \succ c'$, there exists $p \in \mathcal{C}^\sigma \cap [E, 1]$ with $\mathbb{E}_p u(f_E d') < u(d')$.

Let $x^\sigma_E = \lambda_f$ in Case 1 and $x^\sigma_E = 1$ in Case 2. By the previous observations, for every $f \in \mathcal{A}$, $\min_{p \in \mathcal{C}^\sigma \cap [E, x^\sigma_E]} \mathbb{E}_p u(f_E c) \geq u(c)$, where $f_E c \sim^\sigma_E c$, and for any $d \succ c$, $\min_{p \in \mathcal{C}^\sigma \cap [E, x^\sigma_E]} \mathbb{E}_p u(f_E d) < u(d)$. It follows from the continuity of the maximin-EU functional that $\min_{p \in \mathcal{C}^\sigma \cap [E, x^\sigma_E]} \mathbb{E}_p u(f_E c) = u(c)$ for all $f \in \mathcal{A}$ with $f_E c \sim^\sigma_E c$. By Consequentialism, for every $f \in \mathcal{A}$, $f \sim^\sigma_E c$ for $c \in \mathcal{A}^c$ such that $f_E c \sim^\sigma_E c$, so the preferences $\succeq^\sigma_E$ are represented by $V(f) = u(c)$ such that $f_E c \sim^\sigma_E c$. Since:

$$\min_{p\in\mathcal{C}^\sigma\cap[E,x_E^\sigma]}\mathbb{E}_p u(f_E c)=u(c)\Leftrightarrow\min_{p\in\mathcal{C}^\sigma\cap[E,x_E^\sigma]}\big(p(E)(\mathbb{E}_{p(\bullet/E)}u(f))+(1-p(E))u(c)\big)=u(c)$$
$$\Leftrightarrow\min_{p\in\mathcal{C}^\sigma\cap[E,x_E^\sigma]}\mathbb{E}_{p(\bullet/E)}u(f)=u(c)$$
$$\Leftrightarrow\underline{\quad\min\quad}_{p\in\left(\mathcal{C}^\sigma\cap[E,x_E^\sigma]\right)_E}\mathbb{E}_p u(f)=u(c)$$

This establishes the representation.

As concerns the uniqueness of x_E^σ, it is clear from the proof that, if there exist $f\in\mathcal{A}$ with $c\in\mathcal{A}^c$ such that $f_E c\sim_E^\sigma c$ but $f_E c\nsim^\sigma c$, then $x_E^\sigma=\inf\Lambda_f^\sigma$ for any such $f\in\mathcal{A}$ and $c\in\mathcal{A}^c$ in Case 1, and $x_E^\sigma=1$ if Case 2 holds. Since Λ_f^σ is uniquely defined on the basis of preferences, this implies that x_E^σ is unique. If $f_E c\sim_E^\sigma c$ whenever $f_E c\sim^\sigma c$, and for no such f,c there exists $e,d\in\mathcal{A}^c$ with $e\succ d\succ c$ and $f_E e\succeq^\sigma d$, then by the analysis of this case 1.ii., $\min_{p\in\mathcal{C}^\sigma\cap[E,x]}\mathbb{E}_p u(f_E c)=u(c)$ iff $f_E c\sim_E^\sigma c$, for all $x\in[0,1]$, as required. Finally, if $f_E c\sim_E^\sigma c$ whenever $f_E c\sim^\sigma c$ but for some $f\in\mathcal{A}$ and $c\in\mathcal{A}^c$, there exists $e,d\in\mathcal{A}^c$ with $e\succ d\succ c$ and $f_E e\succeq^\sigma d$, then by the observations about Λ_f^σ (case 1.ii.), $\min_{p\in\mathcal{C}^\sigma\cap[E,x]}\mathbb{E}_p u(g_E d)=u(d)$ iff $g_E d\sim_E^\sigma d$, whenever $x<\lambda$ for all $\lambda\in\Lambda_f^\sigma$, as required. □

Define the function ϕ_E relating E-resilient stakes levels to values in $[0,1]$ as follows:

1. If σ satisfies the conditions of clause 1. of Lemma 1, then $\phi_E(\sigma)=x_E^\sigma$ such that (13) holds.

2. If σ satisfies the conditions of clause 2. of Lemma 1, then $\phi_E(\sigma)=\sup\{\phi_E(\sigma'):\sigma'>\sigma\}$.[29]

3. If σ satisfies the conditions of clause 3. of Lemma 1, then $\phi_E(\sigma)=\max\left\{\sup\{\phi_E(\sigma'):\sigma'>\sigma\},\overline{x_E^\sigma}\right\}$, where $\overline{x_E^\sigma}$ is as in Lemma 1.

By definition and Lemma 1, (13) holds for $\phi_E(\sigma)$ for every E-resilient stakes level σ.

Claim 2. For every E-resilient σ',σ'' with $\sigma''>\sigma'$, $\phi_E(\sigma'')\leq\phi_E(\sigma')$.

Proof. Let $\Lambda_{f'}^{\sigma'}$ and $\Lambda_{f''}^{\sigma''}$ be defined as in the proof of Lemma 1, for appropriate f',f''. By the proof of that Lemma, if the stakes levels σ', σ'' satisfy the conditions of clause 1. (ie. there exists $f\in\mathcal{A}$ and $c\in\mathcal{A}^c$ with $f_E c\sim_E^{\sigma'} c$ but $f_E c\nsim^{\sigma'} c$ and similarly for σ"), then $x_E^{\sigma'}=\inf\Lambda_{f'}^{\sigma'}$ (under case 1 in the proof of the Lemma) or $x_E^{\sigma'}=1$ (in case 2), and similarly for $x_E^{\sigma''}$. By Probability Consistency and Claim 1, for any $\lambda\notin\Lambda_{f''}^{\sigma''}$, $\lambda\notin\Lambda_{f'}^{\sigma'}$, so if $x_E^{\sigma''}=1$, then $x_E^{\sigma'}=1$ (both stakes levels are in case 2), and if $x_E^{\sigma''}=\inf\Lambda_{f'}^{\sigma''}<1$, $x_E^{\sigma'}=\min\left\{\inf\Lambda_{f'}^{\sigma'},1\right\}\geq\inf\Lambda_{f'}^{\sigma''}=x_E^{\sigma''}$. If σ' satisfies the conditions of clause 1 and is in case 1 of Lemma 1 (so

[29]We adopt the convention that the infimum over the empty set is 1.

$x_E^{\sigma'} = \inf \Lambda_{f'}^{\sigma'} < 1$) and σ'' satisfies the conditions of clause 3. (in particular, for some $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$, there exists $e, d \in \mathcal{A}^c$ with $e \succ d \succ c$ and $f_E e \succeq^{\sigma''} d$), $\overline{x_E^{\sigma''}} = \inf \Lambda_{f''}^{\sigma''}$ for appropriate f'', and $x_E^{\sigma'} = \inf \Lambda_{f'}^{\sigma'} \geq \inf \Lambda_{f''}^{\sigma''} = \overline{x_E^{\sigma''}}$ by Probability Consistency and Claim 1, which implies, in the light of the previous analysis of case of clause 1, that $\phi_E(\sigma'') \leq \phi_E(\sigma')$. If σ' satisfies the conditions of clause 1 and is in case 2 of Lemma 1 (so $x_E^{\sigma'} = 1$), then by Probability Consistency and Claim 1 and the argument in case 2 of Lemma 1, σ'' does not satisfy the conditions of clause 3. Given the previous two cases, if σ' satisfies the conditions of clause 1. and σ'' satisfies the conditions of clause 2., then it follows from clause 2. and the fact that $\phi_E(\sigma_1) \leq \phi_E(\sigma_2)$ for all $\sigma_1 > \sigma_2$ satisfying the conditions of clause 1 or 3, that $\phi_E(\sigma'') \leq \phi_E(\sigma')$. If σ'' satisfies the conditions of clauses 2 or 3, then the result is immediate. □

For every E-resilient σ, let $x_E^\sigma = \phi_E(\sigma)$. Let $\mathcal{D} = \bigcap_{\sigma'\ E-resilient} \overline{\left(\mathcal{C}^{\sigma'} \cap [E, x_E^{\sigma'}]\right)_E}$ and $y_E = \sup_{\sigma\ E-resilient} \phi_E(\sigma)$. As noted above, there exists an E-resilient stakes level, so $\mathcal{D} \neq \emptyset$. By Confidence Consistency, for any stakes level σ'' that is not E-resilient, $\sigma' > \sigma''$ for every E-resilient stakes level σ'. It follows from the confidence representation (3) that $\mathcal{C}'' \subseteq \mathcal{D}$ for every $\mathcal{C}''$ representing $\succeq_E^{\sigma''}$ according to (12).

Claim 3. Under Information-based Learning, for any stakes level σ'', if σ'' is not E-resilient, then $\succeq_E^{\sigma''}$ is represented by $\mathcal{D}$.

Proof. Let σ'' be a stakes level that is not E-resilient, and let $\mathcal{C}''$ be a closed convex set representing $\succeq_E^{\sigma''}$ according to (12). (Such a set exists by the representation (3).) As noted above, $\mathcal{C}'' \subseteq \mathcal{D}$). Suppose that the inverse containment does not hold, so there exists $p \in convcl(\mathcal{D}) \setminus \mathcal{C}''$. By a separating hyperplane argument, there exists $f \in \mathcal{A}$, $c \in \mathcal{A}^c$ such that $\mathbb{E}_q u(f) \geq u(c)$ for all $q \in \mathcal{C}''$ whereas $\mathbb{E}_p u(f) < u(c)$. It follows that $f \not\succeq_E^{\sigma'} c$ for all E-resilient σ' but $f \succeq_E^{\sigma''} c$, contradicting Information-based Learning. So $\mathcal{C}'' = \mathcal{D}$ and $\mathcal{D}$ represents $\succeq_E^{\sigma''}$, as required. □

Claim 4. For any stakes level σ'' that is not E-resilient and any $\mathcal{C}''$ representing $\succeq^{\sigma''}$ according to (12), $\mathcal{C}'' \cap \{p \in \Delta(\Sigma) : \ p(E) \geq y_E\} = \emptyset$.

Proof. Consider a non-E-resilient σ'', and let $\mathcal{C}''$ represent $\succeq^{\sigma''}$. By Claim 3, $\mathcal{D} = \bigcap_{\sigma'\ E-resilient} \overline{\left(\mathcal{C}^{\sigma'} \cap [E, x_E^{\sigma'}]\right)_E} = \overline{\left(\bigcap_{\sigma'\ E-resilient} \left(\mathcal{C}^{\sigma'} \cap [E, x_E^{\sigma'}]\right)\right)_E}$ represents $\succeq_E^{\sigma''}$; however, by Confidence Consistency and the confidence representation, $\mathcal{C}'' \subseteq \bigcap_{\sigma'\ E-resilient} \mathcal{C}^{\sigma'}$. So if $\mathcal{C}'' \cap \{p \in \Delta(\Sigma) : \ p(E) \geq y_E\} \neq \emptyset$, then $\mathcal{C}'' \cap \bigcap_{\sigma'\ E-resilient} \left(\mathcal{C}^{\sigma'} \cap [E, x_E^{\sigma'}]\right) = \mathcal{C}'' \cap \bigcap_{\sigma'\ E-resilient} \mathcal{C}^{\sigma'} \cap \bigcap_{\sigma'\ E-resilient} [E, x_E^{\sigma'}] = \mathcal{C}'' \cap \bigcap_{\sigma'\ E-resilient} \mathcal{C}^{\sigma'} \cap [E, y_E] \neq \emptyset$, and hence, for every $f \in \mathcal{A}$, $c \in \mathcal{A}^c$, if $f_E c \succeq^{\sigma''} c$, then $f_E c \succeq_E^{\sigma''} c$ by the reasoning in the proof of Lemma 1, contradicting the assumption that σ'' is not E-resilient. Hence $\mathcal{C}'' \cap \{p \in \Delta(\Sigma) : \ p(E) \geq y_E\} = \emptyset$ as required. □

Define $\rho_E : \Xi \rightrightarrows [0, 1]$ as follows:[30]

[30]Recall from Section 2.3 that stakes levels are defined as sets (equivalence classes) of acts.

$$\rho_E(\mathcal{C}) = \begin{cases} \{x_E^\sigma : D^{-1}(\mathcal{C}) \cap \sigma \neq \emptyset,\ \sigma\ E-resilient\} & if\ \exists E-resilient\ \sigma\ s.t.\ D^{-1}(\mathcal{C}) \cap \sigma \neq \emptyset \\ y_E^\sigma & otherwise \end{cases} \tag{14}$$

Since $x_E^{\sigma'} \geq x_E^{\sigma''}$ whenever $\sigma' \leq \sigma''$ with σ', σ'' E-resilient, and since D respects $\geqq$, ρ_E is a decreasing correspondence. By the fact that, for every E-resilient σ, $\succeq_E^\sigma$ is represented by $\overline{\left(\mathcal{C}^{\sigma'} \cap [E, x_E^{\sigma'}]\right)_E}$ and by Claims 3 and 4, Ξ_E, defined with respect to ρ_E as in (5), represents $\succeq_E$. Hence $\succeq_E$ is a confidence update of $\succeq$, as required.

Necessity of the axioms is straightforward, given, in the cases of Probability Consistency and Null consistency, the insights involved in Lemma 1 and its proof. □

Proof of Proposition 1. Lemma 1 implies that x_E^σ is uniquely defined under clause 1, which immediately implies the second part of the uniqueness clause, taking $\mathcal{C}$ to be the $\mathcal{D}$ in the proof of the Theorem. As for the first part, it follows from the fact that ϕ_E in the proof of Theorem 2was defined to take the highest admissible value for each stakes level, and the fact that there a unique highest admissible value for each stakes level, by Lemma 1. □

Proof of Proposition 2. Use the same reasoning as the proof of Theorem 2 and Proposition 1, relying on the following strengthening of Lemma 1.

Lemma 2. *Under the conditions in Theorem 2 and Strong Probability Consistency, for all non-null $E, F \in \Sigma$ and for every stakes level σ that is both E- and F-resilient, there exists $x^\sigma \in [0,1]$ such that $\succeq_E^\sigma$ and $\succeq_F^\sigma$ are represented according to (13) with x^σ. (I.e. $V_E^\sigma(f) = \min_{p \in \overline{(C^\sigma \cap [E, x^\sigma])_E}} \mathbb{E}_p u(f)$ represents $\succeq_E^\sigma$ and $V_F^\sigma(f) = \min_{p \in \overline{(C^\sigma \cap [F, x^\sigma])_F}} \mathbb{E}_p u(f)$ represents $\succeq_F^\sigma$.) Moreover, the uniqueness of x^σ is as in Lemma 1.*

Proof. The proof employs the same reasoning as the proof of Lemma 1, with the definition of cases by (for instance) "there exists $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$ with $f_E c \sim_E^\sigma c$ but $f_E c \nsim^\sigma c$ such that there exists $e \in \mathcal{A}^c$ with $e \succ c$ and $f_E e \succ^\sigma c$" replaced by "there exists $f \in \mathcal{A}$ and $c \in \mathcal{A}^c$ with $f_E c \sim_E^\sigma c$, $f_E c \nsim^\sigma c$ and $f_E e \succ^\sigma c$ for some $e \in \mathcal{A}^c$ with $e \succ c$, or with $f_F c \sim_F^\sigma c$, $f_F c \nsim^\sigma c$ and $f_F e \succ^\sigma c$ for some $e \in \mathcal{A}^c$ with $e \succ c$" (and similarly for the other cases). □

□

Proof of Proposition 3. Fix a non-null event E, and let Ξ_E, respectively Ξ'_E be the confidence rankings and D_E and D'_E the cautiousness coefficients representing $\succeq_E$ and $\succeq'_E$ and obtained by confidence update according to Theorem 2. Let ϕ_E and ϕ'_E be as defined prior to Claim 2 in the proof of Theorem 2 for decision makers $\succeq$ and $\succeq'$ respectively. By (Hill, 2013, Thm 2 and the arguments used in its proof), (i) iff for every stakes level σ that is E-resilient according to

$\succeq$, $D'_E(f) \supseteq D_E(f)$ for every $f \in \sigma$. Note that it follows that any such stakes level is also E-resilient according to $\succeq'$. Since, by Theorem 2 and its proof, $D_E(f) = \overline{(D(f) \cap [E, \phi_E(\sigma_f)])_E}$, and similarly for $D'_E(f)$, the previous containement holds iff $\phi_E(\sigma) \geq \phi'_E(\sigma)$ for every such stakes level σ. By the definition of the maximal correspondences ρ_E and ρ'_E representing the updates, this holds iff (ii), as required. □

A.2 Proofs of other results

Proof of Proposition 4 . As is well-known (Hammond, 1994), when the state space is finite, p^{CPS} is equivalent to a sequence $(p_1, \ldots, p_n)$ of (ordinary) probability measures, with disjoint supports, in the following sense: for every $E_1, E_2 \in \Sigma$ with $E_2 \neq \emptyset$, $p^{CPS}(E_1/E_2) = p_j(E_1/E_2)$ where $p_j(E_2) \neq 0$ and $p_k(E_2) = 0$ for all $k < j$. Define the confidence ranking $\Xi(p^{CPS}) = \{\{p_i : i \leq k\} : k = 1, \ldots n\}$. This is a well-defined min-closed confidence ranking. Taking, for each $E \in \Sigma$, ρ_E with $\rho_E(\mathcal{C}) = 0$ for all $\mathcal{C} \in \Xi(p^{CPS})$, it is clear that, for every $E \in \Sigma$, the confidence update $\Xi(p^{CPS})_E = \{\{p_i(\bullet/E)\} : p_i(E) > 0\}$, whose centre $p_{\Xi(p^{CPS})_E} = p_j(\bullet/E)$ where $p_k(E) = 0$ for all $k < j$. Hence $p_{\Xi(p^{CPS})_E} = p^{CPS}(\bullet/E)$, and the confidence update exhibits the same conditional probabilities are the conditional probability system p^{CPS}, as required. □

References

ALCHOURRON, C. E., P. GÄRDENFORS, AND D. MAKINSON (1985): "The Logic of Theory Change: Partial Meet Contraction and Revision Functions," *The Journal of Symbolic Logic*, 50, 510–530.

ANSCOMBE, F. J. AND R. J. AUMANN (1963): "A Definition of Subjective Probability," *The Annals of Mathematical Statistics*, 34, 199–205.

BATTIGALLI, P. AND M. SINISCALCHI (2002): "Strong belief and forward induction reasoning," *Journal of Economic Theory*, 106, 356–391.

BLUME, L., A. BRANDENBURGER, AND E. DEKEL (1991a): "Lexicographic Probabilities and Choice under Uncertainty," *Econometrica*, 59, 61 – 79.

——— (1991b): "Lexicographic probabilities and equilibrium refinements," *Econometrica: Journal of the Econometric Society*, 81–98.

BRADLEY, R. (2005): "Radical Probabilism and Mental Kinematics," *Philosophy of Science*, 72, 342–364.

——— (2017): *Decision theory with a human face*, Cambridge University Press.

BRADLEY, S. (2014): "Imprecise Probabilities," in *The Stanford Encyclopedia of Philosophy*, ed. by E. N. Zalta, winter 2014 ed.

CABALLERO, R. J. AND A. KRISHNAMURTHY (2008): "Collective risk management in a flight to quality episode," *The Journal of Finance*, 63, 2195–2230.

CAMERER, C. F., G. LOEWENSTEIN, AND M. RABIN (2011): *Advances in behavioral economics*, Princeton University Press.

COX, L. A. T. (2012): "Confronting deep uncertainties in risk analysis," *Risk Analysis*, 32, 1607–1629.

DE BONDT, W. F. AND R. THALER (1985): "Does the stock market overreact?" *The Journal of finance*, 40, 793–805.

DE BONDT, W. F. AND R. H. THALER (1987): "Further evidence on investor overreaction and stock market seasonality," *Journal of finance*, 557–581.

DEKEL, E. AND M. SINISCALCHI (2015): "Epistemic game theory," in *Handbook of Game Theory with Economic Applications*, Elsevier, vol. 4, 619–702.

DEMPSTER, A. P. (1967): "Upper and Lower Probabilities Induced by a Multivalued Mapping," *Annals of Mathematical Statistics*, 38, 325–339.

DIETRICH, F., C. LIST, AND R. BRADLEY (2016): "Belief revision generalized: A joint characterization of Bayes' and Jeffrey's rules," *Journal of Economic Theory*, 162, 352–371.

ELLSBERG, D. (1961): "Risk, Ambiguity, and the Savage Axioms," *Quart. J. Econ.*, 75, 643–669.

EPSTEIN, L. G. AND M. LE BRETON (1993): "Dynamically Consistent Beliefs Must Be Bayesian," *Journal of Economic Theory*, 61, 1–22.

EPSTEIN, L. G. AND M. SCHNEIDER (2003): "Recursive multiple-priors," *Journal of Economic Theory*, 113, 1–31.

——— (2007): "Learning under ambiguity," *The Review of Economic Studies*, 74, 1275–1303.

GAJDOS, T., T. HAYASHI, J.-M. TALLON, AND J.-C. VERGNAUD (2008): "Attitude toward imprecise information," *J. Econ. Theory*, 140, 27 – 65.

GALLAGHER, J. (2014): "Learning about an Infrequent Event: Evidence from Flood Insurance Take-Up in the United States," *American Economic Journal: Applied Economics*, 6, 206–233.

GARDENFORS, P. (1988): *Knowledge in Flux : Modeling the Dynamics of Epistemic States*, Cambridge, MA: MIT Press.

GHIRARDATO, P. (2002): "Revisiting Savage in a conditional world," *Economic Theory*, 20, 83–92.

GHIRARDATO, P. AND M. MARINACCI (2002): "Ambiguity Made Precise: A Comparative Foundation," *J. Econ. Theory*, 102, 251–289.

GIACOMINI, R., V. SKRETA, AND J. TUREN (2019): "Heterogeneity, Inattention and Bayesian Updates," *American Economic Journal: Macroeconomics*.

GILBOA, I. AND M. MARINACCI (2013): "Ambiguity and the Bayesian Paradigm," in *Advances in Economics and Econometrics: Volume 1, Economic Theory: Tenth World Congress*, Cambridge University Press, vol. 49, 179.

GILBOA, I., S. MINARDI, AND L. SAMUELSON (2017): "Cases and Scenarios in Decisions Under Uncertainty," Tech. rep., HEC Paris.

GILBOA, I., A. POSTLEWAITE, AND D. SCHMEIDLER (2009): "Is it always rational to satisfy Savage's axioms?" *Economics and Philosophy*, 285–296.

——— (2011): "Rationality of belief," *Synthese*.

GILBOA, I. AND D. SCHMEIDLER (1989): "Maxmin expected utility with non-unique prior," *J. Math. Econ.*, 18, 141–153.

——— (1993): "Updating Ambiguous Beliefs," *Journal of Economic Theory*, 59, 33–49.

GRETHER, D. M. (1980): "Bayes Rule as a Descriptive Model: The Representativeness Heuristic," *Quarterly Journal of Economics*, 95, 537–557.

——— (1992): "Testing bayes rule and the representativeness heuristic: Some experimental evidence," *Journal of Economic Behavior & Organization*, 17, 31–57.

GRIFFIN, D. AND A. TVERSKY (1992): "The weighing of evidence and the determinants of confidence," *Cognitive Psychology*, 24, 411–435.

GUL, F. AND W. PESENDORFER (2018): "Evaluating Ambiguous Random Variables and Updating by Proxy," Tech. rep., Princeton.

HAMMOND, P. J. (1994): "Elementary non-Archimedean representations of probability for decision theory and games," in *Patrick Suppes: scientific philosopher*, Springer, 25–61.

HANANY, E. AND P. KLIBANOFF (2007): "Updating preferences with multiple priors," *Theoretical Economics*, 2, 261–298.

HILL, B. (2013): "Confidence and decision," *Games and Economic Behavior*, 82, 675–692.

——— (2015): "Confidence as a Source of Deferral," HEC Paris Research Paper ECO/SCD-2014-1060, HEC Paris, April 11, 2015 version.

——— (2016): "Incomplete preferences and confidence," *Journal of Mathematical Economics*, 65, 83–103.

——— (2018): "Dynamic Consistency and Ambiguity: A Reappraisal," HEC Paris Research Paper ECO/SCD-2013-983, HEC Paris, July 13, 2018 version.

——— (2019): "Confidence in Beliefs and Rational Decision Making," *Economics & Philosophy*, 35, 223–258.

ILUT, C. L. AND M. SCHNEIDER (2014): "Ambiguous business cycles," *American Economic Review*, 104, 2368–99.

JEFFREY, R. C. (1972): *The Logic of Decision*, Chicago: University of Chicago Press, 2nd ed.

——— (1992): *Probability and the Art of Judgment*, Cambridge University Press.

JOYCE, J. M. (2010): "A defense of imprecise credences in inference and decision making," *Philosophical perspectives*, 24, 281–323.

KAHNEMAN, D., P. SLOVIC, AND A. TVERSKY (1982): *Judgment under Uncertainty: Heuristics and Biases*, Cambridge University Press.

KOVACH, M. (2015): "Ambiguity and Partial Bayesian Updating," Mimeo, Caltech.

LEITGEB, H. (2017): *The stability of belief: How rational belief coheres with probability*, Oxford University Press.

MANSKI, C. F. (2013): *Public Policy in an Uncertain World : Analysis and Decisions*, Cambridge, Mass: Harvard University Press.

MANSKI, C. F. AND D. S. NAGIN (1998): "Bounding disagreements about treatment effects: A case study of sentencing and recidivism," *Sociological methodology*, 28, 99–137.

MYERSON, R. B. (1986): "Multistage games with communication," *Econometrica: Journal of the Econometric Society*, 323–358.

ORTOLEVA, P. (2012): "Modeling the Change of Paradigm: Non-Bayesian Reactions to Unexpected News," *American Economic Review*, 102, 2410–2436.

——— (2014): "Hypothesis Testing and Ambiguity Aversion," *Rivista di Politica Economica*, 3, 45–64.

PEARCE, D. G. (1984): "Rationalizable strategic behavior and the problem of perfection," *Econometrica: Journal of the Econometric Society*, 1029–1050.

PIRES, C. P. (2002): "A Rule For Updating Ambiguous Beliefs," *Theory and Decision*, 53, 137–152.

RENY, P. J. (1992): "Backward induction, normal form perfection and explicable equilibria," *Econometrica: Journal of the Econometric Society*, 627–649.

RÉNYI, A. (1955): "On a new axiomatic theory of probability," *Acta Mathematica Hungarica*, 6, 285–335.

STALNAKER, R. (1998): "Belief revision in games: forward and backward induction," *Mathematical Social Sciences*, 36, 31–56.

VALLINDER, A. (2017): "Imprecise Bayesianism and Global Belief Inertia," *The British Journal for the Philosophy of Science*, 69, 1205–1230.

WALLEY, P. (1991): *Statistical reasoning with imprecise probabilities*, London: Chapman and Hall.