Decision under Uncertainty

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Abstract

A series of famous examples casts doubt on the standard, Bayesian account of belief and decision in situations of considerable uncertainty. They have spawned a significant literature in economics, and to a lesser extent philosophy. The short article surveys some of this literature, with an emphasis on the normative issue of rational decision.

1 INTRODUCTION

Imagine that you are faced with two urns each containing only black and white balls. For one of the urns (the unknown urn), that is all you know; for the other (the known urn), you have counted the balls, and know that exactly half are black. What would you say if asked for the probability that the next ball drawn from the known urn was black? And what about the unknown urn? And, if you had to place a bet on the next ball drawn from one of the urns being black, would you bet on the known or the unknown urn? And what if the bets were on the next ball being white?

This example crops up in several disciplines. Keynes uses it to motivate his notion of *weight of evidence*: 'It is evident that in either case the probability of drawing a white ball is 1/2, but that the weight of the argument in favor of this conclusion is greater [for the known urn]' (1921). Replacing direct observation of the composition of the known urn with sampling yields something close to Popper's 'paradox of ideal evidence'. In economics, the example is associated with Ellsberg (1961), and has spawned a significant literature on *decision on uncertainty*[1] over the last 30 years.

Its importance lies in the challenge it poses to the standard account of rational belief and decision, Bayesianism (see chapter 8.2 in this volume), and in particular to the:

- **Bayesian thesis about rational belief:** it can be represented by a function assigning a single number (between 0 to 1) to each proposition or event, which satisfies the laws of probability;
- **Bayesian thesis about rational decision:** the chosen action in any decision is that which maximises the *expected utility* or *desirability* on the basis of the agent's beliefs.

Though not formulated as such, Keynes's and Popper's points appear to challenge the first thesis: representing belief by probabilities cannot, allegedly, capture the weight of evidence supporting a belief. As often noted, this argument is not watertight: there are several differences between one's Bayesian probabilities concerning the two urns, including one's beliefs about their composition or the robustness of beliefs to further observations (Joyce, 2005).

By contrast, Ellsberg's version involves decision. He observed a tendency (borne out in subsequent experiments; (Camerer & Weber, 1992)) to prefer betting on the ball drawn from the known urn over that drawn from the unknown urn, whatever the colour: a pattern of behaviour that has come

[1] We follow economists in using 'decision to uncertainty' to refer to cases where the probabilities of the various outcomes are not provided, as opposed to 'decision under risk', where they are given.

to be known as *uncertainty aversion* or *ambiguity aversion*. (The field often qualifies the unknown urn as involving *ambiguity* or *Knightian uncertainty*.[2]) Such preferences conflict with the Bayesian theses. A Bayesian would choose to bet on black from the known urn (call this event B_K) over black in the unknown urn (B_U) only if $p(B_K) > p(B_U)$ where p represents her beliefs. However, she would have the same preference over bets on white only if $p(W_K) > p(W_U)$. Clearly no probability function can satisfy these two inequalities. So the Bayesian must condemn these so-called *Ellsberg preferences* as irrational. Some find this drastic. This opens up the question of what more permissive account should or can replace it.

Although distinct, the points concerning belief representation and decision are intimately related. The difference in your ignorance about the two urns – or in the 'weight of evidence' in the two cases – seems to justify a preference between them (Ellsberg, 1961; Gilboa, Postlewaite, & Schmeidler, 2009; Levi, 1986). Indeed, Ellsberg's point about decision adds a twist to the debate about weight of evidence: whatever ways there are of capturing something like weight in the framework of Bayesian probability, they remain irrelevant for choice. Indeed, the two criticisms are perhaps strongest when combined: *Bayesianism*, it appears, *reserves no role for the weight of evidence (or similar factors) in choice*.

This survey will present some of the main responses to this challenge, particularly in the economic field of decision under uncertainty, but also in philosophy. Whilst the focus will be on decision, the relationship with belief representation means that this issue cannot be ignored. Indeed, it will structure the survey: different belief representations (replacing Bayesian probabilities) naturally require different, though not unrelated rules (replacing expected utility). The emphasis will be solely on rational decision, though the reader should bear in mind that some proposals were developed with other goals (e.g. tractability for economic modelling, descriptive accuracy) in mind. For ease of exposition, we shall largely eschew technicalities, at times abstracting liberally from precise formulations to focus on the gist. For details, discussions of dimensions other than the normative one or for more on the relevant economic literature, the reader is referred to excellent and more complete existing surveys such as (Gilboa & Marinacci, 2013).

Table 1 lists the main terminology and definitions used throughout the paper, which are standard in the economic branch of decision theory (though little depends on the use of this framework). For simplicity, we assume a fixed utility function throughout the paper, and use **1**, **x**, **0** to refer to consequences yielding utility 1, x, 0 according to this function. We assume sets to be finite as far as possible (so as to use sums instead of integrals). Some notions are illustrated on the previous example in Table 2. The columns correspond to the states of the world, each specifying the resolution of all

Terminology	Explanation
S	The set of states of the world (state space)
X	The set of consequences (consequence space)
f, g, etc.	Acts (objects of choice): functions from S to X
E, F etc.	Events: subsets of S
Δ	The set of probability measures over the state space S (probability space)
$EU_p(f)$	The expected utility of act f calculated with probability measure $p \in \Delta$ and utility function u : $EU_p(f) = \sum_{s \in S} p(s)u(f(s))$
1, **x**, **0**…	Consequence yielding utility value 1, x, 0 etc.
$\mathbf{1}_E\, \mathbf{0}$	Bet yielding consequence **1** if event E occurs and consequence **0** otherwise
A functional V *represents* a preference relation $\succcurlyeq$	For every pair of acts f, g, $f \succcurlyeq g$ if and only if $V(f) \geq V(g)$.

Table 1: Terminology and definitions

[2] After the notion of uncertainty introduced by (Knight, 1921).

relevant uncertainty (the colour of the ball drawn from each urn). The rows correspond to the acts – in this case, bets yielding 1 unit of utility if won and 0 otherwise. The entries in the table are the consequences obtained under each act and in each state (e.g. if you bet on B_K and the state B_K& B_U realises, you obtain a consequence worth 1 utility unit).

	B_K & B_U (Black ball drawn from both urns)	B_K & W_U (Black drawn from known urn, white from unknown urn)	W_K & B_U	W_K & W_U
$\mathbf{1}_{B_K}\mathbf{0}$ (Bet on B_K)	**1**	**1**	**0**	**0**
$\mathbf{1}_{B_U}\mathbf{0}$ (Bet on B_U)	**1**	**0**	**1**	**0**
$\mathbf{1}_{W_K}\mathbf{0}$ (Bet on W_K)	**0**	**0**	**1**	**1**
$\mathbf{1}_{W_U}\mathbf{0}$ (Bet on W_U)	**0**	**1**	**0**	**1**

Table 2: Ellsberg choices

2 Multiple Priors

A popular reaction to the opening example focusses on the *precision* of the Bayesian representation of beliefs. In the unknown urn, there is no evidence to justify a particular call on the probability of black on the next draw. Yet, by insisting that a precise probability value must be assigned to this event, Bayesianism has – the thought goes – no way of expressing this ignorance. A natural move is thus to allow imprecision in valuations, by using *sets of probabilities measures* as representations of belief. Bayesianism amounts to the special case where the agent's set is a singleton. Such a representation has been discussed and defended under names such as *credal sets* in philosophy (Joyce, 2011; Levi, 1986), *imprecise probabilities* in statistics (Walley, 1991); in economics, one speaks of *multiple priors* or *sets of priors*.

How could or should one decide on the basis of a set of priors $\mathcal{C}$? One of the earliest theories in economics – *maximin Expected Utility or maximin-EU*, developed by (Gilboa & Schmeidler, 1989)[3] – looks at the worst-case expected utility across the set; that is, it considers the representation of preferences by

$$\min_{p \in C} EU_p f \qquad (2.1)$$

The rule is *cautious* or *pessimistic*, insofar as it bases preferences on the worst the act can do, according to the probability measures in the set. As such, it can straightforwardly account for the Ellsberg preferences. (The interested reader may verify this using the set $C^* = \{p \in \Delta: 0.3 \leq p(B_U) \leq 0.7; p(B_k) = 0.5\}$.)

One purportedly restrictive aspect of this rule is the focus on the worst case; an apparently less extreme alternative is the *α-maximin-EU* ou *Hurwicz* criterion, suggested by (Hurwicz, 1951; Jaffray, 1988):

$$\alpha \min_{p \in C} EU_p f + (1-\alpha) \max_{p \in C} EU_p f \qquad (2.2)$$

where α is a number between 0 and 1. This rule contains maximin-EU as a special case (where $\alpha = 1$), but goes beyond the arguably extreme case of total caution, by taking into consideration both the worst and the best case. Other generalisations in this direction allow the α to depend more or less strongly on the act f being evaluated (Ghirardato, Maccheroni, & Marinacci, 2004).

[3] Similar decision rules have been defended in philosophy (Gärdenfors & Sahlin, 1982) and developed in robust statistics, where they are called Γ-maximin (Berger, 1985).

Another approach focusses on what an agent can say 'for sure' on the basis of her set of priors. Under a popular proposal, a preference for one act over another is formed only if all probability measures in the set 'agree' that the former has higher expected utility. In other words, for all acts $f, g,\ f \succcurlyeq g$ if and only if

$$EU_p f \geq EU_p g \qquad for\ all\ p \in C$$

(2.3)

This *unanimity rule* has been defended in economics (Bewley, 1986), in statistics under the name *maximality* (Walley, 1991), as well as by several philosophers (S. Bradley & Steele, 2016).[4] Note that it does not order certain pairs of acts: the reader is invited to check, for instance, that it does not order the Ellsberg bets under the set of priors C^* above. Related rules include Levi's E-admissibility (1986), which picks out acts that are best according to at least one probability measure in the set.

Beyond intuitive remarks about their reasonableness, how can decision rules such as these be evaluated on normative grounds? Two potentially relevant families of results have been developed in the field; we shall illustrate them on maximin-EU.

2.1 EVALUATION 1: IMPLICATIONS FOR CHOICE

One way of evaluating the normative credentials of a decision rule is in terms of its *implications for choice*. If it leads to unpalatable choices, such as choosing to obtain a sure loss, then this provides good reason for skepticism; if its consequences for choice seem sensible, then this may provide arguments in its favor. Any particular choice results from the combination of the decision rule (such as (2.1)) and the agent's attitudes (the set of priors and utility function). To focus evaluation on the rule, decision theory considers the implications it has *no matter* the agent's attitudes. For instance, the maximin-EU rule generates transitive preferences (see below) for any C and *u* used. The central results in decision under uncertainty – *representation theorems* – fully characterize the implications of the decision rule for choice. That is, they provide necessary and sufficient conditions on preferences – called *axioms* – for there to be some specification of the agent's attitudes representing them according to the rule. Figure 1 presents the general schema on maximin-EU (see (Gilboa, 2009; Gilboa & Marinacci, 2013) for details).

Preferences $\succcurlyeq$ satisfy a (particular) set of axioms	if and only if	There exists a set of priors C and utility function *u* such that $\succcurlyeq$ is represented by (2.1) calculated with C and *u*.
Moreover, the representing C and *u* are suitably unique.		

Figure 1: Representation Theorem

These results help pinpoint the properties of preferences that distinguish decision rules from one another. For instance, the Bayesian expected utility rule satisfies (see chapters 8.1 and 8.2 in this volume):[5]

***Weak Order (WO)*:** the preference relation is transitive (for all *f, g, h*, $f \succcurlyeq g$ and $g \succcurlyeq h$ imply $g \succcurlyeq h$) and complete (for all *f, g*, $f \succcurlyeq g$ or $f \preccurlyeq g$).

***Sure Thing Principle (STP)*:** the preference across any pair of acts is independent of what the acts yield on events where they agree.

For instance, in Table 2, since $\mathbf{1}_{B_K}\mathbf{0}$ and $\mathbf{1}_{B_U}\mathbf{0}$ agree on the event that the balls drawn from the two urns are the same colour (i.e. the first and fourth states), STP implies that preferences over these acts should be independent of their consequences on this event. The same goes for $\mathbf{1}_{W_K}\mathbf{0}$ and $\mathbf{1}_{W_U}\mathbf{0}$. However,

[4] Different versions vary according to whether the inequalities are weak (as (2.3)) or strict.

[5] Throughout, we retain the standard names for axioms used in the literature. The reader is refereed to e.g. (Gilboa, 2009; Gilboa & Marinacci, 2013) for further details.

since on the complement event (i.e. the event consisting of the second and third states), $\mathbf{1}_{B_K}\mathbf{0}$ coincides with $\mathbf{1}_{W_U}\mathbf{0}$ and $\mathbf{1}_{B_U}\mathbf{0}$ coincides with $\mathbf{1}_{W_K}\mathbf{0}$, STP implies that $\mathbf{1}_{B_K}\mathbf{0} \succcurlyeq \mathbf{1}_{B_U}\mathbf{0}$ if and only if $\mathbf{1}_{W_U}\mathbf{0} \succcurlyeq \mathbf{1}_{W_K}\mathbf{0}$. Hence, the Ellsberg preferences (for $\mathbf{1}_{B_K}\mathbf{0}$ over $\mathbf{1}_{B_U}\mathbf{0}$ and for $\mathbf{1}_{W_K}\mathbf{0}$ over $\mathbf{1}_{W_U}\mathbf{0}$) violate STP.

In terms of choice, the maximin-EU and unanimity rules take complementary approaches: the former weakens STP, whilst retaining WO; the latter drops the completeness clause of WO, whilst holding onto STP. Various arguments have been proposed, in both the philosophical and economic literatures, that violating one or other of these axioms leads to unsavoury consequences, in particular in *dynamic contexts*. They have been used by some to argue for, say, dropping WO rather than STP (S. Bradley & Steele, 2016; Seidenfeld, 1988). They have also been used to criticise any divergence from Bayesianism – and hence all of the approaches discussed in this survey – as irrational (Al Najjar & Weinstein, 2009; Elga, 2010; Hammond, 1988).[6] This is currently the main battleground between Bayesianism and critics, and deserves a survey in itself. Note that, even if these arguments are correct about the weaknesses of non-Bayesian approaches in dynamic contexts, these need to be traded off against their strengths in dealing with evidence in choice; some have claimed that the latter outweigh the former (Gilboa et al., 2009).

Apart from dropping STP, maximin-EU retains two axioms satisfied by expected utility:

P4: the preference for a bet on *E* over a bet on *F* with the same stakes is independent of the stakes.

Uncertainty Aversion (UA): for any pair of disjoint events *E*, *F* with $\mathbf{1}_E\mathbf{0} \sim \mathbf{1}_F\mathbf{0}$, $\frac{\mathbf{1}}{\mathbf{2}}{}_{E\cup F}\mathbf{0} \succcurlyeq \mathbf{1}_E\mathbf{0}$.

Preferences over bets on events – for a bet on B_K over a bet on B_U (Table 2) – reflect agents' 'willingness to bet', and are typically considered to be related to their beliefs (Savage, 1954). P4 states that the relative willingness to bet on different events is independent of the stakes involved – that is, of the consequences of winning or losing the bet, assuming that these are the same for both bets.[7] Uncertainty Aversion translates the caution of maximin-EU. $\frac{\mathbf{1}}{\mathbf{2}}{}_{E\cup F}\mathbf{0}$ 'hedges' the uncertainty involved in the bets on *E* and *F*: the uncertainty is 'halved', insofar as the payoff depends on whether *E* or *F* realizes, not which one does; but the winnings are halved as well. For an agent who dislikes uncertainty, such hedging can only be attractive, and hence the hedge is (weakly) preferred to the initial bets. This axiom is often taken as the property characterizing *uncertainty (or ambiguity) aversion*. However, it seems that people are not universally ambiguity averse (Wakker, 2010); for instance, if there were ten possible colors in the running example rather than two, then there is a tendency to bet on the unknown urn. Were such preferences to be deemed reasonable, this would be a blow for the rational credentials of maximin-EU. The α-maximin-EU rule retains P4 but drops UA, and hence can accommodate these preferences.

2.2 EVALUATION 2: BELIEFS, TASTES, AND UNCERTAINTY ATTITUDES

Another evaluation criterion concerns the capacity of an approach to neatly separate beliefs from tastes or values. Consider the following oft-cited criticism of maximin-EU. In our running example, you know nothing about the composition of the unknown urn, so any probability measure is possible; the set $\mathcal{C}_0 = \{p \in \Delta;\ 0 \leq p(B_U) \leq 1, B_K = 0.5\}$ captures this. However, according to maximin-EU with this set, a bet on the unknown urn is considered worse than a bet on black from an urn known to contain 1 black ball and 99 white ones. Such caution seems extreme.

This objection relies on two assumptions: firstly, that the set of priors involved in the decision rule represents beliefs; and secondly, that the beliefs should perfectly match available information.

6 Whilst these arguments focus on decision, they are related to 'dilation' arguments that focus specifically on learning under imprecise probabilities (Joyce, 2011; Seidenfeld & Wasserman, 1993).

7 This axiom, which was introduced by (Savage, 1954), whose nomenclature we adopt, is related to Gilboa-Schmeidler's C-Independence (1989).

Concerning the latter assumption, the set $\mathcal{C}_0$ may reflect the objectively available information, but it does not capture a Bayesian agent's beliefs, for these must be precise; so why should things be different for non-Bayesians? Indeed, the set of priors is often understood to represent the agent's state of belief, which incorporates but may go beyond the 'objective' information (see also (R. Bradley, 2017)). Indeed, many economists subscribe to the revealed preference paradigm (see chapter 8.1 in this volume), and take the set of priors to be the uniquely determined set specified in the appropriate representation theorem (Figure 1).

Turning to the first assumption, suppose that $\mathcal{C}_1 = \{\mathrm{p} \in \Delta;\ 0.4 \leq \mathrm{p}(\mathrm{B_U}) \leq 0.6\}$ represents your preferences (according to (2.1)), so you do not hold the preference claimed in the objection.[8] Since this set goes beyond the 'objective' information, one can ask why you are using it. Is it because you have further information, or an inclination to 'believe' in the principle of insufficient reason? Or is it because you are not so cautious as to use $\mathcal{C}_0$, but rather have a higher tolerance of uncertainty? These two possibilities are radically different: the former concerns beliefs (and their formation), the latter values or tastes for bearing uncertainty. Decision theory has developed formal tools that can shed light on such questions.

The central concept is comparative uncertain aversion: between two agents, which (if any) is *more* uncertainty averse. The following is a widely-accepted behavioral definition of the concept:

Comparative Uncertainty Aversion. Agent 1 is *more uncertainty averse* as 2 if whenever 1 weakly prefers an act *f* over a sure ***c***, then so does 2.

If Ann prefers an uncertain act *f* to a sure (utility) amount ***c*** then she is not so averse to uncertainty to consider bearing the uncertainty involved in *f* to be worse that getting the 'non-uncertain' ***c***. If Bob exhibits the same preference in all such cases, then he is not that averse to uncertainty either; so, he is (weakly) less uncertainty averse than Ann.

Influenced by the analogy with risk attitude, economists have generally considered uncertainty attitude as a taste (for bearing uncertainty) and been interested in its consequences for, say, investment decisions. In particular, they have characterized comparative uncertainty aversion in terms of the parameters of decision models, with results such as that in Figure 2 for maximin-EU (Ghirardato & Marinacci, 2002). It tells us that differences in uncertainty attitude correspond to differences in the set of priors. The fact that a comparison of tastes or values – as uncertainty attitudes are understood to be – translates to a difference in the set of priors casts doubt on the claim that this set reflects only beliefs. Indeed, the conclusion often drawn is that there is no clean interpretation of the set of priors: it reflects aspects of both belief and uncertainty attitude (Klibanoff, Marinacci, & Mukerji, 2005). The maximin-EU account, it seems, does not have the resources to determine whether and to what extent the set $\mathcal{C}_1$ reflects enhanced beliefs or uncertainty tolerance.

For agents with the same utility function, 1 is more uncertainty averse than 2	if and only if	2's set of priors is a subset of 1's set of priors.

Figure 2: Comparative Uncertainty Aversion

These interpretational subtleties signal a perhaps more severe departure from the Bayesian benchmark than first imagined. Bayesianism vaunts a clear separation of beliefs (captured by the probability measure) and tastes (reflected in the utility function); maximin-EU apparently does not. To the extent that such a separation is desirable, this could bode ill for its normative credentials. For instance, in public decisions (e.g. concerning climate policy), it is standard for one group to supply the factual judgements (e.g. climate experts) and another to provide the values (e.g. society or its representatives); without the separation of beliefs and tastes, this division of labor is compromised.

[8] Applying the maxmin-EU rule with this set of priors evaluates a bet on black from the unknown urn using the worst-case probability for black – 0.4 – and this is better than the probability of getting black from an urn with one black ball out of 100.

Whilst current arguments against value-free science tend to focus on difficulties faced *in practice* (see (Douglas, 2009) and chapter 14.2 in this volume), using sets of priors to report uncertainty risks ruling out such a possibility *in principle*. Moreover, the separation allows questions of theoretical rationality (e.g. learning) to be treated independently of those of practical rationality (e.g. decision; see chapter 2.2 in this volume). But if an agent's set of priors reflects not only the evidence acquired but also her uncertainty attitude, then both will play a role in the formation of such sets; any theory of belief update will thus also have to incorporate value considerations related to uncertainty tolerance. To date, work on belief updating for sets of priors does not seem to have grappled with this issue.

2.3 ACROSS MULTIPLE PRIOR RULES

In comparing the various rules for choosing on the basis of sets of priors, the belief-taste separation issue seems not to favor any particular one, but rather concerns the multiple prior representation in general: results à la Figure 2 for the unanimity rule suggests that it suffers from similar problems.[9] As concerns implications for choice, whilst some dispute the normative credentials of the preferences promoted by rules retaining WO (such as maximin-EU), few dispute the preference orderings yielded by the unanimity rule: if it recommends against an act as being worse than another for all probability measures, then it would seem like a bad idea to choose it. However, as noted previously, the rule may remain silent on some comparisons between acts; the issue is what to do in these cases.

Some suggest that there is nothing more to be said about rational decision: the unanimity rule provides all the guidance there is, and in cases where it is silent, there is no more guidance to be had (S. Bradley & Steele, 2016). Others invoke 'mechanisms' which are specific to such cases of 'indecision': for instance, choosing a (contextually provided) status quo option, taking a deferral option, or choosing at random. Some have suggested 'picking' a precise probability in the set of priors or 'sharpening' for the purposes of decision (Joyce, 2011). Such a procedure needs to be carried out in a coherent way across decisions, to avoid agents making chains of decisions that yield sure losses. Rules such as maximin-EU can be thought of as providing principles for 'picking' a probability: it always chooses a probability measure that evaluates the act as badly as possible among those in the set. Though the relevant probability measure differs according to the act evaluated (the measure used to evaluate a bet on white may not be relevant for a bet on black), the representation theorem tells us that the rule is invulnerable to the aforementioned problems. Indeed, any of the aforementioned rules satisfying WO can be thought of as ways of 'complementing' the unanimity approach in situations where it remains silent. The possibility of complementing approaches violating WO – seen as a strong rational base – by invoking considerations such as 'security' – which may not enjoy the same interpersonally valid rational credentials – has been discussed by (Levi, 1986). (Gilboa, Maccheroni, Marinacci, & Schmeidler, 2010) invoke a similar intuition, distinguishing 'objectively' from 'subjectively rational' preferences, represented according to the unanimity and maximin-EU rules respectively. They provide axioms characterizing the maximin-EU 'completion' of given unanimity preferences.

3 NON-ADDITIVE PROBABILITIES

Another reaction to Bayesianism's apparent difficulties in accounting for evidence focusses on the additivity of probability functions. A range of proposals, including Dempster-Shafer belief functions, possibility functions or Shackle's degrees of surprise (see chapter 4.7 in this volume), employ real-valued functions assigning a number in [0,1] to each event, and satisfying a mild monotonicity condition: bigger events do not get a lower value. Such (monotonic real-valued set) functions are called *capacities* (other terms used, in various literatures, include *fuzzy*, *confidence* or *plausibility measures*).

[9] For α-maximin-EU, despite some indications of belief-taste separation (Ghirardato, Maccheroni, & Marinacci, 2004), there are reasons for skepticism (Klibanoff, Marinacci, & Mukerji, 2005). It is complicated by the interdependence between the representing set of priors and the α (the uniqueness is weaker than in Figure 1), which, without controversial assumptions, is unconducive to a clean separation.

A first-pass decision rule involving capacities would keep the expected utility formula (Table 1), but with capacities in the place of probabilities. It has long been known that such a rule has very unattractive behavioral properties: any decision maker using it will strictly prefer some dominated act (i.e. one that does worse in all states than another available act; see (Quiggin, 1982; Wakker, 2010)). Basically, the main decision rule under uncertainty using capacities that avoids such problems is the *Choquet Expected Utility* or *CEU* rule proposed by (Schmeidler 1989), which evaluates an act f according to:

$$\sum_{x_i} v\big(\{s\colon u(f(s)) \geq x_i\}\big)[x_i - x_{i+1}] \tag{3.1}$$

where v is the capacity, u is the utility function, and x_i are the utility values of consequences of f, in decreasing order. Beyond being the main rule suitable for the sorts of belief representation mentioned above, it has proved popular descriptively (see §5 below), partly because of its implications for choice: the rule does not assume UA (though it does satisfy P4), which, as noted, is sometimes violated.

However, UA is satisfied whenever the capacity is convex: that is, whenever $v(E \cup F) + v(E \cap F) \geq v(E) + v(F)$ for all events E, F. In this case, the Choquet integral coincides with the maximin-EU evaluation using the set of probability measures dominating the capacity (called the *core*), i.e. $\{p \in \Delta\colon \text{for all } E \subseteq S, p(E) \geq \text{v(E)}\}$. Since several non-additive probability representations – such as Dempster-Shafer belief functions – are convex capacities, this means that the decision rule for them is (equivalent to) maximin-EU. So the previous remarks also hold for them. More generally, results à la Figure 2 indicate that the capacity in the CEU rule reflects attitude to uncertainty (as well as, potentially, beliefs), so the separation issue remains problematic.

4 CONFIDENCE & OTHER SECOND-ORDER REPRESENTATIONS

A third approach focusses on a purported multi-dimensional character of beliefs: to use Keynesian vocabulary, beyond the *balance* of evidence supporting a probabilistic judgement, there is also its *weight*. The thought that both of these 'dimensions' are involved in the representation of beliefs dates back at least as far as (Peirce, 1878) : 'to express the proper state of our belief, not one number but two are requisite, the first depending on the inferred probability, the second on the amount of knowledge on which that probability is based'. Similar distinctions have more recently played a prominent role in cognitive psychology (Griffin & Tversky, 1992). To capture this idea, representations generally employ some structure over the probability space Δ.

A simple example is a (weak) order over the probability space, or equivalently, a nested family of sets of probability measures, denoted Ξ (Figure 3). (R. Bradley, 2017; Hill, 2013, 2019) argue that such a structure can capture an agent's *confidence in her beliefs*. Each set in the family represents the beliefs or probability judgements held at a given level of confidence: the probability judgements that hold for all measures in the set are those that the agent holds with (at least) the corresponding amount of confidence. For instance, an agent represented according to Figure 3 has high confidence in the probability judgement of 0.5 for B_K, but only low confidence in the probability judgement of 0.5 for B_U. Larger sets in the family (for which fewer judgements hold) correspond to higher levels of confidence. Conceptually, confidence in a probability judgement is related to the evidence underpinning it (R. Bradley, 2017), so this approach relates to the aforementioned tradition. Technically, the representation is simply the 'system of spheres' representation from belief revision and conditional logic (see chapters 5.2 and 6.1 in this volume), applied over the probability space rather than the state space. Belief representations in this spirit have been proposed by (Gärdenfors & Sahlin, 1982; Nau, 1992), though they assume that the second-order structure is cardinal: a number is assigned to each probability measure. Naturally, every cardinal structure over the probability space induces the ordinal structure described above (by ordering probability measures according to the value assigned). Similar uncertainty representations have been promoted in econometrics (Manski, 2013).

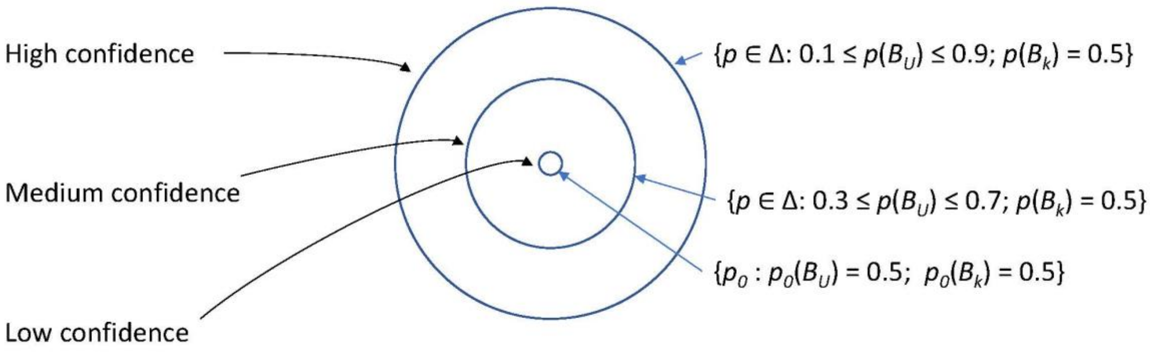

Figure 3: Confidence in beliefs

Like multiple priors (§2), several decision rules operate with this representation of beliefs. A notable family encapsulates the arguably reasonable maxim: *the higher the stakes involved in the decision, the more confidence is required in a belief for it to play a role*. They involve a function *D* assigning to each decision a confidence level – formally, a set in the nested family Ξ representing the agent's beliefs – and evaluate acts on the basis of the set picked out as appropriate for the decision by *D*. For each multiple prior decision rule (§2), there is a 'confidence version', involving the same rule but allowing the set of priors involved to vary according to the decision. For instance, the maximin-EU member of this family represents preferences according to:

$$\min_{p \in D(f)} EU_p f$$

(4.1)

The main difference for choice with respect to maximin-EU lies in P4, which directly clashes with the intuition that for different stakes, different levels of confidence may be appropriate and hence different beliefs may inform the decision, leading to potentially different willingness to bet. (4.1) weakens this requirement, allowing the willingness to bet to change with the stakes in line with the aforementioned maxim. Unlike maximin-EU, results à la Figure 2 suggest a clean separation between beliefs and values (Hill, 2013): the family of sets Ξ represents the beliefs while the function *D* represents the agent's uncertainty aversion, or taste for choosing on the basis of limited confidence. Drawing on similar results for the confidence version of the unanimity rule, (Hill, 2019) argues that these two points – the mild yet motivated divergence from multiple prior models in terms of choice implications, and the clean separation of beliefs and tastes – are general properties of the approach.

As noted previously, any cardinal representation of confidence in beliefs can be used with this sort of decision procedure, by 'forgetting' the numbers and using only the order. However, other decision rules have been developed that make specific use of the cardinal structure. Prominent ones are the *variational preferences* rule (Maccheroni, Marinacci, & Rustichini, 2006), which represents preferences by:

$$\min_{p \in \Delta}(EU_p\, f + c(p))$$

(4.2)

where c is a real-valued function on Δ, and the *confidence preferences* rule (Chateauneuf & Faro, 2009):

$$\min_{p \in \Delta} \frac{1}{\varphi(p)} EU_p\, f$$

(4.3)

where φ is a function from Δ to [0,1]. The former rule is motivated by an important literature on robustness in macroeconomics: a central model developed there (Hansen & Sargent, 2008) is a special case. The latter was motivated by and technically related to the literature on fuzzy sets: φ is a fuzzy set of probability measures. Both approaches diverge from maximin-EU by weakening P4, without motivating the divergence with a maxim similar to that behind (4.1). However, these models do not cleanly separate beliefs and tastes: results à la Figure 2 suggest that c and φ capture uncertainty attitudes and hence cannot be thought of as pure representations of belief.

5 PROBABILITIES AFTER ALL

Finally, some strive to retain the probabilistic representation of beliefs. Inspired by work on decision under risk, one approach assumes that agents assign a precise probability $p(E)$ to each event E, but that in choice these probability values are 'deformed' by a 'weighting function' *w*, so that the decision weight attached to the event is *w(p(E))*. Since the composition of *w* and *p* is a capacity, this approach, popular in Prospect Theory ((Wakker, 2010) ; see chapter 8.3 in this volume), uses the CEU rule (§3). A different approach reverts to a second-order probability μ over the space of first-order probabilities *Δ*. To the extent that this is a second-order structure, it can be thought of as related to the confidence approaches in §4 (Marinacci, 2015).

The main challenge for these approaches lies in the incompatibility between the Ellsberg preferences and *probabilistic sophistication*: roughly, the principle that all the decision-relevant information about an event is summarized in the probability assigned to it (Machina & Schmeidler, 1992). For instance, in the Ellsberg example, if the agent assigns probability 0.5 to both B_K and B_U, as is often deemed reasonable, then the deformed weight *w(0.5)* assigned to the two events is the same, and the CEU rule predicts indifference between the bets. Likewise, a second-order probability μ generates a 'reduced' probability $\sum p\mu(p)$, which can be used in the context of the expected utility rule; doing so is equivalent to following evaluation, which applies expected utility at both levels:

$$\int_{\Delta} EU_p f \, d\mu(p) \tag{5.1}$$

However, as noted at the outset, expected utility cannot accommodate the Ellsberg preferences.

Faced with this situation, a common reply is to treat (the probabilities of) different events differently in decision according to the type of event, and in particular the *source of uncertainty* to which it belongs. The Ellsberg known and unknown urns are different sources of uncertainty and so, the idea goes, the same probability assignment with respect to events from the different sources can be treated differently in decision.

Source dependence was introduced in the behavioral literature (Tversky & Fox, 1995), and continues to play a central role in one of most important descriptive theories of decision, Prospect Theory ((Abdellaoui, Baillon, Placido, & Wakker, 2011; Wakker, 2010); see chapter 8.3 in this volume). Recent versions incorporate source preference by using the CEU rule with weighting functions that depend on the source of uncertainty, so probabilities about events about the known urn are weighted differently from those concerning the unknown urn.

For second-order probabilities, the following *smooth ambiguity* decision rule has been proposed (Klibanoff et al., 2005):[10]

[10] For related approaches, see (Ergin & Gul, 2009; Nau, 2006; Seo, 2009).

$$\int_{\Delta} \varphi(EU_p f)\, d\mu(p)$$

(5.2)

with φ a real-valued function on utility values. The transformation function φ translates the difference in attitude to first-order and second-order uncertainty; interpreting the former as 'physical uncertainty' and the latter as 'model uncertainty', it thus reflects different attitudes to these two sources of uncertainty (Marinacci, 2015). In particular, whenever φ is non-linear, the attitudes differ, and the rule can accommodate Ellsberg preferences. This is one of the prominent models in the literature admitting a clean separation of beliefs from uncertainty attitudes: the second-order probability μ can be understood as a representation of the agent's state of belief, whereas φ reflects her uncertainty attitude (Klibanoff et al., 2005). This separation, combined with the tractability and familiarity of the largely Bayesian framework, have contributed to its increasing popularity in economic modelling.

Whilst some approaches invoking source dependence, particularly in the behavioral literature, have plainly descriptive aims, others seem to harbor normative ambitions (Marinacci, 2015), and hence call for an appraisal of their rational credentials. Criticisms have focused on their resolution of the challenge of reserving a role for weight of evidence in decision (§1). They insist that the Bayesian representation captures all relevant aspects of belief, but introduce another element (e.g. the φ in (5.2)) to account for the role apparently played by weight of evidence. Some have suggested that this tension – in particular the use of a parameter representing uncertainty attitude (φ) to capture evidence, which is a factor pertaining to belief – is damaging for the rational credentials of approaches invoking source dependence (Hill, 2019).

6 References

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