Prices vs Quantities under Severe Uncertainty*

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Abstract

The consensus among economists in favour of carbon taxes over emissions permits is based on a groundbreaking result due to Weitzman (1974). It assumes, however, a probability distribution over abatement costs and damages. As many have argued, current climate uncertainties are far more severe, and do not justify any such distribution. This paper reconsiders the tax-permit comparison in the presence of severe or Knightian uncertainty, drawing on the workhorse maxmin-EU model from the literature on decision under ambiguity (Gilboa and Schmeidler, 1989). Our results show that optimally set permits are strictly more efficient than optimal taxes when uncertainty concerning the slope of marginal abatement costs is severe. They suggest that, given the uncertainty reported in the latest IPCC report, permit policies should be preferred.

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1 Introduction

In 2019, 28 leading economists published a petition calling for a carbon tax to tackle climate change ('Economists' Statement on Carbon Dividends', 2019). They reflect a wide consensus in the profession. The economic arguments in favour of market-based instruments – be they by fixing a price for carbon via taxes, or a quantity via permits – are well-known. However, economists typically go beyond, and incorporate a preference for taxes over permits in the case of climate change. The reason draws on a pioneering theorem due to Weitzman (1974), which suggests that, in the absence of complete information, the comparison between the two tools hinges on the comparison of the slopes of the marginal abatement cost and damage curves. Since climate change is a stock problem, the former is typically steeper than the latter, whence the superiority of taxes, and the economic consensus (Nordhaus, 2007; Metcalf, 2009).

However, Weitzman's result assumes that uncertainty about abatement costs and damages can be represented by a probability distribution over possible abatement cost and damage functions. In other words, his analysis applies in what decision theorists call situations of *risk*. By contrast, many have noted that climate uncertainties are considerably more severe, especially as concerns the abatement cost and damage functions (Pindyck, 2013; Stern, 2013; Stiglitz, 2019; Stern et al., 2022; Blanchard et al., 2023). Probability distributions over possible (marginal) damage or abatement cost functions are typically not provided by the current state of the science. The situation is thus closer to what is often referred to as *(Knightian) uncertainty*, *severe uncertainty* or *ambiguity* (Gilboa and Marinacci, 2013). This article revisits the comparison between prices and quantities under severe uncertainty, by analysing the problem under one of the main ambiguity models in the decision theory literature, Gilboa and Schmeidler's (1989) maxmin-Expected Utility (maxmin-EU). As discussed in Section 2.3, this rule reflects a concern for robustness of policy evaluation and choice in the face of uncertainty; for instance, it nests one of the robustness rules promoted by Hansen and Sargent (2008).

In the climate context, this model incorporates a set of probability distributions over the possible abatement cost and damage functions, and evaluates a policy by the worst-case total cost (abatement costs plus damages) across all distributions in the set. It is thus rich enough to cover the spectrum of types of uncertainty between two extremes. On one side, when a single probability distribution has been established, the set used in the rule contains only this distribution, and maxmin-EU rule reduces to standard expected utility. The analysis thus collapses to Weitzman's comparison of prices and quantities under risk. At the other extreme, the set could contain every probability distribution with support contained in a given set of abatement cost or damage functions. This reflects situations where it is known that the abatement cost function, say, belongs to a certain family or satisfies certain constraints, but nothing more is known, for instance about the probabilities of various family members. In such cases, the maxmin-EU rule evaluates a policy by the worst case across all feasible abatement cost and damage functions. Then, we show, the comparison of climate policies is diametrically opposed to the message typically drawn from Weitzman's Theorem: the optimal quantity policy always performs at least as well as the optimal pricing one. Moreover, by identifying a condition for a strict ranking, we establish that whenever current knowledge does not provide specific information about the slope of the marginal abatement costs, the optimal permit policy strictly outperforms optimally set taxes.

Our general result applies between these extremes: that is, to cases where the state of the science may yield some probabilistic conclusions about (marginal) abatement costs and damages, without necessarily pinning down a single probability distribution over abatement cost and damage functions. Roughly, the result introduces indices of risk and (severe) uncertainty, and partially characterises the relative performance of quantity vs. pricing policies in terms of them. When the former dominates, we are close to the Weitzman risk-only case, and taxes are more likely to be preferable. When the extent of (severe) uncertainty outweighs risk, then the term behind our previous worst-case result dominates, and permits outperform taxes. We obtain results not only for uncertainty pertaining directly to abatement costs, damages, and their (higher-order) derivatives, but also when the analyst adopts a parametric family for the abatement cost or damage functions, and the uncertainty concerns parameter values.

Our results highlight the importance of carefully considering the extent and character of uncertainty when formulating and recommending economic policies for effective emissions mitigation. The justification of taxes that relies on the assumption that the current state of the science provides a probability distribution over abatement cost curves will lead to suboptimal policies whenever this assumption is incorrect. Moreover, subsequent sensitivity analysis in the application of these policies – for instance, when setting the carbon price – cannot fully correct this suboptimality. By applying an evaluation approach that incorporates the actual state of uncertainty from the outset, according to the principles established by decades of research in decision theory, our results provide the bedrock for an arguably more robust analysis of policies for carbon mitigation. They suggest that for uncertainties similar to those that we actually face, quantity policies are more efficient than pricing ones.

The paper is organised as follows. Section 2 introduces the setup and some preliminary notions and notation, including the representation and types of uncertainty, and policy evaluation with them. Section 3 sets out the optimisation problems for quantity and pricing problems under (severe) uncertainty. Section 4 contains our theoretical results and discussion. Proofs and other material are contained in the Appendix.

2 Setup

2.1 Emissions, abatement costs and damages

Let the set of potential carbon emissions reductions be $[0, E^{max}] \subseteq \mathbb{R}$, where E^{max} is the maximal emissions reduction (leading to zero emissions). Where necessary, it is taken to be equipped with the Lebesgue measure. $\Delta([0, E^{max}])$ is the space of probability distributions over $[0, E^{max}]$. For every $\epsilon \in \Delta([0, E^{max}])$, if ϵ is a Dirac distribution with full weight on E, we denote it, with slight abuse of notation, by E. For a fixed $\delta \in \mathbb{R}$, $\epsilon^{+\delta}$ is the distribution defined by $\epsilon^{+\delta}(E + \delta) = \epsilon(E)$, whenever such a distribution exists. It is the result of shifting ϵ to the right by δ.

We follow Weitzman (1974) in considering a simple abstract problem, with (abatement) costs and damages.[1] Let Θ denote the space of possible *abatement cost functions*. For each $\theta \in \Theta$, $C(\bullet, \theta)$ is a positive-real-valued function defined on $[0, E^{max}]$: $C(E, \theta)$ is the abatement cost of emission reduction E under the function θ. Θ contains all increasing, differentiable, strictly convex functions on $[0, E^{max}]$ that take the value 0 at zero emissions reductions. These are standard assumptions for abatement cost functions in the literature.[2]

We use Ξ to denote the space of *damage functions*: $D(E, \xi)$ are the damages brought about by emissions reduction E under function $\xi \in \Xi$. Analogous with the case of abatement costs, Ξ contains all decreasing, differentiable, strictly convex functions on $[0, E^{max}]$ that take the value 0 at E^{max}.

The *total costs* of emissions reductions E under abatement cost and damage functions θ and ξ are given by $T(E, \theta, \xi) = C(E, \theta) + D(E, \xi)$.

2.2 Uncertainty

Let $\Delta(\Theta)$ be the set of probability distributions over Θ, and similarly for $\Delta(\Xi)$. Scientific knowledge about abatement costs can be summarised by a (non-empty) set of probability distributions $\mathcal{C}$ over the space of possible cost functions Θ; i.e. a set $\mathcal{C} \subseteq \Delta(\Xi)$. Similarly, the set $\mathcal{D} \subseteq \Delta(\Xi)$ represents knowledge about damages. Without loss of generality for the evaluation of policies, and following standard practice (Gilboa and Marinacci, 2013), we assume that $\mathcal{C}$ and $\mathcal{D}$ are closed and convex.[3]

As standard, for a distribution $p \in \Delta(\Theta)$, $\operatorname{supp} p$ is the support of p: $\operatorname{supp} p = \{\theta \in \Theta : p(\theta) > 0\}$. The support of a set of priors $\mathcal{C}$ is defined similarly: $\operatorname{supp} \mathcal{C} = \bigcup_{p \in \mathcal{C}} \operatorname{supp} p$. For $\theta \in \Theta$, δ_θ denotes δ_θ denotes the Dirac distribution with weight on θ: i.e. such that $\delta_\theta(\theta) = 1$.

We illustrate some possible types of uncertainty about abatement costs, and corresponding forms for $\mathcal{C}$. Similar examples and terminology can be developed for damages.

Example 1 (Risk)**.** There is *risk* when scientific knowledge identifies a single probability distribution representing uncertainty about abatement costs. In this case, $\mathcal{C}$ is a singleton containing this distribution. This is the standard assumption in cost-benefit analysis, and the case considered by Weitzman (1974).

Example 2 (Generated by constraints on costs)**.** Scientific knowledge could establish a set of (categorical) conclusions such as 'the abatement cost at E is less than x', 'the marginal abatement cost at E is greater than y' and so on. These can be summarised by: for every E, the abatement costs for emissions reduction level E lie in a range $[m(E), M(E)]$, marginal abatement costs lie in the range $[m^1(E), M^1(E)]$ and the slope of marginal abatement costs lie in the

[1] He talks of costs and benefits; we use climate terminology.

[2] Strict convexity is convenient insofar as it implies that inverses of first-order derivatives are unique; it can be weakened at the price of more complexity, with no particular impact on the results.

[3] Whilst not needed for our results, it is standard to work with the weak* topology on $\Delta(\Theta)$ and $\Delta(\Xi)$.

range $[m^2(E), M^2(E)]$, for increasing functions $m, M, m^1, M^1, m^2, M^2 : [0, E^{max}] \to \mathbb{R}_{\geq 0}$. In such cases, we say that uncertainty comes in the form of (categorical) *constraints* on abatement costs, marginal costs and the slope of marginal costs. This can be represented by $\mathcal{C}_{m,M,m^1,M^1,m^2,M^2}$, the set of priors generated by $\{m, M, m^1, M^1, m^2, M^2\}$, defined as follows:

$$\mathcal{C}_{m,M,m^1,M^1,m^2,M^2} = \left\{ p \in \Delta(\Theta) : \forall \theta \in \operatorname{supp} p, \begin{array}{l} C(E,\theta) \in [m(E), M(E)] \\ C_1(E,\theta) \in [m^1(E), M^1(E)] \\ C_{11}(E,\theta) \in [m^2(E), M^2(E)] \end{array} \right\}$$

Note that for any $C(\bullet, \theta)$ satisfying all the constraints, the Dirac probability measure $\delta_\theta \in \mathcal{C}_{m,M,m^1,M^1,m^2,M^2}$. Sets of priors incorporating constraints on higher-order derivatives can be defined similarly: for instance, $\mathcal{C}_{[m,M],[m^1,M^1],[m^2,M^2],[m^3,M^3]}$ is the set defined as above with the added condition that the third derivative of the abatement cost at E lies in the interval $[m^3(E), M^3(E)]$.

As we shall see, much scientific knowledge about abatement costs arguably comes in this form.

Example 3 (Generated by constraints on parameters)**.** Scientific knowledge could establish that abatement cost functions belong to a certain parametric family (or families) of functions $f(E, \mathbf{a})$ for some vector of parameters $\mathbf{a}$. Moreover, it could establish that $\mathbf{a}$ lies in a set A. This can be represented by $\mathcal{C}_A$, the set of priors generated by this family and these vectors, defined as:

$$\mathcal{C}_{f,A} = \{p \in \Delta(\Theta) : \forall \theta \in \operatorname{supp} p,\ C(E,\theta) = f(E, \mathbf{a}) \text{ for some } \mathbf{a} \in A\}$$

Example 4 (Generated by probabilistic constraints on costs)**.** Scientific knowledge could establish probabilistic conclusions such as 'the probability that the abatement cost at E is greater than x is greater than ρ', 'the probability that the marginal abatement cost at E is less than y is greater than ρ'' and so on. The Intergovernmental Panel on Climate Change (IPCC) use an uncertainty language involving probability intervals, and emit conclusions that effectively amount to statements of this sort (Mastrandrea et al., 2010; IPCC, 2023).[4] Such conclusions can be summarised by: for every probability value $\rho \in (0, 1]$ and every E, the probability is greater than ρ that abatement costs for emissions reduction level E lie in a range $[m^{(\rho)}(E), M^{(\rho)}(E)]$, marginal abatement costs lie in the range $[m^{(\rho)1}(E), M^{(\rho)1}(E)]$ and the slope of marginal abatement costs lie in the range $[m^{(\rho)2}(E), M^{(\rho)2}(E)]$, where $m^{(\rho)}, M^{(\rho)}, m^{(\rho)1}, M^{(\rho)1}, m^{(\rho)2}, M^{(\rho)2} : [0, E^{max}] \to \mathbb{R}_{\geq 0}$, for $\rho \in (0, 1]$, is a nested family of increasing functions.[5] We say that uncertainty comes in the form of *probabilistic constraints* on the abatement cost, marginal cost and the slope of the marginal cost. This can be represented by $\mathcal{C}_{\{m^{(\rho)},M^{(\rho)},m^{(\rho)1},M^{(\rho)1},m^{(\rho)2},M^{(\rho)2}\}}$,

[4]For instance, they state (2023, §3.1.1) that the probability that equilibrium climate sensitivity is between 2.5°C and 4°C is greater than 66%.

[5]A nested family is one such that $[m^{(\rho)}, M^{(\rho)}] \subseteq [m^{(\rho')}, M^{(\rho')}]$ whenever $\rho \leq \rho'$ and similarly for the derivatives.

the set of priors generated by the family $\{m^{(\rho)}, M^{(\rho)}, m^{(\rho)1}, M^{(\rho)1}, m^{(\rho)2}, M^{(\rho)2}\}$, defined as:

$$\left\{ p \in \Delta(\Theta) : \forall \rho \in (0,1],\ p\left(\left\{\theta \in \Theta : \begin{array}{l} C(E,\theta) \in [m^{(\rho)}(E), M^{(\rho)}(E)] \\ C_1(E,\theta) \in [m^{(\rho)1}(E), M^{(\rho)1}(E)] \\ C_{11}(E,\theta) \in [m^{(\rho)2}(E), M^{(\rho)2}(E)] \end{array} \right\}\right) \geq \rho \right\}$$

Uncertainty involving higher derivatives or generated by probabilistic constraints on parameters can be defined similarly.

Scientific knowledge may establish combinations of the types of uncertainty just set out, which can be represented by taking intersections of the corresponding sets of priors.

2.3 Policy evaluation

To choose among policies, we employ the maxmin-EU decision rule (Gilboa and Schmeidler, 1989), which evaluates an uncertain option by the worst-case expected utility of the outcomes of that option. In the current context it selects the policy P, leading to a total cost $T(P,\theta,\xi)$ under abatement cost function θ and damage function ξ, that minimises:

$$\max_{p\in\mathcal{C}} \max_{q\in\mathcal{D}} \mathbb{E}_p \mathbb{E}_q T(P,\theta,\xi) \tag{1}$$

where $\mathbb{E}_p$ denotes the expectation taken with respect to p (and similarly for q). Eq. (1) picks out, for each policy, its 'worst-case' – i.e. maximum – total cost over all probability distributions in $\mathcal{C}$ and $\mathcal{D}$. As such, it provides an evaluation that is *robust* to the uncertainty concerning abatement costs and damages, as reflected in $\mathcal{C}$ and $\mathcal{D}$. Indeed, the constraint preferences in the robustness framework developed by Hansen and Sargent (2008) are a special case of maxmin-EU preferences. Many other decision rules in the ambiguity literature (e.g. Maccheroni et al., 2006; Klibanoff et al., 2005) are generalisations, to which our central conclusions extend. Several researchers in economics, philosophy and climate science have argued that such rules are normatively more appropriate than Expected Utility for policy decisions in the face of severe uncertainty (Stainforth et al., 2007; Manski, 2013; Gilboa and Marinacci, 2013; Hill, 2019; Bradley and Steele, 2015; Bradley, 2017; Bradley et al., 2017; Berger and Marinacci, 2020).

Note moreover that maxmin-EU policy evaluation includes both Expected Utility and worst-case scenario analysis as special cases. If uncertainty comes in form of risk (Example 1), then $\mathcal{C}$ and $\mathcal{D}$ are singletons. Eq. (1) then evaluates policies by their expected total cost, as under standard Expected Utility-based cost-benefit analysis. At the other end of the spectrum, if uncertainty is generated by (categorical) constraints (Examples 2 and 3), then Eq. (1) reduces to the maximum total cost across all abatement cost and damage functions in $\operatorname{supp}\mathcal{C}$ and $\operatorname{supp}\mathcal{D}$ respectively.[6] This is the sort of evaluation used in worst-case analysis, insofar as it uses the abatement cost and damage functions yielding the highest total cost.

[6]This is a well-known consequence of the fact that maximisation over $\mathcal{C}$ coincides with maximisation over $\operatorname{supp}\mathcal{C}$ when $\delta_\theta \in \mathcal{C}$ for every $\theta \in \operatorname{supp}\mathcal{C}$.

For future reference, define the functions $\hat{C}, \hat{D} : [0, E^{max}] \to \mathbb{R}$ by:

$$\hat{C}(E) = \max_{p \in \mathcal{C}} \mathbb{E}_p C(E, \theta)$$
$$\hat{D}(E) = \max_{q \in \mathcal{D}} \mathbb{E}_q D(E, \xi)$$

$\hat{C}$ gives the worst-case (i.e. highest) expected abatement costs under the relevant uncertainty (i.e. over all distributions in $\mathcal{C}$), for every emissions reductions level; similarly for $\hat{D}$ and worst-case damages. Since all functions in Θ and Ξ are increasing and strictly convex, it follows from standard convex analysis results that $\hat{C}$ and $\hat{D}$ are increasing and strictly convex. However, they need not be differentiable, so do not necessarily belong to Θ and Ξ.

As a final point of notation, define, for each $r \in (0, 1]$, the function $\widehat{C^r} : [0, E^{max}] \to \mathbb{R}$ by

$$\widehat{C^r}(E) = \max_{p \in \mathcal{C}} \sup \{x : p(\{\theta : C(E, \theta) \geq x\}) \geq r\}$$

$\widehat{C^r}$ yields the r-quantile worst-case cost for each emissions reduction level. More precisely, for every r and emissions level E, $\widehat{C^r}(E)$ is the value such that, according to the set $\mathcal{C}$, the probability that emissions reduction E yields a cost of $\widehat{C^r}(E)$ or higher is at most r. By Proposition A.1 in the Appendix, $\widehat{C^r}$ is strictly convex for all r.

2.4 Differentiability: notions and terminology

In the interests of generality, while we assume that all members of Θ are differentiable, we do not assume differentiability of the generated worst-case abatement cost and damage functions defined above. At points, we thus require results and notions from convex analysis, including generalisations of the notion of differentiability to such functions. For a convex function $f : [0, E^{max}] \to \mathbb{R}$ and $x \in \mathbb{R}$, we use $\partial f(x) \subseteq \mathbb{R}$ to denote the subgradient of f at x, which is a generalisation of the notion of derivative for non-differentiable functions (Rockafellar, 1970, Ch 25). If $\partial f(x)$ contains one element, then f is differentiable at x and the standard derivative $f'(x)$ is the unique member of $\partial f(x)$. By Rademacher's Theorem, since f is convex, it is differentiable almost everywhere, i.e. at all points except a set of measure zero. Moreover, by (Rockafellar, 1970, Theorems 23.5.1, 26.1 & 26.3), if f is strictly convex, its derivative has a well-defined (i.e. single-valued) inverse: for every y, there exists at most one x such that $y \in \partial f(x)$.

For a convex function $f : \mathbb{R} \to \mathbb{R}$, $d \in \partial f(x)$, and $x, y \in \mathbb{R}$ the *average second-order derivative* of f between x and y by d is

$$f''^{d}_{x,y} = \frac{2}{(y - x)^2} \left(f(y) - f(x) - d(y - x)\right) \tag{2}$$

Note that, if f is twice differentiable, this is indeed an average of second-order derivatives (see Remark A.1 in the Appendix). When f is differentiable at x, we omit the superscript d.

Finally, for a convex function f, by the Alexandrov Theorem, f has a second derivative almost everywhere; we denote it by $f''(x)$ at all points x where it exists.

3 Policies

Our results will compare quantity and pricing policies under the maxmin-EU evaluation (1). We first set out the application of policy evaluation to the two policies in turn.

3.1 Quantity policy

Under a quantity (or permit) policy, the policy maker chooses an emissions reduction level L. Under Eq. (1), the optimal policy satisfies:

$$T_{quant} = \min_{L} \max_{p \in \mathcal{C}} \max_{q \in \mathcal{D}} \mathbb{E}_p \mathbb{E}_q \left(D(L, \xi) + C(L, \theta) \right) \tag{3}$$

I.e. it is the policy option leading to the lowest total cost under the maxmin-EU evaluation with $\mathcal{C}$ and $\mathcal{D}$. The optimal policy is L^* such that there exists x with $x \in \partial \max_{q \in \mathcal{D}} \mathbb{E}_q D(L, \xi)$ and $-x \in \partial \max_{q \in \mathcal{D}} \mathbb{E}_q D(L, \xi)$.[7] In other words, the optimality condition is that $\hat{D}$ and $\hat{C}$ share a common subgradient (derivative), up to negation. Given the previously noted facts about $\hat{D}$, $\hat{C}$ and strictly convex functions (Sections 2.3 and 2.4), there is a unique quantity optimum – and all other emissions levels have a strictly worse total cost under Eq. (1).

3.2 Pricing policy

Under a pricing (or tax) policy, the policy maker chooses a tax level τ. Under Eq. (1), the optimal policy satisfies:

$$T_{price} = \min_{\tau} \max_{p \in \mathcal{C}} \max_{q \in \mathcal{D}} \mathbb{E}_p \mathbb{E}_q \left(D(h(\tau, \theta), \xi) + C(h(\tau, \theta), \theta) \right)$$

where $h(\tau, \theta)$ is the emissions reductions level resulting from tax τ, under cost function θ, in equilibrium. Following Weitzman (1974), note that under standard market assumptions and the assumption that agents learn the parameter θ, we have:

$$C_1(h(\tau, \theta), \theta) = \tau$$

So the optimisation problem becomes:

$$T_{price} = \min_{\tau} \max_{p \in \mathcal{C}} \max_{q \in \mathcal{D}} \mathbb{E}_p \mathbb{E}_q \left(D(C_1^{-1}(\tau, \theta), \xi) + C(C_1^{-1}(\tau, \theta), \theta) \right) \tag{4}$$

[7] This reduces to the standard equality of (absolute values of) derivatives when the functions involved are differentiable.

where $C_1^{-1}(E, \theta)$ is the inverse of $C_1(\bullet, \theta)$, considered as a function of the first coordinate at E. Since all cost functions are differentiable and strictly convex, $C_1^{-1}(\bullet, \theta)$ is a well-defined function on $[0, E^{max}]$ for all $\theta \in \Theta$.

For every tax level τ and $p \in \Delta(\Theta)$, p generates a distribution $\epsilon_{\tau,p} \in \Delta([0, E^{max}])$ on the emissions reduction levels under τ, defined by $\epsilon_{\tau,p}([0, E]) = p(\{\theta : C_1(E', \theta) = \tau \text{ for some } E' \leq E,\})$ for all $E \in [0, E^{max}]$. For each abatement cost function θ, tax τ will lead to a unique level of emissions reduction; $\epsilon_{\tau,p}$ is the distribution over emissions reductions levels generated by τ, under the distribution p over possible abatement cost functions.

Under tax τ, define, for each $r \in (0, 1]$, E_r to be such that $\tau \in \partial\widehat{C^r}(E_r)$.[8] Recall (Section 2.3) that the function $\widehat{C^r}$ gives the r-quantile worst-case (highest) abatement cost; E_r is the emissions reduction level under tax τ for this r-quantile worst case. This defines a distribution over emissions levels by $\epsilon_\tau([0, E]) = \inf\{r : E_r \leq E\}$. Under tax τ, ϵ_τ is the distribution over emissions levels generated by cost functions yielding worst-case abatement costs of emissions under uncertainty $\mathcal{C}$. Note that, since the worst-case (highest) costs are involved, there is a sense in which this distribution is as far to the left as possible, among all emissions reductions distributions under τ consistent with uncertainty $\mathcal{C}$.

For tax τ, let $E_\tau = \mathbb{E}_{\epsilon_\tau} E$ and $\sigma_\tau^2 = \mathbb{E}_{\epsilon_\tau}(E - \mathbb{E}_{\epsilon_\tau} E)^2$. The former is the expected emissions reduction under τ, under the worst-case abatement costs consistent with uncertainty $\mathcal{C}$. The latter is the variance of this worst-case: it is a proxy for the *risk* (as opposed to the uncertainty) concerning emissions reductions levels resulting from tax τ.

Define:

$$\varepsilon_\tau = \max\{\delta : \epsilon_\tau^{+\delta} \in \{\epsilon_{\tau,p} : p \in \mathcal{C}\}\} \tag{5}$$

ε_τ is an indication of the 'width' of the set of priors $\mathcal{C}$, when projected down onto the consequences of tax τ in terms of emissions reduction. It tracks how far the previous emissions reduction distribution can be shifted to the right whilst remaining consistent with the set of priors $\mathcal{C}$. For instance, if $\mathcal{C}$ contains a single probability distribution, which corresponds to the risk-only case studied by Weitzman, then $\varepsilon_\tau = 0$. By contrast, large ε_τ indicate a large variety in expected emissions reductions that could result from tax τ, according to $\mathcal{C}$. In the light of this, we refer to ε_τ as the *uncertainty* concerning emissions reductions levels resulting from tax τ.

4 Policy comparisons

4.1 Worst-case analysis: Constraints on costs

Figure 1, which is drawn from the chapter on long-term mitigation pathways in the latest IPCC report, provides a starting point for consideration of uncertainty. It plots the marginal abatement

[8] By the strict convexity of $\widehat{C^r}$ and the observations in Section 2.4, $\partial\widehat{C^r}$ has a well-defined inverse, so E_r is uniquely defined.

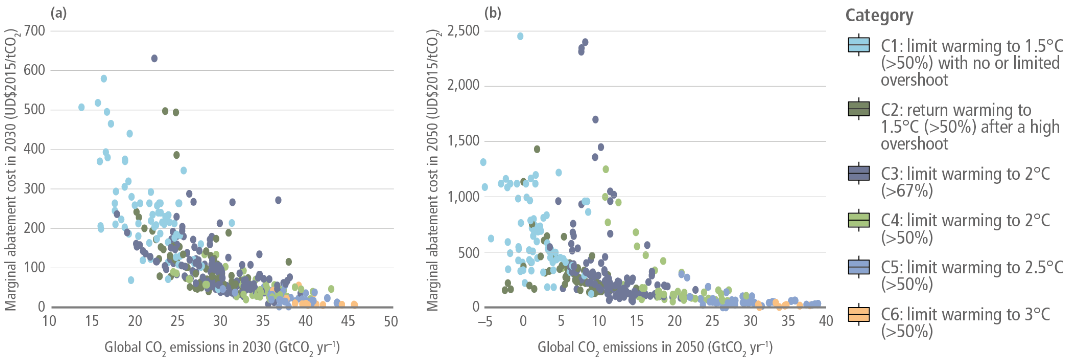

Figure 1: Marginal abatement cost of carbon in 2030 (a) and 2050 (b).
IPCC (2023, Figure 3.33). Note that marginal abatement costs are plotted against emissions, whereas the setup introduced in Section 2 works with (marginal) abatement costs as a function of emissions reductions.

cost against emissions for a range of mitigation pathways and climate models in the IPCC AR6 scenario database, consistent with different global warming levels.[9] As such, it summarises the state of scientific knowledge about marginal abatement costs at the time of the last IPCC report.

The Figure suggests that marginal abatement costs at a given time can, at best, be reasonably bounded between two functions, following roughly the upper and lower limits of the range of points. In the terminology introduced in Section 2.2, these are (categorical) *constraints* on the marginal abatement costs. However, there is little more that can be said with any confidence, as confirmed by the accompanying text (IPCC, 2023, Ch 3.6.1.1). For instance, whereas it states that marginal abatement costs increase with the extent of emissions reduction, no (other) bounds are reported on the slope of the marginal abatement cost curve.[10] Indeed, the Figure is consistent with a range of ‘precise’ marginal abatement cost functions, with differing slopes at different points, as long as these functions lie within the upper and lower limits mentioned above. Likewise, the Figure (and accompanying text) offers little to justify robust probabilistic constraints on marginal abatement costs. All of which poses the question: if this summarises the current state of knowledge about abatement costs, how do pricing and quantity policies compare under this extent of uncertainty?

In the terms introduced in Section 2.2, the IPCC report would seem to imply, at best, uncertainty about abatement costs generated by constraints (upper and lower limits) $[m', M']$ on the

[9]As emphasised by the IPCC, the points are thus not random draws from an underlying distribution, nor even the results of systematic runs of the same model; this limits the relevance of classical statistical analyses conducted on them, and inhibits drawing strong conclusions about the shape of marginal abatement cost curves from them.

[10]In personal communication, a lead author of this chapter who was willing to comment on the slope of the marginal abatement cost curve ‘did not believe that there were any established results on [it]’. Several lead authors, in personal communication, emphasised that Figure 1 cannot be used to draw strong conclusions on marginal abatement costs, suggesting that it was tenuous to use the Figure to bound marginal abatement costs, much less their slope. They emphasised the relevance of systematic comparisons of marginal abatement costs with the same model; the two studies they provided are discussed in Section 4.3 below. Only one lead author expressed a stronger opinion on marginal abatement costs, stating that ‘the slope of the marginal abatement costs is generally assumed to be positive and increasing with emissions reductions’. Section 4.3 examines the consequences of incorporating such constraints on the third derivative of abatement costs, establishing an extension of Theorem 1 below.

marginal abatement costs, and a lower bound of 0 for the slope of the marginal abatement curve. Our first result concerns the more general case of constraints $[m, M]$, $[m', M']$ on the abatement and marginal abatement costs, and a single lower constraint $\underline{C''}$ on the slope of marginal costs. Using the notation introduced previously, such uncertainty concerning abatement costs is represented by the set of priors $\mathcal{C}_{[m,M],[m',M'],[\underline{C''},\infty)}$.

Theorem 1. *Consider any $\mathcal{D} \subseteq \Delta(\Xi)$ characterising uncertainty about damages, and suppose that uncertainty about abatement costs is characterised by $\mathcal{C}_{[m,M],[m',M'],[\underline{C''},\infty)}$ for some $[m, M]$, $[m', M']$, $\underline{C''}$. Then $T_{price} \geq T_{quant}$.*

Moreover, the inequality is strict whenever $\varepsilon_{\tau^*} > 0$ for optimal tax τ^*, $\hat{D}$ and $\hat{C}$ are second-order differentiable at the optimal quantity level L^* and $\hat{D}''(L^*) > \frac{\hat{C}''(L^*)\underline{C''}}{\hat{C}''(L^*)-\underline{C''}}$.

Theorem 1 paints a markedly different picture of the comparison between taxes and permits in the world of (severe) uncertainty, as opposed to risk. For one, under uncertainty characterised by constraints on the abatement cost function, quantity policies always do at least as well as pricing policies: there are no cases in which the latter outperform the former in a robust worst-case analysis.

Moreover, the Theorem identifies a condition under which quantity policies are guaranteed to perform strictly better than pricing policies.[11] The first clause essentially demands there is some uncertainty, in the sense specified in Section 3.2, i.e. a spread in the emissions reductions distributions consistent with the state of knowledge and uncertainty $\mathcal{C}_{[m,M],[m',M'],[\underline{C''},\infty)}$, under tax τ^*. The second clause – $\hat{D}''(L^*) > \frac{\hat{C}''(L^*)\underline{C''}}{\hat{C}''(L^*)-\underline{C''}}$ – covers some notable special cases, and calls for interpretation.

Note firstly that if the lower bound on the second derivative is 0, then this condition is automatically satisfied (since $\hat{D}$ is strictly convex; Section 2.3). This corresponds to the absence of known constraints on the slope of the marginal cost, apart from that it is positive: the result says that quantity policies always fair better than pricing policies under such uncertainty. To return to the previous example, if, as the IPCC report and Figure 1 suggest, our knowledge can be summarised by constraints on marginal abatement costs with nothing about the slope of marginal cost apart from the fact that it is positive, then the corresponding uncertainty is characterised by $\mathcal{C}_{(-\infty,\infty),[m',M'],[0,\infty)}$. Since this corresponds to a lower bound on the slope of marginal abatement of 0, Theorem 1 shows that, under current uncertainty as reported by the IPCC, quantity policies are more efficient than pricing policies.

More generally, this clause resembles Weitzman's condition pertaining to taxes vs. permits under risk (see Section 4.2) insofar as it compares a slope for marginal damages with a term involving slopes of marginal abatement costs. However, there are several crucial differences. The first concerns the abatement cost term. Instead of, say, comparing $\hat{D}''(L^*)$ with the slope of the marginal worst-case costs ($\hat{C}''(L^*)$), the condition involves the lower known bound on the slope, $\underline{C''}$. Moreover, the term decreases as the lower bound decreases, but also as the difference

[11] The condition can be strengthened to accommodate non-second-order differentiability of damages and abatement costs at L^*, using notions from convex analysis. Such technical details are omitted.

between the slope of the marginal worst-case costs and the lower bound increases. So, whilst $\hat{C}''(L^*)$ – which could conceivably be gleaned from Figure 1 by looking at the upper limit on the marginal abatement costs – may be larger than $\hat{D}''(L^*)$, if the known lower bound is much lower, the condition in the Theorem may still be satisfied. This suggests that the performance comparison between quantity and pricing policies may be sensitive to the slope of the marginal costs used. In particular, a value reflecting the extent of uncertainty about the slope is more relevant than 'average' slopes drawn, say, from fitting marginal abatement curves to the points in plots such as Figure 1.

Secondly, the condition in Theorem 1, unlike in Weitzman's result, is sufficient but not necessary. This reflects the global nature of our result: the fact that taxes lead to higher worst-case total costs under the specified conditions depends on no local approximations. This is relevant in the current context since, as we have seen, there is little reason to think that the relevant uncertainties are 'small'.

4.2 General case: Probabilistic constraints

The previous result is a special case of a more general one involving probabilistic constraints on costs (Example 4). For readability, we report the most general form of the result in the Appendix (Theorem A.1) and focus here on a simplification obtained by making the following assumption, parts of which are comparable to assumptions made by Weitzman (1974).

Assumption 1. For optimal tax level τ^*, suppose that $\widehat{Cr}''^{\tau^*}_{E_r,E_{\tau^*}}$ (considered as a random variable varying with r) is independent from $(E_r - E_{\tau^*})^2$, and that the slope of $\widehat{Cr}'$ is relatively stable in the region of E_r and E_{τ^*}, so that, for all sufficiently small δ, $\widehat{Cr}''^{\tau^*}_{E_r,E_{\tau^*}}, \widehat{Cr}''^{\tau^*}_{E_r,E_r-\delta} \approx \widehat{Cr}''(E_{\tau^*})$, which exists for all $r \in (0,1]$.[12]

Similarly, suppose that $\hat{D}'$ is relatively stable in the region of $E_{\tau^*}+\varepsilon_{\tau^*}$, so the expected slope of the marginal damages over distributions with mean $E_{\tau^*} + \varepsilon_{\tau^*}$ is approximately $\hat{D}''(E_{\tau^*} + \varepsilon_{\tau^*})$, which exists. Suppose also that $\hat{D}$ is differentiable at E_{τ^*}.

Finally, suppose that the $E_{\tau^*} = L^*$, the optimal quantity policy.

As above, we consider the case of a single lower bound on the slope of marginal abatement costs, which in the terminology of Section 2.2 corresponds to $\mathcal{C}_{\{m^{(\rho)},M^{(\rho)},m^{(\rho)1},M^{(\rho)1},m^{(\rho)2},M^{(\rho)2}\}}$ for some family of probabilistic constraints $\{m^{(\rho)}, M^{(\rho)}, m^{(\rho)1}, M^{(\rho)1}, m^{(\rho)2}, M^{(\rho)2}\}$ with $[m^{(\rho)2}, M^{(\rho)2}] = [\underline{C''}, \infty)$ for all $\rho \in (0,1]$.

Theorem 2. *Consider any $\mathcal{D} \subseteq \Delta(\Xi)$ characterising uncertainty about damages, and suppose that uncertainty about abatement costs is characterised by $\mathcal{C}_{\{m^{(\rho)},M^{(\rho)},m^{(\rho)1},M^{(\rho)1},m^{(\rho)2},M^{(\rho)2}\}}$ for some family of probabilistic constraints $\{m^{(\rho)}, M^{(\rho)}, m^{(\rho)1}, M^{(\rho)1}, m^{(\rho)2}, M^{(\rho)2}\}$ with*

[12]The first assumption corresponds, in our framework, to the independence stipulated in Weitzman (1974, p486, footnote 1); the second allows replacement of average second-order derivatives (Section 2.4) with second-order derivatives, as under local approximations.

$[m^{(\rho)2}, M^{(\rho)2}] = [\underline{C''}, \infty)$ *for all* $\rho \in (0,1]$. *Under Assumption 1:*

$$\begin{aligned} & T_{price} - T_{quantity} \\ & \geq \underbrace{\frac{\sigma^2_{\tau^*}}{2}\left(\hat{D}''(L^* + \varepsilon_{\tau^*}) - \hat{C}''(L^*)\right)}_{\textit{Risk}} \\ & \quad + \underbrace{\frac{\varepsilon^2_{\tau^*}}{2}\left(\hat{D}''_{L^*, L^*+\varepsilon_{\tau^*}} - \mathbb{E}_u \frac{\underline{C''}\widehat{C^r}''(L^*)}{\widehat{C^r}''(L^*) - \underline{C''}}\right)}_{\textit{Uncertainty}} \end{aligned} \tag{6}$$

where τ^* *is the optimal tax,* L^* *the optimal quota, and* $\mathbb{E}_u$ *is the expectation over* r *under the uniform probability distribution* u *over* $[0,1]$.[13]

Note first of all that this Theorem generalises the previous one involving non-probabilistic uncertainty, as well as Weitzman's result concerning risk. When there is only risk $\varepsilon_{\tau^*} = 0$, so the optimal tax policy leads to higher cost that the optimal quantity policy whenever $\hat{D}''(L^*) > \hat{C}''(L^*)$. Recalling that the expectation here is taken over resulting emission reductions rather than parameters in the specification of the abatement cost function, this is easily seen to be equivalent to Weitzman's (1974) result. On the other hand, when there is only uncertainty $\sigma^2_{\tau^*} = 0$, so the comparison between tax and quantity policies turns on the 'Uncertainty' term. In the absence of risk (and notably non-trivial probabilistic constraints), this term reduces to a comparison of the sort in Theorem 1. As discussed above, when little is known about the slope of the marginal abatement costs, it will be positive: taxes lead to a higher total cost than permits.

Eq. (6) tells us that between these two extremes, the comparison between optimal taxes and permits turns, largely, on the extent of risk and uncertainty in our current knowledge about abatement costs. More specifically, let's call $PR = \hat{D}''(L^* + \varepsilon_{\tau^*}) - \hat{C}''(L^*)$ the *pure-risk term*, and $PU = \hat{D}''_{L^*, L^*+\varepsilon_{\tau^*}} - \mathbb{E}_u \frac{\underline{C''}\widehat{C^r}''(L^*)}{\widehat{C^r}''(L^*) - \underline{C''}}$ the *pure-uncertainty term*. As noted, PR is very similar to the well-known Weitzman term; economists typically consider that it is negative (Nordhaus, 2007). PU compares the average slope of the marginal damage curve across the uncertainty in expected emissions under τ^* (i.e. between L^* and $L^* + \varepsilon_{\tau^*}$) with the expectation of a term, $\frac{\underline{C''}\widehat{C^r}''(L^*)}{\widehat{C^r}''(L^*) - \underline{C''}}$, that is of a similar form to that in Theorem 1. As argued previously, given current ignorance about slopes of marginal abatement costs, $\underline{C''}$ is can reasonably be considered to be small, so $\frac{\underline{C''}\widehat{C^r}''(L^*)}{\widehat{C^r}''(L^*) - \underline{C''}}$ will be close to zero and the PU term is typically positive.

The comparison between taxes and permits thus depends on the extent to which the limits on our knowledge reflect *bona fide* uncertainty as opposed to risk. If $\sigma^2_{\tau^*} \gg \varepsilon^2_{\tau^*}$ – i.e. the extent of probabilised risk, captured by the variance, dwarfs the non-probabilised uncertainty, as reflected in the 'width' of the set of priors $\mathcal{C}$ – then the pure-risk term dominates. If $\varepsilon^2_{\tau^*} \gg \sigma^2_{\tau^*}$ – the extent of dispersion between priors in $\mathcal{C}$ outstrips the variance under appropriate worst-case priors – then the pure-uncertainty term dominates. In the latter case, our result implies that quantity policies outperform pricing ones. If Figure 1 and the frequent admonitions as to the severity of

[13] The other notation is defined in Sections 2 and 3.

the uncertainty concerning the economics of climate change (Pindyck, 2013; Bradley and Steele, 2015; Stern et al., 2022) are to be believed, this most closely corresponds to the uncertainties facing us today. *Pace* the conclusion typically drawn from Weitzman's Theorem, Theorem 2 would suggest that, under current uncertainty, permits are better than taxes.

4.3 Higher derivatives

Whilst the previous analyses incorporate constraints up to the second derivative of the abatement costs, several parametric forms involve a positive third derivative of the abatement cost curve – i.e. convex marginal abatement costs. Some model-based diagnostic studies (Kriegler et al., 2015; Harmsen et al., 2021) tentatively corroborate this suggestion. We now extend our analysis to incorporate such constraints. To this end, recall the notation mentioned in Example 2 (Section 2.2), under which $\mathcal{C}_{[m,M],[m',M'],[m'',M''],[m''',M''']}$ involves, in addition to the constraints considered in the previous sections, the constraint that the third derivative of the abatement cost always lies in the interval $[m''', M''']$. The following extends the result in Section 4.1 to the case of general constraints on abatement, marginal abatement costs and their slopes, and a single lower constraint $\underline{C'''} \geq 0$ on the convexity of marginal abatement costs.

Theorem 3. *Consider any $\mathcal{D} \subseteq \Delta(\Xi)$ characterising uncertainty about damages, and suppose that uncertainty about abatement costs is characterised by $\mathcal{C}_{[m,M],[m',M'],[m'',M''],[\underline{C'''},\infty)}$ for some $[m, M]$, $[m', M']$, $[m'', M'']$, $\underline{C'''}$. Then $T_{price} \geq T_{quant}$.*

Moreover, if the third derivative of $\hat{C}$ is approximately constant, then the inequality is strict whenever

$$\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon_{\tau^*}} + \hat{C}''(L^*) + \frac{1}{3}\varepsilon_{\tau^*}\hat{C}'''(L^*) > \frac{\left(2\hat{C}''(L^*) + \varepsilon_{\tau^*}\hat{C}'''(L^*)\right)^{\frac{3}{2}}}{3\varepsilon_{\tau^*}^{\frac{1}{2}}\left(\hat{C}'''(L^*) - \underline{C'''}\right)^{\frac{1}{2}}} \tag{7}$$

The assumption concerning the third derivative of $\hat{C}$ is merely for simplicity. The proof delivers a version of the result without this assumption, where $\hat{C}'''(L^*)$ is replaced by the average third derivative $\hat{C}$ à la Eq. (2) (Section 2.4); see Appendix A.3 for details. The points below thus hold in the absence of this assumption, using the appropriate replacement.

As for the case considered in Section 4.1, under constraints on abatement costs and higher derivatives, optimal quantity policies never perform worse than optimal pricing ones. Moreover, Theorem 3 provides a sufficient condition for them to do strictly better. Whilst they are a few studies suggesting convex marginal abatement costs (Kriegler et al., 2015; Harmsen et al., 2021), no lower bound to its convexity has been proposed, suggesting that $\underline{C'''}$ reflecting the current state of knowledge can be taken close to zero, so $\frac{\varepsilon_{\tau^*}\hat{C}'''(L^*)^{\frac{3}{2}}}{3\left(\hat{C}'''(L^*)-\underline{C'''}\right)^{\frac{1}{2}}} \approx \frac{1}{3}\varepsilon_{\tau^*}\hat{C}'''(L^*)$. Hence, as ε_{τ^*} increases, the term on the right tends to $\frac{1}{3}\varepsilon_{\tau^*}\hat{C}'''(L^*)$. Given that the other terms on the left are positive, it follows that the condition in the Theorem holds when ε_{τ^*} is not too small. The general conclusion in the case of convex marginal abatement costs is thus as in the previous analyses: in the context of significant uncertainty, permits are better than taxes.[14]

4.4 Parametric constraints

Another potential form of uncertainty concerning abatement costs consists in certainty that cost functions belong to a particular family (or families), and knowledge about the range of possible or probable parameter values (Section 2.2, Example 3). Similar results can be obtained in this case (see Theorem A.2 in Appendix A.4 for a general result). We illustrate them by considering the logarithmic marginal abatement cost family used by Nordhaus (1991): $\mathcal{C} = \mathcal{C}_{f,A}$, where each $\theta \in \operatorname{supp}\mathcal{C}$ is such that $C_1(E,\theta) = f(E,a,b) = a - b\ln(E^{max} - E)$ for positive $(a,b) \in A$. We consider the case where there is a lowest marginal cost curve in $\operatorname{supp}\mathcal{C}$, characterised by $(\underline{a}, \underline{b})$, and a highest marginal cost curve, characterised by $(\overline{a}, \overline{b})$, i.e. there exists $(\underline{a}, \underline{b}), (\overline{a}, \overline{b}) \in A$ such that for all $(a,b) \in A$ and all $E \in [0, E^{max}]$, $f(E, \underline{a}, \underline{b}) \leq f(E, a, b) \leq f(E, \overline{a}, \overline{b})$.

Proposition 1. *Consider any $\mathcal{D} \subseteq \Delta(\Xi)$ characterising uncertainty about damages, and suppose that uncertainty about abatement costs is characterised by parametric constraints, concerning the family $f(E,a,b) = a - b\ln(E^{max} - E)$, with highest and lowest marginal cost curves $(\overline{a}, \overline{b})$ and $(\underline{a}, \underline{b})$. Then $T_{price} \geq T_{quant}$.*

Moreover, the inequality is strict whenever

$$\hat{D}''^{\tau^*}_{L^*, L^*+\varepsilon_{\tau^*}} > \frac{2\left[-\left(\underline{b}(L^* + \varepsilon_{\tau^*}) - \overline{b}L^*\right) + E^{max}(\overline{a} - \overline{b}\ln E^{max} - (\underline{a} - \underline{b}\ln E^{max}))\right]}{\varepsilon^2_{\tau^*}} \tag{8}$$

where L^* is the optimal quantity level and τ^* is the optimal tax level.

The central message of Theorem 1 thus continues to hold when one assumes uncertainty to be characterised by constraints on parametric forms for abatement cost functions. On the one hand, under parametric constraints, permits never fare worse than taxes. On the other hand, there is a condition for permits to strictly outperform taxes, which differs from that in Theorem 1 and involves two terms on the right-hand side of the inequality. To interpret the second term, note that $\overline{a} - \overline{b}\ln E^{max} - (\underline{a} - \underline{b}\ln E^{max})$ is the difference in the marginal cost of the first unit of emissions reduction under the highest and lowest marginal cost curves consistent with the uncertainty $\mathcal{C}$. If this marginal cost is well understood (as suggested by Figure 1), uncertainty concerning it will be low, and the second term in (8) will be small. As for the first term, if uncertainty concerning the marginal costs for the first unit of emissions reduction is low, $\overline{a} - \overline{b}\ln E^{max} \approx \underline{a} - \underline{b}\ln E^{max}$, whence it follows from the specification that $\overline{b} \geq \underline{b}$. Substituting in the parametric expressions for L^* and $L^* + \varepsilon_{\tau^*}$ yields:

$$\underline{b}(L^* + \varepsilon_{\tau^*}) - \overline{b}L^* = \underline{b}(E^{max} - \exp - \frac{\tau^* - \underline{a}}{\underline{b}}) - \overline{b}(E^{max} - \exp - \frac{\tau^* - \overline{a}}{\overline{b}})$$

[14] Whilst we have considered constraints on costs for illustration here, extensions of Theorem 2 comparable to Theorem 3 are obtainable by similar techniques (see Appendix A.3), so the conclusion of Section 4.2 continues to hold when incorporating higher derivatives.

This is greater than zero under the previous specifications; moreover, it is decreasing in $\underline{b}$ throughout most of the range, for any fixed $\overline{b}$. Hence the first term on the right-hand side of (8) is negative whenever $\varepsilon_{\tau^*} > 0$. Notice moreover that $\frac{\overline{b}}{\underline{b}}$ is the ratio of slopes of the highest vs. lowest marginal cost at any emissions reduction level, and hence an indication of the uncertainty concerning the slope of the marginal cost curve. So the first term in (8) is decreasing in the uncertainty about the slope of marginal cost. As for the case discussed in Section 4.1, when there is significant uncertainty – as opposed to risk – about the slope of the marginal cost curve, quantity policies are strictly better than pricing ones.

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A Appendix: Proofs

A.1 General Result

We first state and prove a general result involving uncertainty characterised by (probabilistic) constraints on costs. The results in Sections 4.1 and 4.2 will be corollaries, as shown in the next Section. As a point of notation, for a tax τ, let

$$T_\tau = \max_{p\in\mathcal{C}} \max_{q\in\mathcal{D}} \mathbb{E}_p\mathbb{E}_q \left(D(C_1^{-1}(\tau,\theta),\xi) + C(C_1^{-1}(\tau,\theta),\theta)\right)$$

We consider a general probabilistic lower constraint on the slope of marginal abatement costs, which can be represented by $\mathcal{C}$ generated as in Example 4 from a family $\{m^{(\rho)}, M^{(\rho)}, m^{(\rho)1}, M^{(\rho)1}, m^{(\rho)2}, M^{(\rho)2}\}$ with $[m^{(1-t)2}, M^{(1-t)2}] = [\underline{C''^t}, \infty)$ for all $t \in [0,1)$, for some $\underline{C''^t}$ increasing in t.

Theorem A.1. *Consider any $\mathcal{D} \subseteq \Delta(\Xi)$ characterising uncertainty about damages, and suppose that uncertainty about abatement costs is characterised by $\mathcal{C}_{\{m^{(\rho)},M^{(\rho)},m^{(\rho)1},M^{(\rho)1},m^{(\rho)2},M^{(\rho)2}\}}$ for some family of probabilistic constraints $\{m^{(\rho)}, M^{(\rho)}, m^{(\rho)1}, M^{(\rho)1}, m^{(\rho)2}, M^{(\rho)2}\}$ with $[m^{(1-t)2}, M^{(1-t)2}] = [\underline{C''^t}, \infty)$ for all $t \in [0,1)$, for some $\underline{C''^t}$ increasing in t. For any τ such that $\tau \in \partial\hat{C}(E)$ for some $E \in [0, E^{max}]$, and any $\hat{D}' \in \partial\hat{D}(E_\tau)$:*

$$\begin{aligned} T_\tau \geq & \hat{C}(E_\tau) + \hat{D}(E_\tau) + \varepsilon_\tau(\tau + \hat{D}') \\ & + \frac{\sigma_\tau^2}{2}\left(\mathbb{E}\hat{D}'' - \left(\mathbb{E}_u \widehat{C^r}''^{\tau}_{E_r,E_\tau} + \frac{1}{\sigma_\tau^2} cov(\widehat{C^r}''^{\tau}_{E_r,E_\tau}, (E_r - E_\tau)^2)\right)\right) \\ & + \frac{\varepsilon_\tau^2}{2}\left[\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau} - \mathbb{E}_{u_r}\mathbb{E}_{u_t} \frac{\underline{C''^t}\,\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}}}{\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}} - \underline{C''^t}}\right] \end{aligned} \tag{A.1}$$

where $\mathbb{E}\hat{D}''$ is an expected average second derivative over the support of $\epsilon_\tau^{+\varepsilon_\tau}$ under a worst-case distribution over damage functions (see Eqn (A.12) and surrounding discussion below); for each $r, t \in (0,1]$, $\delta_{r,t}$ is defined implicitly by $\max \partial\widehat{C^r}(E_r - \delta_{r,t}) = \tau - (\varepsilon_\tau + \delta_{r,t})\underline{C''^t}$; $\mathbb{E}_{u_r}\mathbb{E}_{u_t}$ are expectations taken over r and t according to independent uniform distributions $u_r, u_t \in \Delta([0,1])$ over $[0,1]$.

Proof. Consider τ such that $\tau \in \partial\hat{C}(E)$ for some $E \in [0, E^{max}]$. For each $r \in (0,1]$, recall that $\tau \in \partial\widehat{C^r}(E_r)$ and (by the definition in Eqn (2))

$$\widehat{C^r}''^{\tau}_{E_r,E_\tau} = \frac{2}{(E_\tau - E_r)^2}\left(\widehat{C^r}(E_\tau) - \tau(E_\tau - E_r) - \widehat{C^r}(E_r)\right)$$

It follows that

$$\widehat{C^r}(E_r) = \widehat{C^r}(E_\tau) - (\tau + \widehat{C^r}''^{\tau}_{E_r,E_\tau}(E_\tau - E_r))(E_\tau - E_r) + \frac{\widehat{C^r}''^{\tau}_{E_r,E_\tau}}{2}(E_\tau - E_r)^2$$

For each τ, and every $r \in (0,1]$, let $\theta_r \in \Theta$ be such that $C_1(E_r, \theta_r) = \tau$ and $C(E_r, \theta_r) = \widehat{C^r}(E_r)$. (By the definition of E_r etc, such θ_r exist.) We have:

$$\begin{aligned}\mathbb{E}_u C(E_r, \theta_r) =& \mathbb{E}_u \widehat{C^r}(E_\tau) - \mathbb{E}_u \left(\frac{\widehat{C^r}''^{\tau}_{E_r, E_\tau}}{2} (E_r - E_\tau)^2 \right) \\ =& \hat{C}(E_\tau) - \frac{1}{2} \left(\mathbb{E}_u \widehat{C^r}''^{\tau}_{E_r, E_\tau} \sigma_\tau^2 + cov(\widehat{C^r}''^{\tau}_{E_r, E_\tau}, (E_r - E_\tau)^2) \right)\end{aligned} \tag{A.2}$$

since $\mathbb{E}_u E_r = E_\tau$ and $\mathbb{E}_u (E_r - E_\tau)^2 = \sigma_\tau^2$. In the second equation, $\mathbb{E}_u \widehat{C^r}''^{\tau}_{E_r, E_\tau}$ is the expected average second order derivative in that area, across the worst-case cost functions, and the last term is the covariance with the square difference in emissions with respect to E_τ.

For each $r, t \in (0,1]$, note that there exists δ such that:

$$\tau - (\varepsilon_\tau + \delta) \underline{C''^t} \in \partial \widehat{C^r}(E_r - \delta) \tag{A.3}$$

Moreover, by the strict convexity of $\widehat{C^r}$, such δ is unique. By the convexity of $\widehat{C^r}$ and Rademacher's Theorem (e.g. Rockafellar, 1970, Theorem 25.4), $\partial \widehat{C^r}$ is single-valued except on a set of measure zero. So, for every $r, t \in (0,1]$ such that $\partial \widehat{C^r}(E_r - \delta)$ is not a singleton (i.e. $\widehat{C^r}$ is not differentiable at $E_r - \delta$) for δ satisfying the expression above, there exists a small $\rho_{r,t} > 0$ and δ' such that $\tau - (\varepsilon_\tau + \delta')(\underline{C''^t} + \rho_{r,t}) \in \partial \widehat{C^r}(E_r - \delta')$ and $\partial \widehat{C^r}(E_r - \delta')$ is a singleton.

For each $r, t \in (0,1]$, let $\delta_{r,t}$ be the solution to:

$$\tau - (\varepsilon_\tau + \delta_{r,t}) \underline{C''^t} = \widehat{C^r}'(E_r - \delta_{r,t}) \tag{A.4}$$

if such $\delta_{r,t}$ exists (i.e. when $\widehat{C^r}$ is differentiable at $E_r - \delta_{r,t}$), and the solution to

$$\tau - (\varepsilon_\tau + \delta_{r,t})(\underline{C''^t} + \rho_{r,t}) = \widehat{C^r}'(E_r - \delta_{r,t})$$

otherwise. As noted above, $\delta_{r,t}$ is well-defined, and for each r, it satisfies (A.4) for all t except a set of measure zero.

Let $\hat{\theta}_r$ be any cost function such that $C_1(E_r - \delta_{r,t}, \hat{\theta}_r) = \widehat{C^r}'(E_r - \delta_{r,t})$ and $C(E_r - \delta_{r,t}, \hat{\theta}_r) = \widehat{C^r}(E_r - \delta_{r,t})$; such a function exists by the definition of $\widehat{C^r}$ (and the fact that it is differentiable at the points involved). Consider the function $G : [0, E_r + \varepsilon_\tau] \to \mathbb{R}$ defined by

$$G(E) = \begin{cases} C(E, \theta_r) & \text{if } E \leq E_r - \delta_{r,t} \\ C(E_r - \delta_{r,t}, \hat{\theta}_r) + \widehat{C^r}'(E_r - \delta_{r,t})(E - E_r + \delta_{r,t}) \\ + \frac{\underline{C''^t}}{2}(E - E_r + \delta_{r,t})^2 & \text{if } E_r - \delta_r < E \leq E_r + \varepsilon_\tau \end{cases}$$

This hits the $\widehat{C^r}$ curve at a point, and then increases with the t-lowest second-order derivative until the marginal abatement cost reaches τ, which occurs at $E_r + \varepsilon_\tau$. Note that G is increasing, differentiable, strictly convex and satisfies all the constraints in $\mathcal{C}$ on its domain. Hence there exists $\underline{\theta_{r,t}} \in \mathcal{C}$ with $C(E, \underline{\theta_{r,t}}) = G(E)$ for all $E \in [0, E_r + \varepsilon_\tau]$. Take any such $\underline{\theta_{r,t}}$.

Note that, since

$$\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}} = \frac{2}{\delta_{r,t}^2}\left(\widehat{C^r}(E_r - \delta_{r,t}) - \widehat{C^r}(E_r) + \tau\delta_{r,t}\right) \tag{A.5}$$

is the average second-order derivative of $\widehat{C^r}$ in this range (Eqn (2)), we have

$$\widehat{C^r}(E_r - \delta_{r,t}) = \widehat{C^r}(E_r) - \tau\delta_{r,t} + \frac{\delta_{r,t}^2}{2}\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}}$$

from which it follows that

$$\widehat{C^r}'(E_r - \delta_{r,t}) = \tau - \delta_{r,t}\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}} \tag{A.6}$$

Plugging this into (A.4) yields:

$$\delta_{r,t} = \underline{C''^t}\frac{\varepsilon_\tau}{\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}} - \underline{C''^t}} \tag{A.7}$$

for all $(r, t) \in [0, 1]^2$ except a set of measure zero. $\delta_{r,t} = (\underline{C''^t} + \rho_{r,t})\frac{\varepsilon_\tau}{\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}} - (\underline{C''^t} + \rho_{r,t})}$ otherwise.

It follows, substituting these equations in appropriately, that

$$\begin{aligned}
C(E_r + \varepsilon_\tau, \underline{\theta_{r,t}}) = {} & C(E_r - \delta_{r,t}, \underline{\theta_{r,t}}) + \widehat{C^r}'(E_r - \delta_{r,t})(\varepsilon_\tau + \delta_{r,t}) + \frac{\underline{C''^t}}{2}(\varepsilon_\tau + \delta_{r,t})^2 \\
= {} & C(E_r, \theta_r) - \tau\delta_{r,t} + \frac{\delta_{r,t}^2}{2}\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}} \\
& + (\tau - \delta_{r,t}\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}})(\varepsilon_\tau + \delta_{r,t}) + \frac{\underline{C''^t}}{2}(\varepsilon_\tau + \delta_{r,t})^2 \\
= {} & C(E_r, \theta_r) - \frac{\delta_{r,t}^2}{2}\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}} + \tau\varepsilon_\tau \\
& - \delta_{r,t}\varepsilon_\tau\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}} + \frac{\underline{C''^t}}{2}(\varepsilon_\tau + \delta_{r,t})^2 \\
= {} & C(E_r, \theta_r) + \tau\varepsilon_\tau + \frac{\varepsilon_\tau^2}{2(\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}} - \underline{C''^t})^2}\Big[-\underline{C''^t}^2\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}} \\
& - 2\underline{C''^t}\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}}(\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}} - \underline{C''^t}) + \underline{C''^t}\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}}{}^2\Big] \\
= {} & C(E_r, \theta_r) + \tau\varepsilon_\tau - \frac{\varepsilon_\tau^2}{2}\frac{\underline{C''^t}\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}}}{\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_{r,t}} - \underline{C''^t}}
\end{aligned} \tag{A.8}$$

for all $(r, t) \in [0, 1]^2$ except a set of measure zero.

For two independent uniform distributions u_r, u_t over $[0, 1]$, let $\underline{p}$ over $\{\theta_{r,t}\}$ be the distribution over cost functions that they generate; i.e. $\underline{p}(\{\theta_{r,t} : r \leq r', t \leq t'\}) = u_r([0, r']).u_t([0, t'])$

for all r', t'. This belongs to $\mathcal{C}$ because by construction it satisfies all the probabilistic constraints. Note moreover that, by construction, $\epsilon_{\tau,\underline{p}} = \epsilon_\tau^{+\varepsilon_\tau}$. So

$$\begin{aligned}
\mathbb{E}_{\underline{p}} C(C^{-1}(\tau,\theta),\theta) = & \mathbb{E}_{u_r}\mathbb{E}_{u_t} C(E_r + \varepsilon_\tau, \underline{\theta_{r,t}}) \\
= & \mathbb{E}_{u_r} C(E_r, \theta_r) + \tau\varepsilon_\tau - \frac{\varepsilon_\tau^2}{2}\mathbb{E}_{u_r}\mathbb{E}_{u_t} \frac{\underline{C''^t}\widehat{C^r}''^{\tau}_{E_r, E_r - \delta_{r,t}}}{\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_{r,t}} - \underline{C''^t}} \\
= & \hat{C}(E_\tau) - \frac{1}{2}\left(\mathbb{E}_u \widehat{C}''\sigma_\tau^2 + cov(\widehat{C}'', \sigma_\tau^2)\right) + \tau\varepsilon_\tau \\
& - \frac{\varepsilon_\tau^2}{2}\mathbb{E}_{u_r}\mathbb{E}_{u_t} \frac{\underline{C''^t}\widehat{C^r}''^{\tau}_{E_r, E_r - \delta_{r,t}}}{\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_{r,t}} - \underline{C''^t}}
\end{aligned} \tag{A.9}$$

by (A.2), the definition of $\underline{E_r}$ and the fact that (A.8) holds for all $(r,t) \in [0,1]^2$ except a set of measure zero.

Take any $\hat{D}' \in \partial\hat{D}(E_\tau)$ and recall that:

$$\hat{D}''^{\hat{D}'}_{E_\tau, E_\tau+\varepsilon_\tau} = \frac{2}{(\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E - E_\tau)^2}\left(\hat{D}(E_\tau) + \hat{D}'(\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E - E_\tau) - \hat{D}(\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E)\right)$$

is the average second order derivative of $\hat{D}$ between E_τ and $\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E = E_\tau + \varepsilon_\tau$. Hence:

$$\hat{D}(\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E) = \hat{D}(E_\tau) + \hat{D}'\varepsilon_\tau + \frac{\hat{D}''^{\hat{D}'}_{E_\tau, E_\tau+\varepsilon_\tau}}{2}\varepsilon_\tau^2 \tag{A.10}$$

by the specification of $\underline{p}$.

Moreover, for any $q \in \mathcal{D}$ with $\mathbb{E}_q D(\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E, \xi) = \hat{D}(\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E)$,

$$\begin{aligned}
\mathbb{E}_q\mathbb{E}_{\epsilon_{\tau,\underline{p}}} D(E,\xi) = & \mathbb{E}_q D(\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E, \xi) + \mathbb{E}_q\mathbb{E}_{\epsilon_{\tau,\underline{p}}} D_1(\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E, \xi)(E - \mathbb{E}_{\epsilon_{\tau,\underline{p}}} E) \\
& + \mathbb{E}_q\mathbb{E}_{\epsilon_{\tau,\underline{p}}} \frac{D''(\xi)}{2}(E - \mathbb{E}_{\epsilon_{\tau,\underline{p}}} E)^2 \\
= & \hat{D}(\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E) + \frac{\mathbb{E}_q D''(\xi)}{2}\sigma_\tau^2
\end{aligned} \tag{A.11}$$

since $\epsilon_{\tau,\underline{p}} = \epsilon_\tau^{+\varepsilon}$ and so $\mathbb{E}_{\epsilon_{\tau,\underline{p}}}(E - \mathbb{E}_{\epsilon_{\tau,\underline{p}}} E)^2 = \sigma_\tau^2$, and where, for each $\xi \in \Xi$

$$D''(\xi) = \frac{2}{\mathbb{E}_{\epsilon_{\tau,\underline{p}}}(E - \mathbb{E}_{\epsilon_{\tau,\underline{p}}} E)^2}\left(\mathbb{E}_{\epsilon_{\tau,\underline{p}}} D(E,\xi) - D(\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E, \xi) - \mathbb{E}_{\epsilon_{\tau,\underline{p}}} D_1(\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E, \xi)(E - \mathbb{E}_{\epsilon_{\tau,\underline{p}}} E)\right)$$

Note that

$$D''(\xi) = \frac{\mathbb{E}_{\epsilon_{\tau,\underline{p}}} D(\bullet, \xi)''^{D_1(\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E, \xi)}_{\mathbb{E}_{\epsilon_{\tau,\underline{p}}} E, E}}{\mathbb{E}_{\epsilon_{\tau,\underline{p}}}(E - \mathbb{E}_{\epsilon_{\tau,\underline{p}}} E)^2} \tag{A.12}$$

so $D''(\xi)$ is an expectation, over $\epsilon_{\tau,\underline{p}}$, of the average second derivative of the damage function $D(\bullet,\xi)$ between $\epsilon_{\tau,\underline{p}}$ and each point the support of $\epsilon_{\tau,\underline{p}}$. So $\mathbb{E}_q D''(\xi)$ is an expected average second derivative over $\epsilon_{\tau,\underline{p}}$, under the distribution q.

Combining (A.11) and (A.10), and denoting $\mathbb{E}_q D''(\xi)$ by $\mathbb{E}\hat{D}''$, yields:

$$\begin{aligned}\max_{q\in\mathcal{D}} \mathbb{E}_p \mathbb{E}_{\epsilon_{\tau,p}} D(E,\xi) \geq & \mathbb{E}_q \mathbb{E}_{\epsilon_{\tau,p}} D(E,\xi) \\ = & \hat{D}(E_\tau) + \hat{D}'\varepsilon_\tau + \frac{\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau}}{2}\varepsilon_\tau^2 + \frac{\mathbb{E}\hat{D}''}{2}\sigma_\tau^2\end{aligned} \tag{A.13}$$

Combining (A.9) and (A.13), we obtain:

$$\begin{aligned}T_{price} = & \max_{p\in\mathcal{C}}\left(\max_{q\in\mathcal{D}} \mathbb{E}_q \mathbb{E}_{\epsilon_{\tau,p}} D(E,\xi) + \mathbb{E}_p C(C_1^{-1}(\tau,\theta),\theta)\right) \\ \geq & \max_{q\in\mathcal{D}} \mathbb{E}_q \mathbb{E}_{\epsilon_{\tau,\underline{p}}} D(E,\xi) + \mathbb{E}_{\underline{p}} C(C_1^{-1}(\tau,\theta),\theta) \\ \geq & \hat{C}(E_\tau) + \hat{D}(E_\tau) + \varepsilon_\tau(\tau + \hat{D}') \\ & + \frac{\sigma_\tau^2}{2}\left(\mathbb{E}\hat{D}'' - \left(\mathbb{E}_u \widehat{C^\tau}''^{\tau}_{E_r,E_\tau} + \frac{1}{\sigma_\tau^2} cov(\widehat{C^\tau}''^{\tau}_{E_r,E_\tau}, (E_r - E_\tau)^2)\right)\right) \\ & + \frac{\varepsilon_\tau^2}{2}\left[\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau} - \mathbb{E}_{u_r}\mathbb{E}_{u_t} \frac{\underline{C''^t}\widehat{C^\tau}''^{\tau}_{E_r,E_r-\delta_{r,t}}}{\widehat{C^\tau}''^{\tau}_{E_r,E_r-\delta_{r,t}} - \underline{C''^t}}\right]\end{aligned} \tag{A.14}$$

as required. □

Let L^* be the optimal quantity policy. Under the tax policy τ,

$$\begin{aligned}& T_\tau - T_{quantity} \\ \geq & \underbrace{\left(\hat{C}(E_\tau) + \hat{D}(E_\tau)\right) - \left(\hat{C}(L^*) + \hat{D}(L^*)\right)}_{\text{Misspecification}} + \underbrace{\varepsilon_\tau(\tau + \hat{D}')}_{\text{1st order}} \\ & \underbrace{+ \frac{\sigma_\tau^2}{2}\left(\mathbb{E}\hat{D}'' - \left(\mathbb{E}_u \widehat{C^\tau}''^{\tau}_{E_r,E_\tau} + \frac{1}{\sigma_\tau^2} cov(\widehat{C^\tau}''^{\tau}_{E_r,E_\tau}, (E_r - E_\tau)^2)\right)\right)}_{\text{Risk}} \\ & \underbrace{+ \frac{\varepsilon_\tau^2}{2}\left[\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau} - \mathbb{E}_{u_r}\mathbb{E}_{u_t} \frac{\underline{C''^t}\widehat{C^\tau}''^{\tau}_{E_r,E_r-\delta_{r,t}}}{\widehat{C^\tau}''^{\tau}_{E_r,E_r-\delta_{r,t}} - \underline{C''^t}}\right]}_{\text{Uncertainty}}\end{aligned} \tag{A.15}$$

At this level of generality, the optimal tax policy τ^* does not necessarily yield the optimal quota for emissions reduction in expectation: i.e. E_{τ^*} may differ from L^*. The first 'Misspecification' term is the difference between the expected total costs under the expected emissions reduction level generated by the pricing and quantity policies. Since the quantity policy provides the emissions reductions level minimising expected total costs, this term is always positive. Relatedly, the marginal damages at the expected emissions reduction level E_{τ^*} need not match (the negation of) the optimal tax level τ^*; whence the second, '1st-order' term. Whenever the expected worst-case emissions reduction under tax τ^* is equal to the optimal emissions reduction level L^*, these two terms reduce to zero. By specifying this in Assumption 1, we concentrate on this case in the text, hence focusing on the 'Risk' and 'Uncertainty' terms.

A.2 Proofs of Theorems 1 and 2

Proof of Theorem 1. Let τ^* be an optimal tax level. Suppose that $\tau^* \in \partial\hat{C}(E)$ for some $E \neq L^*$. By Lemma A.1 (Appendix A.5), then there exists $\theta \in \operatorname{supp}\mathcal{C}$ with $C_1(E,\theta) = \tau^*$ and $C(E,\theta) = \hat{C}(E)$, so:

$$\begin{aligned}\hat{D}(C_1^{-1}(\tau^*,\theta)) + C(C_1^{-1}(\tau^*,\theta),\theta) =& \hat{D}(E) + \hat{C}(E) \\ >& \hat{D}(L^*) + \hat{C}(L^*) = T_{quant}\end{aligned}$$

since, by the strict convexity of $\hat{D}$ and $\hat{C}$ (Section 2.3), L^* is the unique quantity optimum. If $\tau^* \in \partial\hat{C}(L^*)$, then by a similar argument there exists $\theta \in \operatorname{supp}\mathcal{C}$ with $\hat{D}(C_1^{-1}(\tau^*,\theta)) + C(C_1^{-1}(\tau^*,\theta),\theta) = \hat{D}(L^*) + \hat{C}(L^*) = T_{quant}$. So $T_{price} \geq T_{quant}$.

As concerns the second clause of the theorem, note first that the reasoning in the proof of Theorem A.1 applies for any $\varepsilon \leq \varepsilon_{\tau^*}$. Noting that when $\mathcal{C}$ is generated by constraints $\sigma^2_{\tau^*} = 0$ and that, since L^* is the quantity optimum, $\tau^* \in \partial\hat{D}(L^*)$, the reasoning in the proof of Theorem A.1 implies that, for $\tau^* \in \partial\hat{C}(L^*)$ and any $\varepsilon \in [0, \varepsilon_{\tau^*}]$:

$$T_{\tau^*} \geq \hat{C}(L^*) + \hat{D}(L^*) + \frac{\varepsilon^2}{2}\left[\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon} - \frac{\underline{C''}\hat{C}''^{\tau^*}_{L^*,L^*-\delta}}{\hat{C}''^{\tau^*}_{L^*,L^*-\delta} - \underline{C''}}\right] \tag{A.16}$$

$$\tag{A.17}$$

where δ satisfies $\delta = \underline{C''}\frac{\varepsilon}{\hat{C}''^{\tau^*}_{L^*,L^*-\delta} - \underline{C''}}$. Since, for sufficiently small ε, $\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon} \approx \hat{D}''(L^*)$ and $\hat{C}''^{\tau^*}_{L^*,L^*+\varepsilon} \approx \hat{C}''(L^*)$ whenever these functions are second-order differentiable, $T_{\tau^*} > T_{quantity}$ whenever $\hat{D}''(L^*) > \frac{\hat{C}''(L^*)\underline{C''}}{\hat{C}''(L^*) - \underline{C''}}$, as required. □

Proof of Theorem 2. Under Assumption 1, the 'Misspecification' and 'Risk' terms reduce to zero. Moreover $\widehat{C^r}''^{\tau^*}_{E_r,E_{\tau^*}} \approx \widehat{C^r}''(E_{\tau^*})$ so $\mathbb{E}_u\widehat{C^r}''^{\tau^*}_{E_r,E_{\tau^*}} \approx \mathbb{E}_u\widehat{C^r}''(E_{\tau^*}) = \hat{C}''(E_{\tau^*})$, noting that $\hat{C}''(E_{\tau^*})$ exists because $\widehat{C^r}''(E_{\tau^*})$ does for all r. Substituting this in, proceeding similarly for $\mathbb{E}\hat{D}''$ and incorporating the assumption on the covariance yields the 'Risk' term. The substitution yielding the 'Uncertainty' term follows similarly. □

A.3 Higher derivatives

Proof of Theorem 3. The proof of the first clause is essentially identical to the proof of Theorem 1. The proof of the second clause is similar to that of Theorems 1 and A.1, with the passage in the latter theorem concerning the function G (and up to Eq. (A.8)) replaced by the following reasoning. (For notational simplicity, we remove mention of t, and work with the single lower bound $\underline{C'''}$ on the third derivative. The rest of the notation is as in the proof of Theorem A.1.)

For each $r \in (0,1]$, let δ_r be such that

$$\tau - (\varepsilon_\tau + \delta_r)\widehat{C^r}''(E_r - \delta_r) - \frac{1}{2}(\varepsilon_\tau + \delta_r)^2 \underline{C'''} = \widehat{C^r}'(E_r - \delta_r) \tag{A.18}$$

if such δ_r exists (i.e. when $\widehat{C^r}$ is twice differentiable at $E_r - \delta_r$), and the solution to

$$\tau - (\varepsilon_\tau + \delta_r)\widehat{C^r}''(E_r - \delta_r) - \frac{1}{2}(\varepsilon_\tau + \delta_r)^2 (\underline{C'''} + \rho_r) = \widehat{C^r}'(E_r - \delta_r)$$

for some small $\rho_r > 0$ such that such a solution exists, otherwise . By the reasoning in the proof of Theorem A.1 (and the fact that, by the third-derivative constraint, the marginal abatement curve is convex), δ_r is well-defined, and for each r, it satisfies (A.18) except a set of measure zero.

Let $\hat{\theta}_r$ be any cost function such that $C_1(E_r - \delta_r, \hat{\theta}_r) = \widehat{C^r}'(E_r - \delta_r)$, $C(E_r - \delta_r, \hat{\theta}_r) = \widehat{C^r}(E_r - \delta_r)$ and $C_{11}(E_r - \delta_r, \hat{\theta}_r) = \widehat{C^r}''(E_r - \delta_r)$; such a function exists by the definition of $\widehat{C^r}$ (and the fact that it is twice differentiable at the points involved). Consider the function $G : [0, E_r + \varepsilon_\tau] \to \mathbb{R}$ defined by

$$G(E) = \begin{cases} C(E, \theta_r) & \text{if } E \le E_r - \delta_r \\ C(E_r - \delta_r, \hat{\theta}_r) + \widehat{C^r}'(E_r - \delta_r)(E - E_r + \delta_r) & \\ + \frac{\widehat{C^r}''(E_r - \delta_{r,t})}{2}(E - E_r + \delta_r)^2 + \frac{\underline{C'''}}{6}(E - E_r + \delta_r)^3 & \text{if } E_r - \delta_r < E \le E_r + \varepsilon_\tau \end{cases}$$

This hits the $\widehat{C^r}$ curve at a point, and then increases with the t-lowest third-order derivative until the marginal abatement cost reaches τ, which occurs at $E_r + \varepsilon_\tau$. Note that G is increasing, twice differentiable, strictly convex and satisfies all the constraints in $\mathcal{C}$ on its domain. Hence there exists $\underline{\theta_r} \in \mathcal{C}$ with $C(E, \underline{\theta_r}) = G(E)$ for all $E \in [0, E_r + \varepsilon_\tau]$. Take any such $\underline{\theta_r}$.

Analogous to the definition of average second-order derivative (Section 2.4), define the average third-order derivative $\widehat{C^r}'''^{\tau}_{E_r, E_r - \delta_r}$ to be such that:

$$\widehat{C^r}(E_r - \delta_r) = \widehat{C^r}(E_r) - \tau\delta_r + \frac{\delta_r^2}{2}\widehat{C^r}''(E_r) - \frac{\delta_r^3}{6}\widehat{C^r}'''^{\tau}_{E_r, E_r - \delta_r}$$

from which it follows that

$$\widehat{C^r}'(E_r - \delta_r) = \tau - \delta_r \widehat{C^r}''(E_r) + \frac{\delta_r^2}{2}\widehat{C^r}'''^{\tau}_{E_r, E_r - \delta_r} \tag{A.19}$$

and

$$\widehat{C^r}''(E_r - \delta_r) = \widehat{C^r}''(E_r) - \delta_r \widehat{C^r}'''^{\tau}_{E_r, E_r - \delta_r} \tag{A.20}$$

Plugging this into (A.18) yields:

$$(\delta_r+\varepsilon_\tau)^2 = 2\frac{\varepsilon_\tau \widehat{C^r}''(E_r) + \frac{\varepsilon_\tau^2}{2}\widehat{C^r}'''^\tau_{E_r,E_r-\delta_r}}{\widehat{C^r}'''^\tau_{E_r,E_r-\delta_r} - \underline{C'''}} \tag{A.21}$$

for all $r \in [0,1]$ except a set of measure zero. A similar expression holds, with $\underline{C'''}$ replaced by $\underline{C''} + \rho_r$, otherwise. Hence

$$\delta_r = (X-1)\varepsilon_\tau \tag{A.22}$$

for $X = \left(\frac{\frac{2}{\varepsilon_\tau}\widehat{C^r}''(E_r)+\widehat{C^r}'''^\tau_{E_r,E_r-\delta_r}}{\widehat{C^r}'''^\tau_{E_r,E_r-\delta_r}-\underline{C'''}}\right)^{\frac{1}{2}} > 0$.

It follows, substituting these equations in appropriately, that

$$\begin{aligned}
C(E_r+\varepsilon_\tau,\underline{\theta_r}) =& C(E_r-\delta_r,\underline{\theta_r}) + \widehat{C^r}'(E_r-\delta_{r,t})(\varepsilon_\tau+\delta_r) \\
&+ \frac{\widehat{C^r}''(E_r-\delta_r)}{2}(\varepsilon_\tau+\delta_r)^2 + \frac{\underline{C'''}}{6}(\varepsilon_\tau+\delta_r)^3 \\
=& C(E_r,\theta_r) - \tau\delta_r + \frac{\delta_r^2}{2}\widehat{C^r}''(E_r) - \frac{\delta_r^3}{6}\widehat{C^r}'''^\tau_{E_r,E_r-\delta_r} \\
&+ (\tau - \delta_r\widehat{C^r}''(E_r) + \frac{\delta_r^2}{2}\widehat{C^r}'''^\tau_{E_r,E_r-\delta_r})(\varepsilon_\tau+\delta_r) \\
&+ \frac{\widehat{C^r}''(E_r) - \delta_r\widehat{C^r}'''^\tau_{E_r,E_r-\delta_r}}{2}(\varepsilon_\tau+\delta_r)^2 + \frac{\underline{C'''}}{6}(\varepsilon_\tau+\delta_r)^3 \\
=& C(E_r,\theta_r) + \tau\varepsilon_\tau + \frac{\varepsilon_\tau^2}{2}\widehat{C^r}''(E_r) \\
&+ \frac{\varepsilon_\tau^3}{2}\widehat{C^r}'''^\tau_{E_r,E_r-\delta_r}\left(-\frac{1}{3}(X-1)^3 + (X-1)^2X - X^2(X-1)\right) + \frac{\varepsilon_\tau^3}{2}\frac{\underline{C'''}}{3}X^3 \\
=& C(E_r,\theta_r) + \tau\varepsilon_\tau + \frac{\varepsilon_\tau^2}{2}\widehat{C^r}''(E_r) \\
&- \frac{\varepsilon_\tau^3}{6}\left[X^3\left(\widehat{C^r}'''^\tau_{E_r,E_r-\delta_r} - \underline{C'''}\right) - \widehat{C^r}'''^\tau_{E_r,E_r-\delta_r}\right] \\
=& C(E_r,\theta_r) + \tau\varepsilon_\tau + \frac{\varepsilon_\tau^2}{2}\left[\widehat{C^r}''(E_r)\right. \\
&-\frac{1}{3}\frac{1}{\left(\widehat{C^r}'''^\tau_{E_r,E_r-\delta_r} - \underline{C'''}\right)^{\frac{1}{2}}}\left[\left(2\varepsilon_\tau^{-\frac{1}{3}}\widehat{C^r}''(E_r) + \varepsilon_\tau^{\frac{2}{3}}\widehat{C^r}'''^\tau_{E_r,E_r-\delta_r}\right)^{\frac{3}{2}}\right. \\
&\left.\left.-\varepsilon_\tau\widehat{C^r}'''^\tau_{E_r,E_r-\delta_r}\left(\widehat{C^r}'''^\tau_{E_r,E_r-\delta_r} - \underline{C'''}\right)^{\frac{1}{2}}\right]\right]
\end{aligned} \tag{A.23}$$

for all $r \in [0,1]$ except a set of measure zero. Continuing as in the proof of Theorem A.1 yields an expression like Eq. (A.1) except that the uncertainty term (see Eq. (A.15)) is replaced by (inserting the subscript t back in):

$$\frac{\varepsilon_\tau^2}{2}\Bigg[\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau}+\mathbb{E}_{u_r}\mathbb{E}_{u_t}\Bigg(\widehat{C^r}''(E_r)$$

$$-\frac{1}{3\left(\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}-\underline{C'''}\right)^{\frac{1}{2}}}\left(\left(2\varepsilon_\tau^{-\frac{1}{3}}\widehat{C^r}''(E_r)+\varepsilon_\tau^{\frac{2}{3}}\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}\right)^{\frac{3}{2}}-\varepsilon_\tau\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}\left(\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}-\underline{C'''}\right)^{\frac{1}{2}}\right)\Bigg)\Bigg]$$

$$=\frac{\varepsilon_\tau^2}{2}\left[\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau}+\mathbb{E}_{u_r}\mathbb{E}_{u_t}\left(\widehat{C^r}''(E_r)+\frac{1}{3}\varepsilon_\tau\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}-\frac{\varepsilon_\tau^{-\frac{1}{2}}\left(2\widehat{C^r}''(E_r)+\varepsilon_\tau\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}\right)^{\frac{3}{2}}}{3\left(\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}-\underline{C'''}\right)^{\frac{1}{2}}}\right)\right] \tag{A.24}$$

Noting, as in the proof of Theorem 1, the simplifications implied by the problem under constraints, we obtain that $T_{\tau^*} > T_{quantity}$ whenever

$$\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon_{\tau^*}}+\hat{C}''(L^*)+\frac{1}{3}\varepsilon_{\tau^*}\hat{C}'''(L^*)-\frac{\left(2\hat{C}''(L^*)+\varepsilon_{\tau^*}\hat{C}'''(L^*)\right)^{\frac{3}{2}}}{3\varepsilon_{\tau^*}^{\frac{1}{2}}\left(\hat{C}'''(L^*)-\underline{C'''}\right)^{\frac{1}{2}}}>0$$

as required.

We note finally that this proof also establishes a generalisation of Theorem A.1 to incorporate uncertainty in the third derivative of abatement costs. □

A.4 Constraints on parameters

The following is the general result concerning uncertainty generated by parametric constraints. The Proposition presented in the text is a corollary, as shown below.

Theorem A.2. *Consider any $\mathcal{D} \subseteq \Delta(\Xi)$ characterising uncertainty about damages, and suppose that uncertainty about abatement costs is characterised by probabilistic parametric constraints (Section 2.2). For any τ such that $\tau \in \partial\hat{C}(E)$ for some $E \in [0, E^{max}]$, any $\hat{D}' \in \partial\hat{D}(E_\tau)$, any $\underline{p} \in \mathcal{C}$ such that $\epsilon_{\tau,\underline{p}}E = \epsilon_\tau^{+\varepsilon_\tau}$ and any family $\theta_r, \underline{\theta_r} \in \operatorname{supp}\mathcal{C}$ such that $C_1(E_r,\theta_r) = \tau$, $C(E_r,\theta_r) = \widehat{C^r}(E_r)$, $C_1(E_r+\varepsilon_\tau,\underline{\theta_r}) = \tau$ and $\underline{\theta_r} \in \operatorname{supp}\underline{p}$ for every $r \in (0,1]$:*

$$T_\tau \geq \hat{C}(E_\tau)+\hat{D}(E_\tau)+\varepsilon_\tau(\tau+\hat{D}') \tag{A.25}$$

$$+\frac{\sigma_\tau^2}{2}\left(\mathbb{E}\hat{D}''-\left(\mathbb{E}_u\widehat{C^r}''^{\tau}_{E_r,E_\tau}+\frac{1}{\sigma_\tau^2}cov(\widehat{C^r}''^{\tau}_{E_r,E_\tau},(E_r-E_\tau)^2)\right)\right) \tag{A.26}$$

$$+\frac{\varepsilon_\tau^2}{2}\left[\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau}+\frac{2}{\varepsilon_\tau^2}\mathbb{E}_u\left(\int_0^{E^r}e\left(C_{11}(e,\theta_r)-C_{11}(e,\underline{\theta_r})\right)de-\int_{E_r}^{E_r+\varepsilon_\tau}eC_{11}(e,\underline{\theta_r})de\right)\right] \tag{A.27}$$

where all the terms are as in Theorem A.1.

Proof. The proof is identical to the proof of Theorem A.1, apart from the part introducing $\delta_{r,t}$ (and finishing with Eqn (A.9)), which should be replaced by the following.

Now consider $\underline{p} \in \mathcal{C}$ with $\epsilon_{\tau,\underline{p}}E = \epsilon_{\tau}^{+\varepsilon_\tau}$ (by the definition of ε_τ, such $\underline{p}$ exists), and $\underline{\theta_r} \in \operatorname{supp} \underline{p}$ as specified in the statement of the Theorem. For each such $\underline{\theta_r}$, we have:

$$\begin{aligned}
C(E_r + \varepsilon_\tau, \underline{\theta_r}) &= C(E_r, \theta_r) + \left(\int_0^{E_r+\varepsilon_\tau} C_1(e, \underline{\theta_r}) de - \int_0^{E_r} C_1(e, \theta_r) de \right) \\
&= C(E_r, \theta_r) + \left(\tau(E_r + \varepsilon_\tau) - \int_0^{E_r+\varepsilon_\tau} eC_{11}(e, \underline{\theta_r}) de - \tau E_r + \int_0^{E_r} eC_{11}(e, \theta_r) de \right) \\
&= C(E_r, \theta_r) + \tau\varepsilon_\tau + \int_0^{E_r} e \left(C_{11}(e, \theta_r) - C_{11}(e, \underline{\theta_r}) \right) de - \int_{E_r}^{E_r+\varepsilon_\tau} eC_{11}(e, \underline{\theta_r}) de
\end{aligned}$$

where the second inequality follows by integration by parts, and θ_r is as specified in the statement of the Theorem (as well as in the proof of Theorem A.1). Whence

$$\begin{aligned}
\mathbb{E}_{\underline{p}} C(C_1^{-1}(\tau, \theta), \theta) &= \mathbb{E}_u C(E_r + \varepsilon_\tau, \underline{\theta_r}) \\
&= \mathbb{E}_u C(E_r, \theta_r) + \mathbb{E}_u \tau\varepsilon_\tau \\
&+ \mathbb{E}_u \left(\int_0^{E_r} e \left(C_{11}(e, \theta_r) - C_{11}(e, \underline{\theta_r}) \right) de - \int_{E_r}^{E_r+\varepsilon_\tau} eC_{11}(e, \underline{\theta_r}) de \right) \\
&= \hat{C}(E_\tau) + \tau\varepsilon_\tau - \frac{1}{2} \left(\mathbb{E}\widehat{C}''\sigma_\tau^2 + cov(\widehat{C}'', \sigma_\tau^2) \right) \\
&+ \mathbb{E}_u \left(\int_0^{E_r} e \left(C_{11}(e, \theta_r) - C_{11}(e, \underline{\theta_r}) \right) de - \int_{E_r}^{E_r+\varepsilon_\tau} eC_{11}(e, \underline{\theta_r}) de \right)
\end{aligned} \tag{A.28}$$

where the third equality comes from substituting in Eq. (A.2).

□

Proof of Proposition 1. The proof of the first clause is essentially identical to the proof of Theorem 1, noting that $\hat{C}$ is second-order differentiable (for $E < E^{max}$) since $\hat{C}'(E) = \overline{a} - \overline{b}\ln(E^{max} - E)$ for all E. For the second clause, first note that, for $\tau^* = \hat{C}'(L^*)$, $f(L^*, \overline{a}, \overline{b}) = \tau^*$ since this is the highest marginal cost function; similarly, by the definition of ε_{τ^*}, $f(L^* + \varepsilon_{\tau^*}, \underline{a}, \underline{b}) = \tau^*$. Taking Theorem A.2, applying the reductions employed in the proof of Theorem 1 and substituting the parametric expressions for $C(\bullet, \theta)$ yields, for L^* the quantity optimum:

$$
\begin{aligned}
T_{\tau^*} \geq& \hat{C}(L^*) + \hat{D}(L^*) + \frac{\varepsilon_{\tau^*}^2}{2}\left[\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon_{\tau^*}} + \frac{2}{\varepsilon_{\tau^*}^2}\left((L^*+\varepsilon_{\tau^*})\underline{b} - L^*\overline{b}\right.\right. \\
&\left.\left. - E^{max}(\overline{a} - \overline{b}\ln E^{max} - (\underline{a} - \underline{b}\ln E^{max}))\right)\right] \\
=& \hat{C}(L^*) + \hat{D}(L^*) + \frac{\varepsilon_{\tau^*}^2}{2}\left[\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon_{\tau^*}} + \frac{2}{\varepsilon_{\tau^*}}\frac{\left((L^*+\varepsilon_{\tau^*})\underline{b} - L^*\overline{b}\right)}{\varepsilon_{\tau^*}}\right. \\
&\left. - \frac{2}{\varepsilon_{\tau^*}^2}E^{max}(\overline{a} + \overline{b}\ln E^{max} - (\underline{a} + \underline{b}\ln E^{max}))\right]
\end{aligned}
\tag{A.29}
$$

since $\ln(E^{max} - (L^* + \varepsilon)) = \frac{\tau^* - \underline{a}}{\underline{b}}$ and $\ln(E^{max} - L^*) = \frac{\tau^* - \overline{a}}{\overline{b}}$. The result follows immediately. □

A.5 Other results and remarks

Proposition A.1. *Consider $\mathcal{C}$ generated by probabilistic constraints (be they on costs or parametric). For all $r \in (0, 1]$, $\widehat{C^r}$ is strictly convex.*

Proof. We reason for the case of probabilistic constraints on costs; the case of probabilistic parametric constraints is analogous. Let $\mathcal{C} = \mathcal{C}_{\{m^{(\rho)},M^{(\rho)},m^{(\rho)1},M^{(\rho)1},m^{(\rho)2},M^{(\rho)2}\}}$ and for every $\rho \in (0, 1]$, let $\Theta_\rho \subseteq \Theta$ be $\left\{\theta \in \Theta : \begin{array}{c} C(E,\theta) \in [m^{(\rho)}(E), M^{(\rho)}(E)] \\ C_1(E,\theta) \in [m^{(\rho)1}(E), M^{(\rho)1}(E)] \\ C_{11}(E,\theta) \in [m^{(\rho)2}(E), M^{(\rho)2}(E)] \end{array}\right\}$. Hence, by the definition of $\mathcal{C}$, for every $p \in \mathcal{C}$, $p(\Theta_\rho) \geq \rho$.

Suppose that there exists x and $p \in \mathcal{C}$ with $p(\{\theta : C(E, \theta) \geq x\}) \geq r$. Since, by the previous observation, $p(\Theta_{1-r}) \geq 1-r$, it follows that there exists $\theta \in \Theta_{1-r}$ with $C(E, \theta) \geq x$. So $\{x : \exists p \in \mathcal{C},\ p(\{\theta : C(E, \theta) \geq x\}) \geq r\} \subseteq \{x : \exists\theta \in \Theta_{1-r},\ C(E, \theta) \geq x\}$. Conversely, consider x such that $C(E, \theta) \geq x$ for some $\theta \in \Theta_{1-r}$. By the nestedness of the family of constraints defining $\mathcal{C}$, any p satisfying the constraints for $s < 1 - r$ and putting weight r on θ satisfies all of the constraints (including those for $t \geq 1 - r$); so there exists such $p \in \mathcal{C}$. Since, by construction, $p(\{\theta : C(E, \theta) \geq x\}) \geq r$, we have the converse inclusion. Hence

$$\{x : \exists p \in \mathcal{C},\ p(\{\theta : C(E, \theta) \geq x\}) \geq r\} = \{x : \exists\theta \in \Theta_{1-r},\ C(E, \theta) \geq x\}$$

It thus follows that

$$
\begin{aligned}
\widehat{C^r}(E) &= \max_{p\in\mathcal{C}} \sup\{x : p(\{\theta : C(E, \theta) \geq x\}) \geq r\} \\
&= \sup\{x : \exists p \in \mathcal{C},\ p(\{\theta : C(E, \theta) \geq x\}) \geq r\} \\
&= \sup\{x : \exists\theta \in \Theta_{1-r},\ C(E, \theta) \geq x\} \\
&= \sup\{C(E, \theta) : \theta \in \Theta_{1-r}\}
\end{aligned}
$$

So $\widehat{C^r}$ is the pointwise supremum of strictly convex functions, and hence strictly convex. $\square$

Remark A.1 (Average second-order derivatives). Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable, and consider $f''_{x,y}$ for $x, y, \in \mathbb{R}$, defined as in Eqn (2). Then:

$$\begin{aligned} f(y) &= f(x) + \int_x^y f'(z)dz \\ &= f(x) + \int_x^y \left(f'(x) + \int_x^z f''(u)du \right) dz \end{aligned}$$

So

$$f''_{x,y} = \frac{2}{(y-x)^2} \big(f(y) - f(x) - f'(x)(y-x) \big) \tag{A.30}$$

$$= \frac{\int_x^y \left(\int_x^z f''(u)du \right) dz}{\int_x^y \left(\int_x^z du \right) dz} \tag{A.31}$$

which is a (normalised) expectation over the values of f'' between x and y.

Lemma A.1. *Under the conditions of Theorem 1 ($\mathcal{C}$ generated by constraints), if $y \in \partial\hat{C}(E)$ for some E, then there exists $\theta \in \Theta$ with $\delta_\theta \in \operatorname{supp}\mathcal{C}$ such that $C_1(E,\theta) = y$ and $C(E,\theta) = \hat{C}(E)$.*

Proof. Let $\partial\hat{C}(E) = [x, z]$. By standard results in convex analysis, there exists $\delta_{\underline{\theta}}, \delta_{\overline{\theta}} \in \mathcal{C}$ with $C(E,\underline{\theta}) = C(E,\overline{\theta}) = \hat{C}(E)$, $C_1(E,\underline{\theta}) = x$ and $C_1(E,\underline{\theta}) = z$. Since $y \in \partial\hat{C}(E)$, there exists $\alpha \in [0,1]$ with $y = \alpha x + (1-\alpha)z$. Consider the function $C' : [0, E^{max}] \to \mathbb{R}$ such that $C'(e) = \alpha C(e,\underline{\theta}) + (1-\alpha)C(e,\overline{\theta})$. This function is clearly increasing, differentiable and strictly convex since $C(\bullet,\underline{\theta}), C(\bullet,\overline{\theta})$ are; hence there exists $\theta \in \Theta$ such that $C' = C(\bullet,\theta)$. Moreover, for constraints m, M and every e, $m(e) \leq C(e,\underline{\theta}), C(e,\overline{\theta}) \leq M(e)$, the same holds for $C(E,\theta)$. Since the same holds for constraints on marginal cost and its slope, it follows that θ satisfies all the constraints, so $\delta_\theta \in \operatorname{supp}\mathcal{C}$, establishing the result. $\square$