# Beyond Uncertainty Aversion

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**Abstract**

Although much of the theoretical literature on ambiguity works under the assumption of uncertainty aversion, experimental evidence suggests that it is not a universal behavioural trait. This paper introduces and axiomatises the family of $\alpha$-UA (for $\alpha$-Uncertainty Attitude) preferences: a simple extension of uncertainty averse preferences with a Hurwicz-style mixing coefficient, so as to admit a richer range of uncertainty attitudes. The parameters of the model are uniquely identified in our characterisation. It provides, in the Hurwicz $\alpha$-maxmin EU special case, a new resolution of a long-standing identification problem. It also yields novel models, including extensions of variational and multiplier preferences beyond the assumption of uncertainty aversion. Comparative statics support the interpretation of the mixing coefficient as an index of imprecision aversion. In a standard portfolio problem, the model yields the intuitive relationship between imprecision aversion and investment in an uncertain asset: as the former increases, the latter decreases.

**Keywords:** Ambiguity, uncertainty aversion, imprecision attitude, objective imprecision, multiple priors, $\alpha$-maxmin EU, multiplier preferences.

**JEL codes:** D01, D80, D81

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# 1 Introduction

In one of Ellsberg's classic examples (1961), decision makers regularly prefer betting on the colour of a ball from an urn with known 50 red-50 blue composition to betting on a ball drawn from an urn containing red and blue balls, but in an unknown proportion. This uncertainty averse behaviour has inspired an impressive range of decision models, many of which—such as the maxmin EU, variational and multiplier models (Gilboa and Schmeidler, 1989; Maccheroni et al., 2006; Hansen and Sargent, 2001)—retain the assumption of *uncertainty aversion*. Indeed, economic applications incorporating ambiguity almost exclusively rely on models assuming uncertainty aversion (e.g. Epstein and Wang, 1994; Hansen and Sargent, 2008) or employed in specifications that imply it (e.g. Gollier, 2011; Maccheroni et al., 2013; Ju and Miao, 2012). However, experimental findings (e.g. Wakker, 2010; Abdellaoui et al., 2011; Baillon and Bleichrodt, 2015; Kocher et al., 2018) and casual observation suggest that subjects are rarely universally uncertainty averse. Indeed, as Ellsberg himself noted (2001), if the urns in the example contain ten colours, in equal proportion in the known urn and in unknown proportions in the unknown one, many would prefer betting on a given colour from the unknown urn over the known one—an *uncertainty seeking* behaviour. Some take this to question the relevance and applicability of much of the theoretical literature on ambiguity. By contrast, the aim of this paper is to provide a general and fully identifiable method of extending existing uncertainty averse models to admit a richer range of uncertainty attitudes.

More specifically, it is known that any uncertainty averse model can be written as $V(f) = \inf_{p\in\Delta} J(u(f), p)$ for every act (state-contingent outcome) $f$, where $u$ is a von Neumann-Morgenstern utility and $J$ an appropriate functional (Cerreia-Vioglio et al., 2011b; see also Section 3.4). We characterise the following more liberal representation, as concerns uncertainty attitudes:

$$V(f) = \alpha \inf_{p\in\Delta} J(u(f), p) + (1-\alpha) \sup_{p\in\Delta} \hat{J}(u(f), p) \tag{1}$$

where $\hat{J}$ is the "uncertainty seeking" conjugate of $J$ (in a sense to be defined below) and $\alpha \in [0, 1]$. Under our axiomatisation, $u$, $\alpha$ and $J$ are suitably unique.

Representation (1) generalises the class of uncertainty averse preferences by the addition of a single parameter, $\alpha$, which modulates the strength of the "uncertainty averse" and "uncertainty seeking" components. It thus reflects attitude to or optimism in the face of uncertainty, and more specifically imprecision. We refer to preferences represented by (1) as *$\alpha$-UA preferences* (for $\alpha$-Uncertainty Attitude). Plugging in the appropriate functional forms for $J$ yields one-parameter uncertainty-attitude-permissive generalisations of popular ambiguity models, such as maxmin EU, variational, and multiplier preferences.

For instance, in the special case of maxmin EU (Gilboa and Schmeidler, 1989), our approach yields an axiomatic characterisation of Hurwicz $\alpha$-maxmin EU preferences, which evaluate an act $f$ by:

$$V(f) = \alpha \min_{p \in C} \mathbb{E}_p u(f) + (1 - \alpha) \max_{p \in C} \mathbb{E}_p u(f) \tag{2}$$

where $C$ is a set of priors and $\alpha \in [0, 1]$ can be thought of as regulating uncertainty attitude (for instance, $\alpha = 1$ corresponds to the uncertainty averse maxmin EU). To date, no comprehensive axiomatic foundations, applying in all state spaces and independently of specific assumptions about the form of the set $C$, are known for the Hurwicz $\alpha$-maxmin EU model (see Section 8). A central sticking point is to identify the $\alpha$ parameter separately from the set of priors $C$ (see Sections 4 and 8). Our characterisation identifies these two elements uniquely in generic cases, hence providing the missing foundations. The approach also yields characterisations and unique identifications for the generalisations of the other aforementioned ambiguity models.

To confront the identification problem, our central insight is to use *objective imprecision*, through the concept of a *bi-lottery*: the set of mixtures of a pair of von Neumann-Morgenstern lotteries. These naturally model choice options for which "objective" information is provided about the outcomes in the form of probability ranges, rather than precise probability values. For instance, a prospect yielding a (known) 50% chance of winning \$100, and nothing if not, is a lottery; a prospect where the chance of winning \$100 is between 25% and 75%, and nothing more is known, is a bi-lottery. For a consumer who is told that the probability of car theft is 0.5%, her insurance choice can be modeled as a choice among lotteries; if all that she is told is that the probability is between 0.1% and 1%, the choice is more naturally modeled using bi-lotteries. Whilst the object of some attention in the theoretical, experimental and applied literatures (see Section 8), the innovation in this paper is to use objective imprecision—bi-lotteries—as a tool for eliciting "subjective imprecision".

A standard approach situates acts within a one-dimensional space generated by "objectively uncertain choice objects": invariably, the space of expected utility values of von Neumann-Morgenstern lotteries. For instance, matching probability techniques in behavioural economics (e.g. Abdellaoui et al., 2005) and much theoretical work in the Anscombe and Aumann (1963) framework assign values to an act via its "lottery equivalent"—a lottery that is indifferent to it. The challenge of representations such as (1) is to identify two numbers: the infimum and the supremum of the appropriate functionals. To do this, we develop a way of situating acts in the two-dimensional space generated by bi-lotteries.

To illustrate, consider an event $E$—say, the event that the Fed raises interest rates before the end of the year—and a bet on $E$ yielding \$50 if $E$ and nothing otherwise. To investigate a decision maker Ann's evaluation of this bet, one standardly looks at preferences between the bet and "objective" lotteries. For instance, suppose she strictly prefers a lottery yielding \$50 with probability 0.5 (and nothing otherwise) to the bet. This lottery could be physically realised by a bet on red in the next draw from an urn with a known 50 red-50 blue composition. Suppose moreover that she also strictly prefers this lottery to the \$50 bet on the complementary event $E^c$. This pair of preferences is incompatible with Subjective Expected Utility (SEU), and is a known indication of uncertainty aversion: the uncertainty or imprecision in her evaluation concerning $E$ disqualifies it against the precise probability 0.5, whether she is betting for or against the event. This pair of preferences thus suggest that, under Ann's evaluation, the bet on $E$ is strictly more uncertain—or *more imprecise*—than the 50-50 lottery.

One could also consider Ann's preferences between the bet on $E$ and the bet on red from an Ellsberg unknown urn, containing 100 red and blue balls in an unknown proportion. This bet realises the "objective" bi-lottery yielding \$50 with probability in the range $[0,1]$, and nothing otherwise. Moreover, it could be that she has opposite preferences to those above: she strictly prefers the bet on $E$ over the bet on red in the Ellsberg urn, and the bet on $E^c$ over the bet on blue from the Ellsberg urn. After all, if the 50-50 lottery is deemed *more* attractive because it is precise, then it is natural that the Ellsberg bi-lottery is deemed *less* attractive for its complete lack of precision. Such preferences thus suggest that she evaluates the bet on $E$ as strictly less uncertain—or *more precise*—than the Ellsberg bi-lottery.

Similar reasoning holds for intermediate cases. Consider a partially unknown urn, containing 100 red or blue balls, where it is only known that at least 25 of the balls are red, and at least 25 are blue; nothing is known about the composition of the remaining 50 balls. This is a bi-lottery in the previously specified sense, and we can consider Ann's preference between the bet on $E$ and the bet on red being drawn from this urn, and her preference between the bet on $E^c$ and the bet on blue from the urn. Suppose that, for each of these pairs of bets, she is indifferent; in such cases, we say that this is a *bi-lottery equivalent*. Applying the previous reasoning, it would seem that she considers the bet on $E$ to be both weakly more and weakly less precise than the bi-lottery with winning probability range $[0.25, 0.75]$. In other words, her evaluation of the bet on $E$ matches that of the bi-lottery equivalent. This matching can be used to pin down the worst- and best-case evaluations of the bet, as required for representation (1).

Our main result provides necessary and sufficient axioms for the general representation of the form (1). At its core is an axiom implying the existence of a bi-lottery equivalent for each act. This axiom, Attitude

Coherence, formalises the intuition mooted above: if a decision maker opts for a maximally precise objective bet—a lottery—over the bet on an event and its complement, then she cannot also strictly prefer a maximally imprecise bi-lottery—such as a bet on the Ellsberg urn—over the bet on the event and its complement. For the former preference pattern would imply a distaste for imprecision, while the latter suggests an appetite for imprecision, and hence taken together they indicate an inconsistent valence of imprecision attitude. Our result provides suitably unique identification of the parameters of the model. Moreover, adding standard axioms (e.g. C-Independence, Weak C-Independence) yields generalisations of the corresponding uncertainty averse preferences (maxmin EU, variational preferences) of the form (1). They each incorporate the corresponding identifications, for instance of the set of priors in the $\alpha$-maxmin EU model. We also characterise a Choquet EU special case of (1) which embeds several classes of $\alpha$-maxmin EU representations for which uniqueness has been obtained in the literature, hence pinpointing how our identification result holds more generally.

Comparative statics exercises show that the role of the two elements of the model—the $\alpha$ and the functional $J$—can be separated, with the former corresponding to comparisons in imprecision attitude and the latter to comparisons in evaluation imprecision. Moreover, we show how to define incomplete subrelations that correspond to the "revealed priors" in the underlying uncertainty averse models, and that in this sense extend existing analyses in terms of unambiguous preferences (Ghirardato et al., 2004). Finally, we consider a standard portfolio problem under $\alpha$-UA preferences, showing that, despite their non-convexity, intuitive comparative statics results on the effect of imprecision attitude on investment can be obtained.

The paper is organised as follows. After some technical preliminaries (Section 2), we present and axiomatise the general version of the model (Section 3). We then show how adding (versions of) well-known axioms yields special cases extending some important uncertainty averse preference families (Section 4), and discuss in detail the solution provided to the identification problem for $\alpha$-maxmin EU. Section 5 considers imprecision and imprecision attitude in the context of the model's comparative statics, while Section 6 relates the proposed model to incomplete preferences and its "revealed priors". Section 7 contains a brief study of a portfolio problem under the proposed preferences, and Section 8 discusses remaining issues and related literature.

## 2 Preliminaries

Let $Z$, the set of monetary prizes, be a closed bounded subset $[\mathbf{w}, \mathbf{b}] \subset \mathbb{R}$. A (simple) *lottery* $l$ is a probability distribution with finite support over $Z$.[1] Let $\mathcal{L}$ be the set of lotteries, with the standard mixture operation, and the topology of weak convergence. For $\lambda \in [0,1]$ and $l_1, l_2 \in \mathcal{L}$, $\lambda l_1 + (1-\lambda)l_2$, generally shortened to $(l_1)_\lambda l_2$, is the $\lambda$-mixture of $l_1$ and $l_2$ . For any pair of lotteries $l, m \in \mathcal{L}$, $[l,m]$ denotes the set of mixtures of $l, m$ ($[l,m] = \{\lambda l + (1-\lambda)m : \lambda \in [0,1]\}$), and is called the *bi-lottery generated by* $l, m$. $\mathcal{B}$ is the set of bi-lotteries. Mixtures of bi-lotteries are defined setwise: for $\lambda \in [0,1]$ and $[l_1, m_1], [l_2, m_2] \in \mathcal{B}$, $\lambda[l_1, m_1] + (1-\lambda)[l_2, m_2] = \left\{\lambda l_1^* + (1-\lambda) l_2^* : l_1^* \in [l_1, m_1], l_2^* \in [l_2, m_2]\right\}$; we denote this mixture by $[l_1, m_1]_\lambda [l_2, m_2]$. (It is straightforward to show that this is a mixture operation, in the sense of Herstein and Milnor, 1953.) With slight abuse of notation, we denote the singleton bi-lottery $[l,l]$ by $l$ and use $\mathcal{L}$ to denote the subset of such bi-lotteries in $\mathcal{B}$; similarly, we use $z, w$ etc to denote degenerate lotteries yielding $z, w \in Z$ with probability 1. As explained above, bi-lotteries can be thought of as “objectively imprecise” sources of uncertainty: all the decision maker knows about the bi-lottery $[l,m]$ is that the final obtained outcome will depend on some lottery (distribution) in $[l,m]$.[2]

The setup used here will be precisely as the standard Anscombe-Aumann one (in its Fishburn 1970 adaptation) except that $\mathcal{B}$, rather than just $\mathcal{L}$, is the set of consequences. Let $\mathcal{S}$ be a non-empty set of states, with a $\sigma$-algebra $\Sigma$ of subsets of $\mathcal{S}$, called *events*. $\Delta$ is the space of finitely additive probabilities on $\Sigma$, endowed with the weak-* topology. A (simple) act $f$ is a finite-valued $\Sigma$-measurable function from $S$ to $\mathcal{B}$; $\mathcal{A}$ is the set of simple acts. $\mathcal{A}^l \subseteq \mathcal{A}$ is the set of those acts whose images belong to $\mathcal{L}$ (i.e. are singleton bi-lotteries); we call the elements of $\mathcal{A}^l$ *lottery-acts*. Mixtures of acts are defined pointwise, as standard. For $f, g \in \mathcal{A}$ and $\lambda \in [0,1]$, we use $f_\lambda g$ to denote the $\lambda$-mixture of $f$ and $g$. Similarly, for $f, g \in \mathcal{A}$ and an event $E \in \Sigma$, $f_E g \in \mathcal{A}$ is such that $f_E g(s) = f(s)$ for all $s \in E$ and $f_E g(s) = g(s)$ for all $s \notin E$. With slight abuse of notation, $\mathcal{B}$ will be used to denote the constant acts (i.e. those yielding the same bi-lottery in all states), and similarly for $\mathcal{L}$ (i.e. acts yielding the same singleton bi-lottery in all states). The decision maker's preferences over acts are denoted by $\succeq$; $\succ$ and $\sim$ are the asymmetric and symmetric part of this relation respectively. Throughout, we adopt the convention that a bi-lottery is written as $[l,m]$ only when $l \preceq m$ (i.e. if $m \prec l$, we write $[m,l]$).

A utility function $v : Z \to [-1,1]$ is *normalised* if $v(\mathbf{w}) = -1$ and $v(\mathbf{b}) = 1$. Let $B(\Sigma)$ be the set of

[1] The results extend directly to $Z$ any compact subset of a connected topological space, and similar results can be obtained taking lotteries to be Borel probability measures over $Z$.

[2] The axioms and results extend almost immediately when the set of closed convex sets of lotteries is used in the place of the set of bi-lotteries $\mathcal{B}$.

$\Sigma$-measurable functions on $S$ taking values in $[-1,1]$. The constant function in $B(\Sigma)$ taking value $x \in [-1,1]$ is denoted $x^*$. A function $I : B(\Sigma) \to \mathbb{R}$ is *normalised* if $I(x^*) = x$, *constant additive* if $I(a+x^*) = I(a)+x$, and *positively homogeneous* if $I(\kappa a) = \kappa I(a)$, for all $x \in [-1,1]$, $\kappa > 0$ and $a \in B(\Sigma)$ such that $a + x^* \in B(\Sigma)$ (resp. $\kappa a \in B(\Sigma)$). $I$ is *monotonic* if $a \geqslant b$ implies that $I(a) \geqslant I(b)$ (where $\geqslant$ is the standard statewise order on $B(\Sigma)$). $I$ is *balanced* if, for all $a \in B(\Sigma)$, $I(a) \leqslant -I(-a)$. For any $a \in B(\Sigma)$ and $p \in \Delta$, we write $\mathbb{E}_p a$ for $\int a dp$.

# 3 General case

## 3.1 Precision

To state the axioms, we require several preliminary definitions. The first is that of the *complement* of a bi-lottery or a lottery-act.

**Definition 1.** For every $l, \hat{l} \in \mathcal{L}$, $\hat{l}$ is a *complement* of $l$ if $l_{\frac{1}{2}}\hat{l} \sim \mathbf{b}_{\frac{1}{2}}\mathbf{w}$. For every $[l,m] \in \mathcal{B}$, $[\hat{m},\hat{l}]$ is a *complement* of $[l,m]$ if $\hat{m}$ is a complement of $m$, and $\hat{l}$ is a complement of $l$. For every $f \in \mathcal{A}^l$, $\hat{f}$ is a *complement of f* if $\hat{f}(s)$ is a complement of $f(s)$ for every $s \in S$.

The complement is a sort of conjugate: for a lottery that is better than the midway utility point between the best and worst prizes ($\mathbf{b}_{\frac{1}{2}}\mathbf{w}$), its complement will be just as far below it in utility space. Likewise, the complement of a lottery-act will yield, in each state, a low-utility lottery whenever the original act yields a high-utility one, and vice versa. This generalises the notion of a bet on the complementary event to all lottery-acts: for an event $E$, the bet on $E^c$, $\mathbf{w}_E\mathbf{b}$, is the complement of the bet on $E$, $\mathbf{b}_E\mathbf{w}$. Similarly, for a bi-lottery that is physically realised by a bet on red from an urn about which some information has been provided, its complement is realised by a bet on not red from the same urn. It is straightforward to show (under the basic axioms below) that complements exist for all bi-lotteries and lottery-acts, and that they are unique up to indifference (statewise, for acts). Henceforth, for any lottery-act $f$, we use $\hat{f}$ to denote any complement of $f$ (all statements will be independent of which one) and similarly for lotteries and bi-lotteries.

We introduce the following order on lottery-acts and bi-lotteries (i.e. elements for which complements are defined).

**Definition 2.** For every $f, g \in \mathcal{A}^l \cup \mathcal{B}$, $f \gtrapprox g$ if and only if $f \succeq g$ and $\hat{f} \succeq \hat{g}$.

When $f \gtrapprox g$, then both $f$ and its complement are preferred to $g$ and its complement, respectively. These are the sorts of preferences discussed in the Introduction: the standard Ellsberg (two-colour) preference for a bet on the known urn over the unknown urn, no matter the colour one is betting on, indicates that these bets are ordered under $\gtrapprox$. This is also the case for reverse Ellsberg preferences—where the unknown urn is preferred to the known urn, no matter the colour betted on—with the $\gtrapprox$-order in the other direction. As noted previously, standard Ellsberg preferences involve, on the one hand, the fact that the bet on the known urn is considered more *precise* than the unknown one, and, on the other hand, an aversion to imprecision. Reverse Ellsberg preferences involve the same difference in precision, but with the opposite taste—an appetite for imprecision. So $f \gtrapprox g$ indicates that $f$ and $g$ can be ordered according to perceived precision, with $f$ being more precise if the decision maker is imprecision averse, and more imprecise if the decision maker is imprecision seeking. Accordingly, we call $\gtrapprox$ the *precision relation*. We denote its asymmetric part by $\gg$ (i.e. $f \gg g$ if $f \gtrapprox g$ and $g \not\gtrapprox f$), and its symmetric part by $\approx$ (i.e. $f \approx g$ if $f \gtrapprox g$ and $g \gtrapprox f$).[3] It follows from the previous remarks that $f \approx g$ when $f$ and $g$ are considered as imprecise as each other in the decision makers eyes. In particular, if $f \approx [l,m]$ for a lottery-act $f$ and bi-lottery $[l,m]$, this indicates that $[l,m]$ matches the imprecision of $f$, under the decision maker's subjective evaluation. In this case, we say that $[l,m]$ is a *bi-lottery equivalent* of $f$. Finally, we introduce the following derived relation.

**Definition 3.** For every $f, g \in \mathcal{A}^l \cup \mathcal{B}$, $f \underline{\gtrapprox} g$ if and only if, for every $[l,m] \in \mathcal{B}$ such that $g \gtrapprox [l,m]$, $f \gtrapprox [l',m']$ for some $[l',m'] \in \mathcal{B}$ with $l' \geq l$.

To fix ideas when considering this definition, suppose for the moment that the decision maker is imprecision averse; so, when $f \gtrapprox [l,m]$ for some bi-lottery $[l,m]$, this is an indication that $f$ is considered *more precise* than the bi-lottery. If $f \gtrapprox [l,m]$, then any bi-lottery $[l',m']$ matching $f$'s imprecision, i.e. such that $f \approx [l',m']$, is more precise than $[l,m]$, so its "worst-case lottery" $l'$ will be weakly preferred to $l$. In other words, if $f \gtrapprox [l,m]$, then the decision maker's worst possible evaluation of $f$, as revealed by the lower bound of any bi-lottery equivalent of $f$, cannot be worse than (her evaluation of) the lottery $l$. The *lower precision relation* $\underline{\gtrapprox}$ orders acts by this worst-case evaluation: $f \underline{\gtrapprox} g$ says that $f$'s worst possible evaluation is weakly higher than $g$'s. Whilst this interpretation holds for imprecision averse decision makers, the opposite interpretation holds for imprecision seeking decision makers—$f \underline{\gtrapprox} g$ means that $f$'s worst possible evaluation is weakly lower than $g$'s. We discuss the axiomatic consequences of this difference, and provide a choice-based characterisation of imprecision aversion, below. We use $\underline{\approx}$ to denote the symmetric part of $\underline{\gtrapprox}$.

[3]Note that, under the basic axioms below, $\gtrapprox$ is transitive and reflexive, but not complete.

### 3.2 Axioms

**Basic Axioms.** First consider the following axioms.

**Axiom** (Weak Order)**.** $\succeq$ *is a weak order.*

**Axiom** (Continuity)**.** *For every $f, g, h \in \mathcal{A}$, the sets $\{\beta \in [0,1] : f_\beta g \succeq h\}$ and $\{\beta \in [0,1] : f_\beta g \preceq h\}$ are closed in* $[0,1]$.

**Axiom** (Monetary Monotonicity)**.** *For every $z, w \in Z$, $z \succeq w$ if and only if $z \geqslant w$.*

**Axiom** (Monotonicity)**.** *For every $f, g \in \mathcal{A}$, if $f(s) \succeq g(s)$ for all $s \in S$, then, $f \succeq g$.*

**Axiom** (Objective Independence)**.** *For every $[l,m], [l^*,m^*], [l',m'] \in \mathcal{B}$ and $\lambda \in (0,1)$, $[l,m] \succeq [l^*,m^*]$ if and only if $[l,m]_\lambda[l',m'] \succeq [l^*,m^*]_\lambda[l',m']$.*

**Axiom** (Bi-Monotonicity)**.** *For every $[l,m], [l',m'] \in \mathcal{B}$, if $l \preceq l'$ and $m \preceq m'$, then $[l,m] \preceq [l',m']$.*

Weak Order, Continuity and Monotonicity are standard in decision under uncertainty, and Monetary Monotonicity is standard whenever the domain of prizes is monetary. Objective Independence and Bi-Monotonicity are new axioms for bi-lotteries. The latter is a natural monotonicity property saying that whenever the best and worst lotteries in one bi-lottery are preferred to those of another, the former bi-lottery is preferred. The former is the standard independence axiom for precise, objective lotteries, applied to bi-lotteries. Given our focus on the most conservative extension possible of classic uncertainty averse models, which typically assume independence over lotteries, we retain this version of the standard axiom here.

These axioms can be thought of as the equivalent of the weak axioms for "rational preferences" in the Anscombe-Aumann domain that are adopted in much of the theoretical literature on ambiguity, and studied by Cerreia-Vioglio et al. (2011a). Adapting their terminology, we call preferences satisfying these axioms *MBBA preferences* (for Monotone, Bi-Bernouilli, Archimedean).

**Main Axiom.** For $g \in \mathcal{A}^l$ with $g \approx [l_g, m_g]$, consider the sets:

$$PR_g = \{[l,m] \in \mathcal{B} : (m \sim m_g \text{ and } l \succeq l_g) \text{ or } (l \succeq m_g \text{ and } l \sim m)\}$$

$$IMP_g = \{[l,m] \in \mathcal{B} : (m \succeq m_g \text{ and } l \sim l_g) \text{ or } (m \sim \mathbf{b} \text{ and } l \succeq l_g))\}$$

The bi-lotteries in $PR_g$ are either standard (precise) lotteries or correspond to bi-lotteries that are subsets of the bi-lottery equivalent of $g$, $[l_g, m_g]$, with the same maximal lottery (ie. of the form $[l, m_g]$ for $l \succeq l_g$).

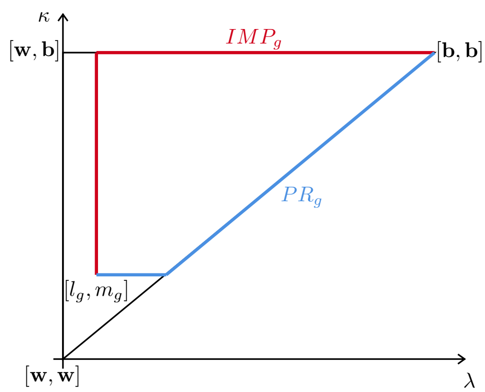


Figure 1: $PR_g$ and $IMP_g$, for $g \approx [l_g, m_g]$.
The (black) triangle represents the set of bi-lotteries of the form $[\mathbf{b}_\lambda\mathbf{w}, \mathbf{b}_\kappa\mathbf{w}]$ for $0 \leqslant \lambda \leqslant \kappa \leqslant 1$, with the point $(\lambda, \kappa)$ representing the bi-lottery $[\mathbf{b}_\lambda\mathbf{w}, \mathbf{b}_\kappa\mathbf{w}]$. Under the MBBA preference axioms, each bi-lottery is associated to a unique bi-lottery of the form $[\mathbf{b}_\lambda\mathbf{w}, \mathbf{b}_\kappa\mathbf{w}]$, and hence to a unique point in the triangle (Appendix A.1). The point corresponding to $[l_g, m_g]$ is indicated. The sets $PR_g$ and $IMP_g$ (or, more specifically, their intersection with the plotted set of bi-lotteries) are indicated in blue and red respectively. $PR_g$ contains only bi-lotteries which are maximally precise whilst having as maximal element a lottery weakly preferred to $m_g$; $IMP_g$ is the set of maximally imprecise bi-lotteries among those whose minimal element is weakly preferred to $l_g$.

Hence, beyond being weakly preferred to $g$ (under Bi-Monotonicity), they are also, in a sense, as *precise as can be* with this property; hence the notation. The opposite holds for the bi-lotteries in $IMP_g$: they are either supersets of the bi-lottery equivalent $[l_g, m_g]$ with the same minimal lottery (i.e. of the form $[l_g, m]$ for $m \geq m_g$), or as imprecise as can be, insofar as the maximal lottery is as high as possible. So, beyond being weakly preferred to $g$ (under Bi-Monotonicity), they are as *imprecise as can be* with this property. These sets are illustrated in Figure 1, which also provides a useful graphical representation of $\mathcal{B}$.

The following is the central novel axiom of our approach.

**Axiom** (Attitude Coherence)**.** *For every $f, g \in \mathcal{A}^l$ with $f(s) \geq g(s)$ for all $s \in S$ and $g \approx [l_g, m_g]$, and for all $p \in PR_g$ and $I \in IMP_g$,*

$$p \gg f \Rightarrow I \not\gg f$$
$$\textit{and} \;\; f \gg p \Rightarrow f \not\gg I$$

As noted in the Introduction, comparisons of acts and their complements with bi-lotteries gives an indication of decision makers' perceived imprecision and their attitude towards it. Let $f = \mathbf{b}_E\mathbf{w}$ be the bet on an event $E$ of interest, say that the Fed will raise interest rates before the end of the year. An SEU decision maker will evaluate such bets consistently with a subjective probability for $E$. For instance, if she prefers the lottery $\mathbf{b}_{0.45}\mathbf{w}$ to $f$, then she prefers the complementary bet $\hat{f} = \mathbf{w}_E\mathbf{b}$—the bet against $E$—to the complementary lottery, $\mathbf{w}_{0.45}\mathbf{b}$. However, imprecision-sensitive decision makers may violate this pattern, for some lotteries. Consider a decision maker who exhibits strict preferences for the lottery and its complement over the bet $f$ and its complement: $\mathbf{b}_{0.45}\mathbf{w} \gg f$, in the notation introduced above. Such preferences indicate a difference in and a sensitivity to precision. As for the difference, the (evaluations of) lotteries must be *more precise or less ambiguous* than (the evaluations of) the bets concerning $E$—since lotteries are maximally precise, they cannot be less precise. As for the sensitivity, the preference for the lotteries indicate a negative attitude towards the imprecision in the bets—that is, *imprecision aversion*. Compare this with a decision maker exhibiting the opposite pattern—strict preference for the bets over the lotteries ($f \gg \mathbf{b}_{0.45}\mathbf{w}$). Here, there is the same difference in precision—the bets are less precise, for they cannot be any more precise than the lotteries—but the imprecision is considered attractive. In other words, this preference pattern suggests an *imprecision seeking* attitude. If $f$ dominates $g$ statewise, then by Monotonicity it is preferred to $g$, so this same reasoning applies to $\gg$-orderings between $f$ and maximally precise bi-lotteries that are guaranteed by Bi-Monotonicity to be better than $g$—that is, the bi-lotteries in $PR_g$.

This reasoning depends on the fact that lotteries are *less ambiguous or more precise* than acts, and hence *ceteris paribus* more (respectively less) attractive to imprecision averse (resp. seeking) decision makers. The same logic holds, but in reverse, for maximally imprecise bi-lotteries. Compare the bet $f$ on $E$ with a bet on red from the Ellsberg unknown urn, with 100 balls, each of which is red or blue, but where nothing is known about the proportion. This bet realises the bi-lottery $[\mathbf{w}, \mathbf{b}]$. SEU decision makers remain consistent: if they prefer $f$ to $[\mathbf{w}, \mathbf{b}]$, then they prefer the complementary bi-lottery $\widehat{[\mathbf{w}, \mathbf{b}]}$ (which can be realised by the bet on blue from the urn) to the complementary bet $\hat{f} = \mathbf{w}_E\mathbf{b}$ (against $E$). Any deviations from this SEU behaviour are related to a difference in perceived precision, but in this case, it is the bi-lotteries which are *less precise* or *more ambiguous* than the bets concerning $E$—since they are maximally imprecise, they cannot be more precise. So a strict preference for the bi-lottery and its complement over the bet on $E$ and its complement—$[\mathbf{w}, \mathbf{b}] \gg f$—indicates that the decision maker values the increased imprecision in the bi-lottery positively: it signals *imprecision seeking*. It is the opposite deviation—a preference for the bets concerning $E$ over the bi-lotteries, $f \gg [\mathbf{w}, \mathbf{b}]$—which corresponds to *imprecision aversion*: $f$, if anything, is less imprecise than the maximally-imprecise bi-lottery. As for the case of lotteries, a similar point holds for bi-lotteries in $IMP_g$, which are the maximally imprecise bi-lotteries dominating a bi-lottery equivalent of $g$.

In the light of this, Attitude Coherence merely says that, for each act $f$, the *valences of the imprecision* *attitude* with respect to $f$, as indicated by the comparison with maximally precise and maximally imprecise bi-lotteries, agree: if one implies imprecision aversion (in terms of the precision orderings), then the other does not imply imprecision seeking, and vice versa. As such, it is a basic consistency axiom guaranteeing a coherent notion of imprecision attitude for each act. Note that it makes no assumptions on how attitudes vary across acts or whether the attitude is one of aversion or appetite for imprecision.

**Weak Uncertainty Aversion** For the final axiom, recall the standard Uncertainty Aversion axiom due to Schmeidler (1989):

**Axiom** (Uncertainty Aversion)**.** *For all* $f, g \in \mathcal{A}^l$ *and* $\beta \in (0,1)$*, if* $f \sim g$*, then* $f_\beta g \succeq f$*.*

As discussed, this axiom will not be imposed here; instead, we adopt the following weakening.

**Axiom** (Weak Uncertainty Aversion)**.** *For all* $f, g \in \mathcal{A}^l$*, and* $\beta \in (0,1)$*, if* $f \underline{\approx} g$*, then* $f_\beta g \underline{\gtrsim} f$*.*

Weak Uncertainty Aversion is just Uncertainty Aversion, but formulated with the lower precision relation (Definition 3) in the place of the preference relation. A similar intuition justifies it, but considering the worst possible evaluation of acts (as revealed by $\underline{\gtrsim}$; see Section 3.1), rather that their all-things-considered assessment (according to $\succeq$). Uncertainty Aversion is often motivated by a preference for hedging the uncertainty in $f$ and $g$: this translates into a preference for the mixture of the two acts over each of them. This hedging preference is justified only when the decision maker's evaluations of $f$ and $g$ are the same: hence the indifference condition in the axiom. However, a decision maker could be indifferent between the acts although her evaluation of one act does not coincide with that of the other: for instance, the indifference could result "fortuitously" from the weighting of the acts' best- and worst-case evaluations. In such cases, Uncertainty Aversion risks applying the hedging rationale spuriously. Weak Uncertainty Aversion avoids such spurious cases by only recognising a positive effect of hedging on the worst possible evaluation of the acts: the mixture $f_\beta g$ cannot do any worse than one could have done from $f$ and $g$. As such, it retains the hedging motivation of the original axiom, whilst correcting for situations where that axiom, arguably, may apply it incorrectly. Like Uncertainty Aversion, Weak Uncertainty Aversion is an axiom imposing quasiconcavity (in the representation); unlike Uncertainty Aversion, it will not impose it for the functional representing preferences.

## 3.3 Base Result

The following is the central technical result of the paper, and underlies the characterisations of the main models in the sequel.

**Proposition 1.** *Let $\succeq$ be a preference relation on $\mathcal{A}$. The following are equivalent:*

*i. $\succeq$ is a MBBA preference satisfying* Attitude Coherence*;*

*ii. There exists a normalised, strictly increasing utility function $v : Z \to [-1, 1]$, $\alpha \in [0, 1]$ and a normalised, continuous, monotonic, balanced functional $I : B(\Sigma) \to \mathbb{R}$ such that $\succeq$ is represented by:*

$$V(f) = \alpha I(u \circ f) + (1 - \alpha)(-I(-u \circ f)) \tag{3}$$

*where $u : \mathcal{B} \to \mathbb{R}$ is given by:*

$$u([l, m]) = \alpha \min_{l' \in [l,m]} \mathbb{E}_{l'} v + (1 - \alpha) \max_{l' \in [l,m]} \mathbb{E}_{l'} v \tag{4}$$

*Moreover:*

*i. If $\alpha > 0.5$, $\succeq$ satisfies* Weak Uncertainty Aversion *if and only if $I$ is quasiconcave.*

*ii. If $\alpha < 0.5$, $\succeq$ satisfies* Weak Uncertainty Aversion *if and only if $I$ is quasiconcave and quasiconvex.*

*Furthermore, $v$ and $\alpha$ are unique, and whenever $\alpha \neq 0.5$, $I$ is unique.*

Representation (3) has a general $\alpha$-mixture form, where the mixture is taken over a functional $I$ over acts, and the "conjugate" of this functional (which coincides with the negation of the value of the complement act, when defined). The properties of $I$ are standard in the literature, with the exception of balancedness, which guarantees that $I$ always takes lower values than the conjugate $-I(-\bullet)$—so the former can coherently be thought of as the worst-case evaluation, and the latter as the best case. As we shall see, these functionals are the key to obtaining generalisations of known uncertainty averse representations.

Preferences over bi-lotteries (represented by $u$) follow a Hurwicz-style representation (4), mixing the lowest and highest expected utilities among the lotteries in the bi-lotteries.[4] The mixing coefficient $\alpha$ is the same for general acts and bi-lotteries (constant acts): that is, in (3) and (4). This models a decision maker whose attitude to imprecision—which will turn out to be captured by the mixture coefficient $\alpha$ (Section 5.1)—is independent of whether the imprecision is "objective" (as in the case of bi-lotteries) or "subjective" (as in the case of general acts). This can be thought of as the analogue, for the case of imprecision, of the standard assumption that the same utility function is involved in the evaluation of both "objective" uncertainty (i.e. vNM lotteries) and "subjective" uncertainty (i.e. general acts).

[4]Under the convention that the notation $[l, m]$ implies that $l \leq m$; (4) can be simplified to $u([l, m]) = \alpha\mathbb{E}_l v + (1 - \alpha)\mathbb{E}_m v$. We present the more general form (4) to emphasise that this convention plays no role in the result.

A central characteristic of Proposition 1 is that the mixture coefficient $\alpha$ is determined uniquely, and, whenever it differs from 0.5, so is $I$.[5] Existing work on the Hurwicz $\alpha$-maxmin EU representation has recognised the difficulty in separating out the mixture coefficient from other parameters in that model (Section 4). This result suggests that bi-lotteries provide a solution to this problem. In fact, the case of $\alpha = 0.5$ corresponds to imprecision neutrality (Section 5.2), where bi-lotteries have no extra bite above standard lotteries, and so the insight employed here cannot be used. For that reason, we shall concentrate on decision makers who are not imprecision neutral in the sequel, i.e. for which $\alpha \neq 0.5$.

Moreover, the main case of interest—given the quasiconcavity of all uncertainty averse representations (e.g. Cerreia-Vioglio et al., 2011b)—is when $I$ is (only) quasiconcave, and Weak Uncertainty Aversion implies this only when $\alpha > 0.5$. It will be simpler to restrict subsequent development to this case; we do this by imposing the following axiom, which guarantees that $\alpha > 0.5$. (See Section 5.2, Proposition 4 for a full discussion of the axiom and justification of the name.)

**Axiom** (Imprecision Aversion)**.** *For every $[l, m] \in \mathcal{B}$ with $m > l$, $[l, m] \prec l_{\frac{1}{2}}m$.*

As will be discussed, versions of all results below can be obtained for $\alpha < 0.5$ by the obvious modifications of the axioms.

### 3.4 $\alpha$-UA Preferences

The main connection to the literature on uncertainty averse preferences is provided by the following result, which is a corollary of Proposition 1. We say that a functional $G : [-1, 1] \times \Delta \to (-\infty, \infty]$ is *increasing* if it is increasing in the first coordinate for all $p \in \Delta$; *calibrated* if $\inf_{p\in\Delta} G(t, p) = t$ for all $t \in [-1, 1]$; *linearly continuous* if the map $\psi \to \inf_{p\in\Delta} G(\sum_{s\in S} \psi(s)p(s), p)$ from $[-1, 1]^S$ to $[-\infty, \infty]$ is extended-valued continuous, in the sense of Cerreia-Vioglio et al. (2011b, Section 2.2); *balanced* if $\inf_{p\in\Delta} G(\mathbb{E}_p a, p) \leqslant \sup_{p\in\Delta} -G(-\mathbb{E}_p a, p)$, for all $a \in B(\Sigma)$.

**Theorem 1.** *Let $\succeq$ be a preference relation on $\mathcal{A}$. The following are equivalent:*

*i. $\succeq$ is an MBBA preference satisfying Attitude Coherence, Weak Uncertainty Aversion and Imprecision Aversion;*

*ii. There exists a normalised, strictly increasing utility function $v : Z \to [-1, 1]$, $\alpha \in (0.5, 1]$ and a linearly continuous, quasiconvex, increasing, calibrated, balanced $G : [-1, 1] \times \Delta \to (-\infty, \infty]$ such that $\succeq$ is represented by:*

[5] The uniqueness of $v$ is standard, given that it is normalised (Section 2).

$$V(f) = \alpha \inf_{p \in \Delta} G\left(\mathbb{E}_p(u \circ f), p\right) + (1-\alpha) \sup_{p \in \Delta} -G\left(-\mathbb{E}_p(u \circ f), p\right) \tag{5}$$

*where $u$ satisfies (4).*

*Moreover, $\alpha$ and $v$ are unique, and there is a unique minimal $G$ satsfying (5).*

The case of $\alpha = 1$ corresponds to the uncertainty averse preference representation of Cerreia-Vioglio et al. (2011b). Representation (5) is the natural generalisation beyond uncertainty aversion, involving a Hurwicz-style $\alpha$-mixture of the infimum in the Cerreia-Vioglio et al. (2011b) representation, and the supremum of the "conjugate" function. For this reason, we call MBBA preferences satisfying Attitude Coherence, Weak Uncertainty Aversion and Imprecision Aversion *$\alpha$-UA preferences* (for $\alpha$-Uncertainty Attitude). It is straightforward to check that this family can comfortably accommodate uncertainty-seeking behavior (and violations of the Uncertainty Aversion axiom), such as the 10-colour Ellsberg example mentioned in the Introduction (see also Section 5.2).

The focus on the $\alpha > 0.5$ case here is purely for expositional ease. A version of this result and all subsequent results with $\alpha < 0.5$ can be obtained by replacing Weak Uncertainty Aversion and Imprecision Aversion with the corresponding "Weak Uncertainty Seeking" and "Imprecise Seeking" axioms, where the order of preferences is inverted.

# 4 $\alpha$-maxmin EU, variational, multiplier preferences and beyond

We now show how the proposed approach naturally yields unique identification for the $\alpha$-maxmin EU model, as well as extensions of several other ambiguity models beyond the assumption of uncertainty aversion. The characterisations of these special cases of representation (5) are summarized in Table 1, with the relevant axioms listed in Figure 2. The table is to be read in the context of the following result.

**Theorem 2.** *Let $\succeq$ be a preference relation on $\mathcal{A}$. For each row in Table 1, the following are equivalent:*

*i. $\succeq$ are $\alpha$-UA preferences satisfying the axiom(s) in the left column of Table 1;*

*ii. There exists a normalised, strictly increasing utility function $v : Z \to [-1, 1]$, $\alpha \in (0.5, 1]$ and the elements specified in the middle column of Table 1, such that $\succeq$ is represented as stated in that column, where $u$ satisfies (4).*

*Moreover, $\alpha$ and $v$ are unique, and the uniqueness of the other parameters are as stated in the right column of Table 1.*

**Axiom** (Comonotonic Independence)**.** *For all pairwise comonotonic*[a] $f, g, h \in \mathcal{A}$ *and* $\lambda \in (0, 1)$*,* $f \succeq g$ *if and only if* $f_\lambda h \succeq g_\lambda h$ *.*

**Axiom** (C-Independence)**.** *For all* $f, g \in \mathcal{A}$*,* $c \in \mathcal{B}$ *and* $\lambda \in (0, 1)$*,* $f \succeq g$ *if and only if* $f_\lambda c \succeq g_\lambda c$ *.*

**Axiom** (Weak C-Independence)**.** *For all* $f, g \in \mathcal{A}$*,* $c, d \in \mathcal{B}$ *and* $\lambda \in (0, 1)$*,* $f_\lambda c \succeq g_\lambda c$ *if and only if* $f_\lambda d \succeq g_\lambda d$ *.*

**Axiom** (Weak Monotone Continuity)**.** *If* $f, g \in \mathcal{A}$*,* $l \in \mathcal{L}$ *and* $\{E_n\}_{n\geqslant 1} \in \Sigma$ *with* $E_1 \supseteq E_2 \supseteq \ldots$ *and* $\bigcap_{n\geqslant 1} E_n = \emptyset$*, then* $f \succ g$ *implies that there exists* $n_0$ *with* $l_{E_{n_0}} f \succ g$*.*

**Axiom** (Weak P2)**.** *For all* $f, g, h, h' \in \mathcal{A}^l$ *and* $E \in \Sigma$*,* $f_E h \underline{\gtrapprox} g_E h$ *if and only if* $f_E h' \underline{\gtrapprox} g_E h'$*.*

[a] A pair of acts $f, g \in \mathcal{A}$ are comonotonic if, for no $s, t \in \mathcal{S}$, $f(s) \succ f(t)$ and $g(s) \prec g(t)$.

Figure 2: Some Axioms

We now discuss these cases in turn.

**$\alpha$-Maxmin EU and Choquet EU** Aside from Uncertainty Aversion, the other main axiom in the Gilboa and Schmeidler (1989) axiomatisation of maxmin EU is C-Independence (Figure 2). The result in the first row of Table 1 shows that this is all that is required to axiomatise the maxmin EU special case of our $\alpha$-UA preference family. Hence the C-independence axiom yields the well-known Hurwicz $\alpha$-maxmin EU representation. The chief contribution of this result with respect to the existing literature is the full identification of the set of priors $C$ and the mixture coefficient $\alpha$, in the absence of any specific constraints on the form of $C$. This can be illustrated on the following example.

**Example 1.** Consider an Ellsberg unknown urn with two states of the world, $S = \{b, r\}$ ($b$ for a blue ball being drawn next, $r$ for a red ball), so each probability measure in $\Delta$ is characterised by the probability given to $b$, and $\Delta$ can be indexed by these $p \in [0,1]$. Let $Z = [-1, 1]$ and suppose that $v$ is the identity function. Consider preferences $\succeq$ over $\mathcal{A}^l$ (the set of lottery-acts) represented according to (6) with $C = [0, 1]$ and $\alpha = 0.7$. It is straightforward to check that these preferences are also represented by $C = [0.3, 0.7]$ and $\alpha = 1$.[9] Hence there is a lack of a unique $(\alpha, C)$ pair representing $\succeq$ over $\mathcal{A}^l$.

In the full space of acts $\mathcal{A}$, this lack of uniqueness is resolved using bi-lotteries. In particular, whenever preferences $\succeq$ over $\mathcal{A}$ are represented according to (6) and (4), then it is easy to check that $C = [0, 1]$ if and only if $1_s(-1) \approx [-1, 1]$ (the bi-lottery generated by the degenerate lottery yielding $-1$ and that yielding 1 is

[6] $\int$ is the Choquet integral.

[7] $c : \Delta \to [0, \infty]$ is grounded if its infimum value is 0.

[8] $R$ is the relative entropy.

[9] For any $f \in \mathcal{A}^l$, if $f(s) \leqslant f(t)$, then $0.7 \min_{p\in[0,1]} \mathbb{E}_p(u(f)) + 0.3 \max_{p\in[0,1]} \mathbb{E}_p(u(f)) = 0.7u(f(s)) + 0.3(u(f(t))) = \min_{p\in[0.3,0.7]} \mathbb{E}_p u(f)$, and similarly if $f(s) \geqslant f(t)$.

| Supplementary Axiom(s) | Representation | Uniqueness |
|---|---|---|
| C-Independence | $V(f) = \alpha \min_{p\in C} \mathbb{E}_p(u \circ f) + (1-\alpha) \max_{p \in C} \mathbb{E}_p(u\circ f)$ (6)<br>$C \subseteq \Delta$ : closed convex set of priors | $C$ is unique |
| Comonotonic Independence | $V(f) = \int u(f) d\mu^{\alpha}$ (7)<br>$\mu$: convex capacity<br>$\mu^{\alpha}$: capacity defined by<br>$\mu^{\alpha}(E) = \alpha\mu(E) + (1-\alpha)(1-\mu(E^c))$ for every $E \subseteq \mathcal{S}$[6] | $\mu$ unique |
| Weak C-Independence | $V(f) = \alpha \min_{p\in\Delta} (\mathbb{E}_p(u\circ f) + c(p)) + (1-\alpha)\max_{p\in\cdot} (\mathbb{E}_p(u\circ f) - c(p))$ (8)<br>$c : \Delta \to [0,\infty]$ : grounded, convex, lower semicontinuous function[7] | There is a unique minimal $c$ |
| Weak C-Independence, Weak Monotone Continuity, Weak P2 | $V(f) = \alpha \min_{p\in\Delta} (\mathbb{E}_p(u\circ f) + \theta R(p\Vert q)) + (1-\alpha)\max_{p\in\Delta} (\mathbb{E}_p(u\circ f) - \theta R(p\Vert q))$ (9)<br>$\theta \in (0,\infty]$, $q \in \Delta$[8] | $\theta$ and $q$ are unique. |

Table 1: Special cases

a bi-lottery equivalent for the bet on $s$). Similarly, $C = [0.3, 0.7]$ if and only if $1_s(-1) \approx [-0.4, 0.4]$. Since, under such non-degenerate preferences, one cannot have both $1_s(-1) \approx [-1,1]$ and $1_s(-1) \approx [-0.4, 0.4]$, the representing set $C$ can be at most one of $[0,1]$ and $[0.3, 0.7]$. Similarly, $\alpha = 1$ if and only if $[-1,1] \sim -1$, whereas $\alpha = 0.7$ if and only if $[-1,1] \sim -0.4$. Since at most one of these preferences is possible, the $\alpha$ representing preferences can be at most one of these values. Expanding the consequence space to include bi-lotteries thus resolves the uniqueness issue with the $\alpha$-maxmin EU model, pinning down the $\alpha$ and $C$.

The scope of this contribution can be brought out via the comparison with the result on the second line of the table. It shows that, when the main axiom for Choquet EU—Comonotonic Independence (Schmeidler, 1989)—is added, one obtains a Choquet special case of the $\alpha$-UA preference family, where the capacity in the representation is generated from a convex capacity by an $\alpha$-mixture with its (concave) conjugate. Since the

Choquet integral of a convex capacity coincides with the maxmin-EU functional applied to the core of that capacity (Schmeidler, 1989), preferences represented by (7) correspond to the special case of the $\alpha$-maxmin EU representation (6) where the set of priors $C$ is equal to the core of a (convex) capacity. Some of the main results in the literature that have obtained an $\alpha$-maxmin EU representation with unique $\alpha$ and $C$—notably Chateauneuf et al., 2007, Thm 5.1; Gul and Pesendorfer, 2014, Thms 1 & 5, 2015, Prop 1—are special cases of (7) where the convex capacity is generated by a probability measure (on the state space in the case of Chateauneuf et al. 2007; on a sub-$\sigma$-algebra of an infinite state space in the case of Gul and Pesendorfer 2014, 2015). The contribution of the first row of Table 1 is to obtain the identification without imposing any structure on the capacity, and indeed without assuming that the set of priors corresponds to a capacity at all. The following stylized example emphasises the difference.

**Example 2.** Consider an economist represented by (6) with $\alpha = 0.9$ and $v$ the identity function. She knows that the state of the economy in 2022 may depend on the duration of the protection provided by Covid vaccines. For simplicity, suppose that protection duration can be long ($l$) or short ($sh$), and the economy can be either in state $s$ or $t$. In forming beliefs about the duration of protection provided by vaccines, she adopts the judgement emitted by the Covid scientific committee, according to which the probability of long-duration protection, $p(l) \in [0.2, 0.8]$. Concerning the future state of the economy, she combines this with her judgements on the economic question on which she is a specialist, which can be summarised in the probability of specific future states of the economy given vaccine protection duration. Suppose that she considers these to be $p(s/l) \in [0.7, 0.9]$ and $p(s/sh) = [0.3, 0.6]$. This clearly defines a set of priors, namely $C = \{p \in \Delta : p(l) \in [0.2, 0.8], p(s/l) \in [0.7, 0.9], p(s/sh) = [0.3, 0.6]\}$;[10] it is straightforward to check that this is not equal to the core of any capacity. So, whilst one can identify a capacity $\mu$ representing the agent's preferences over two-outcome bets according to (7), it will not represent her preferences over all acts. In particular, it will not properly represent her preferences concerning bets conditional on protection duration. For instance, the preference $1_{s\cap l}(-1)_{t\cap l}0 \succ 0$ indicates that she prefers the bet on state $s$, given that vaccines provide long-run protection, to a constant 0. Such preferences are central to understanding planning behavior. Moreover, in examples such as these, where the agent's expertise pertains to the issue of the future state of the economy *given* vaccine protection duration, it is her conditional beliefs that are most relevant for identification. However, whilst under (6) with $C$, we have the preference $1_{s\cap l}(-1)_{t\cap l}0 \succ 0$, when (7) is used with the capacity deduced from preferences over two-outcome bets, we have $1_{s\cap l}(-1)_{t\cap l}0 \prec 0$.[11] The

[10] For concreteness, we can take the state space to be $\{l, sh\} \times \{s, t\}$.

[11] More specifically, $C$ generates the capacity $\mu$, with $\mu(s) = 0.38$, $\mu(t) = 0.16$, $\mu(s \cap l) = 0.14$, $\mu((s \cap l)^c) =$

aforementioned approaches cannot identify the agent's set of priors under the preferences involved here, because these preferences do not belong to the special cases of the $\alpha$-maxmin EU model which they use. By contrast, Theorem 2 (first line of Table 1) applies to these preferences and provides unique identification.

**Variational Preferences** Aside from standard Uncertainty Aversion, the other main axiom in the Maccheroni et al. (2006) axiomatisation of variational preferences is Weak C-Independence (Figure 2). The result in the third row of Table 1 shows that this is all that is required to axiomatise the variational special case of the $\alpha$-UA preference family.

Indeed, the $\alpha = 1$ case of the representation in Table 1 corresponds to variational preferences (Maccheroni et al., 2006). So this result provides an extension beyond uncertainty aversion, using the $\alpha$-mixture of variational preferences and the corresponding uncertainty seeking version (with a maximum in place of a minimum, and the $c$ still counting as a cost in the maximum, so bearing a negation sign). Note that this extension preserves the uniqueness of $c$, which is as in the Maccheroni et al. (2006) result. If you will, these are fully identifed *$\alpha$-variational preferences.*

**Multiplier Preferences** A well-known special case of variational preferences are multiplier preferences, proposed by Hansen and Sargent (2001). Adapting Strzalecki's (2011) axiomatisation of this family, the third row of Table 1 extends them beyond the limits of uncertainty aversion. Beyond the axioms for variational preferences (see above), the Strzalecki (2011) axiomatisation invokes Weak Monotone Continuity (Figure 2) and Savage's P2. The extension characterised in Table 1 invokes the same axioms, with P2 replaced by Weak P2. Analogously to the weakening of Uncertainty Aversion (Section 3.2), Weak P2 weakens P2 by applying it on the lower precision relation, rather than the preference relation. Moreover, the parameters are fully identified, just as in the axiomatisation of uncertainty averse case (corresponding to our (9) with $\alpha = 1$).

The characterised representation (9)—*$\alpha$-multiplier preferences*, if you will—are thus a natural generalisation of multiplier preferences to accommodate violations of uncertainty aversion. Multiplier preferences are related to maxmin EU preferences where the set of priors is defined by a relative-entropy-based constraint, with the former obtained via the Lagrangian of the latter (Hansen and Sargent, 2008). Clearly, a similar relationship holds between $\alpha$-multiplier preferences and $\alpha$-maxmin EU preferences with such priors.

0.28, $\mu(sh \cup (s \cap l)) = 0.76$, $\mu((sh \cup (s \cap l))^c) = 0.02$ which represents preferences over two-outcome bets according to (7). By calculation, the evaluation of $1_{s\cap l}(-1)_{t\cap l}0$ according to (6) with $\alpha$,$v$, $C$ as specified is $\alpha \min_{p\in C}(p(s \cap l) - p(t \cap l)) + (1 - \alpha)\max_{p\in C}(p(s \cap l) - p(t \cap l)) = 0.9 \times 0.08 + 0.1 \times 0.64 = 0.136 > 0$. By contrast, its evaluation according to (7) with $\alpha$,$v$, $\mu$ as specified is $\alpha(\mu(s \cap l) - (1 - \mu(sh \cup (s \cap l))) + (1 - \alpha)(1 - \mu((s \cap l)^c) - (1 - (1 - \mu((sh \cup (s \cap l))^c)) = -0.02 < 0$.

**Other models** Similar generalisations beyond uncertainty aversion can be obtained for other families of uncertainty averse preferences, such as confidence (Chateauneuf and Faro, 2009) and confidence-based preferences (Hill, 2013). Indeed, since uncertainty averse smooth ambiguity preferences—that is, preferences represented by $V(f) = \phi^{-1}\left(\int \phi\left(\mathbb{E}_p(u(f))\right) d\mu\right)$ with $\mu$ a countably additive Borel probability measure over $\Delta$ and $\phi$ a concave transformation function (Klibanoff et al., 2005)—are a special case of the uncertainty averse preferences featuring in representation (5) (Cerreia-Vioglio et al., 2011b), the general method for generating non-uncertainty averse extensions can also be applied to them. Plugging them in for $I$ in (3) yields:

$$V(f) = \alpha\phi^{-1}\left(\int \phi\left(\mathbb{E}_p(u(f))\right) d\mu\right) + (1-\alpha)\left[-\phi^{-1}\left(\int \phi\left(-\mathbb{E}_p(u(f))\right) d\mu\right)\right] \tag{10}$$

Whenever the base smooth ambiguity representation (notably, the $\phi$) is smooth, the same is true for (10), justifying the moniker *$\alpha$-smooth* for this representation. The smooth ambiguity model can accomodate non-uncertainty averse behavior by using transformation functions $\phi$ which are neither convex nor concave. Representation (10) provides an alternative way of generating non-uncertainty averse behaviour from an (uncertainty averse) smooth ambiguity model base, which retains the use of familiar concave transformation functions (such as exponential or power functions), but generalises the representation. As shall be discussed in Section 7, it may yield interesting comparative statics in some portfolio choice applications.

# 5 Imprecision and uncertainty attitudes

We now consider attitudes to uncertainty and imprecision under $\alpha$-UA preferences. Recall that one of the contributions is to separate out the role of whatever underlies the $I$ (e.g. set of priors in (6), "ambiguity indices" in (8)), which is sometimes related to something of the nature of "ambiguity", or "belief", from the parameter $\alpha$, which seems to regulate the degree of pessimism or caution in the face of the ambiguity. So, rather than a single notion of ambiguity or uncertainty attitude, the model will support two notions—of imprecision and of attitude to imprecision.

## 5.1 Comparative attitudes

The following is a popular notion of ambiguity aversion (Ghirardato and Marinacci, 2002).

**Definition 4.** $\succeq^1$ is *more ambiguity averse than* $\succeq^2$ if, for all $f \in \mathcal{A}^l$ and $l \in \mathcal{L}$:

$$f \geq^1 l \Rightarrow f \geq^2 l \tag{11}$$

In our enriched framework, one can home in on the attitude to the "objective imprecision"—in the bi-lotteries—in isolation from considerations specific to acts (which also involve beliefs and ambiguity concerning the state of the world). Following the previous definition, this yields:

**Definition 5.** $\geq^1$ is *more imprecision averse than* $\geq^2$ if, for all $[l', m'] \in \mathcal{B}$ and $l \in \mathcal{L}$:

$$[l', m'] \geq^1 l \Rightarrow [l', m'] \geq^2 l \tag{12}$$

This follows closely the intuition behind the previous notion of ambiguity aversion (and indeed, Yaari's (1969) notion of risk aversion on which it is based): the more imprecision averse decision maker evaluates every bi-lottery more pessimistically—i.e. it is preferred to fewer precise lotteries.

On the other hand, one can home in on the rest: that is, on preferences between acts and bi-lotteries, which as noted (Introduction and Section 3), reveal the "subjective imprecision" perceived by the decision maker. Consider the following two notions of comparative imprecision.

**Definition 6.** $\geq^1$ is *more imprecise than* $\geq^2$ if, for all $f \in \mathcal{A}^l$ and $l \in \mathcal{L}$:

$$f \underline{\gtrapprox}^1 l \Rightarrow f \underline{\gtrapprox}^2 l \tag{13}$$

**Definition 7.** $\geq^1$ is *strongly more imprecise than* $\geq^2$ if, for all $f \in \mathcal{A}^l$ and $[l, m] \in \mathcal{B}$:

$$f \approx^1 [l, m] \Rightarrow f \gtrapprox^2 [l, m] \tag{14}$$

Imprecision follows the standard notion of ambiguity aversion in comparing acts to lotteries, but according to their lower precision (Definition 3) rather than the preference between them. Under $\alpha$-UA preferences, Anne's evaluation of acts may be more imprecise than Bob's, but nevertheless she may not be more ambiguity averse in the sense of Definition 4, because she may be less averse to imprecision. However, one would expect that, if a decision maker rules out any lottery worse than $l$ as a possible evaluation for $f$—as implied by $f \underline{\gtrapprox}^1 l$ (Section 3.1)—then a decision maker whose evaluations are more precise would do the same. Definition 6 states precisely this as the notion of comparative imprecision.

Strong imprecision involves the precision relation (Definition 2). It says that the less imprecise decision maker considers $f$ to be more precise than any bi-lottery equivalent of $f$ according to the more imprecise decision maker. As we shall see, this notion brings in decision makers' attitudes to imprecision in a way that the previous one does not. It follows from the definition of the lower precision relation and the $\alpha$-UA representation that if decision maker 1 is strongly more imprecise than 2, then she is more imprecise.

The following result maps these notions into the primitives of the model.

**Proposition 2.** *Let $\succeq^1, \succeq^2$ be $\alpha$-UA preferences, represented by $(v^1, \alpha^1, I^1)$ and $(v^2, \alpha^2, I^2)$ respectively according to (3) and (4). Then:*

*i. $\succeq^1$ is more imprecision averse than $\succeq^2$ if and only if $v^1 = v^2$ and $\alpha^1 \geqslant \alpha^2$ ;*

*ii. $\succeq^1$ is more imprecise than $\succeq^2$ if and only if $v^1 = v^2$ and, for all $a \in B(\Sigma)$, $I^1(a) \leqslant I^2(a)$;*

*iii. $\succeq^1$ is strongly more imprecise than $\succeq^2$ if and only if $v^1 = v^2$ and, for all $a \in B(\Sigma)$, $\alpha^2 I^1(a) + (1 - \alpha^2)(-I^1(-a)) \leqslant \alpha^2 I^2(a) + (1 - \alpha^2)(-I^2(-a))$.*

Comparisons of imprecision aversion thus correspond to differences in $\alpha$, with the more imprecision averse decision maker having a higher $\alpha$ (recall that the uncertainty averse extreme of the family is when $\alpha = 1$). This suggests $\alpha$ as an index of *imprecision aversion.* On the other hand, imprecision comparisons correspond to the most intuitive notion: more imprecise decision makers have lower "worst-case evaluation" $I$.[12] Note that this implies that the intervals of possible evaluations generated by the functional $I$ are larger for more imprecise decision makers: $[I^2(a), -I^2(-a)] \subseteq [I^1(a), -I^1(-a)]$. Indeed, clause ii. of this proposition gives immediate and natural corollaries for the models considered in Section 4.

**Corollary 1.** *Let $\succeq^1, \succeq^2$ be $\alpha$-UA preferences. Then:*

*i. If $\succeq^1, \succeq^2$ are $\alpha$-maxmin EU, then $\succeq^1$ is more imprecise than $\succeq^2$ if and only if $v^1 = v^2$ and $C_2 \subseteq C_1$,* where these are as in (6).

*ii. If $\succeq^1, \succeq^2$ are $\alpha$-variational, then $\succeq^1$ is more imprecise than $\succeq^2$ if and only if $v^1 = v^2$ and $c_1 \leqslant c_2$,* where these are the unique minimal functions in (8).

*iii. If $\succeq^1, \succeq^2$ are $\alpha$-multiplier, then $\succeq^1$ is more imprecise than $\succeq^2$ if and only if $v^1 = v^2$, $q_1 = q_2$ and $\theta_1 \leqslant \theta_2$*, where these are as in (9).

[12] The notion of comparative ambiguity aversion in Definition 4 only orders decision makers if they share the same utility function, and the notions defined here retain this property. Versions avoiding this implication can be proposed, employing the technique, developed in Hill, 2019; Wang, 2019, of using acts yielding lotteries over best and worst prizes as consequences in the place of general acts.

*iv.* $\succeq^1$ *is more imprecise than* $\succeq^2$ *if and only if* $v^1 = v^2$ *and* $G_1 \leqslant G_2$, where these are the unique minimal functionals in (5).

In all cases, imprecision comparisons correspond to what one would expect: more imprecise decision makers have larger sets of priors, lower ambiguity indices, and so on. We shall sometimes refer to $I$ or the relevant elements in the special cases ($C$, $c$, and so on) as reflecting *imprecision*.

An improvement in precision need not be attractive, even for an imprecision averse decision maker. Under the $\alpha$-maxmin EU representation, for instance, the set of priors $[0.05, 0.1]$ is more precise than $[0, 1]$, insofar as it is a subset; however, it is not necessarily preferable. This is analogous to the situation in decision under risk: a sure \$5 payment is less risky than a 50-50 lottery between \$50 and \$0, but that doesn't mean that it is preferred. There, notions such as mean-preserving spread "control" which risk comparisons are made. Strong imprecision can be thought of as exerting a similar control, using the imprecision attitude of the strongly less imprecise decision maker ($\alpha^2$). The final clause in Proposition 2 says that the strongly less imprecise decision maker always has a higher "effective" worst-case evaluation—where the $I$ values are weighted by her imprecision aversion parameter.

This difference is relevant for the relationship between these notions and the standard notion of ambiguity aversion (Definition 4), as clear in the following Proposition, where $\succeq^1$ is as imprecise (imprecision averse) as $\succeq^2$ if each is more imprecise (resp. more imprecision averse) than the other.

**Proposition 3.** *Let* $\succeq^1, \succeq^2$ *be* $\alpha$*-UA preferences. Then:*

*i. if* $\succeq^1$ *is more imprecision averse and strongly more imprecise than* $\succeq^2$ *then it is more ambiguity averse;*

*ii. if* $\succeq^1$ *is as imprecision averse as* $\succeq^2$*, it is strongly more imprecise if and only if it is more ambiguity averse;*

*iii. if* $\succeq^1$ *is as imprecise as* $\succeq^2$*, it is more imprecision averse if and only if it is more ambiguity averse.*

The $\alpha$-UA representation involves a separation of the role of imprecision from attitudes to imprecision. It is thus to be expected that revealed ambiguity attitude in the sense of Definition 4 is impacted both by the imprecision on the side of the decision maker's evaluations and by her attitude to that imprecision—and this is what Proposition 3 says. On the one hand, holding imprecision fixed, the ranking of ambiguity aversion follows that of imprecision aversion. This is similar to existing results showing that ambiguity aversion follows the "attitude" parameter in various models (Ghirardato et al., 2004; Klibanoff et al., 2005) under an assumption about fixed beliefs or ambiguity. In the other direction, holding imprecision aversion fixed, ambiguity attitude co-varies with the strong imprecision ranking; results of this sort are rarer in the literature. Strong imprecision rather than imprecision is relevant here, because, as noted, a decision maker can be more precise but extreme (e.g. concentrated at the bottom of the interval), and hence unranked by ambiguity aversion.

In sum, the Ghirardato and Marinacci (2002) notion of ambiguity aversion can be thought of as "factorising" into imprecision aversion and strong imprecision: ambiguity aversion requires at least one of them, though may "trade them off".

## 5.2 Absolute attitudes

The most well-known notion of (absolute) ambiguity aversion is doubtless the Uncertainty Aversion axiom introduced by Schmeidler (1989) (Section 3.2). As noted at the outset, $\alpha$-UA preferences do not satisfy this axiom in general. The Ellserg 10-colour example in the Introduction illustrates this: a preference for betting on a colour (out of ten) for the next ball drawn from an unknown urn over betting on an urn with an equal proportion of each colour can be accommodated by the $\alpha$-maxmin EU model (6) with $\alpha = 0.8$ and the full set of admissible probability distributions, but not by a model satisfying Uncertainty Aversion. Moreover, part of the identification problem for $\alpha$-maxmin EU (Example 1, Section 4) is that Uncertainty Aversion can support representations of the form (6) with $\alpha \neq 1$; so adding it to those above does not impose $\alpha = 1$.

According to another absolute notion of ambiguity attitude in the literature (due to Ghirardato and Marinacci, 2002), a decision maker is ambiguity averse if she is more ambiguity averse than a Subjective Expected Utility (SEU) decision maker, in the sense of Definition 4. As for Schmeidler's notion, $\alpha$-UA preferences are not ambiguity averse in this sense in general. But again, the two comparative notions from Section 5.1 permit a decomposition into two corresponding absolute notions.

On the one hand, a decision maker can be said to be *imprecision averse* if she is more imprecision averse than some SEU decision maker, in the sense of Definition 5. SEU has not been specifically defined on bi-lotteries, but under the reasonable assumption that, in evaluating a bi-lottery $[l, m]$, a Bayesian decision maker would assume a uniform distribution over the $l' \in [l, m]$, we obtain the condition that a decision maker is (strictly) *imprecision averse* if $[l, m] \prec l_{\frac{1}{2}} m$ for every $[l, m] \in \mathcal{B}$ with $m > l$. That is, she prefers the precise "average" lottery to the bi-lottery in all cases. As the following proposition shows, imprecision aversion is completely determined by whether $\alpha > 0.5$, corresponding to the widespread intuition that such values of $\alpha$ reflect "ambiguity aversion" or "pessimism".

**Proposition 4.** *Let $\succeq$ be a MBBA preference satisfying Attitude Coherence, with $\alpha$ as in (3) and (4). Then $\alpha > 0.5$ if and only if $[l,m] \prec l_{\frac{1}{2}}m$ for every $[l,m] \in \mathcal{B}$ with $m > l$.*

Notions of imprecision seeking and imprecision neutrality naturally correspond to a preference for the bi-lottery over the precise average lottery, and an indifference between them respectively. They are characterised by $\alpha < 0.5$ and $\alpha = 0.5$ respectively.

On the other hand, a notion of absolute imprecision can be defined, as being more imprecise than some SEU decision maker, in the sense of Definition 6. All the special cases considered in Section 4 are imprecise in this sense.[13]

**Proposition 5.** *$\alpha$-maxmin EU, $\alpha$-variational, $\alpha$-multiplier and $\alpha$-smooth preferences are imprecise.*

Again, this is as one would expect. SEU is the special case of (3) where $I$ is linear and beliefs are fully "precise"; so it is natural that the other models are more imprecise.

# 6 Incomplete preferences

A natural question often investigated in the ambiguity literature is the relationship between a given (complete) preference and its largest Bewley subrelation, sometimes called the unambiguous preference relation. This is standardly defined as follows.

**Definition 8.** Let $\succeq$ be an $\alpha$-UA preference relation. Its *unambiguous preference relation* $\succeq^*$ on $\mathcal{A}^l$ is defined by: for all $f, g \in \mathcal{A}^l$

$$f \succeq^* g \Leftrightarrow f_\lambda h \succeq g_\lambda h \quad \forall h \in \mathcal{A}^l,\ \lambda \in (0,1]$$

By Cerreia-Vioglio et al. (2011a, Prop 2), $\succeq^*$ is a (generally incomplete) Bewley preference: there exists a utility function $u$ and a closed convex set of priors $C^* \subseteq \Delta$ such that $f \succeq^* g$ if and only if

$$\mathbb{E}_p u \circ f \geqslant \mathbb{E}_p u \circ g \qquad \forall p \in C^* \tag{15}$$

Moreover, it is the largest subrelation of $\succeq$ that is represented according to (15). In order to elucidate the relationship between incomplete Bewley subrelations and $\alpha$-UA preferences, we introduce the following relation.

[13] This is not the case for general $\alpha$-UA preferences (5) because uncertainty averse preferences are not necessarily dominated by a probability distribution (Cerreia-Vioglio et al., 2011a, Example 2), as would be required (see proof of Proposition 5, Appendix A.3).

**Definition 9.** Let $\succeq$ be an $\alpha$-UA preference relation. Its *imprecision-attitude-free* preference relation $\succeq^\circ$ on $\mathcal{A}^l$ is defined by: for all $f, g \in \mathcal{A}^l$

$$f \succeq^\circ g \Leftrightarrow f_\lambda h \underline{\gtrsim} g_\lambda h \quad \forall h \in \mathcal{A}^l,\ \lambda \in (0,1]$$

This relation mimics the standard definition of unambiguous preferences, but uses the lower precision relation in the place of the full preference relation. Unambiguous preferences are designed to pick out comparisons between acts which are unaffected by any hedging motive, or ambiguity: $f \succeq^* g$ is supposed to indicate that ambiguity or hedging motives have no hand in the preference for $f$ over $g$. However, under Hurwicz-style representations, such as (5), it is possible that ambiguity or hedging does drive the preference between $f$ and $g$, but they are nevertheless ordered by the unambiguous preference relation ($f \succeq^* g$) due to a fortuitous interplay with the imprecision aversion parameter $\alpha$. The imprecision-attitude-free preference avoids sanctioning such spurious cases of "unambiguous" comparison by focussing on robustness to hedging with respect to the lower precision relation. As noted previously, this relation is only sensitive to the worst possible evaluation of acts, and so is not open to interference from interplay with the imprecision attitude. $f \succeq^\circ g$ thus indicates that the comparison between the worst possible evaluations of $f$ and $g$ is unaffected by hedging or ambiguity considerations. As such, imprecision-attitude-free preference provides a purer, more restrictive notion of ambiguity-robust comparison.

Imprecision-attitude-free preferences also admit a Bewley multi-prior representation.

**Proposition 6.** *Let $\succeq$ be an $\alpha$-UA preference relation. Then its imprecision-attitude-free preference relation $\succeq^\circ$ is represented by: for all $f, g \in \mathcal{A}^l$, $f \succeq^\circ g$ if and only if*

$$\mathbb{E}_p u \circ f \geqslant \mathbb{E}_p u \circ g \qquad \forall p \in C^\circ \tag{16}$$

*where $C^\circ \subseteq \Delta$ is closed and convex and $u$ is as in Proposition 1. Moreover, $C^\circ$ is unique.*

As suggested by the previous discussion, the imprecise-attitude-free preference relation is a subrelation of unambiguous preferences (Proposition A.5 in Appendix A.4), and hence, by known results (Ghirardato et al., 2004), $C^\circ \supseteq C^*$. This is as to be expected from Example 1 in Section 4: the set of priors representing imprecise-attitude-free preferences could be larger than those picked out by the unambiguous preference relation, because it may be possible to represent the restriction of the preferences to lottery-acts with a larger $\alpha$ and a smaller set of priors.[14]

[14] Relatedly, Example 1 also shows that $\succeq^*$ and $\succeq^\circ$ need not be identical: using the notation introduced there, whenever preferences

Moreover, there is a natural relationship between the imprecision-attitude-free set of priors $C^\circ$ and the parameters in the representation of $\alpha$-UA preferences.

**Proposition 7.** *Let $\succeq$ be an $\alpha$-UA preference relation, and let $C^\circ$ be as in Proposition 6. Then:*

*i.* $C^\circ = cl(dom_\Delta G)$, where $G$ is as in (5).

*ii. If $\succeq$ is $\alpha$-maxmin EU,* $C^\circ = C$, where $C$ is as in (6).

*iii. If $\succeq$ is $\alpha$-variational, then* $C^\circ = cl(supp\ c)$, where $c$ is the unique minimal function in (8).

For each of these models, this is the standard relationship between the set of priors representing the unambiguous preference under the uncertainty averse version of the model on the one hand, and the model parameters on the other. For instance, the former set of priors is the representing set of priors under maxmin-EU, the closure of the support of the ambiguity index under variational preferences, and so on. These are the elements of the models that many consider should remain when attitude (to ambiguity, or imprecision) is removed—indeed, one often finds talk of "revealed ambiguity" (Ghirardato et al., 2004) or "revealed priors or measures" (Cerreia-Vioglio et al., 2011a; Klibanoff et al., 2014), and sometimes a suggestion that they capture something akin to beliefs.

This brings into perspective a central difference between the current approach and approaches to non-uncertainty averse preferences which begin with the Bewley subrelation $\succeq^*$ (Ghirardato et al., 2004; Cerreia-Vioglio et al., 2011a). Those approaches stipulate that the "relevant priors" or "revelant ambiguity" are given by $\succeq^*$, and then use those to construct a representation of $\succeq$ (in the cited papers, generalized Hurwicz representations, where $\alpha$ in (6) may depend on the act being evaluated). The set of priors "involved" in the representation is automatically the one representing $\succeq^*$, by construction; on the other hand, a general form of uniqueness (independently of $\succeq^*$) is not guaranteed. The approach taken here starts by eliciting the $\alpha$ and the minimum functional in the representation, and so the relevant priors come out as a consequence of the representation, rather than as an input. In this way, $\succeq^\circ$ and $C^\circ$ seem to pick up more carefully the priors that are relevant—in particular, by the lights of the uncertainty averse models of which ours provides extensions. If this reading is correct, it might suggest that $C^\circ$, rather than $C^*$, is more aptly interpreted as the "relevent ambiguity" or "relevant priors" for $\alpha$-UA preferences.

are represented according to (6) and (4) with $C = [0,1]$ and $\alpha = 0.7$, then $1_s - 1 \succeq^* -0.4$ but $1_s - 1 \not\succeq^\circ -0.4$.

# 7 $\alpha$ and Portfolio choice

The results in Section 5 suggest that a decision maker's imprecision aversion is captured by the mixture coefficient $\alpha$ in (3). It is natural to ask what effect changes in this attitude have on portfolio choice. This issue is rendered particularly subtle under the proposed representation, because the aim was to go beyond uncertainty aversion, and that means losing the concavity (or quasiconcavity) that is so useful in solving optimisation problems. In this section, we present a preliminary analysis of a standard portfolio problem.

Consider a static problem, involving a safe asset with return $r$ and an uncertain asset with uncertain return $x$ taking values in a closed bounded interval $R \subset \mathbb{R}$, where $r \in R$. An investor is initially endowed with wealth $w$, which she is to allocate between the two assets. The investor's final wealth after investing $a \in [0, w]$ in the uncertain asset is $w.r + a(x - r)$ under return $x$. The investor has $\alpha$-UA preferences, represented according to (3) with functional $I$, imprecision aversion index $\alpha$ and utility function $v$, defined on $Z \supset w.R$ with $v(w.r) = 0$. The portfolio problem can thus be written as:

$$\max_{a \in [0,w]} \alpha I(v(w.r + a(x - r)) + (1 - \alpha)\left[-I(-v(w.r + a(x - r))\right] \tag{17}$$

Suppose moreover that $v$ is concave and continuously differentiable—so investors are risk averse—and $I$ is differentiably concavifiable—that is, there exists a differentiable function $\psi : [-1, 1] \to [-1, 1]$ with everywhere positive derivative such that $\psi \circ I$ is concave. All the special cases of interest considered in Section 4 involve differentiably concavifiable $I$.[15] Then we have:

**Proposition 8.** *Let $v$ be concave and continuously differentiable, and $I$ be differentiably concavifiable, and consider investors with preferences represented by $(v, I, \alpha)$ and $(v, I, \alpha')$ respectively according to (3). Then, for every optimal portfolio allocation $a^* \in [0, w]$ for investor $(v, I, \alpha)$, if $\alpha' > \alpha$ (respectively, $\alpha' < \alpha$), then there exists a optimal portfolio allocation $a^{*\prime}$ for investor $(v, I, \alpha')$ with $a^{*\prime} \leqslant a^*$ (resp. $a^{*\prime} \geqslant a^*$).*

Going beyond uncertainty aversion—as has been the aim in this paper—implies going beyond concave (or quasiconcave) preference functionals, which complicates the study of optima. In this context, this result yields perhaps the most reassuring message one could hope for about the effect of imprecision aversion. Whenever there are unique global optima, more imprecision aversion—higher $\alpha$—leads to lower investment in the uncertain asset. If there are several global optima, more imprecision aversion will lead to a lower

[15] Precisely: all the models considered in that section involve concave $I$, except for the smooth ambiguity model under uncertainty aversion; whenever the $\phi$ in that model is smooth with positive derivative throughout the relevant range—as is typically the case for popular specifications—$\phi \circ I(\bullet) = \int \phi(\mathbb{E}_p \bullet) d\mu$ is concave, since $\phi$ is, so $I$ is differentiably concavifiable.

investment in the uncertain asset in some optimum, and less imprecision aversion will lead to a higher investment in the uncertain asset in some optimum. The result is also extremely general, applying to the $\alpha$-UA version of every uncertainty averse model used in the practice (including the $\alpha$-smooth model, when it is indeed smooth). It is all the more striking that, despite its intuitiveness, results of this sort do not hold in general under every ambiguity model and notion of ambiguity aversion. For instance, Gollier (2011) has shown that, even in the uncertainty averse case, an increase in the ambiguity aversion parameter in the smooth ambiguity model may lead to strictly higher investment in the uncertain asset.[16]

# 8 Discussion and remaining related literature

This paper proposes and provides foundations for Hurwicz-style extensions of several major uncertainty averse decision models beyond the assumption of uncertainty aversion. A central challenge—which is well-known for the $\alpha$-maxmin EU special case—is identification, and in particular the separation of the mixture coefficient $\alpha$ from the "ambiguity- or belief-side" functional ($I$ in representation (3)). Our basic insight for resolving this issue is to enrich the Anscombe-Aumann consequence space to include *objective imprecision*, in the form of bi-lotteries—sets of mixtures of pairs of lotteries. Enriching the structure of choice objects has a long tradition in decision theory, following von Neumann and Morgenstern's introduction of the concept of lottery, and our approach can be thought of as belonging to that tradition.

The notion of bi-lottery is a natural generalisation of that of lottery, with arguably growing relevance in economic modelling. In domains as varied as climate reporting (Mastrandrea et al., 2010), earning forecasts (Du and Budescu, 2005) and central bank projections (Carney et al., 2019), there is an increasing use of ranges rather than point estimates to report values, including probability values. Some have defended range over point reporting for probabilities in econometric and statistical analyses, especially in situations of partial identification (Manski, 2003, 2013). Bi-lotteries, which can roughly be thought of as ranges of lotteries, can represent the provided information in many such cases.

More importantly for the foundational ambitions of this paper, bi-lotteries can be feasibly implemented in laboratory contexts, via analogous treatments to those currently used for von Neumann-Morgenstern lotteries. Just as a lottery corresponds to, and can be physically realised by a draw from an urn with known composition, a bi-lottery corresponds to a draw from an urn where the composition is only partially known (e.g. there are at

[16] Maccheroni et al. (2013) show that "robust mean-variance" preferences, which approximate smooth preferences for "small uncertainties", exhibit a negative correlation between ambiguity aversion and investment. Dziewulski and Quah (2016) study the comparative statics under changes in the set of priors $C$ with constant $\alpha$ under the $\alpha$-maxmin EU model (6).

least 10 and at most 60 red balls out of 100). Indeed, several papers in psychology and behavioral economics have studied preferences over what we call bi-lotteries (e.g. Budescu et al., 2002; Du and Budescu, 2005; Abdellaoui et al., 2019; Burghart et al., 2020), for instance via elicitation of their certainty equivalents.[17] Whilst these studies attest to the interest of the notion of objective imprecision and the feasibility of the concept, insofar as they realise it in the lab, none use it to probe or provide foundations for richer non-uncertainty averse representations of preferences under subjective uncertainty. A sister experimental paper (Abdellaoui et al., 2021) uses bi-lotteries to elicit multiple priors.

Existing theoretical papers on bi-lotteries include Olszewski (2007), who studies preferences over general sets of lotteries, obtaining a representation similar to (4), and a comparative static result similar to Proposition 2.i. Vierø (2009) works in an enriched Anscombe-Aumann framework similar to ours, and Ghirardato (2001) uses something similar for the Savage framework. Both obtain representations which, at the level of states, coincides with Subjective Expected Utility (unlike our (3)), and at the level of consequences involve a Hurwicz evaluation of the form (4), with potential state- or set-dependence of $\alpha$. Gajdos et al. (2008) provide an ambiguity model in which sets of probability measures (over the state space), interpreted as capturing objective imprecise information, feature among the primitives. Here, probability intervals feature in the consequences (bi-lotteries), but no objectively-given information about probabilities over the state space appears in the model.

As mentioned, one contribution of the present paper is to provide a treatment of the identification problem for $\alpha$-maxmin EU preferences. One existing approach to this problem is simply to dictate that $C$ is determined by the unambiguous preference relation (Ghirardato et al., 2004), though this approach has only been successfully demonstrated for infinite state spaces (Eichberger et al., 2011). Another proposed approach involves enrichening the state space to exhibit an infinite product structure, and imposing symmetry axioms. This allows identification of a set of "relevant priors" (Klibanoff et al., 2014), and has recently been used to develop an axiomatisation of a version of the $\alpha$-maxmin EU model adapted to such state spaces and symmetry assumptions (Klibanoff et al., 2021). A final approach, discussed in Section 4, obtains identification for special cases of the $\alpha$-maxmin EU representation (6) where the set of priors is generated by a probability measure over the whole state space (Chateauneuf et al., 2007) or over a sub-algebra (Gul and Pesendorfer, 2015). The approach developed here is complementary, employing an enrichening of the consequence space, without requiring any particular properties of the state space, and relying on no specific assumptions about

[17] Chew et al. (2017) elicit certainty equivalents of bi-lotteries in the context of a wider study involving other types of objectively-given partial information about the probability distribution, as well as compound lotteries.

the shape of the set of priors. Drawing a parallel with the famous representation theorems for SEU, ours can be thought of as an "Anscombe-Aumann" analogue of these "Savage-style" approaches to $\alpha$-maxmin EU. It has the added advantage of yielding not only an axiomatisation of $\alpha$-maxmin EU, but also of providing and characterising $\alpha$-UA versions of other existing uncertainty averse preferences.

Finally, we assume a linear representation of preferences over bi-lotteries (4) to keep as close as possible to the traditional ambiguity literature, which assumes expected utility over Anscombe-Aumann conseqeunces (Section 3.2). Whilst there is some evidence in favour of this representation—Burghart et al. (2020), for instance, find that around 60% of subjects have preferences over bi-lotteries represented by (4)—a potential direction for future work would be to explore weakenings. Similarly, Grant et al. (2019) characterise an "ordinal Hurwicz EU" representation of preferences over acts, which retains the set of priors obtained in the Gul-Pesendorfer representation, but generalises beyond the linear form in $\alpha$-maxmin EU. An analogous generalisation of the results obtained in this paper would be an interesting avenue for future research.

# 9 Conclusion

Casual observation and experimental evidence suggest that uncertainty aversion is not as universal as might be expected from the focus on it in the decision theory literature. In this paper, we have provided and axiomatised extensions of many known ambiguity models—including maxmin EU, variational preferences, multiplier preferences and uncertainty averse preferences—beyond the assumption of uncertainty aversion. They all involve the addition of a single Hurwicz-style mixture coefficient, which, as we demonstrate, captures imprecision aversion. Moreover, our representation results pin down this parameter and the rest of the functional uniquely, resolving an open identification problem for the $\alpha$-maxmin EU model.

The proposed $\alpha$-UA family supports comparative statics analysis, separating imprecision—which captures the relevant parameters of the base uncertainty averse model (e.g. the set of priors under maxmin EU, or the "ambiguity index" for variational preferences)—and imprecision aversion, as captured by the mixture coefficient. A derived incomplete preference subrelation—imprecision-attitude-free preferences—can be defined in a way similar to well-known unambiguous preferences, and picks up more closely the "relevant priors" of the base uncertainty averse model.

Finally, we show, in a simple portfolio problem with a sure and an uncertain asset, that in a general case of our model there is an intuitive relationship between imprecision attitude and investment: more imprecision aversion leads to less investment in the uncertain asset.

# A Proofs

## A.1 Proof of results in Section 3

This section presents the proof of Proposition 1. The only other result in Section 3—Theorem 1—is a direct corollary of Proposition 1, Proposition 2 and (Cerreia-Vioglio et al., 2011b, proof of Thm 3).

*Proof of Proposition 1.* As a point of notation, we sometimes use $b$ below to denote generic bi-lotteries (members of $B$), and $\max b$, resp. $\min b$, to denote the maximal (resp. minimal) element of $b$; so $b = [\min b, \max b]$.

Weak Order, Continuity, Objective Independence, Monetary Monotonicity imply that the restriction of $\succeq$ to $\mathcal{L}$ satisfies the von Neumann-Morgenstern axioms, and hence that there exists a utility function $v : Z \to \mathbb{R}$ representing this restriction according to expected utility. Without loss of generality, we suppose that $v(\mathbf{w}) = -1$ and $v(\mathbf{b}) = 1$. Moreover, for each $[l, m] \in \mathcal{B}$, it follows from Bi-Monotonicity that $m \succeq [l, m] \succeq l$, whence, by Continuity, there exists $l' \in \mathcal{L}$ with $[l, m] \sim l'$.

**Lemma A.1.** *There exists a unique $\alpha \in [0, 1]$ such that the restriction of $\succeq$ to $\mathcal{B}$ is represented by:*

$$u(b) = \alpha \min_{l \in b} \mathbb{E}_l v + (1 - \alpha) \max_{l \in b} \mathbb{E}_l v \tag{18}$$

*Proof.* By Monetary Monotonicity and Bi-Monotonicity, $\mathbf{b} \succeq [\mathbf{w}, \mathbf{b}] \succeq \mathbf{w}$, whence by Continuity and Objective Independence, there exists a unique $\gamma \in [0, 1]$ with $(\mathbf{b})_\gamma \mathbf{w} \sim [\mathbf{w}, \mathbf{b}]$. For any $[\mathbf{b}_\delta \mathbf{w}, \mathbf{b}_\epsilon \mathbf{w}] \in \mathcal{B}$ (with $\delta \leqslant \epsilon$), note that $[\mathbf{b}_\delta \mathbf{w}, \mathbf{b}_\epsilon \mathbf{w}] = (1 - (\epsilon - \delta))\mathbf{b}_{\frac{\delta}{1-(\epsilon-\delta)}} \mathbf{w} + (\epsilon - \delta)[\mathbf{w}, \mathbf{b}]$, so, by Objective Independence, $[\mathbf{b}_\delta \mathbf{w}, \mathbf{b}_\epsilon \mathbf{w}] \sim (1-(\epsilon-\delta))\mathbf{b}_{\frac{\delta}{1-(\epsilon-\delta)}} \mathbf{w} + (\epsilon - \delta)\mathbf{b}_\gamma \mathbf{w} = \mathbf{b}_{\gamma\epsilon+(1-\gamma)\delta}\mathbf{w}$. Since, by the standard argument, for each $l \in \mathcal{L}$, $l \sim \mathbf{b}_\epsilon \mathbf{w}$ for some $\epsilon \in [0, 1]$, it follows that, for every $b \in \mathcal{B}$, $b \sim \mathbf{b}_{\gamma\epsilon+(1-\gamma)\delta}\mathbf{w}$ where $\min b \sim \mathbf{b}_\delta \mathbf{w}$ and $\max b \sim \mathbf{b}_\epsilon \mathbf{w}$. By the EU representation of $\succeq$ on $\mathcal{L}$, $\min_{l\in b} \mathbb{E}_l v = \mathbb{E}_{\min b} v = 2\delta - 1$, $\max_{l\in b} \mathbb{E}_l v = \mathbb{E}_{\max b} v = 2\epsilon - 1$, and $u(b) = \mathbb{E}_{\mathbf{b}_{\gamma\epsilon+(1-\gamma)\delta}\mathbf{w}} v = (1-\gamma)(2\delta-1) + \gamma(2\epsilon-1) = (1-\gamma)\min_{l\in b} \mathbb{E}_l v + \gamma \max_{l\in b} \mathbb{E}_l v$ represents $\succeq$ on $\mathcal{B}$. Setting $\alpha = 1 - \gamma$ yields the desired representation. Since $\gamma$ is unique, so is $\alpha$. □

**Lemma A.2.** *For each $f \in \mathcal{A}^l$, there exists a bi-lottery $[l^f, m^f] \in \mathcal{B}$ with $f \approx [l^f, m^f]$.* Moreover, if $\alpha \neq 0.5$ for $\alpha$ as in (18), then for any bi-lottery $b'$ such that $f \approx b'$, $\max b' \sim m^f$ and $\min b' \sim l^f$. Finally, if $\alpha = 0.5$ for $\alpha$ as in (18), then there exists $[l^f, m^f] \in \mathcal{B}$ with $f \approx [l^f, m^f]$ and $l^f \sim m^f$.

*Proof.* Consider any $f \in \mathcal{A}^l$. Since, by Monetary Monotonicity and Monotonicity, $\mathbf{b} \succeq f \succeq \mathbf{w}$, it follows from Continuity and Objective Independence that there exists a unique $\gamma \in [0, 1]$ such that $f \sim \mathbf{w}_\gamma \mathbf{b}$. Call

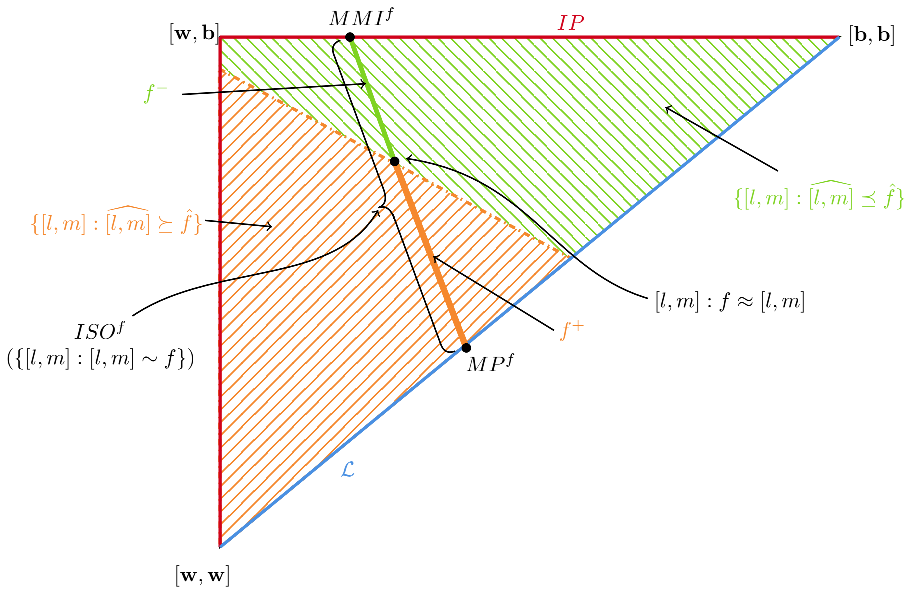


Figure 3: Illustration of $\mathcal{B}$ and the construction in the proof of Lemma A.2
The triangle represents the set of bi-lotteries of the form $[\mathbf{b}_\lambda\mathbf{w}, \mathbf{b}_\kappa\mathbf{w}]$ for $0 \leqslant \lambda \leqslant \kappa \leqslant 1$, with the point $(\lambda, \kappa)$ representing the bi-lottery $[\mathbf{b}_\lambda\mathbf{w}, \mathbf{b}_\kappa\mathbf{w}]$.

this element $MP^f$. Now consider the set $IP = \{[\mathbf{w}_\delta\mathbf{b}, \mathbf{w}_\epsilon\mathbf{b}] \in \mathcal{B} : \delta, \epsilon \in [0,1], \delta = 1 \textit{ or } \epsilon = 0\}$. By Objective Independence, if $[\mathbf{w}_\delta\mathbf{b}, \mathbf{b}] \prec \mathbf{b}$, then for every $\delta, \delta' \in [0,1]$, $[\mathbf{w}_\delta\mathbf{b}, \mathbf{b}] \preceq [\mathbf{w}_{\delta'}\mathbf{b}, \mathbf{b}]$ if and only if $\delta \geqslant \delta'$ (consider the mixture of $[\mathbf{w}_\delta\mathbf{b}, \mathbf{b}]$ with $\mathbf{b}$, when $\delta \geqslant \delta'$). Similarly, if $[\mathbf{w}, \mathbf{w}_\epsilon\mathbf{b}] \succ \mathbf{w}$, then, for every $\epsilon, \epsilon' \in [0,1]$, $[\mathbf{w}, \mathbf{w}_\epsilon\mathbf{b}] \preceq [\mathbf{w}, \mathbf{w}_{\epsilon'}\mathbf{b}]$ if and only if $\epsilon \leqslant \epsilon'$. By Continuity, and the fact that $\mathbf{b} \succeq f \succeq \mathbf{w}$, there exists $[\mathbf{w}_\delta\mathbf{b}, \mathbf{w}_\epsilon\mathbf{b}] \in IP$ such that $[\mathbf{w}_\delta\mathbf{b}, \mathbf{w}_\epsilon\mathbf{b}] \sim f$. Moreover, by the previous observations, this element is unique whenever $f \nsim \mathbf{b}$ and $f \nsim \mathbf{w}$. Set $MMI^f$ to be this element whenever it is unique; otherwise, set $MMI^f = \mathbf{b}$ if $f \sim \mathbf{b}$ and $MMI^f = \mathbf{w}$ if $f \sim \mathbf{w}$. These points are indicated on Figure 3, which illustrates the constructions involved in this proof.

Let $ISO_f = \{[l,m] \in \mathcal{B} : [l,m] \sim f\}$. By Objective Independence, this set are closed under mixture. Let $f^+ = \left\{[l,m] \in ISO_f : \widehat{[l,m]} \preceq \hat{f}\right\}$ and $f^- = \left\{[l,m] \in ISO_f : \widehat{[l,m]} \succeq \hat{f}\right\}$; by Continuity, these sets are closed. We now show that they are both non-empty. If $\widehat{MP^f} \sim \hat{f}$ or $\widehat{MMI^f} \sim \hat{f}$, then this is the case, so suppose not. Consider the case where $\widehat{MP^f} \succ \hat{f}$, so $f^-$ is non-empty. Suppose for reductio that $\widehat{MMI^f} \succ \hat{f}$, so that (by Objective Independence and Weak Order) $f^+$ is empty. By Continuity, there exists $l \in \mathcal{L}$ with $f \sim MP^f \prec l$ and $\hat{l} \succ \hat{f}$. Similarly, there exists $I \in IP$ with $f \sim MMI^f \prec I$ and $\hat{I} \succ \hat{f}$. Since $l \in PR_{\mathbf{w}}$ and $I \in IMP_{\mathbf{w}}$ (where these sets are as defined in Section 3.2), the existence of such $l$ and $I$ contradicts Attitude Coherence. So $\widehat{MMI^f} \preceq \hat{f}$ and $f^+$ is non-empty. A similar argument establishes the claim for the case of

$\widehat{MP^f} \prec \hat{f}$.

Since $f^+$ and $f^-$ are both non-empty and closed, there exists $[l, m] \in f^+ \cap f^-$. For each $[l, m]$, by construction $[l, m] \sim f$ and $\widehat{[l, m]} \sim \hat{f}$, so $f \approx [l, m]$. Moreover, if there exist $[l, m], [l', m'] \in f^+ \cap f^-$ with $l \nsim l'$ or $m \nsim m'$, then by Objective Independence, we have that $[l_\lambda l', m_\lambda m'] \sim f$ and $[l_\lambda \widehat{l', m_\lambda} m'] \sim \hat{f}$ for all $\lambda \in [0, 1]$. So the indifference curves in $\mathcal{B}$, $\{[l, m] \in \mathcal{B} : [l, m] \sim f\}$ and $\left\{[l, m] \in \mathcal{B} : \widehat{[l, m]} \sim \hat{f}\right\}$ are parallel. By Lemma A.1, this can only be the case if $\alpha = 0.5$ in (18). So when $\alpha \neq 0.5$ in (18), $[l, m]$ is unique up to indifference in the minimum and maximum elements: for any $[l', m']$ such that $f \approx [l', m']$, $l \sim l'$ and $m \sim m'$. Moreover, for any $b' \in \mathcal{B}$ such that $f \approx b'$, since, as noted above, $b' \sim [\min b', \max b']$ and $\hat{b}' \sim [\widehat{\max b'}, \widehat{\min b'}]$, $f \approx [\min b', \max b']$; whence $\max b' \sim m^f$ and $\min b' \sim l^f$ as required. Finally, if $\alpha = 0.5$ in (18), then the indifference curves in $\mathcal{B}$, $\{[l, m] \in \mathcal{B} : [l, m] \sim f\}$ and $\left\{[l, m] \in \mathcal{B} : \widehat{[l, m]} \sim \hat{f}\right\}$ are parallel. Since the intersection of these curves is non-empty, and each contain $[l', m']$ with $l' \sim m'$, there exist $[l', m'] \in \mathcal{B}$ with $f \approx [l', m']$ and $l' \sim m'$, as required. □

Note that, for any $l, l' \in \mathcal{L}$, if $l \sim l'$, then $\hat{l}_{\frac{1}{2}} l \sim \hat{l}_{\frac{1}{2}} l' \sim \hat{l'}_{\frac{1}{2}} l$ by Objective Independence and Definition 1, whence $\hat{l} \sim \hat{l'}$. It follows from this and Monotonicity that, for all $f, f' \in \mathcal{A}^l$, if $f(s) \sim f'(s)$ for all $s \in S$, then $f \approx f'$. For each $f \in \mathcal{A}$, let $f^l \in \mathcal{A}^l$ be any lottery-act such that $f(s) \sim f^l(s)$ for all $s \in S$.

Define the pair of functionals $V^-, V^+ : \mathcal{A} \to \mathbb{R}$ as follows. If $\alpha \neq 0.5$, for every $f \in \mathcal{A}$, $V^-(f) = \mathbb{E}_{l^f} v$ and $V^+(f) = \mathbb{E}_{m^f} v$ where $f^l \approx [l^f, m^f]$; if $\alpha = 0.5$, for every $f \in \mathcal{A}$, $V^-(f) = \mathbb{E}_{l^f} v = V^+(f) = \mathbb{E}_{m^f} v$ where $f^l \approx [l^f, m^f]$ with $l^f \sim m^f$. By the previous remarks, these functions are well-defined (in particular, they are independent of the choice of $f^l$ and $[l^f, m^f]$ satisfying the previously stated conditions). By definition $V^-(f) \leqslant V^+(f)$ for all $f \in \mathcal{A}$. Since $f^l \approx [l^f, m^f]$, $f \sim [l^f, m^f]$ for every $f \in \mathcal{A}$, so by Lemma A.1, the functional

$$V(f) = \alpha V^-(f) + (1 - \alpha) V^+(f) \tag{19}$$

represents $\succeq$.

Moreover, since $f^l \approx [l^f, m^f]$ implies $\hat{f}^l \approx [\hat{m^f}, \hat{l^f}]$ (because $\widehat{\widehat{f(s)}} \sim f(s)$ for all $s \in S$), and since $\mathbb{E}_{l^f} v = -\mathbb{E}_{\hat{l^f}} v$ (by Definition 1 and the normalisation of $v$) it follows that $V^-(f) = V^-(f^l) = -V^+(\hat{f}^l)$ and $V^+(f) = V^+(f^l) = -V^-(\hat{f}^l)$.

**Lemma A.3.** *$V^- = I \circ u$ where $I : B(\Sigma) \to \mathbb{R}$ is a balanced, normalised, monotone, continuous functional and $u$ is as in Lemma A.1.*

*Proof.* For each $a \in B(\Sigma)$, there exists $f \in \mathcal{A}^l$ such that $a = u \circ f$ (it suffices to take the act $f$ with $f(s) = \mathbf{w}_{\frac{1-a}{2}}\mathbf{b}$ for all $s \in S$). Define $I : B(\Sigma) \to \mathbb{R}$ by $I(a) = V^-(f)$ for any such $f \in \mathcal{A}$. By Monotonicity, $I$ is well-defined (it is independent of the choice of $f$). By construction, $V^- = I \circ u$. Since $V^-(f) \leqslant V^+(f) = -V^-(\hat{f})$ and $u(\hat{f}(s)) = -u(f(s))$ for all $s \in S$, $I$ is balanced. Now we consider the remaining properties for the case of $\alpha \neq 0.5$. The case of $\alpha = 0.5$ is immediate for monotonicity, and similar for the other properties.

**Monotonicity.** Consider $a, b \in B(\Sigma)$ with $a \geqslant b$. By the previous remarks, there exist $f, g \in \mathcal{A}^l$ with $a = u \circ f$ and $b = u \circ g$. So $f(s) \succeq g(s)$ for every $s \in S$. By Lemma A.2, there exist $[l^f, m^f], [l^g, m^g] \in \mathcal{B}$ with $f \approx [l^f, m^f]$ and $g \approx [l^g, m^g]$. By Monotonicity, $f \succeq g$, so, by the arguments in the proof of Lemma A.2, there exist $p^f \in PR_g$ and $I^f \in IMP_g$ with $p^f \sim I^f \sim f$. By the argument in the proof of Lemma A.2 applied to $ISO_f = \{[l, m] \in \mathcal{B} : l \succeq l^g, m \succeq m^g, [l, m] \sim f\}$, there exists $[l, m] \in \mathcal{B}$ with $l \succeq l^g$, $m \succeq m^g$ and $f \approx [l, m]$. By the uniqueness properties of $[l^f, m^f]$ such that $f \approx [l^f, m^f]$ (Lemma A.2), it follows that $l^f \succeq l^g$ and $m^f \succeq m^g$ and so $I(a) \geqslant I(b)$, as required.

**Normalisation.** Follows immediately from the fact that, for every constant lottery act $l \in \mathcal{L}$, $V^-(l) = u(l)$ (by definition and (18)).

**Continuity.** An immediate consequence of Continuity, the monotonicity of $I$, and Cerreia-Vioglio et al. (2011b, Prop 43). □

This establishes the sufficiency of the axioms for the representation. The proof of necessity is straighforward. In the case $\alpha = 0.5$, Attitude Coherence is trivially satisfied because, for all $f \in \mathcal{A}^l$, there exist no $b \in \mathcal{B}$ with $b \ggg f$ or $f \ggg b$. In the case $\alpha \neq 0.5$, Attitude Coherence follows immediately once one notices that (3) and (4) imply (by basic algebra) that, for every $f \in \mathcal{A}^l$ and $b \in \mathcal{B}$, $f \approx b$ if and only if $I(u \circ f) = \min_{l \in b} b$ and $-I(-u \circ f) = \max_{l \in b} b$. This observation, combined with the transitivity of $\gtrapprox$ and the fact that, for all $b, b' \in \mathcal{B}$ $b \approx b'$ if and only if $\min b \sim \min b'$ and $\max b \sim \max b'$, guarantees the uniqueness of $I$; the uniqueness of $\alpha$ is established by Lemma A.1 and that of $v$ by standard arguments.

Now we turn to the remaining clauses of the Proposition.

**Lemma A.4.** *If $\alpha > 0.5$, where $\alpha$ is as in the representations (3) and (4), then for every $f \in \mathcal{A}^l$, $l \in \mathcal{L}$, $f \underline{\gtrapprox} l$ if and only if $l' \succeq l$ for all $l', m' \in \mathcal{L}$ with $f \approx [l', m']$. If $\alpha < 0.5$, then for every $f \in \mathcal{A}^l$, $l \in \mathcal{L}$, $f \underline{\gtrapprox} l$ if and only if $l' \preceq l$ for all $l', m' \in \mathcal{L}$ with $f \approx [l', m']$.*

*Proof.* Consider $f \in \mathcal{A}^l$, and any $l', m' \in \mathcal{L}$ with $f \approx [l', m']$. For any $[l, m] \in \mathcal{B}$, by the definition of $\gtrapprox$, representation (4) and the fact that $\mathbb{E}_{\hat{l}} v = -\mathbb{E}_l v$, $f \gtrapprox [l, m]$ iff $f \succeq [l, m]$ and $\hat{f} \succeq \widehat{[l, m]}$ iff $[l', m'] \succeq [l, m]$ and

$\widehat{[l',m']} \succeq \widehat{[l,m]}$ iff $\alpha\mathbb{E}_{l'}v + (1-\alpha)\mathbb{E}_{m'}v \geqslant \alpha\mathbb{E}_{l}v + (1-\alpha)\mathbb{E}_{m}v$ and $\alpha\mathbb{E}_{\hat{m}'}v + (1-\alpha)\mathbb{E}_{\hat{l}'}v \geqslant \alpha\mathbb{E}_{\hat{m}}v + (1-\alpha)\mathbb{E}_{\hat{l}}v$ iff $\alpha\mathbb{E}_{l'}v + (1-\alpha)\mathbb{E}_{m'}v \geqslant \alpha\mathbb{E}_{l}v + (1-\alpha)\mathbb{E}_{m}v$ and $\alpha\mathbb{E}_{m'}v + (1-\alpha)\mathbb{E}_{l'}v \leqslant \alpha\mathbb{E}_{m}v + (1-\alpha)\mathbb{E}_{l}v$. It follows that

$$(\mathbb{E}_l v - \mathbb{E}_{l'}v) \geqslant \frac{\alpha}{1-\alpha}(\mathbb{E}_{m'}v - \mathbb{E}_m v) \geqslant \left(\frac{\alpha}{1-\alpha}\right)^2 (\mathbb{E}_l v - \mathbb{E}_{l'}v)$$

If $\alpha > 0.5$, $\frac{\alpha}{1-\alpha} > 1$, so this is only the case if $\mathbb{E}_l v \leqslant \mathbb{E}_{l'}v$, and hence if $l \preceq l'$; if $\alpha < 0.5$, this is only the case if $\mathbb{E}_l v \geqslant \mathbb{E}_{l'}v$, and hence if $l \succeq l'$, hence establishing the result. □

**Lemma A.5.** *If $\alpha \neq 0.5$, $\gtrapprox$ is a mixture continuous weak order (i.e. it satisifies Weak Order, Continuity). Moreover, if $\alpha > 0.5$, then $\gtrapprox$ is monotonic (it satisfies Monotonicity).*

*Proof.* Weak order is an immediate corollary of Lemma A.4 and Weak Order; (mixture) continuity is an immediate corollary of Lemmas A.3 and A.4, as is monotonicity in the case of $\alpha > 0.5$. □

**Lemma A.6.** *If $\alpha > 0.5$, where $\alpha$ is as in the representations (3) and (4), and $\succeq$ satisfies Weak Uncertainty Aversion, then $I$ is quasiconcave.*

*Proof.* By the definition of $u$ and $I$, it suffices to show that, for every $l \in \mathcal{L}$, $\{f \in \mathcal{A}^l : I(u \circ f) \geqslant u(l)\}$ is convex. By the construction of $I$, $I(u \circ f) = u(l')$ for every $l', m' \in \mathcal{L}$ with $f \approx [l', m']$. By Lemma A.4, it follows that $\{f \in \mathcal{A}^l : I(u \circ f) \geqslant u(l)\} = \left\{f \in \mathcal{A}^l : f \gtrapprox l\right\}$. By Lemma A.5, it suffices to show that $f \gtrapprox g$ implies $f_\beta g \gtrapprox g$ for all $\beta \in (0,1)$. Suppose for reductio that this is not the case, and that $f \gtrapprox g$ but $f_\delta g \lessapprox g$ and not $f_\delta g \gtrapprox g$ for some $\delta \in (0,1)$. Since $\gtrapprox$ is continuous (Lemma A.5), $\left\{\gamma \in [0,1] : f_\gamma g \lessapprox g\right\}$ is compact; let $\beta = \max\left\{\gamma \in [0,1] : f_\gamma g \lessapprox g\right\}$. We now show that $f_\beta g \approxeq g$. If $\beta = 1$, this is immediate (from the fact that $f \gtrapprox g$), so suppose not. If it were not the case that $f_\beta g \approxeq g$, then $\beta \in \left\{\gamma \in [0,1] : not(f_\gamma g \gtrapprox g)\right\}$ which is open (by the continuity of $\gtrapprox$), so there exists $\beta' > \beta$ such that it is not the case that $f_\gamma g \gtrapprox g$, and hence, by Lemma A.5 such that $f_{\beta'} g \lessapprox g$, contradicting the maximality of $\beta$. So $f_\beta g \approxeq g$. It follows from Weak Uncertainty Aversion, for every $\gamma \in (0,1)$, $f_{\beta\gamma} g \gtrapprox g$. Taking $\gamma = \frac{\delta}{\beta}$ it follows that $f_\delta g \gtrapprox g$, contradicting the assumption that this is not the case. So $f \gtrapprox g$ implies $f_\beta g \gtrapprox g$ for all $\beta \in (0,1)$ as required, and $\{f \in \mathcal{A}^l : I(u(f)) \geqslant u(l)\}$ is convex.

□

**Lemma A.7.** *If $\alpha < 0.5$ and $\succeq$ satisfies Weak Uncertainty Aversion, then $I$ is quasiconcave and quasiconvex.*

*Proof.* By a similar reasoning to the previous case, $\{f \in \mathcal{A}^l : I(u(f)) \leqslant u(l)\}$ is convex for every $l \in \mathcal{L}$. For every $x \in [-1,1]$, it follows that $LCS_I(x) = \{a \in B(\Sigma) : I(a) \leqslant x\}$ and $UCS_{\hat{I}}(x) = \{a \in B(\Sigma) : -I(-a) \geqslant x\}$

are convex. (Note that if $u \circ f = a$, then $u \circ \hat{f} = -a$.) So the intersection $LCS_I(x) \cap UCS_{\hat{I}}(x)$ is convex; it is non-empty because it contains the constant function $x^*$. We show that, for all $x \in (-1, 1)$ and $a \in LCS_I(x) \cap UCS_{\hat{I}}(x)$, $I(a) = -I(-a)$. Suppose that this is not the case, and consider any $x \in (-1, 1)$ and $a \in LCS_I(x) \cap UCS_{\hat{I}}(x)$ with $I(a) \neq -I(-a)$. Take any such $a$ with $I(a)$ maximal; by definition, $I(a) = x$. So $-I(-a) > I(a) = x$, because $I$ is balanced. By the continuity of $I$, there exists $\gamma \in (0, 1)$ with $\gamma a + (1-\gamma)(-1)^* \in UCS_{\hat{I}}(x)$. By the convexity of $LCS_I(x)$, $\gamma a + (1-\gamma)(-1)^* \in LCS_I(x) \cap UCS_{\hat{I}}(x)$. We consider three cases.

*Case* 1. $I(a) > I(-a)$. Because of this, the fact that $I(1^*) \geqslant I(a)$ and continuity, there exists $\delta \in [0, 1)$ with $I(\delta(-a) + (1-\delta)1^*) = I(a) = x$. Since $-I(-(\delta(-a) + (1-\delta)1^*)) \geqslant I(\delta(-a) + (1-\delta)1^*)$, it follows that $\delta(-a)+(1-\delta)1^* \in LCS_I(x) \cap UCS_{\hat{I}}(x)$. By the convexity of this set, it follows that $\frac{\delta}{1+\delta}a + \frac{1}{1+\delta}(\delta(-a) + (1-\delta)1^*) = \left(\frac{1-\delta}{1+\delta}\right)^* \in LCS_I(x) \cap UCS_{\hat{I}}(x)$. Since $I$ is normalised, this is the case if and only if $\delta = \frac{1-x}{1+x}$. So $\frac{1-x}{1+x}(-a) + \frac{2x}{1+x}1^* \in LCS_I(x) \cap UCS_{\hat{I}}(x)$. By the convexity of $LCS_I(x) \cap UCS_{\hat{I}}(x)$, $\frac{1-x}{1-x+\gamma+\gamma x}(\gamma a + (1-\gamma)(-1)^*) + (1 - \frac{1-x}{1-x+\gamma+\gamma x})\left(\frac{1-x}{1+x}(-a) + \frac{2x}{1+x}1^*\right) = \left(\frac{x+\gamma+\gamma x-1}{1-x+\gamma+\gamma x}\right)^* \in LCS_I(x) \cap UCS_{\hat{I}}(x)$. However, since $\frac{x+\gamma+\gamma x-1}{1-x+\gamma+\gamma x} < x$ whenever $\gamma < 1$, $\left(\frac{x+\gamma+\gamma x-1}{1-x+\gamma+\gamma x}\right)^* \notin UCS_{\hat{I}}(x)$, which is a contradiction. So there exists no $x \in (-1, 1)$ and $a \in LCS_I(x) \cap UCS_{\hat{I}}(x)$ with $I(a) \neq -I(-a)$, as required.

*Case* 2. $I(a) < I(-a)$. This is treated as the previous case, using a mixture of $-a$ and $(-1)^*$.

*Case* 3. $I(a) = I(-a)$. It follows that $\frac{1}{2}a + \frac{1}{2}(-a) = 0^* \in LCS_I(x) \cap UCS_{\hat{I}}(x)$, so $x = 0$. But, by convexity of the set, $\frac{1}{2}(\gamma a + (1-\gamma)(-1)^*) + \frac{1}{2}(-a) = \left(\frac{1}{2}(\gamma - 1)\right)^* \in LCS_I(x) \cap UCS_{\hat{I}}(x)$, which is a contradiction, since $\gamma < 1$.

So for all $x \in (-1, 1)$ and $a \in LCS_I(x) \cap UCS_{\hat{I}}(x)$, $I(a) = -I(-a)$. If $I(a) = 1$, then since $1 \geqslant -I(-a) \geqslant I(a) = 1$, $I(a) = -I(-a)$. If $-I(-a) > -1$, $a \in LCS_I(-I(-a)) \cap UCS_{\hat{I}}(-I(-a))$, so $I(a) = -I(-a) > -1$; hence $I(a) = -I(-a)$ whenever $I(a) = -1$. Since, for every $a \in B(\Sigma)$, $a \in LCS_I(I(a)) \cap UCS_{\hat{I}}(I(a))$, it follows that $I(a) = -I(-a)$ for every $a \in B(\Sigma)$. So $UCS_I(x) = \{a \in B(\Sigma) : I(a) \geqslant x\} = UCS_{\hat{I}}(x)$ for all $x \in [-1, 1]$ and hence is convex. Hence $I$ is quasi-concave and quasi-convex, as required. □

□

## A.2 Proofs of results in Section 4

*Proof of Theorem 2.* Note firstly that the functionals $I$ involved in representations (6), (7), (8) and (9) are concave, and concavity of $I$ implies balancedness. (Concavity implies, for all $a \in B(\Sigma)$, that $\frac{1}{2}I(a) + \frac{1}{2}I(-a) \leqslant I(\frac{1}{2}a + \frac{1}{2}(-a)) = I(0) = 0$, whence $I(a) \leqslant -I(-a)$.) Theorem 2, first row of Table 1, follows from this fact and Proposition A.2 below, by the proof of Gilboa and Schmeidler (1989, Theorem 1). Theorem 2, second row of Table 1, follows from it, Proposition A.2, the fact that Comonotonic Independence implies C-Independence, Proposition A.4 and Schmeidler (1986, Corollary). Theorem 2, third row of Table 1, follows from it and Proposition A.1, by the proof of Maccheroni et al. (2006, Theorem 1). As for the fourth row of Table 1, note that, by Lemmas A.4 and A.3 above, Weak P2 is satisfied if and only if that $I$ satisfies P2. Propositions A.1 and A.3 show that $I$ is constant additive and satisfies Weak Monotone Continuity; the result is a direct corollary of the proof of Strzalecki (2011, Theorem 1). □

**Proposition A.1.** *An $\alpha$-UA preference $\succeq$ satisfies Weak C-Independence iff $I$ in (3) is constant additive.*

*Proof.* We show sufficiency of the axiom for constant additivity; necessity is straightforward. We first show the result for every $a \in int(B(\Sigma))$.[18] For each such $a$, there exists an $a' \in B(\Sigma)$ and $\lambda \in (0,1)$ with $a = \lambda a'$. Let $f' \in \mathcal{A}^l$ be such that $u \circ f' = a'$ (by the proof of Proposition 1, such an $f'$ exists); by construction, $u \circ (f_\lambda(\mathbf{w}_{\frac{1}{2}}\mathbf{b})) = a$ and $u \circ (\hat{f}_\lambda(\mathbf{w}_{\frac{1}{2}}\mathbf{b})) = -a$. By Monotonicity, $(\mathbf{b})_\lambda(\mathbf{w}_{\frac{1}{2}}\mathbf{b}) \succeq f_\lambda(\mathbf{w}_{\frac{1}{2}}\mathbf{b})$, so by Continuity, there exists $d \in \mathcal{B}$ with $f_\lambda(\mathbf{w}_{\frac{1}{2}}\mathbf{b}) \sim d_\lambda(\mathbf{w}_{\frac{1}{2}}\mathbf{b})$; let $y \in [-1,1]$ be such that $u \circ (d_\lambda(\mathbf{w}_{\frac{1}{2}}\mathbf{b})) = y^*$. Similarly, there exists $y' \in [-1,1]$ with $u \circ d'_\lambda(\mathbf{w}_{\frac{1}{2}}\mathbf{b}) = y'^*$ where $d' \in \mathcal{B}$ is such that $\hat{f}_\lambda(\mathbf{w}_{\frac{1}{2}}\mathbf{b}) \sim d'_\lambda(\mathbf{w}_{\frac{1}{2}}\mathbf{b})$. Now consider any $x \in [-1,1]$ and let $c \in \mathcal{B}$ be such that $u \circ c = x^*$. Note that $u \circ (f_\lambda c) = a + (1-\lambda)x^*$, and $u \circ (\hat{f}_\lambda \hat{c}) = -a - (1-\lambda)x^*$. By Weak C-Independence, $f_\lambda c \sim d_\lambda c$ and $\hat{f}_\lambda \hat{c} \sim d'_\lambda \hat{c}$. By (3) and (4) and the fact that $I$ is normalised, we have that

$$
\begin{aligned}
\alpha I(a) + (1-\alpha) - I(-a) &= y \\
\alpha I(a + (1-\lambda)x^*) + (1-\alpha)(-I(-a-(1-\lambda)x^*)) &= y + (1-\lambda)x \\
\alpha I(-a) + (1-\alpha) - I(a) &= y' \\
\alpha I(-a-(1-\lambda)x^*) + (1-\alpha)(-I(a+(1-\lambda)x^*)) &= y' - (1-\lambda)x
\end{aligned}
\tag{20}
$$

It follows from the first two equations that:

[18] For a set $X$, $int(X)$ is the interior of $X$.

$$I(a+(1-\lambda)x^*)-I(a)-(1-\lambda)x=\frac{1-\alpha}{\alpha}\left(I(-a-(1-\lambda)x^*)-I(a)+(1-\lambda)x\right) \tag{21}$$

and from the latter two that:

$$I(a+(1-\lambda)x^*)-I(a)-(1-\lambda)x=\frac{\alpha}{1-\alpha}\left(I(-a-(1-\lambda)x^*)-I(a)+(1-\lambda)x\right) \tag{22}$$

Whenever $\alpha \neq 0.5$, (21) and (22) can only hold simultaneously if both sides are zero. So $I(a+(1-\lambda)x^*) = I(a)+(1-\lambda)x$. Since this holds for all $\lambda \in (0,1)$, we have that $I(a+x^*) = I(a)+x$ for all $x \in [-1,1]$ and $a \in int(B(\Sigma))$ with $a+x^* \in B(\Sigma)$. By the continuity of $I$, this holds for all $a \in B(\Sigma)$, as required. □

**Proposition A.2.** *An $\alpha$-UA preference $\succeq$ satisfies C-Independence iff $I$ is constant additive and positively homogeneous.*

*Proof.* Sufficiency of the axiom for constant additivity follows from Proposition A.1 and the fact that C-Independence implies Weak C-Independence. Sufficiency of the axiom for positive homogeneity is established applying a similar argument to that in the proof of Proposition A.1 to $f_v(\mathbf{w}_{\frac{1}{2}}\mathbf{b})$ and $\hat{f}_v(\mathbf{w}_{\frac{1}{2}}\mathbf{b})$ (where $u \circ f = a$). Necessity is straightforward. □

**Proposition A.3.** *An $\alpha$-UA preference $\succeq$ satisfies Weak Monotone Continuity iff, for all $a,b \in B(\Sigma)$, $x \in [-1,1]$ and $\{E_n\}_{n\geqslant 1} \in \Sigma$ with $E_1 \supseteq E_2 \supseteq \ldots$ and $\bigcap_{n\geqslant 1} E_n = \varnothing$, $I(a) > I(b)$ implies that there exists $n_0$ with $I(x^*_{E_{n_0}}a) > I(b)$ (where $x^*_{E_{n_0}}a$ is defined in an analogous way to $f_E g$).*

*Proof.* We first assume Weak Monotone Continuity. Suppose $I(a) > I(b)$, and take $f,g \in \mathcal{A}^l$ with $u \circ f = a$ and $u \circ g = b$. By the proof of Proposition 1, $f \gtrapprox g$ and $g \not\gtrapprox f$. Let $g \approx [l,m]$ (such $[l,m]$ exists by the proof of Proposition 1); so $I(b) = u(l)$ and $f \ggcurly [l,m']$ for some $m' \in \mathcal{L}$. Since $\alpha > 0.5$, this can only be the case if $f \succ [l,m']$ and $\hat{f} \succ \widehat{[l,m']}$ for some $m' \in \mathcal{L}$. By Weak Monotone Continuity, there exists $n_1$ such that $l^x_{E_{n_1}} f \succ [l,m']$, where $l^x \in \mathcal{L}$ and $u(l^x) = x$. Similarly, by Weak Monotone Continuity, there exists $n_2$ such that $\hat{l}^x{}_{E_{n_2}}\hat{f} \succ \widehat{[l,m']}$. Hence, for $n_0 = \max\{n_1,n_2\}$, $l^x_{E_{n_0}} f \succ [l,m']$ and $\widehat{l^x_{E_{n_0}} f} = \hat{l}^x{}_{E_{n_0}}\hat{f} \succ \widehat{[l,m']}$, whence $l^x_{E_{n_0}} f \ggcurly [l,m']$. So $l^x_{E_{n_0}} f \gtrapprox g$ and $l^x_{E_{n_0}} f \not\lessapprox g$, and hence $I(x^*_{E_{n_0}}a) > I(b)$, as required. The converse implication is established using a similar argument. □

**Proposition A.4.** *If an $\alpha$-UA preference $\succeq$ satisfies Comonotonic Independence, then $I$ satisfies: for all pairwise comonotonic $a,b,c \in B(\Sigma)$ and $\lambda \in (0,1)$, if $I(a) > I(b)$, then $I(\lambda a+(1-\lambda)c) > I(\lambda b+(1-\lambda)c)$.*

*Proof.* We show sufficiency; necessity is straightforward. First, since (4) implies (4), it follows from Proposition (A.2) that $I$ is constant additive and positively homogeneous. Fix $\lambda \in (0,1)$, let $a,b,c \in B(\Sigma)$ be pairwise comonotonic, and let $f,g,h \in \mathcal{A}^l$ be such that $u \circ f = a$, $u \circ g = b$, $u \circ h = c$ (by the proof of Proposition 1, such acts exist). By construction, $u \circ \hat{f} = -a$, $u \circ \hat{g} = -b$, $u \circ \hat{h} = -c$. By Monotonicity, $\mathbf{b} \succeq f,g,h$, so by Continuity, there exist $c_f, c_g, c_h \in \mathcal{B}$ with $f \sim c_f$, $g \sim c_g$, $h \sim c_h$; let $x,y,z \in [-1,1]$ be such that $u \circ c_f = x^*$, $u \circ c_g = y^*$, $u \circ c_h = z^*$. Similarly, there exist $c'_f, c'_g, c'_h \in \mathcal{B}$ and $x',y',z' \in [-1,1]$ with $\hat{f} \sim c'_f$, $\hat{g} \sim c'_g$, $\hat{h} \sim c'_h$ and $u \circ c'_f = x'^*$, $u \circ c'_g = y'^*$, $u \circ c'_h = z'^*$. Note that $u \circ (f_\lambda h) = \lambda a + (1-\lambda)c$, $u \circ (\hat{f}_\lambda \hat{h}) = -\lambda a - (1-\lambda)c$, $u \circ (f_\lambda(\mathbf{b}_{\frac{1}{2}}\mathbf{w})) = \lambda a$, $u \circ (\hat{f}_\lambda \widehat{(\mathbf{b}_{\frac{1}{2}}\mathbf{w})}) = -\lambda a$, and similarly for $g$, $x^*$, $y^*$ in the place of $f$. By Comonotonic Independence, and the fact that $f,g,h$ and all constant acts are pairwise comonotonic, as are $\hat{f}, \hat{g}, \hat{h}$ and all constant acts, $f_\lambda(\mathbf{b}_{\frac{1}{2}}\mathbf{w}) \sim (c_f)_\lambda(\mathbf{b}_{\frac{1}{2}}\mathbf{w})$, $\hat{f}_\lambda\widehat{(\mathbf{b}_{\frac{1}{2}}\mathbf{w})} \sim (c'_f)_\lambda\widehat{(\mathbf{b}_{\frac{1}{2}}\mathbf{w})}$, $f_\lambda h \sim (c_f)_\lambda h \sim (c_f)_\lambda(c_h)$, $\hat{f}_\lambda \hat{h} \sim (c'_f)_\lambda \hat{h} \sim (c'_f)_\lambda(c'_h)$, and similarly for $g$. By (3) and (4) and the fact that $I$ is normalised, we have that

$$
\begin{aligned}
\alpha I(\lambda a) + (1-\alpha) - I(-\lambda a) &= \lambda x \\
\alpha I(\lambda a + (1-\lambda)c) + (1-\alpha)(-I(-\lambda a - (1-\lambda)c)) &= \alpha I(\lambda x^* + (1-\lambda)c) + (1-\alpha)(-I(-\lambda x^* - (1-\lambda)c)) \\
\alpha I(\lambda b) + (1-\alpha) - I(-\lambda b) &= \lambda y \\
\alpha I(\lambda b + (1-\lambda)c) + (1-\alpha)(-I(-\lambda b - (1-\lambda)c)) &= \alpha I(\lambda y^* + (1-\lambda)c) + (1-\alpha)(-I(-\lambda y^* - (1-\lambda)c)) \\
\alpha I(-\lambda a) + (1-\alpha) - I(\lambda a) &= \lambda x' \\
\alpha I(-\lambda a - (1-\lambda)c) + (1-\alpha)(-I(\lambda a + (1-\lambda)c)) &= \alpha I(\lambda x'^* - (1-\lambda)c) + (1-\alpha)(-I(-\lambda x^* + (1-\lambda)c)) \\
\alpha I(-\lambda b) + (1-\alpha) - I(\lambda b) &= \lambda y' \\
\alpha I(-\lambda b - (1-\lambda)c) + (1-\alpha)(-I(\lambda b + (1-\lambda)c)) &= \alpha I(\lambda y'^* - (1-\lambda)c) + (1-\alpha)(-I(-\lambda y^* + (1-\lambda)c))
\end{aligned}
$$

It follows from the first four equations that:

$$
\alpha \begin{pmatrix} [I(\lambda a + (1-\lambda)c) - I(\lambda b + (1-\lambda)c)] \\ -[I(\lambda a) - I(\lambda b)] \\ -[I(\lambda x^* + (1-\lambda)c) - I(\lambda y^* + (1-\lambda)c)] \\ +[\lambda x - \lambda y] \end{pmatrix} = (1-\alpha) \begin{pmatrix} [I(-\lambda a - (1-\lambda)c) - I(-\lambda b - (1-\lambda)c)] \\ -[I(-\lambda a) - I(-\lambda b)] \\ -[I(-\lambda x^* - (1-\lambda)c) - I(-\lambda y^* - (1-\lambda)c)] \\ -[\lambda x - \lambda y] \end{pmatrix}
$$

from which it follows, by the constant additivity of $I$:

$$\alpha\left(\begin{array}{l}[I(\lambda a+(1-\lambda)c)-I(\lambda b+(1-\lambda)c)]\\ -[I(\lambda a)-I(\lambda b)]\end{array}\right)=(1-\alpha)\left(\begin{array}{l}[I(-\lambda a-(1-\lambda)c)-I(-\lambda b-(1-\lambda)c)]\\ -[I(-\lambda a)-I(-\lambda b)]\end{array}\right) \tag{23}$$

From the last four equations, by a similar reasoning, it follows that:

$$\alpha\left(\begin{array}{l}[I(-\lambda a-(1-\lambda)c)-I(-\lambda b-(1-\lambda)c)]\\ -[I(-\lambda a)-I(-\lambda b)]\end{array}\right)=(1-\alpha)\left(\begin{array}{l}[I(\lambda a+(1-\lambda)c)-I(\lambda b+(1-\lambda)c)]\\ -[I(\lambda a)-I(\lambda b)]\end{array}\right) \tag{24}$$

Whenever $\alpha \neq 0.5$, (23) and (24) can only hold simultaneously if both sides are zero. So $I(\lambda a+(1-\lambda)c)-I(\lambda b+(1-\lambda)c)=I(\lambda a)-I(\lambda b)=\lambda(I(a)-I(b))$, where the last equality holds due to the positive homogeneity of $I$. The conclusion follows immediately. □

### A.3 Proofs of results in Section 5

*Proof of Proposition 2.* Clause i. follows from representation (4) by basic algebra. The necessity direction of clause ii. follows from representation (3) and Lemma A.4. As for sufficiency, first note that, for $l, m \in \mathcal{L}$ $l \gtrapprox m$ if and only if $l \succeq m$, so if $\succeq^1$ is more imprecise than $\succeq^2$, then $l \succeq^1 m$ implies $l \succeq^2 m$ for all $l, m \in \mathcal{L}$, whence, by standard arguments (e.g. Ghirardato et al., 2004, Section B.4), the normalised utility functions are the same. The rest of this direction follows from Lemma A.4 and the construction of $I$ in the proof of Proposition 1. The necessity direction of clause iii. is immediate from the representation. As for the sufficiency part, suppose that $\succeq^1$ is strongly more imprecise than $\succeq^2$. Note firstly that, for all $l, m \in \mathcal{L}$ $l \gtrapprox m$ if and only if $l \approx m$ if and only if $l \sim m$; so $l \sim^2 \mathbf{w}_\beta\mathbf{b}$ whenever $l \sim^1 \mathbf{w}_\beta\mathbf{b}$ and hence the normalised utility functions are the same. For every $a \in B(\Sigma)$, by previous arguments, $f \in \mathcal{A}^l$ defined by $f(s) = \mathbf{w}_{\frac{1-a}{2}}\mathbf{b}$ is such that $u^1 \circ f = a$ and there exists $[\mathbf{w}_\beta\mathbf{b}, \mathbf{w}_\gamma\mathbf{b}] \in \mathcal{B}$ with $f \approx^1 [\mathbf{w}_\beta\mathbf{b}, \mathbf{w}_\gamma\mathbf{b}]$. By the representation (see also proof of Proposition 1), $I^1(a) = \mathbb{E}_{(\mathbf{w})_\beta\mathbf{b}}\nu$ and $-I^1(-a) = \mathbb{E}_{(\mathbf{w})_\gamma\mathbf{b}}\nu$. It follows from strong imprecision aversion that $f \gtrapprox^2 [\mathbf{w}_\beta\mathbf{b}, \mathbf{w}_\gamma\mathbf{b}]$; so, by the representation $\alpha^2 I^2(a) + (1-\alpha^2) - I^2(-a) \geqslant \alpha^2\mathbb{E}_{\mathbf{w}_\beta\mathbf{b}}\nu + (1-\alpha^2)\mathbb{E}_{\mathbf{w}_\gamma\mathbf{b}}\nu$, which implies the desired inequality. □

*Proof of Proposition 3.* Since, under representation (3), $\succeq^1$ is more ambiguity averse than $\succeq^2$ iff $\nu_1 = \nu_2$ and $\alpha^1 I^1(a) + (1-\alpha^1) - I^1(-a) \leqslant \alpha^2 I^2(a) + (1-\alpha^2) - I^2(-a)$ for all $a \in B(\Sigma)$, the clauses follow immediately from Proposition 2. □

*Proof of Proposition 4.* Immediate from representation (4). □

*Proof of Proposition 5.* By Proposition 2, an $\alpha$-UA preference is imprecise if and only if the representing functional $I$ in (3) is dominated by the expectation of a probability distribution over $S$ (i.e. there exists $p \in \Delta$ with $\mathbb{E}_p a \geqslant I(a)$ for all $a \in B(\Sigma)$). The result follows from known results (Ghirardato and Marinacci, 2002; Maccheroni et al., 2006; Klibanoff et al., 2005). □

## A.4 Proofs of results in Section 6

*Proof of Proposition 6.* By standard arguments (Ghirardato et al., 2004, proof of Prop 4), $\succeq^\circ$ is reflexive, transitive, non-degenerate and satisfies Independence and C-Completeness (i.e. it is complete on $\mathcal{L}$). Since, whenever $f(s) \succeq g(s)$ for all $s \in S$, $\lambda f(s) + (1-\lambda)h(s) \succeq \lambda g(s) + (1-\lambda)h(s)$ for all $\lambda \in (0,1)$ and $h \in \mathcal{A}^l$, it follows from Lemmas A.3 and A.4 that $f \succeq^\circ g$, so $\succeq^\circ$ also satisfies Monotonicity. Moreover, by the same Lemmas, for all $f, f', g \in \mathcal{A}^l$, $\left\{\beta \in [0,1] : (f_\beta f')_\lambda h \underset{\approx}{\gtrsim} g_\lambda h \;\forall h \in \mathcal{A}^l,\ \lambda \in (0,1)\right\} = \left\{\beta \in [0,1] : I((f_\beta f')_\lambda h) \geqslant I(g_\lambda h) \;\forall h \in \mathcal{A}^l,\ \lambda \in (0,1)\right\}$, which, by the continuity of $I$, is closed; and similarly for $\left\{\beta \in [0,1] : (f_\beta f')_\lambda h \underset{\approx}{\lesssim} g_\lambda h \;\forall h \in \mathcal{A}^l,\ \lambda \in (0,1)\right\}$. So $\succeq^\circ$ is mixture continuous (it satisfies Continuity). The representation follows from known results (e.g. Ghirardato et al., 2004; Gilboa et al., 2010). □

*Proof of Proposition 7.* Let $\succeq$ be represented according to (3) by $(v, I, \alpha)$, and consider the preference $\succeq^1$ represented by $(v, I, 1)$: this is an uncertainty averse preference, in the sense of Cerreia-Vioglio et al. (2011b) (see Section 3.4). By Lemma A.4, for every $f, g \in \mathcal{A}^l$, $f \succeq^\circ g$ if and only if, for every $\lambda \in (0,1)$ and $h \in \mathcal{A}^l$, whenever $f_\lambda h \approx [l, m]$ and $g_\lambda h \approx [l', m']$, then $l \geq l'$. By representations (3) and (4) (see also the construction of $I$ in the proof of Proposition 1), this holds if and only if $I(f_\lambda h) \geq I(g_\lambda h)$ for every $\lambda \in (0,1)$ and $h \in \mathcal{A}^l$, where $I$ is as in representation (3). Since $\succeq^1$ is represented by this $I$ and $\alpha = 1$, this is the case if and only if $f \succeq^{1*} g$. So $\succeq^{1*} = \succeq^\circ$. The proposition follows from known results for the unambiguous preference of uncertainty averse versions of the various models (Cerreia-Vioglio et al., 2011b; Ghirardato et al., 2004; Maccheroni et al., 2006). □

**Proposition A.5.** $\succeq^\circ \subseteq \succeq^*$.

*Proof of Proposition A.5.* By Proposition 6 and the fact that $\mathbb{E}_p u(f) \geqslant \mathbb{E}_p u(g)$ if and only if $\mathbb{E}_p u(\hat{g}) \geqslant \mathbb{E}_p u(\hat{f})$, $f \succeq^\circ g$ if and only if $\hat{g} \succeq^\circ \hat{f}$. It follows, by the reasoning in the proof of Proposition 7, that, if $f \succeq^\circ g$, then $\alpha I(f_\lambda h) \geqslant \alpha I(g_\lambda h)$ for all $\lambda \in (0,1)$ and $h \in \mathcal{A}^l$, and similarly for $-I(\widehat{f_\lambda h})$, so $\alpha I(f_\lambda h) + (1-$

$\alpha)(-I(\widehat{f_\lambda h})) \geqslant \alpha I(g_\lambda h) + (1-\alpha)(-I(\widehat{g_\lambda h}))$ for all $\lambda \in (0,1)$ and $h \in \mathcal{A}^l$, whence, by representation (3), $f \succeq^* g$, as required. □

## A.5 Proofs of results in Section 7

*Proof of Proposition 8.* Consider investors $(v, I, \alpha)$ and $(v, I, \alpha')$ with preferences as specified, and suppose that $\alpha' > \alpha$. It suffices to show that, for every optimal portfolio allocation for $a^* \in [0, w]$ for investor $(v, I, \alpha)$, there exists an optimal portfolio allocation $a^{*\prime}$ for investor $(v, I, \alpha')$ with $a^{*\prime} \leqslant a^*$, and for every optimal portfolio allocation $a^{*\prime} \in [0, w]$ for investor $(v, I, \alpha')$, there exists an optimal portfolio allocation $a^*$ for investor $(v, I, \alpha)$ with $a^{*\prime} \leqslant a^*$.

To formally connect with the framework used previously, take the state space $S = R$ with the Borel $\sigma$-algebra, with each state $x$ yielding return $x$ for the uncertain asset. Since $v$ is concave, the function $\bar{u} : [0, w] \to B(\Sigma)$ defined by $\bar{u}(a)_x = v(w.r + a(x - r))$ is concave in each coordinate. So the optimisation problem for $(v, I, \alpha)$ can be written as

$$\max_{a \in [0,w]} \alpha(I \circ \bar{u})(a) + (1-\alpha)(-I \circ (-\bar{u}))(a) \tag{25}$$

and similarly for $(v, I, \alpha')$.

Since $I$ is differentiably concavifiable, there exists a differentiable $\psi$ with everywhere positive derivative and a concave $\bar{I}$ with $I = \psi^{-1} \circ \bar{I}$. Since $\bar{I}$ is monotonic and concave, $\bar{I} \circ \bar{u}$ is concave: for every $x, y \in [0, w]$ and $\lambda \in [0, 1]$

$$\begin{aligned} \bar{I}(\bar{u}(\lambda x + (1-\lambda)y)) &\geqslant \bar{I}(\lambda \bar{u}(x) + (1-\lambda)\bar{u}(y)) \\ &\geqslant \lambda \bar{I}(\bar{u}(x)) + (1-\lambda)\bar{I}(\bar{u}(y)) \end{aligned}$$

where the first inequality holds by the concavity of $\bar{u}$ in each coordinate and the monotonicity of $\bar{I}$, and the second by the concavity of $\bar{I}$. Similarly, $-I = \phi \circ -\bar{I}$, where $\phi(x) = -\psi^{-1}(-x)$ for all $x \in [-1, 1]$, with $\phi$ differentiable with everywhere positive derivative and $-\bar{I}$ convex; by similar reasoning, $-\bar{I} \circ -\bar{u}$ is convex.

Since $\bar{I}$ and $\bar{u}$ are concave, they are both locally Lipschitz; since $\psi$ is differentiable with everywhere positive derivative, $\psi^{-1}$ is differentiable and hence locally Lipschitz. So $I \circ \bar{u}$ is locally Lipschitz, and likewise for $-I \circ (-\bar{u})$. So the Clarke-Rockafellar differentials $\partial(I \circ \bar{u})$ and $\partial(-I \circ (-\bar{u}))$ exist at all points in $(0, w)$, and are well-defined (Clarke, 1990, Ch 2).

Since $I \circ \bar{u} = \psi^{-1} \circ \bar{I} \circ \bar{u}$ with $\psi^{-1}$ differentiable, it follows from (Clarke, 1990, Thm 2.3.9) that

$$\partial(I \circ \bar{u})(x) = (\psi^{-1})'(\bar{I} \circ \bar{u})(x).\partial(\bar{I} \circ \bar{u})(x) \tag{26}$$

Since $\bar{I} \circ \bar{u}$ is concave, $\partial(-(\bar{I} \circ \bar{u}))$ is monotone (in the sense of Rockafellar 1970, Ch 24): for every $a, b \in [0, w]$, $a^* \in \partial(\bar{I} \circ \bar{u})(a)$, $b^* \in \partial(\bar{I} \circ \bar{u})(b)$, we have that $(b^* - a^*).(b - a) \leqslant 0$.

Define $\hat{\bar{u}} : [-w, w] \to \mathbb{R}^S$, by:

$$\hat{\bar{u}}(x) = \begin{cases} \bar{u}(x) & x \geqslant 0 \\ -\bar{u}(-x) & x < 0 \end{cases} \tag{27}$$

Note that $(\bar{I} \circ \bar{u})(x) = (\bar{I} \circ \hat{\bar{u}})(x)$ and $(-\bar{I} \circ (-\bar{u}))(x) = (-\bar{I} \circ \hat{\bar{u}})(-x)$ for all $x \in [0, w]$.

Since $v$ and hence $\bar{u}$ is continuously differentiable, it follows straightforwardly from the definition that $\hat{\bar{u}}$ is. From this and the fact that $\bar{I}$ is continuous and concave, and hence regular in the sense of Clarke (1990, Defn 2.3.4), it follows from (Clarke, 2013, Thm10.19) (see also Clason 2017, Thm 8.14) that $\partial(\bar{I} \circ \hat{\bar{u}})(x) =< \partial\bar{I}(\hat{\bar{u}}(x)), \hat{\bar{u}}'(x) >$ for all $x \in (-w, w)$. It follows in particular that, for every $x \in (0, w)$, $\partial(\bar{I} \circ \bar{u})(x) =< \partial\bar{I}(\bar{u}(x)), \bar{u}'(x) >$ and $\partial(-\bar{I} \circ (-\bar{u}))(x) = - < \partial\bar{I}(-\bar{u}(x)), -\bar{u}'(x) >$, and that $\partial(\bar{I} \circ \hat{\bar{u}})(0) =< \partial\bar{I}(\hat{\bar{u}}(0)), \hat{\bar{u}}'(0) >=< \partial I(\mathbf{0}), \bar{u}'(0) >$, where $\bar{u}(0) = \mathbf{0}$, the zero element in $B(\Sigma)$. By the continuity of $\bar{u}'$ and of the superdifferential $\partial\bar{I}$ for continuous concave $\bar{I}$ (Rockafellar, 1970, Thm 24.4), $\lim_{x \to 0} < \partial\bar{I}(\bar{u}(x)), \bar{u}'(x) > \subseteq < \partial\bar{I}(\bar{u}(0)), \bar{u}'(0) >$, and similarly for $- < \partial\bar{I}(-\bar{u}(x)), -\bar{u}'(x) >$.

We now consider cases, according to the properties of $< \partial I(\mathbf{0}), \bar{u}'(0) >$:

*Case* 1. $< \partial I(\mathbf{0}), \bar{u}'(0) > \cap (-\infty, 0] = \varnothing$. It follows by the aforementioned convergence of superdifferentials that as $x \to 0$ that $\partial(\bar{I} \circ \bar{u})(x) \cap (-\infty, 0] = \varnothing$ for all $x \in (0, \epsilon)$ for some $\epsilon > 0$. By the concavity of $\bar{I} \circ \bar{u}$ and the corresponding monotonicity property of $\partial(\bar{I} \circ \bar{u})$ cited above, $x^* > 0$ for $x^* \in \partial(\bar{I} \circ \bar{u})(x)$ for $x > 0$ sufficiently small, but there may exist $x \in [0, w]$ with $x^* \leqslant 0$ for some $x^* \in \partial(\bar{I} \circ \bar{u})(x)$. If there exists such $x$, let $y$ be the infimum such element; by concavity, $x^* > 0$ for all $x^* \in \partial(\bar{I} \circ \bar{u})(x)$ and every $x < y$. Since $\psi^{-1}$ has positive derivative everywhere (because $\psi$ does), it follows from (26) that $x^* > 0$ for all $x^* \in \partial(I \circ \bar{u})(x)$ and every $x < y$. On the other hand, by a similar convergence argument and the convexity of $-\bar{I} \circ -\bar{u}$, $x^* > 0$ for every $x^* \in \partial(-\bar{I} \circ (-\bar{u}))(x)$ and every $x \in (0, w)$. Again, since $\phi$ has positive derivative, $-I \circ -\bar{u}$ is strictly increasing throughout $[0, w]$.

*Case* 2. $\partial(\bar{I} \circ \bar{u})(0) \cap [0, \infty) = \varnothing$. By similar reasoning to case 1, $I \circ \bar{u}$ is strictly decreasing throughout $[0, w]$, and $-I \circ -\bar{u}$ is initially strictly decreasing, with perhaps a minimum after which it is increasing (or constant).

*Case* 3. $0 \in \partial(\bar{I} \circ \bar{u})(0)$. By similar reasoning to case 1, $I \circ \bar{u}$ is decreasing (or constant) and $-I \circ -\bar{u}$ is increasing (or constant) throughout $[0, w]$.

In all cases, there exists $y \in [0, w]$ such that: i. $I \circ \bar{u}$ and $-I \circ -\bar{u}$ are both either strictly increasing or strictly decreasing on $[0, y)$ and ii. $I \circ \bar{u}$ is decreasing and $-I \circ -\bar{u}$ is increasing on $(y, w]$.

Now consider any (globally) optimal allocation $a^*$ for the optimisation problem (25) for investor $(\nu, I, \alpha)$. If $a^* = w$, then any optimal allocation for the problem for $(\nu, I, \alpha')$ is weakly lower, as required. Now suppose that $a^* < w$. If $a^* < y$, then it is in the region where $I \circ \bar{u}$ and $-I \circ -\bar{u}$ are both strictly increasing or strictly decreasing. If they are both strictly increasing (case 1), then for any $a^* < a < y$, $\alpha(I \circ \bar{u})(a) + (1 - \alpha)(-I \circ -\bar{u})(a) > \alpha(I \circ \bar{u})(a^*) + (1 - \alpha)(-I \circ -\bar{u})(a^*)$, contradicting the optimality of $a^*$; so $a^* \geqslant y$ in this case. By similar reasoning, if $I \circ \bar{u}$ and $-I \circ -\bar{u}$ are both strictly decreasing at 0 (case 2), then $a^* = 0$ or $a^* \geqslant y$.

We first consider the case where $a^* \geqslant y$, so $I \circ \bar{u}$ is decreasing and $-I \circ -\bar{u}$ is increasing above $a^*$. Hence, for every $a > a^*$, $[(I \circ \bar{u})(a^*) - (I \circ \bar{u})(a)] \geqslant 0 \geqslant [(-I \circ (-\bar{u}))(a^*) - (-I \circ (-\bar{u}))(a)]$. Moreover, since $a^*$ is an optimum for $(\nu, I, \alpha)$, $\alpha\left[(I \circ \bar{u})(a^*) - (I \circ \bar{u})(a)\right] + (1-\alpha)\left[(-I \circ (-\bar{u}))(a^*) - (-I \circ (-\bar{u}))(a)\right] \geqslant 0$ for all $a > a^*$. Since $\alpha' > \alpha$, it follows that, for every $a > a^*$, $\alpha'\left[(I \circ \bar{u})(a^*) - (I \circ \bar{u})(a)\right] + (1 - \alpha')\left[(-I \circ (-\bar{u}))(a^*) - (-I \circ (-\bar{u}))(a)\right] \geqslant 0$. For every $a > a^*$, if either of the inequalities $[(I \circ \bar{u})(a^*) - (I \circ \bar{u})(a)] \geqslant 0 \geqslant [(-I \circ (-\bar{u}))(a^*) - (-I \circ (-\bar{u}))(a)]$ are strict (as will be the case in cases 1 and 2), then $\alpha'\left[(I \circ \bar{u})(a^*) - (I \circ \bar{u})(a)\right] + (1-\alpha')\left[(-I \circ (-\bar{u}))(a^*) - (-I \circ (-\bar{u}))(a)\right] > 0$, so $a$ is not an optimum for $(\nu, I, \alpha')$. So $\alpha'\left[(I \circ \bar{u})(a^*) - (I \circ \bar{u})(a)\right] + (1 - \alpha')\left[(-I \circ (-\bar{u}))(a^*) - (-I \circ (-\bar{u}))(a)\right] = 0$ for some $a > a^*$ only if $(I \circ \bar{u})(a^*) = (I \circ \bar{u})(a)$ and $(-I \circ (-\bar{u}))(a^*) = (-I \circ (-\bar{u}))(a)$, whence $a$ is (also) a global optimum for $(\nu, I, \alpha)$. (Since they are decreasing and increasing respectively, this only occurs is the functions are both constant at $a^*$, and hence only in case 3.) Hence either all global optima for $(\nu, I, \alpha')$ are less than $a^*$, or $a > a^*$ and $a^*$ are both global optima for both $(\nu, I, \alpha')$ and $(\nu, I, \alpha)$; in both cases, the optima for $(\nu, I, \alpha')$ are weakly lower than some optima for $(\nu, I, \alpha)$. In all cases, $a'^* \leqslant a^*$ for some global optimum $a'^*$ for $(\nu, I, \alpha')$, and for every global optimum $a'^*$ for $(\nu, I, \alpha')$, either $a'^* \leqslant a^*$ or $a'^*$ is also a global optimum for $(\nu, I, \alpha)$, as required.

The remaining case to be considered is where $a^* = 0$ and $I \circ \bar{u}$ and $-I \circ -\bar{u}$ are both strictly decreasing from 0 to a point $y \in (0, w]$ (case 2). If they are strictly decreasing up to $w$, then clearly 0 is the optimum for $(v, I, \alpha')$ and the desired result holds. If not, then there exists $y \in (0, w)$ above which $I \circ \bar{u}$ is decreasing and $-I \circ -\bar{u}$ is increasing. If $(-I \circ -\bar{u})(a) \leqslant 0$ for all $a \in [y, w]$, then $\alpha'(I \circ \bar{u})(a) + (1 - \alpha')(-I \circ -\bar{u})(a) < 0 = \alpha'(I \circ \bar{u})(0) + (1 - \alpha')(-I \circ -\bar{u})(0)$ and 0 is the optimum for $(v, I, \alpha')$. Suppose not, and let $y' = \inf\{a \in [y, w] : (-I \circ -\bar{u})(a) > 0\}$. Take $\bar{a} \in \arg\max_{a \in [y', w]} \alpha(I \circ \bar{u})(a) + (1 - \alpha)(-I \circ -\bar{u})(a)$ and $\bar{a}' \in \arg\max_{a \in [y', w]} \alpha'(I \circ \bar{u})(a) + (1 - \alpha')(-I \circ -\bar{u})(a)$. Since 0 is a global optimum for $(v, I, \alpha)$, $0 = \alpha(I\circ\bar{u})(0) + (1-\alpha)(-I\circ-\bar{u})(0) \geqslant \alpha(I\circ\bar{u})(\bar{a}) + (1-\alpha)(-I\circ-\bar{u})(\bar{a}) \geqslant \alpha(I\circ\bar{u})(\bar{a}') + (1-\alpha)(-I\circ-\bar{u})(\bar{a}')$. Since $\alpha' > \alpha$, $(I \circ \bar{u})(\bar{a}') < (I \circ \bar{u})(0) = 0$ and $(-I \circ -\bar{u})(\bar{a}') \geqslant 0$, $\alpha(I \circ \bar{u})(\bar{a}') + (1 - \alpha)(-I \circ -\bar{u})(\bar{a}') > \alpha'(I \circ \bar{u})(\bar{a}') + (1 - \alpha')(-I \circ -\bar{u})(\bar{a}')$, whence $\bar{a}'$ is not a global optimum for $(v, I, \alpha')$, so 0 is the only global optimum. Since $0 \leqslant a^*$ for every global optimum for $(v, I, \alpha)$, this yields the required result.

□

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