Prices vs Quantities under Severe Uncertainty*

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Abstract

The consensus among economists in favour of carbon taxes over emissions permits is based on a groundbreaking result due to Weitzman (1974). It assumes, however, a probability distribution over abatement costs, and similarly for damages. As many have argued, current climate uncertainties are far more severe, and do not justify any such distributions. This paper reconsiders the tax-permit comparison in the presence of severe or Knightian uncertainty, drawing on the workhorse maxmin-EU model from the literature on decision under uncertainty (Gilboa and Schmeidler, 1989). Our results show that optimally set permits are strictly more efficient than optimal taxes when uncertainty concerning the slope of marginal abatement costs is severe. They suggest that, given the uncertainty reported in the latest IPCC report, permit policies are more efficient.

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1 Introduction

In 2019, over 45 leading economists, including 28 Nobel prize laureates, published a petition calling for a carbon tax to tackle climate change ('Economists' Statement on Carbon Dividends', *Wall Street Journal*, 2019). They reflect a wide consensus in the profession. The economic arguments in favour of market-based instruments – be they by fixing a price for carbon via taxes, or a quantity via permits – are well-known. However, economists typically go beyond, and incorporate a preference for taxes over permits in the case of climate change. The reason draws on a pioneering theorem due to Weitzman (1974), which suggests that, in the absence of complete information, the relative efficiency of the two tools hinges on the comparison between the slopes of the marginal abatement cost and damage curves. Since climate change is a stock problem, the former is typically steeper than the latter, whence the superiority of taxes, and the economic consensus (Nordhaus, 2007; Metcalf, 2009).

However, Weitzman's result assumes that uncertainty about abatement costs can be represented by a probability distribution over possible abatement cost functions, and similarly for damages. In other words, his analysis applies in what decision theorists call situations of *risk*. By contrast, many have noted that climate uncertainties are considerably more severe, especially as concerns the abatement cost and damage functions (Pindyck, 2013; Stern, 2013; Stiglitz, 2019; Stern et al., 2022; Blanchard et al., 2023). Probability distributions over possible (marginal) abatement cost or damage functions are typically not provided by the current state of the science. The situation is thus closer to *(Knightian) uncertainty*, *severe uncertainty* or *ambiguity* (Gilboa and Marinacci, 2013). This article revisits the comparison between prices and quantities under severe uncertainty, by analysing the problem under one of the main ambiguity models in the decision theory literature, Gilboa and Schmeidler's (1989) maxmin-Expected Utility (maxmin-EU) model. As discussed in Sections 2.3 and 5.4, this rule reflects a concern for robustness in policy evaluation in the face of uncertainty; for instance, it nests a robustness rule promoted by Hansen and Sargent (2008).

In the climate context, the maxmin-EU model incorporates a set of probability distributions over possible abatement cost functions, and likewise for damage functions. It evaluates a policy by the worst-case expected total cost (abatement costs plus damages) across all distributions in the sets. It is thus rich enough to cover the spectrum of types of uncertainty between two extremes. On one side, when a single probability distribution has been established, the set used in the rule contains only this distribution, and maxmin-EU reduces to standard expected utility. The analysis thus collapses to Weitzman's comparison of prices and quantities under risk. At the other extreme, the set could be comprised of every probability distribution with support contained in a given set of abatement cost functions. This reflects situations where it is known that the abatement cost function belongs to a certain family or satisfies certain constraints, but nothing more is known about the probabilities of various family members. In such cases, the maxmin-EU rule evaluates a policy by the worst case across all potential abatement cost functions. Then, we show, the comparison of climate policies is diametrically opposed to the message typically drawn from Weitzman's Theorem: the optimal quantity policy is always at least as efficient as the optimal pricing one. Moreover, by identifying a condition for a strict ranking, we establish that whenever current knowledge does not provide specific information about the slope of the marginal abatement costs, the optimal permit policy strictly outperforms optimally set taxes. Based on the chapter of the latest IPCC report dedicated to abatement costs and correspondence with its lead authors, this seems a fair description of the current state of scientific knowledge.

Our general result applies between these extremes: that is, to cases where the science may provide some probabilistic conclusions about (marginal) abatement costs or damages, without necessarily pinning down a single probability distribution over abatement cost or damage functions. Roughly, the result introduces indices of risk and (severe) uncertainty, and partially characterises the relative performance of quantity vs. pricing policies in terms of them. When the risk index dominates, we are close to the Weitzman case, and taxes are more efficient. When the extent of (severe) uncertainty outweighs the risk, then the term behind our previous worst-case result dominates, and permits outperform taxes. The aforementioned admonitions as to the severity of uncertainty in this context (Pindyck, 2013; Stern, 2013; Stiglitz, 2019; Stern et al., 2022) suggest that the latter case corresponds more closely to our current state of knowledge.

We also consider simple extensions incorporating different assumptions about our knowledge concerning (higher) derivatives of marginal abatement costs, uncertainty over parameter values for given parametric families of the abatement cost functions, weaker assumptions about market reaction to carbon prices, or different attitudes to uncertainty. Roughly, they show that the main message of our baseline results – that the purported superiority of taxes over permits is seriously undermined in the presence of (severe) uncertainty – is robust to such factors.

Our results highlight the importance of carefully considering the extent and character of uncertainty when formulating and recommending economic policies for effective emissions mitigation. The justification of taxes that relies on the assumption that the current state of the science provides a probability distribution over abatement cost curves will lead to suboptimal policies whenever this assumption is incorrect. Moreover, subsequent sensitivity analysis in the application of these policies – for instance, when setting the carbon price – cannot fully correct this suboptimality. By applying an evaluation approach that incorporates the actual state of uncertainty from the outset, according to the principles established by decades of research in decision theory, our results provide the bedrock for an arguably more robust analysis of policies for carbon mitigation. They suggest that for uncertainties similar to those that we actually face, quantity policies may be more efficient than pricing ones.

The paper is organised as follows. Section 2 introduces the setup and some preliminary notions and notation, including the representation and types of uncertainty, and policy evaluation with them. Section 3 sets out the optimisation problems for quantity and pricing problems under (severe) uncertainty. Section 4 contains our main theoretical results and discussion, whilst Section 5 presents and discusses a range of extensions. Proofs and other material are contained in the Appendix.

2 Setup

2.1 Emissions, abatement costs and damages

Let the set of potential carbon emissions reductions be $[0, E^{max}] \subseteq \mathbb{R}$, where E^{max} is the maximal emissions reduction (leading to zero emissions). $\Delta([0, E^{max}])$ is the space of probability distributions over $[0, E^{max}]$. For a fixed $\delta \in \mathbb{R}$ and $\epsilon \in \Delta([0, E^{max}])$, $\epsilon^{+\delta}$ is the distribution defined by $\epsilon^{+\delta}(E + \delta) = \epsilon(E)$, whenever such a distribution exists. It is the result of shifting ϵ to the right by δ.

We follow Weitzman (1974) in considering a simple abstract problem, with (abatement) costs and damages.[1] Let Θ denote the space of possible aggregate *abatement cost functions*. For each $\theta \in \Theta$, $C(\bullet, \theta)$ is a real-valued function defined on $[0, E^{max}]$: $C(E, \theta)$ is the aggregate abatement cost of emissions reduction E under the function θ. Θ contains all increasing, differentiable, strictly convex functions on $[0, E^{max}]$ that take the value 0 at zero emissions reductions. These are standard assumptions for abatement cost functions in the literature.[2]

We use Ξ to denote the space of *damage functions*: $D(E, \xi)$ are the damages brought about by emissions reduction E under function $\xi \in \Xi$. Analogous with the case of abatement costs, Ξ contains all decreasing, differentiable, strictly convex functions on $[0, E^{max}]$ that take the value 0 at E^{max}.

The *total costs* of emissions reduction E under abatement cost and damage functions θ and ξ are given by $T(E, \theta, \xi) = C(E, \theta) + D(E, \xi)$.

2.2 Uncertainty

Let $\Delta(\Theta)$ be the set of probability distributions over Θ, and similarly for $\Delta(\Xi)$. Scientific knowledge about abatement costs can be summarised by a (non-empty) set of probability distributions $\mathcal{C}$ over the space of possible cost functions Θ; i.e. a set $\mathcal{C} \subseteq \Delta(\Theta)$. Similarly, the set $\mathcal{D} \subseteq \Delta(\Xi)$ represents knowledge about damages. Without loss of generality for the evaluation of policies, and following standard practice (Gilboa and Marinacci, 2013), we assume that $\mathcal{C}$ and $\mathcal{D}$ are closed and convex.[3]

As standard, for a distribution $p \in \Delta(\Theta)$, $\operatorname{supp} p$ is the support of p: $\operatorname{supp} p = \{\theta \in \Theta : p(\theta) > 0\}$. The support of a set of priors $\mathcal{C}$ is defined similarly: $\operatorname{supp} \mathcal{C} = \bigcup_{p \in \mathcal{C}} \operatorname{supp} p$. For $\theta \in \Theta$, δ_θ denotes δ_θ denotes the Dirac distribution with weight on θ: i.e. such that $\delta_\theta(\theta) = 1$.

2.3 Policy evaluation

To choose among policies, we employ the maxmin-EU decision rule (Gilboa and Schmeidler, 1989), which evaluates an uncertain option by the worst-case expected utility of the outcomes

[1] He talks of costs and benefits; we use climate terminology.

[2] Strict convexity is convenient, but can be weakened at the price of more complexity, with no particular impact on the results.

[3] Whilst not needed for our results, it is standard to work with the weak* topology on $\Delta(\Theta)$ and $\Delta(\Xi)$.

of that option. In the current context it selects the policy P, leading to a total cost $T(P, \theta, \xi)$ under abatement cost function θ and damage function ξ, that minimises:

$$\max_{p \in \mathcal{C}} \max_{q \in \mathcal{D}} \mathbb{E}_p \mathbb{E}_q T(P, \theta, \xi) \tag{1}$$

where $\mathbb{E}_p$ denotes the expectation taken with respect to p (and similarly for q). Eq. (1) picks out, for each policy, its worst-case – i.e. maximum – expected total cost over all probability distributions in $\mathcal{C}$ and $\mathcal{D}$. As such, it provides an evaluation that is *robust* to the uncertainty concerning abatement costs and damages, as reflected in $\mathcal{C}$ and $\mathcal{D}$. Indeed, the constraint preferences in the robustness framework developed by Hansen and Sargent (2008) are a special case of maxmin-EU preferences. Several researchers in economics, philosophy and climate science have argued that rules such as these are normatively more appropriate than Expected Utility for policy decisions in the face of severe uncertainty (Stainforth et al., 2007; Manski, 2013; Gilboa and Marinacci, 2013; Marinacci, 2015; Hill, 2019; Bradley and Steele, 2015; Bradley, 2017; Bradley et al., 2017; Berger and Marinacci, 2020; Berger et al., 2021).

Translating the 'worst-case' maxmin-EU evaluation, define the functions $\hat{C}, \hat{D} : [0, E^{max}] \rightarrow \mathbb{R}$ by:

$$\hat{C}(E) = \max_{p \in \mathcal{C}} \mathbb{E}_p C(E, \theta)$$
$$\hat{D}(E) = \max_{q \in \mathcal{D}} \mathbb{E}_q D(E, \xi)$$

$\hat{C}$ gives the *worst-case* (i.e. highest) *expected abatement costs* under the relevant uncertainty (i.e. over all distributions in $\mathcal{C}$), for every emissions reductions level; similarly for $\hat{D}$ and worst-case damages. These functions are strictly convex[4] but need not be differentiable, so do not necessarily belong to Θ and Ξ.

Similarly, for each $r \in (0, 1]$, the *r-quantile worst-case cost* is given by the function $\widehat{C^r} : [0, E^{max}] \rightarrow \mathbb{R}$ where

$$\widehat{C^r}(E) = \max_{p \in \mathcal{C}} \sup \{x : p(\{\theta : C(E, \theta) \geq x\}) \geq r\}$$

Each probability distribution p generates a distribution over costs for each emissions reduction level E, and $\sup \{x : p(\{\theta : C(E, \theta) \geq x\}) \geq r\}$ is its rth quantile. So $\widehat{C^r}(E)$ is the cost such that, according to the set $\mathcal{C}$, the probability that emissions reduction E yields a cost of $\widehat{C^r}(E)$ or higher is at most r.

2.4 Illustrations

To illustrate the flexibility of the framework, we consider some possible types of uncertainty, and the corresponding policy evaluations under Eq. (1).

[4]This follows from the strict convexity of the functions in Θ and Ξ and standard convex analysis.

Example 1 (Risk)**.** There is *risk* when scientific knowledge identifies a single probability distribution representing uncertainty about, say, abatement costs. In this case, $\mathcal{C}$ is a singleton containing this distribution. When $\mathcal{C}$ and $\mathcal{D}$ are singletons, Eq. (1) evaluates policies by their expected total cost, and hence reduces to standard Expected Utility-based cost-benefit analysis. Weitzman (1974) considers this case, showing, roughly, that optimal taxes are more efficient if $\hat{D}''(L^*) < \hat{C}''(L^*)$, where L^* is the optimal quantity policy.

Example 2 (Categorical Uncertainty)**.** Scientific knowledge could establish a set of *categorical* conclusions that delimit a subset $\underline{\Theta} \subseteq \Theta$ of abatement cost functions. For instance, this set could be defined by a set of constraints on abatement costs or (higher) derivatives at each emissions reductions level, translating scientific knowledge (e.g. that 'the marginal abatement cost at E is greater than y'). Or it could result from evidence justifying a specific parametric family of abatement cost functions, the set corresponding to all members of the family with parameters in a particular scientifically established range.

For such uncertainty, $\mathcal{C}$ is the set of all probability distributions with support in $\underline{\Theta}$. When $\mathcal{C}$ and $\mathcal{D}$ are generated from such sets $\underline{\Theta}$ and $\underline{\Xi} \subseteq \Xi$ in this way, Eq. (1) reduces to the maximum total cost across all abatement cost and damage functions in $\underline{\Theta}$ and $\underline{\Xi}$ respectively.[5] This is the sort of evaluation used in worst-case analysis, insofar as it uses the abatement cost and damage functions yielding the highest total cost.

2.5 Differentiability: notions and terminology

As noted above, we do not assume differentiability of the generated worst-case abatement cost and damage functions (Section 2.3). At points, we thus require concepts from convex analysis, including generalisations of the notion of differentiability to such functions. For a convex function $f : [0, E^{max}] \to \mathbb{R}$ and $x \in \mathbb{R}$, $\partial f(x) \subseteq \mathbb{R}$ denotes the subgradient of f at x, which is a generalisation of the notion of derivative for non-differentiable functions (Rockafellar, 1970, Ch 25). If $\partial f(x)$ contains one element, then f is differentiable at x and the standard derivative $f'(x)$ is the unique member of $\partial f(x)$. By Rockafellar (1970, Theorems 23.5.1, 26.1 & 26.3), if f is strictly convex, its derivative has a well-defined (i.e. single-valued) inverse: for every y, there exists at most one x such that $y \in \partial f(x)$.

For a convex function $f : \mathbb{R} \to \mathbb{R}$, $d \in \partial f(x)$, and $x, y \in \mathbb{R}$ the *average second-order derivative* of f between x and y by d is

$$f''^{d}_{x,y} = \frac{2}{(y-x)^2} \left(f(y) - f(x) - d(y-x) \right) \tag{2}$$

Note that, if f is twice differentiable, this is indeed an average of second-order derivatives (see Remark A.2 in Appendix A.7). When f is differentiable at x, we omit the superscript d.

[5]This follows from the fact that maximisation over $\mathcal{C}$ coincides with maximisation over $\operatorname{supp}\mathcal{C}$ when $\delta_\theta \in \mathcal{C}$ for every $\theta \in \operatorname{supp}\mathcal{C}$.

3 Policies

Our main results compare quantity and pricing policies under the maxmin-EU evaluation (1). We first set out the application of policy evaluation to the two policies in turn.

3.1 Quantity policy

Under a quantity (or permit) policy, the policy maker chooses an emissions reduction level L. Under Eq. (1), the evaluation of the optimal quantity policy is:

$$T_{quant} = \min_{L} \max_{p \in \mathcal{C}} \max_{q \in \mathcal{D}} \mathbb{E}_p \mathbb{E}_q \left(D(L, \xi) + C(L, \theta) \right) \tag{3}$$

The optimal policy is the reductions level leading to the lowest total cost under the maxmin-EU evaluation with $\mathcal{C}$ and $\mathcal{D}$. There is a unique quantity optimum L^*, and all other emissions levels have a strictly higher total cost under Eq. (1). For differentiable $\hat{D}$ and $\hat{C}$, the optimum satisfies the standard condition $\hat{C}'(L^*) = -\hat{D}'(L^*)$.[6]

3.2 Pricing policy

Under a pricing (or tax) policy, the policy maker chooses a tax level τ. Under Eq. (1), the evaluation of the optimal pricing policy is:

$$T_{price} = \min_{\tau} \max_{p \in \mathcal{C}} \max_{q \in \mathcal{D}} \mathbb{E}_p \mathbb{E}_q \left(D(h(\tau, \theta), \xi) + C(h(\tau, \theta), \theta) \right)$$

where $h(\tau, \theta)$ is the emissions reductions level resulting from tax τ, under cost function θ, in equilibrium. Following Weitzman (1974), note that under standard market assumptions and the assumption that agents learn θ, we have:

$$h(\tau, \theta) = C_1^{-1}(\tau, \theta)$$

where $C_1^{-1}(E, \theta)$ is the inverse of $C_1(\bullet, \theta)$, considered as a function of the first coordinate at E.[7] So the optimal pricing policy solves:

$$T_{price} = \min_{\tau} \max_{p \in \mathcal{C}} \max_{q \in \mathcal{D}} \mathbb{E}_p \mathbb{E}_q \left(D(C_1^{-1}(\tau, \theta), \xi) + C(C_1^{-1}(\tau, \theta), \theta) \right) \tag{4}$$

3.3 Risk and Uncertainty under tax τ

As just noted, for each abatement cost function, tax τ will lead to a unique level of emissions reduction. Similarly, under a distribution over possible abatement cost functions, the tax will

[6] In the absence of differentiability, L^* is such that there exists x with $x \in \partial \max_{p \in \mathcal{C}} \mathbb{E}_p C(L^*, \xi)$ and $-x \in \partial \max_{q \in \mathcal{D}} \mathbb{E}_q D(L^*, \xi)$. The strict convexity of $\hat{C}$ and $\hat{D}$ (Section 2.3) implies that subgradients exist; moreover, combined with the facts noted in Section 2.5, it implies uniqueness of the quantity optimum.

[7] Since all cost functions are differentiable and strictly convex, $C_1^{-1}(\bullet, \theta)$ is a well-defined function on $[0, E^{max}]$ for all $\theta \in \Theta$.

result in a distribution over emissions reduction levels. For probability distribution over abatement costs $p \in \Delta(\Theta)$, the distribution on the emissions reduction levels $\epsilon_{\tau,p} \in \Delta([0, E^{max}])$ generated by tax level τ is defined by $\epsilon_{\tau,p}([0, E]) = p(\{\theta : C_1(E', \theta) = \tau \text{ for some } E' \leq E\})$ for all $E \in [0, E^{max}]$.

Similarly, under the r-quantile worst-case cost $\widehat{C^r}$ (Section 2.3), tax τ results in an emissions reduction level which we denote by E_r: i.e. for each $r \in (0, 1]$, E_r is such that $\tau \in \partial\widehat{C^r}(E_r)$.[8] This generates a distribution over emissions levels, defined by $\epsilon_\tau([0, E]) = \inf\{r : E_r \leq E\}$. ϵ_τ is the distribution resulting from tax τ under the 'worst-case' distribution over abatement costs, according to uncertainty $\mathcal{C}$. Note that, since the worst-case (i.e. highest) costs are involved, there is a sense in which this distribution is as far to the left as possible, among all emissions reductions distributions resulting from τ and consistent with uncertainty $\mathcal{C}$.

For tax τ, we denote the expected emissions reduction resulting from τ under the worst-case abatement costs consistent with uncertainty $\mathcal{C}$ by $E_\tau = \mathbb{E}_{\epsilon_\tau} E$. The variance of this worst-case, $\sigma_\tau^2 = \mathbb{E}_{\epsilon_\tau}(E - \mathbb{E}_{\epsilon_\tau} E)^2$, is a proxy for the *risk* (as opposed to the uncertainty) concerning emissions reductions levels resulting from tax τ. For instance, for categorical uncertainty (Example 2), $\sigma_\tau^2 = 0$.

Finally, define:

$$\varepsilon_\tau = \max\{\delta : \epsilon_\tau^{+\delta} \in \{\epsilon_{\tau,p} : p \in \mathcal{C}\}\} \tag{5}$$

ε_τ tracks how far the worst-case emissions reductions distribution can be shifted to the right whilst remaining consistent with the uncertainty $\mathcal{C}$. As such, it is an indication of the 'width' of $\mathcal{C}$, when projected onto the consequences of tax τ in terms of emissions reductions. For instance, if $\mathcal{C}$ contains a single probability distribution (Example 1), then $\varepsilon_\tau = 0$. By contrast, large ε_τ indicates a large variety in expected emissions reductions that could result from tax τ, according to $\mathcal{C}$. In the light of this, we refer to ε_τ as the *uncertainty* concerning emissions reductions levels resulting from tax τ.

4 Policy comparisons

4.1 Worst-case analysis: Categorical Constraints

Figure 1, which is drawn from the chapter on long-term mitigation pathways in the latest Intergovernmental Panel on Climate Change (IPCC) report, provides a starting point for consideration of uncertainty. It plots the marginal abatement cost against emissions for a range of mitigation pathways and climate models in the IPCC AR6 scenario database, consistent with different global warming levels.[9] As such, it summarises the state of scientific knowledge about

[8] By Proposition A.3 in the Appendix, $\widehat{C^r}$ is strictly convex for all r; it follows from the observations in Section 2.5 that E_r is uniquely defined.

[9] As emphasised by the IPCC, the points are not random draws from an underlying distribution, nor even the results of systematic runs of the same model; this undermines the relevance of classical statistical analyses conducted on them, and inhibits drawing strong conclusions about the shape of marginal abatement cost curves from them.

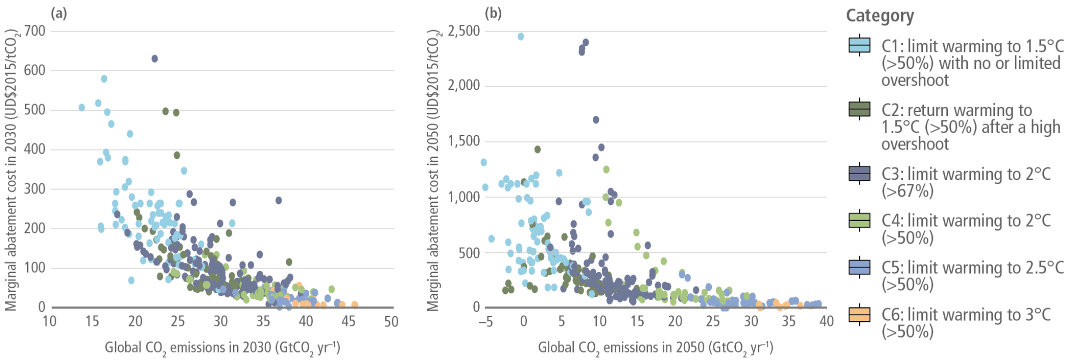

Figure 1: Marginal abatement cost of carbon in 2030 (a) and 2050 (b).
IPCC (2023, Figure 3.33). Note that marginal abatement costs are plotted against emissions, whereas the setup introduced in Section 2 works with (marginal) abatement costs as a function of emissions reductions.

marginal abatement costs at the time of the last IPCC report.

The Figure suggests that marginal abatement costs at a given time can, at best, be reasonably bounded between two functions, following roughly the upper and lower limits of the range of points. However, there is little more that can be said with any confidence, as confirmed by the accompanying text (IPCC, 2023, Ch 3.6.1.1). For instance, whereas it states that marginal abatement costs increase with the extent of emissions reduction, no (other) bounds are reported on the slope of the marginal abatement cost curve.[10] Indeed, the Figure is consistent with a range of 'precise' marginal abatement cost functions, with differing slopes at different points, as long as these functions lie within the upper and lower limits mentioned above. Likewise, the Figure (and accompanying text) offers little to justify robust probabilistic conclusions regarding marginal abatement costs. All of which poses the question: if this summarises the current state of knowledge about abatement costs, how do pricing and quantity policies compare under this extent of uncertainty?

The IPCC report would seem to imply a particular form of categorical uncertainty (Example 2, Section 2.4): at best, scientific knowledge provides a set of constraints on the marginal abatement costs – i.e. upper and lower bounds for each emissions reduction level – and a lower bound of 0 for the slope of the marginal abatement curve. Our first result concerns the more general case of constraints on the abatement and marginal abatement costs, and single lower and upper bounds on the slope of marginal costs. This can be represented by sets of priors of form:

[10] In personal communication, the lead author of this section of the chapter stated that they did 'not believe that there were any established results on the second derivative of the abatement cost'. Several lead authors (we contacted 14 out of the 16 lead and coordinating lead authors, of which 5 replied) emphasised that Figure 1 cannot be used to draw strong conclusions on marginal abatement costs, suggesting that it was tenuous to use it to bound marginal abatement costs, much less their slope. They emphasised the relevance of systematic comparisons of marginal abatement costs with the same model, though, as one lead author put it, 'the resulting [marginal abatement costs] vary significantly across models'. The only studies of this sort they provided are discussed in Section 5.1 below. Only one lead author expressed a stronger opinion on marginal abatement costs, stating that 'the slope of the marginal abatement costs is generally assumed to be positive and increasing with emissions reductions'. Section 5.1 examines the consequences of incorporating such constraints on the third derivative of abatement costs, establishing an extension of the Theorem in this section.

$$\mathcal{C}_{[m,M],[m^1,M^1],[\underline{C''},\overline{C''}]} = \left\{ p \in \Delta(\Theta) : \forall\theta \in \operatorname{supp} p, \begin{array}{l} C(E,\theta) \in [m(E), M(E)] \\ C_1(E,\theta) \in [m^1(E), M^1(E)] \\ C_{11}(E,\theta) \in [\underline{C''}, \overline{C''}] \end{array} \right\} \quad (6)$$

for increasing functions $m, M, m^1, M^1 : [0, E^{max}] \to \mathbb{R}_{\geq 0}$ reflecting the bounds on abatement costs and marginal abatement costs, and lower bound $\underline{C''} \in \mathbb{R}_{\geq 0}$ and upper bound $\overline{C''} > \underline{C''}$ on their slope.

Theorem 1. *Consider any $\mathcal{D} \subseteq \Delta(\Xi)$ characterising uncertainty about damages, and suppose that uncertainty about abatement costs is characterised by $\mathcal{C}_{[m,M],[m^1,M^1],[\underline{C''},\overline{C''}]}$ for some $[m, M]$, $[m^1, M^1]$ and $\underline{C''} < \overline{C''}$. Then $T_{price} \geq T_{quant}$.*

Moreover, if $\hat{D}$ and $\hat{C}$ are second-order differentiable at the optimal quantity level $L^* \in (0, E^{max})$, then $T_{price} > T_{quant}$ whenever $\varepsilon_{\tau^*} > 0$ for optimal tax τ^* and

$$\hat{D}''(L^*) > \frac{\underline{C''}}{\hat{C}''(L^*) - \underline{C''}} \hat{C}''(L^*) \quad (7)$$

Theorem 1 paints a markedly different picture of the comparison between taxes and permits in the world of (severe) uncertainty, as opposed to risk. For one, under uncertainty characterised by constraints on abatement costs and derivatives, quantity policies are always at least as efficient as pricing policies: there are no cases in which the latter outperform the former in a robust worst-case analysis.

Moreover, the Theorem identifies a condition under which quantity policies are guaranteed to be strictly more efficient than pricing ones.[11] The first clause essentially demands there is some uncertainty, in the sense specified in Section 3.3, i.e. a spread in the emissions reductions distributions consistent with the state of knowledge and uncertainty $\mathcal{C}_{[m,M],[m^1,M^1],[\underline{C''},\overline{C''}]}$, under tax τ^*. The second clause covers some notable special cases, and calls for interpretation.

Note first that if the lower bound on the second derivative is 0, then condition (7) is automatically satisfied (since $\hat{D}$ is strictly convex; Section 2.3). This corresponds to the absence of known constraints on the slope of the marginal abatement costs, apart from that it is positive: the result says that quantity policies always outperform pricing policies under such uncertainty. To return to the previous discussion, if, as the IPCC report and Figure 1 suggest, our knowledge can be summarised by constraints on marginal abatement costs with nothing about their slope apart from the fact that it is positive, then the corresponding uncertainty is characterised by a set of the sort defined in Eq. (6) with form $\mathcal{C}_{(-\infty,\infty),[m^1,M^1],[0,\infty)}$. Theorem 1 thus shows that, under current uncertainty about abatement costs as reported by the IPCC and any uncertainty about damages, quantity policies are more efficient than pricing policies.[12]

[11] The condition can be strengthened to accommodate non-second-order differentiability of damages and abatement costs at L^*, using notions from convex analysis. Such technical details are omitted.

[12] Note that the result holds independently of the constraints $[m^1, M^1]$ on marginal abatement costs; in particular, it holds even if, as suggested by some experts cited in footnote 10, they are far wider than suggested by Figure 1.

More generally, (7) resembles Weitzman's condition pertaining to taxes vs. permits under risk (Example 1) insofar as it involves slopes of marginal damages and abatement costs. However, instead of comparing $\hat{D}''(L^*)$ directly with the slope of the marginal worst-case costs $\hat{C}''(L^*)$, the condition involves the multiplicative factor $\frac{\underline{C''}}{\hat{C}''(L^*)-\underline{C''}}$. This factor decreases as the lower bound $\underline{C''}$ decreases, but also as the difference between the slope of the marginal worst-case costs and the lower bound increases. So, whilst $\hat{C}''(L^*)$ – which could conceivably be gleaned from Figure 1 by looking at the upper limit on the marginal abatement costs – may be larger than $\hat{D}''(L^*)$, if the scientifically established lower bound is significantly lower, the condition in the Theorem may still be satisfied. This suggests that the performance comparison between quantity and pricing policies may be sensitive to the slope of the marginal costs used. In particular, a value reflecting the extent of uncertainty about the slope is more relevant than 'average' slopes drawn, say, from fitting marginal abatement curves to the points in plots such as Figure 1.

Note finally that, unlike in Weitzman's result, the condition in Theorem 1 is sufficient but not necessary. This reflects the global nature of our result, which, unlike Weitzman's, depends on no local approximations or assumptions that the uncertainty is 'small'. This is relevant in the current context since, as has been noted, such assumptions are largely unjustified.

Remark 1. The central intuition behind Theorem 1 – and indeed many of our results – is illustrated in Figure 2. In the presence of uncertainty, there will be a gap between the 'highest' (marginal) abatement cost function according to $\mathcal{C}$ ($\hat{C}$; Section 2.3) – depicted in black in the Figure – and 'lower' functions consistent with $\mathcal{C}$, such as the green one. But then, abatement cost functions which follow the 'highest' one up to a point and then increase with a slope of marginal abatement cost equal to the lower bound $\underline{C''}$ – such as the red curve in the Figure – are possible according to the state of knowledge $\mathcal{C}$. Evaluation of the worst-case total cost for the optimal quantity policy L^* involves the black 'highest' abatement cost curve; by contrast, both the black and red ones are relevant for evaluation of the tax τ^*. Under the black marginal abatement cost curve, the tax has the same total costs as the quota: so, in worst-case, τ^* cannot have a strictly lower total cost than L^*, yielding the first clause of the Theorem. The net pink shaded area (negative above the red curve; positive below) indicates the difference in total cost between the evaluation of the tax under the red abatement cost function and the worst-case damage function on the one hand, and the worst-case evaluation of the quota on the other. If it is strictly positive – as it clearly will be for small $\underline{C''}$ – then the total cost under tax τ^* will be strictly higher than under L^*: so, under (1), the optimal tax policy will be strictly less efficient than optimal quotas. This yields the second clause of the Theorem. The reader is referred to Appendices A.1 and A.2 for further details.[13]

[13] It is not difficult to come up with narratives corresponding to (marginal) abatement cost functions such as the red one in Figure 2, for instance in terms of expensive technological breakthroughs which unlock subsequent low-cost emissions mitigation possibilities. For policy evaluation, the question is what current scientific knowledge says about such possibilities: as is evident in the preceding discussion of the state of the science as reported by the IPCC, it seems that it can't rule them out.

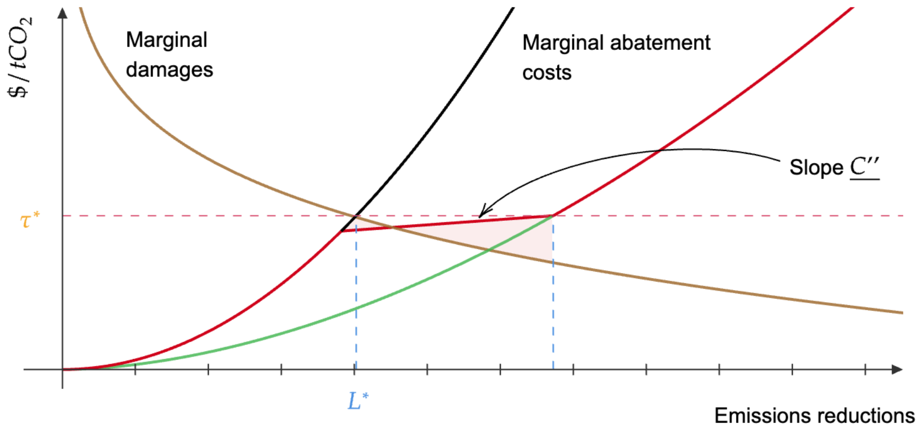

Figure 2: Theorem 1: intuition.
The black and brown lines depict the 'highest' marginal abatement cost and marginal damage functions in $\mathcal{C}$ and $\mathcal{D}$, respectively. The green line depicts a 'lower' marginal abatement cost function in $\mathcal{C}$. The red line depicts a marginal abatement cost function that satisfies all the constraints (in particular its slope is at least $\underline{C''}$), so it is also in $\mathcal{C}$. The shaded pink area indicates the difference between the total cost with tax τ^* under the red marginal abatement cost function and the worst-case total cost under the optimal quantity policy L^*. When this is positive, the worst-case total cost under τ^* is higher than under L^*.

4.2 General case: Probabilistic constraints

Beyond the categorical constraints considered above, scientific knowledge could establish probabilistic conclusions, such as 'the probability that the marginal abatement cost at E is between m and M is greater than ρ'. On some issues, the IPCC emit conclusions that effectively amount to statements of this sort (Mastrandrea et al., 2010; IPCC, 2023).[14] Uncertainty involving such *probabilistic constraints* on abatement costs, marginal abatement costs and their slopes can be represented by sets of priors $\mathcal{C}_{[m^{(\rho)},M^{(\rho)}],[m^{(\rho)1},M^{(\rho)1}],[m^{(\rho)2},M^{(\rho)2}]} \subseteq \Delta(\Theta)$ of the form:

$$\left\{ p \in \Delta(\Theta) : \forall \rho \in (0,1],\ p\left(\left\{ \theta \in \Theta : \begin{array}{l} C(E,\theta) \in [m^{(\rho)}(E), M^{(\rho)}(E)] \\ C_1(E,\theta) \in [m^{(\rho)1}(E), M^{(\rho)1}(E)] \\ C_{11}(E,\theta) \in [m^{(\rho)2}(E), M^{(\rho)2}(E)] \end{array} \right\} \right) \geq \rho \right\} \tag{8}$$

for nested families of increasing functions[15] $m^{(\rho)}, M^{(\rho)}, m^{(\rho)1}, M^{(\rho)1}, m^{(\rho)2}, M^{(\rho)2} : [0, E^{max}] \to \mathbb{R}_{\geq 0}$, for each $\rho \in (0,1]$ summarising the upper and lower probabilistic bounds on abatement costs, marginal abatement costs and their slopes. As in the previous section, we consider the case of single lower and upper bounds on the slope of marginal abatement costs; i.e. $\mathcal{C}_{[m^{(\rho)},M^{(\rho)}],[m^{(\rho)1},M^{(\rho)1}],[\underline{C''},\overline{C''}]}$ defined as in Eq. (8) with $m^{(\rho)2}(E) = \underline{C''} < \overline{C''} = M^{(\rho)2}(E)$ for all $\rho \in (0,1]$ and $E \in [0, E^{max}]$.

[14] For instance, they state (2023, §3.1.1) that the probability that equilibrium climate sensitivity is between 2.5°C and 4°C is greater than 66%.

[15] A nested family is one such that $[m^{(\rho)}, M^{(\rho)}] \subseteq [m^{(\rho')}, M^{(\rho')}]$ whenever $\rho \leq \rho'$ and similarly for the derivatives.

Theorem 1 is a special case of a general result involving such probabilistic constraints. For readability, we report the most general form in Appendix A.1 (Theorem A.1) and focus here on a simplification obtained by making the following assumption.

Assumption 1. For optimal tax level τ^*, suppose that $\widehat{Cr}''^{\tau^*}_{E_r,E_{\tau^*}}$ (considered as a random variable varying with r) is independent from $(E_r - E_{\tau^*})^2$, and that for all sufficiently small δ, $\widehat{Cr}''^{\tau^*}_{E_r,E_{\tau^*}}, \widehat{Cr}''^{\tau^*}_{E_r,E_r-\delta} \approx \widehat{Cr}''(E_{\tau^*})$, which exists for all $r \in (0,1]$.

Moreover, suppose that $\hat{D}$ is differentiable at E_{τ^*} and that the expected slope of the marginal damages over distributions with mean $E_{\tau^*} + \varepsilon_{\tau^*}$ is approximately $\hat{D}''(E_{\tau^*} + \varepsilon_{\tau^*})$, which exists.

Finally, suppose that $E_{\tau^*} = L^* \in (0, E^{max})$, the optimal quantity policy.

The first assumption concerning abatement costs corresponds to the independence stipulated by Weitzman (1974, p486, footnote 1). The other assumptions concerning abatement costs and damages are simplifications allowing, for example, replacement of average second-order derivatives (Section 2.5) with second-order derivatives. The final assumption rules out potential misspecifications arising from pricing policies (see Appendix A.1), so as to focus on the risk and uncertainty comparisons.

Theorem 2. *Consider any $\mathcal{D} \subseteq \Delta(\Xi)$ characterising uncertainty about damages, and suppose that uncertainty about abatement costs is characterised by $\mathcal{C}_{[m^{(\rho)},M^{(\rho)}],[m^{(\rho)1},M^{(\rho)1}],[\underline{C''},\overline{C''}]}$ for some family $m^{(\rho)}, M^{(\rho)}, m^{(\rho)1}, M^{(\rho)1}$ and $\underline{C''} < \overline{C''}$. Under Assumption 1:*

$$T_{price} - T_{quantity} \geq \frac{\sigma^2_{\tau^*}}{2} \underbrace{\left(\hat{D}''(L^* + \varepsilon_{\tau^*}) - \hat{C}''(L^*)\right)}_{Risk} + \frac{\varepsilon^2_{\tau^*}}{2} \underbrace{\left(\hat{D}''_{L^*,L^*+\varepsilon_{\tau^*}} - \mathbb{E}_u \frac{\underline{C''}\widehat{Cr}''(L^*)}{\widehat{Cr}''(L^*) - \underline{C''}}\right)}_{Uncertainty} \tag{9}$$

where τ^* is the optimal tax, L^* the optimal quantity level, and $\mathbb{E}_u$ is the expectation over r under the uniform probability distribution u over $[0,1]$.[16]

Note first of all that this Theorem generalises Weitzman's (1974) result concerning risk (Example 1), as well as Theorem 1 involving categorical uncertainty. When there is only risk, $\varepsilon_{\tau^*} = 0$, so the optimal tax policy leads to a higher cost that the optimal quantity policy whenever $\hat{D}''(L^*) > \hat{C}''(L^*)$, as in Weitzman's Theorem.[17] On the other hand, when there is only uncertainty, $\sigma^2_{\tau^*} = 0$, so the optimal tax policy has higher cost whenever $\hat{D}''_{L^*,L^*+\varepsilon_{\tau^*}} > \frac{\underline{C''}}{\hat{C}''(L^*) - \underline{C''}} \hat{C}''(L^*)$, which is similar to the condition in Theorem 1. As discussed there, when little is known about the slope of the marginal abatement costs, this inequality will hold: taxes lead to a higher total cost than optimal permits.

Eq. (9) tells us that between these two extremes, the comparison between optimal taxes and permits turns, largely, on the extent of risk and uncertainty in our current knowledge about

[16] The other notation is defined in Sections 2 and 3.

[17] Except for the use of expectations over parameters in the specification of the abatement cost function by Weitzman rather than over resulting emission reductions as here, the $\varepsilon_\tau = 0$ version of (9) is analogous to his result.

abatement costs. As noted, the 'Risk' term in Eq. (9) is similar to the well-known Weitzman term; economists typically consider that it is negative (Nordhaus, 2007). The 'Uncertainty' term is compares the average slope of the marginal damage curve across the uncertainty in expected emissions under τ^* (i.e. between L^* and $L^* + \varepsilon_{\tau^*}$) with the expectation of a term, $\frac{\underline{C''}\widehat{C^r}''(L^*)}{\widehat{C^r}''(L^*)-\underline{C''}}$, that is of an analogous form to that in Theorem 1. As argued in Section 4.1, given current ignorance about slopes of marginal abatement costs, $\underline{C''}$ is can reasonably be considered to be small, so $\frac{\underline{C''}\widehat{C^r}''(L^*)}{\widehat{C^r}''(L^*)-\underline{C''}}$ will be close to zero and the 'Uncertainty' term in Eq. 9 is typically positive.

The comparison between taxes and permits thus depends on the extent to which the limits on our scientific knowledge reflect *bona fide* uncertainty as opposed to risk. If $\sigma^2_{\tau^*} \gg \varepsilon^2_{\tau^*}$ – i.e. the extent of probabilised risk, captured by the variance, dwarfs the non-probabilised uncertainty, as reflected in the 'width' of the set of priors $\mathcal{C}$ – then the negative, first term in Eq. (9) dominates. Here, taxes involve lower total cost. If $\varepsilon^2_{\tau^*} \gg \sigma^2_{\tau^*}$ – the extent of dispersion between priors in $\mathcal{C}$ outstrips the variance under appropriate worst-case priors – then the second, positive term dominates. In this case, permits are more efficient. If Figure 1 and the frequent admonitions as to the severity of the uncertainty concerning the economics of climate change (Stainforth et al., 2007; Pindyck, 2013; Bradley and Steele, 2015; Stiglitz, 2019; Stern et al., 2022) are to be believed, the second case most closely corresponds to the uncertainties facing us today. *Pace* the conclusion typically drawn from Weitzman's result, Theorem 2 would suggest that, under current uncertainty, permits outperform taxes.

5 Robustness and Extensions

Weitzman's (1974) Theorem has spurred a sprawling literature examining and extending the comparison of pricing and quantity policies to a wide range of situations and contexts. This literature is too vast to be surveyed here; suffice it to say that, to the best of our knowledge, this is the first paper to consider the comparison under the sort of uncertainty present in the climate context. Accordingly, we consider the exercise in this paper to be, like Weitzman's, a first step: the focus on an abstract emissions reductions problem brings out some central messages concerning potential consequences of uncertainty, whilst leaving the study of implications for specific markets and contexts for future research.

That said, it is worth briefly considering the robustness of the main messages from the results in Section 4. To this end, we consider simple extensions across four separate dimensions: the incorporation of constraints on higher derivatives, as opposed to the second derivative of the abatement cost considered above; the representation of uncertainty using parametric families of abatement cost functions, as opposed to (probabilistic) constraints; extension beyond the (perfect competition) assumption that market reaction to a policy is dictated by the marginal abatement cost function; and generalisation to more flexible attitudes to uncertainty. In the interests of space, we include the most important results, relegating the statement of others to the Appendices.

5.1 Higher derivatives

Whilst the previous analyses incorporate constraints up to the second derivative of the abatement costs, several parametric forms involve a positive third derivative of the abatement cost curve (i.e. convex marginal abatement costs). Some model-based diagnostic studies (Kriegler et al., 2015; Harmsen et al., 2021) tentatively corroborate this suggestion. We now extend our analysis to incorporate such constraints. To this end, we present a result concerning sets of priors of form $\mathcal{C}_{[m,M],[m^1,M^1],[m^2,M^2],[\underline{C'''},\overline{C'''}]}$, defined as in Eq. (6) with the added constraints that the second derivative of the abatement cost lies in $[m^2(E), M^2(E)]$ for all E, and the third derivative always lies above $\underline{C'''} \geq 0$ and below $\overline{C'''} > \underline{C'''}$.

Theorem 3. *Consider any $\mathcal{D} \subseteq \Delta(\Xi)$ characterising uncertainty about damages, and suppose that uncertainty about abatement costs is characterised by $\mathcal{C}_{[m,M],[m^1,M^1],[m^2,M^2],[\underline{C'''},\overline{C'''}]}$ for some $[m, M]$, $[m^1, M^1]$, $[m^2, M^2]$, and $\underline{C'''} < \overline{C'''}$. Then $T_{price} \geq T_{quant}$.*

Moreover, if the third derivative of $\hat{C}$ is approximately constant, then $T_{price} > T_{quant}$ whenever $\varepsilon_{\tau^*} > 0$ for optimal tax τ^* and

$$\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon_{\tau^*}} > \frac{1}{3}\left[\frac{\varepsilon_{\tau^*}X}{\hat{C}''(L^*)}\left(\left(\frac{2\hat{C}''(L^*)}{\varepsilon_{\tau^*}X} + \frac{\hat{C}'''(L^*)}{X}\right)^{\frac{3}{2}} - \left(\frac{3\hat{C}''(L^*)}{\varepsilon_{\tau^*}X} + \frac{\hat{C}'''(L^*)}{X}\right)\right)\right]\hat{C}''(L^*) \tag{10}$$

where $L^* \in (0, E^{max})$ is the optimal quantity level and $X = \hat{C}'''(L^*) - \underline{C'''}$.

The assumption concerning the third derivative of $\hat{C}$ is merely for simplicity. The proof delivers a version of the result without this assumption, where $\hat{C}'''(L^*)$ is replaced by the average third derivative $\hat{C}$ à la Eq. (2) (Section 2.5); see Appendix A.3 for details. The points below thus hold in the absence of this assumption, using the appropriate replacement.

Theorem 3 extends Theorem 1, showing that, even when incorporating a lower constraint on the convexity of marginal abatement costs, optimal quantity policies are always at least as efficient as optimal pricing ones. Moreover, Theorem 3 provides a sufficient condition for them to be strictly more efficient. Like the condition in Theorem 1, it compares the slope of the marginal damages with a multiple of the slope of the marginal abatement costs, though the multiplicative factor is different here. Whilst there are a few studies suggesting convex marginal abatement costs (Kriegler et al., 2015; Harmsen et al., 2021), no lower bound on its convexity has been proposed, suggesting that $\underline{C'''}$ reflecting the current state of knowledge can be taken close to zero. As $\underline{C'''}$ gets small and ε_{τ^*} increases, the multiplicative factor on the right of (10) tends to zero, suggesting that the condition in the Theorem holds in the presence of significant uncertainty. The general conclusion in the case of convex marginal abatement costs is thus as in the previous analyses: in the presence of significant uncertainty, permits may be more efficient than taxes.[18]

[18]Whilst we have considered categorical constraints for illustration here, extensions of Theorem 2 comparable to Theorem 3 are obtainable by similar techniques (see Appendix A.3). The conclusion of Section 4.2 thus continues to hold when incorporating higher derivatives.

5.2 Parametric abatement costs

Another potential form of uncertainty concerning abatement costs consists in certainty that cost functions belong to a particular family (or families), and knowledge about the range of possible or probable parameter values (Section 2.4, Example 2). Analogous comparison results can be obtained in this case; see Theorem A.2 in Appendix A.4 for a general result. We illustrate them by considering the power marginal abatement cost family, as used in the DICE model (Nordhaus, 1992) and beyond (e.g. Emmerling et al., 2019). More precisely, for any $(a, b) \in \mathbb{R}^2_{>0}$, let $\theta_{f,(a,b)}$ be the cost function in this family defined by these parameters, i.e. such that $C_1(E, \theta_{f,(a,b)}) = f(E, a, b)$ for all $E \in [0, E^{max}]$, where $f(E, a, b) = a \left(\frac{E}{E^{max}}\right)^b$. We consider sets of priors of the form $\mathcal{C}_{f,A}$, defined by:

$$\mathcal{C}_{f,A} = \left\{p \in \Delta(\Theta) : \forall\theta \in \operatorname{supp} p,\ \theta = \theta_{f,(a,b)} \text{ for some } (a, b) \in A\right\}$$

for $A \subset \mathbb{R}^2_{>0}$. These are sets of abatement cost functions in this parametric family. We focus on families $C_{f,A}$ satisfying the following convexity property: for all $\theta_{f,(a^1,b^1)}, \theta_{f,(a^2,b^2)} \in \operatorname{supp} \mathcal{C}_{f,A}$ and every $\theta_{f,(a^3,b^3)} \in \Theta$, if $\min\{C(E, \theta_{f,(a^1,b^1)}), C(E, \theta_{f,(a^2,b^2)})\} \le C(E, \theta_{f,(a^3,b^3)}) \le \max\{C(E, \theta_{f,(a^1,b^1)}), C(E, \theta_{f,(a^2,b^2)})\}$ for all $E \in [0, E^{max}]$, then $\theta_{f,(a^3,b^3)} \in \operatorname{supp} \mathcal{C}_{f,A}$. Such *parameter-convex* sets contain all members of the family consistent with its derived range of abatement costs.

Proposition 1. *Consider any $\mathcal{D} \subseteq \Delta(\Xi)$ characterising uncertainty about damages, and suppose that uncertainty about abatement costs is characterised by a parameter-convex power family $\mathcal{C}_{f,A}$ defined as above. Then $T_{price} \ge T_{quant}$.*

Moreover, if $\hat{D}$ is second-order differentiable at the optimal quantity level $L^* \in (0, E^{max})$, then $T_{price} > T_{quant}$ whenever there exists $(a, b) \in A$ such that $\varepsilon > 0$ and

$$\hat{D}''(L^*) > \frac{2\tau^*}{\varepsilon^2}\left(\frac{b}{b+1}\left(\frac{\tau^*(E^{max})^b}{a}\right)^{\frac{1}{b}} - \frac{\overline{b}}{\overline{b}+1}\left(\frac{\tau^*(E^{max})^{\overline{b}}}{\overline{a}}\right)^{\frac{1}{\overline{b}}}\right) \tag{11}$$

where τ^* is the optimal tax level, ε is such that $f(L^* + \varepsilon, a, b) = \tau^*$, and $(\overline{a}, \overline{b}) \in A$ is such that $f(L^*, \overline{a}, \overline{b}) = \tau^*$ and $f(E, c, d) \le f(E, \overline{a}, \overline{b})$ for all $E \le L^*$ and $(c, d) \in A$ such that $f(L^*, c, d) = \tau^*$.

Proposition 1 suggests that the main messages of Theorem 1 continue to hold when uncertainty is characterised by parametric constraints on abatement cost functions. First, taxes never outperform permits: indeed, this holds for a wide range of parametric families (Remark A.1, Appendix A.4). Second, there is a sufficient condition for permits to strictly outperform taxes, which differs from that in Theorem 1, and is specific to the parametric family under consideration. In the case of the power family, note that the parameter pair $(\overline{a}, \overline{b})$ characterises the highest abatement cost function consistent with the uncertainty $\mathcal{C}_{f,A}$ and relevant for tax τ^*.[19] More-

[19] A simple argument shows that such a pair exists; see the proof of the Proposition, Appendix A.4.

over, $\overline{a}\left(\frac{1}{E^{max}}\right)^{\overline{b}}$ and $a\left(\frac{1}{E^{max}}\right)^{b}$ are the marginal abatement costs of the first unit of emissions under this member of the family and the other member considered. If this marginal cost is well understood (as suggested by Figure 1), uncertainty concerning it will be low, and so $\frac{a\left(\frac{1}{E^{max}}\right)^{b}}{\overline{a}\left(\frac{1}{E^{max}}\right)^{\overline{b}}}$ will be close to 1. For any $b < \overline{b}$ and a chosen so that this ratio is $1+\delta$ for some small $\delta > 0$, condition (11) reduces to

$$\hat{D}''(L^*) > \frac{2\tau^*}{\varepsilon^2}\left(\frac{b}{b+1}\left(\frac{\tau^*(E^{max})^{\overline{b}}}{(1+\delta)\overline{a}}\right)^{\frac{1}{b}} - \frac{\overline{b}}{\overline{b}+1}\left(\frac{\tau^*(E^{max})^{\overline{b}}}{\overline{a}}\right)^{\frac{1}{\overline{b}}}\right) \tag{12}$$

Noting that for such (a, b), $\varepsilon = \left(\frac{\tau^*(E^{max})^{\overline{b}}}{(1+\delta)\overline{a}}\right)^{\frac{1}{b}} - \left(\frac{\tau^*(E^{max})^{\overline{b}}}{\overline{a}}\right)^{\frac{1}{\overline{b}}}$, there clearly exist b such that $\varepsilon > 0$ and the term on the right of Eq. (12) is negative. For small δ, such b can be small, and comfortably within the current range of uncertainty for the exponent in typical applications of the power family.[20] This suggests that, for typical sets of parameters under this family reflecting our current knowledge about (marginal) abatement costs, permits are more efficient than taxes.

This illustration is by no means meant to suggest that one cannot cook up sets of parameters where condition (11) is not satisfied, nor that corresponding conditions for other parametric families will necessarily be automatically satisfied. The aim is rather to show that a main message of the previous sections carries over to the parametric case: uncertainty undermines the purported superiority of pricing over quantity policies.

5.3 Market Imperfections

To this point, we have followed Weitzman (1974) in considering the case where the emission reductions resulting from a carbon price τ are determined by the effective aggregate marginal abatement cost function, i.e. as $C_1^{-1}(\tau, \theta)$ for abatement cost function θ (see Section 3.2). In reality, this assumption often appears overly bold, for instance because of market imperfections or because of agents' uncertainty about the emissions associated with their actions (e.g. firms' uncertainty about the emissions associated with their production decisions).

As a rough gauge of the centrality of this assumption for our results, Appendix A.5 analyses an extension where the 'reaction function' representing market reaction to a carbon price may differ from the effective marginal abatement cost function. We establish a simple result (Proposition A.1) under categorical uncertainty of the sort considered in Section 4.1. It shows that, as in Theorem 1, taxes never outperform quantities; moreover there is a sufficient condition, reminiscent of that in Section 4.1, for quantity policies to be strictly more efficient. As noted in Appendix A.5, for some commonly discussed market imperfections, this condition seems to be satisfied. This suggests that, although diverging from the competitive market case considered by Weitzman involves further complexities, there is little evidence that it changes the fundamental message that uncertainty may have significant consequences for policy comparisons.

[20] For instance, Emmerling et al. (2019, Figure A.4) consider a range between 2 and 4.6.

5.4 Policy evaluation under uncertainty

The policy comparison exercises conducted above use the maxmin-EU decision criterion (1) for evaluation. As discussed in Section 2.4, this framework affords a flexible representation of uncertainty, which can cover the range of cases from risk – where uncertainty is fully summarised by a single probability distribution – to categorical uncertainty – where no probabilities at all need to be provided or postulated to represent the state of knowledge. The capacity to properly reflect categorical uncertainty is particularly relevant for the purposes of this paper, given that the current science provides, at best, a range of possible abatement cost functions with little or no well-established probability judgements over them (Section 4.1).

Nevertheless, criterion (1) is a special case of several notable decision approaches. For instance, recent work in the robustness tradition (Hansen and Sargent, 2008) has emphasised that decision makers may be faced with a range of (structured) models, all of which may be misspecified. This seems a fair description of the science behind the marginal abatement costs plotted in Figure 1. Hansen and Sargent (2022); Cerreia-Vioglio et al. (2020) propose an extension of their multiplier model for such situations, which can easily be applied, for instance to the set of priors $\mathcal{C}_{(-\infty,\infty),[m^1,M^1],[0,\infty)}$ recognised as reflecting the scientific knowledge featured in Figure 1 (Section 4.1). Evaluation (1) corresponds to the special case of their model with no fear of misspecification (Cerreia-Vioglio et al., 2020, Section 4): under misspecification averse evaluations, policies are ranked even worse, suggesting that our results, if anything, are stengthened in this context. More generally, their model is a special case of variational preferences (Maccheroni et al., 2006) and they provide specifications related to the smooth ambiguity model (Klibanoff et al., 2005), suggesting that our results may extend to these families; we leave explorations of such possibilities for further research.

Beyond the issue of the representation of uncertainty, there is that of policy makers' attitude towards it. Both the maxmin-EU criterion (1) and the misspecification one translate aversion towards uncertainty, motivated by the aim of ensuring robustness of the ensuing decision. In Appendix A.6, we reconsider the policy comparison under a simple extension of (1) that allows for a range of uncertainty attitudes, namely the α-maxmin EU model (Ghirardato et al., 2004; Gilboa and Marinacci, 2013; Gul and Pesendorfer, 2015). Our result shows that the previous comparisons do rely on the uncertainty attitude involved in the evaluation, but are not knife-edge: even beyond evaluation (1), as long as there is enough uncertainty aversion, permits will typically outperform taxes. However, in the absence of uncertainty aversion, neither policy systematically outperforms the other: the superiority of taxes cannot be salvaged simply by recognising uncertainty but postulating an appetite for it.

References

(2019). Opinion | Economists' Statement on Carbon Dividends. *Wall Street Journal*. https://www.econstatement.org/.

Barnett, A. H. (1980). The Pigouvian tax rule under monopoly. *The American Economic Review*, 70(5):1037–1041. Publisher: JSTOR.

Berger, L., Berger, N., Bosetti, V., Gilboa, I., Hansen, L. P., Jarvis, C., Marinacci, M., and Smith, R. D. (2021). Rational policymaking during a pandemic. *Proceedings of the National Academy of Sciences*, 118(4). Publisher: National Academy of Sciences Section: Perspective.

Berger, L. and Marinacci, M. (2020). Model uncertainty in climate change economics: A review and proposed framework for future research. *Environmental and Resource Economics*, pages 1–27. Publisher: Springer.

Blanchard, O., Gollier, C., and Tirole, J. (2023). The Portfolio of Economic Policies Needed to Fight Climate Change. *Annual Review of Economics*, 15(1):689–722.

Bradley, R. (2017). *Decision theory with a human face*. Cambridge University Press.

Bradley, R., Helgeson, C., and Hill, B. (2017). Climate Change Assessments: Confidence, Probability, and Decision. *Philosophy of Science*, 84(3):500–522.

Bradley, R. and Steele, K. (2015). Making climate decisions. *Philosophy Compass*, 10(11):799–810.

Cerreia-Vioglio, S., Hansen, L. P., Maccheroni, F., and Marinacci, M. (2020). Making decisions under model misspecification. *arXiv preprint arXiv:2008.01071*.

Emmerling, J., Drouet, L., van der Wijst, K.-I., Van Vuuren, D., Bosetti, V., and Tavoni, M. (2019). The role of the discount rate for emission pathways and negative emissions. *Environmental Research Letters*, 14(10):104008. Publisher: IOP Publishing.

Ghirardato, P., Maccheroni, F., and Marinacci, M. (2004). Differentiating ambiguity and ambiguity attitude. *J. Econ. Theory*, 118(2):133–173.

Gilboa, I. and Marinacci, M. (2013). Ambiguity and the Bayesian Paradigm. In *Advances in Economics and Econometrics: Volume 1, Economic Theory: Tenth World Congress*, volume 49, page 179. Cambridge University Press.

Gilboa, I. and Schmeidler, D. (1989). Maxmin expected utility with non-unique prior. *J. Math. Econ.*, 18(2):141–153.

Gul, F. and Pesendorfer, W. (2015). Hurwicz expected utility and subjective sources. *Journal of Economic Theory*, 159:465–488.

Hansen, L. P. and Sargent, T. J. (2008). *Robustness*. Princeton university press.

Hansen, L. P. and Sargent, T. J. (2022). Structured ambiguity and model misspecification. *Journal of Economic Theory*, 199:105165. Publisher: Elsevier.

Harmsen, M., Kriegler, E., Van Vuuren, D. P., van Der Wijst, K.-I., Luderer, G., Cui, R., Dessens, O., Drouet, L., Emmerling, J., and Morris, J. F. (2021). Integrated assessment model diagnostics: key indicators and model evolution. *Environmental Research Letters*, 16(5):054046. Publisher: IOP Publishing.

Hill, B. (2019). Confidence in Beliefs and Rational Decision Making. *Economics & Philosophy*, 35(2):223–258.

Hurwicz, L. (1951). Some specification problems and applications to econometric models. *Econometrica*, 19(3):343–44.

IPCC (2023). *Climate Change 2022 - Mitigation of Climate Change: Working Group III Contribution to the Sixth Assessment Report of the Intergovernmental Panel on Climate Change*. Cambridge University Press, 1 edition.

IPCC (2023). Climate Change 2023: Synthesis Report. Contribution of Working Groups I, II and III to the Sixth Assessment Report of the Intergovernmental Panel on Climate Change. Technical report, IPCC, Switzerland. [Core Writing Team, H. Lee and J. Romero (eds.)].

Klibanoff, P., Marinacci, M., and Mukerji, S. (2005). A Smooth Model of Decision Making under Ambiguity. *Econometrica*, 73(6):1849–1892.

Kriegler, E., Petermann, N., Krey, V., Schwanitz, V. J., Luderer, G., Ashina, S., Bosetti, V., Eom, J., Kitous, A., and Méjean, A. (2015). Diagnostic indicators for integrated assessment models of climate policy. *Technological Forecasting and Social Change*, 90:45–61. Publisher: Elsevier.

Levin, D. (1985). Taxation within Cournot oligopoly. *Journal of Public Economics*, 27(3):281–290. Publisher: Elsevier.

Maccheroni, F., Marinacci, M., and Rustichini, A. (2006). Ambiguity Aversion, Robustness, and the Variational Representation of Preferences. *Econometrica*, 74(6):1447–1498.

Manski, C. F. (2013). *Public Policy in an Uncertain World : Analysis and Decisions*. Harvard University Press, Cambridge, Mass.

Marinacci, M. (2015). Model uncertainty. *Journal of the European Economic Association*, 13(6):1022–1100.

Mastrandrea, M. D., Field, C. B., Stocker, T. F., Edenhofer, O., Ebi, K. L., Frame, D. J., Held, H., Kriegler, E., Mach, K. J., and Matschoss, P. R. (2010). Guidance note for lead authors of the IPCC fifth assessment report on consistent treatment of uncertainties. *Intergovernmental Panel on Climate Change (IPCC)*.

Metcalf, G. E. (2009). Designing a Carbon Tax to Reduce U.S. Greenhouse Gas Emissions. *Review of Environmental Economics and Policy*, 3(1):63–83.

Nordhaus, W. D. (1991). The Cost of Slowing Climate Change: a Survey. *The Energy Journal*, 12(1):37–66. Publisher: SAGE Publications.

Nordhaus, W. D. (1992). An Optimal Transition Path for Controlling Greenhouse Gases. *Science*, 258(5086):1315–1319.

Nordhaus, W. D. (2007). To Tax or Not to Tax: Alternative Approaches to Slowing Global Warming. *Review of Environmental Economics and Policy*, 1(1):26–44.

Pindyck, R. S. (2013). Climate change policy: what do the models tell us? *Journal of Economic Literature*, 51(3):860–872. Publisher: American Economic Association.

Rockafellar, R. T. (1970). Convex Analysis (Princeton Mathematical Series). *Princeton University Press*, 46:49.

Stainforth, D. A., Allen, M. R., Tredger, E. R., and Smith, L. A. (2007). Confidence, uncertainty and decision-support relevance in climate predictions. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 365(1857):2145–2161.

Stern, N. (2013). The structure of economic modeling of the potential impacts of climate change: grafting gross underestimation of risk onto already narrow science models. *Journal of Economic Literature*, 51(3):838–859. Publisher: American Economic Association.

Stern, N., Stiglitz, J., and Taylor, C. (2022). The economics of immense risk, urgent action and radical change: towards new approaches to the economics of climate change. *Journal of Economic Methodology*, 29(3):181–216.

Stiglitz, J. E. (2019). Addressing climate change through price and non-price interventions. *European Economic Review*, 119:594–612.

Weitzman, M. L. (1974). Prices vs. quantities. *The review of economic studies*, 41(4):477–491.

A Appendix: Proofs and Extensions

A.1 General Result

We first state and prove a general result involving uncertainty characterised by probabilistic constraints. The results in Sections 4.1 and 4.2 will be corollaries, as shown in Appendix A.2. As a point of notation, for a tax τ, let

$$T_\tau = \max_{p\in\mathcal{C}} \max_{q\in\mathcal{D}} \mathbb{E}_p\mathbb{E}_q \left(D(C_1^{-1}(\tau,\theta),\xi) + C(C_1^{-1}(\tau,\theta),\theta)\right)$$

We consider a general probabilistic lower constraint on the slope of marginal abatement costs, which can be represented by $\mathcal{C}$ defined as in Eq. (8) with $[m^{(\rho)2}, M^{(\rho)2}] = [\underline{C''^\rho}, \overline{C''^\rho}]$ for all $\rho \in (0,1]$, for some $\underline{C''^\rho}$ and $\overline{C''^\rho}$ increasing (respectively decreasing) in ρ.

Theorem A.1. *Consider any $\mathcal{D} \subseteq \Delta(\Xi)$ characterising uncertainty about damages, and suppose that uncertainty about abatement costs is characterised by $\mathcal{C}_{[m^{(\rho)},M^{(\rho)}],[m^{(\rho)1},M^{(\rho)1}],[m^{(\rho)2},M^{(\rho)2}]}$ for some family of probabilistic constraints $m^{(\rho)}, M^{(\rho)}, m^{(\rho)1}, M^{(\rho)1}, m^{(\rho)2}, M^{(\rho)2}$ with $[m^{(\rho)2}, M^{(\rho)2}] = [\underline{C''^\rho}, \overline{C''^\rho}]$ for all $\rho \in [0,1)$, for some $\underline{C''^\rho}, \overline{C''^\rho}$ decreasing (respectively, increasing) in ρ. For any τ such that $\tau \in \partial\hat{C}(E)$ for some $E \in [0, E^{max}]$, and any $\hat{D}' \in \partial\hat{D}(E_\tau)$:*

$$\begin{aligned}
T_\tau \geq & \hat{C}(E_\tau) + \hat{D}(E_\tau) + \varepsilon_\tau(\tau + \hat{D}') \\
& + \frac{\sigma_\tau^2}{2}\left(\mathbb{E}\hat{D}'' - \left(\mathbb{E}_u \widehat{C^r}''^\tau_{E_r,E_\tau} + \frac{1}{\sigma_\tau^2} cov(\widehat{C^r}''^\tau_{E_r,E_\tau}, (E_r - E_\tau)^2)\right)\right) \\
& + \frac{\varepsilon_\tau^2}{2}\left[\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau} - \mathbb{E}_u \frac{\underline{C''^r}\,\widehat{C^r}''^\tau_{E_r,E_r-\delta_r}}{\widehat{C^r}''^\tau_{E_r,E_r-\delta_r} - \underline{C''^r}}\right]
\end{aligned} \tag{A.1}$$

where $\mathbb{E}\hat{D}''$ is an expected average second derivative over the support of $\epsilon_\tau^{+\varepsilon_\tau}$ under a worst-case distribution over damage functions (see Eq. (A.12) and surrounding discussion below), for each $r \in (0,1]$, δ_r is impicitly defined for all $r \in (0,1]$ except a set of measure zero by $\tau - (\varepsilon_\tau + \delta_r)\underline{C''^r} \in \partial\widehat{C^r}(E_r - \delta_r)$, and $\mathbb{E}_u$ is the expectation taken over r according to the uniform distribution u over $[0,1]$.

Proof. Consider τ such that $\tau \in \partial\hat{C}(E)$ for some $E \in [0, E^{max}]$. For each $r \in (0,1]$, recall that $\tau \in \partial\widehat{C^r}(E_r)$ and (by the definition in Eq. (2))

$$\widehat{C^r}''^\tau_{E_r,E_\tau} = \frac{2}{(E_\tau - E_r)^2}\left(\widehat{C^r}(E_\tau) - \tau(E_\tau - E_r) - \widehat{C^r}(E_r)\right)$$

It follows that

$$\widehat{C^r}(E_r) = \widehat{C^r}(E_\tau) - (\tau + \widehat{C^r}''^\tau_{E_r,E_\tau}(E_\tau - E_r))(E_\tau - E_r) + \frac{\widehat{C^r}''^\tau_{E_r,E_\tau}}{2}(E_\tau - E_r)^2$$

For each τ, and every $r \in (0,1]$, let $\theta_r \in \Theta$ be such that $C_1(E_r, \theta_r) = \tau$ and $C(E_r, \theta_r) = \widehat{C^r}(E_r)$. (By the argument in the proof of Lemma A.1, Appendix A.7, such θ_r exist.) We have:

$$\begin{aligned}\mathbb{E}_u C(E_r, \theta_r) =& \mathbb{E}_u \widehat{C^r}(E_\tau) - \mathbb{E}_u \left(\frac{\widehat{C^r}''^{\tau}_{E_r, E_\tau}}{2} (E_r - E_\tau)^2 \right) \\ =& \hat{C}(E_\tau) - \frac{1}{2} \left(\mathbb{E}_u \widehat{C^r}''^{\tau}_{E_r,E_\tau} \sigma_\tau^2 + cov(\widehat{C^r}''^{\tau}_{E_r,E_\tau}, (E_r - E_\tau)^2) \right)\end{aligned} \tag{A.2}$$

since $\mathbb{E}_u E_r = E_\tau$ and $\mathbb{E}_u (E_r - E_\tau)^2 = \sigma_\tau^2$. In the second equation, $\mathbb{E}_u \widehat{C^r}''^{\tau}_{E_r,E_\tau}$ is the expected average second order derivative between E_r and E_τ, across the worst-case cost functions, and the last term is the covariance with the square difference in emissions with respect to E_τ.

For each $r \in (0,1]$, note that there exists δ such that:

$$\tau - (\varepsilon_\tau + \delta)\underline{C''^r} \in \partial \widehat{C^r}(E_r - \delta) \tag{A.3}$$

Moreover, by the strict convexity of $\widehat{C^r}$ (Proposition A.3), such δ is unique. By the convexity of $\widehat{C^r}$ and Rademacher's Theorem (e.g. Rockafellar, 1970, Theorem 25.4), $\partial \widehat{C^r}$ is single-valued except on a set of measure zero. So, for every $r \in (0,1]$ such that $\partial \widehat{C^r}(E_r - \delta)$ is not a singleton (i.e. $\widehat{C^r}$ is not differentiable at $E_r - \delta$) for δ satisfying the expression above, there exists a small $\rho_r > 0$ and δ' such that $\tau - (\varepsilon_\tau + \delta')(\underline{C''^r} + \rho_r) \in \partial \widehat{C^r}(E_r - \delta')$ and $\partial \widehat{C^r}(E_r - \delta')$ is a singleton.

For each $r \in (0,1]$, let δ_r be the solution to:

$$\tau - (\varepsilon_\tau + \delta_r)\underline{C''^r} = \widehat{C^r}'(E_r - \delta_r) \tag{A.4}$$

if such δ_r exists (i.e. when $\widehat{C^r}$ is differentiable at $E_r - \delta_r$), and the solution to

$$\tau - (\varepsilon_\tau + \delta_r)(\underline{C''^r} + \rho_r) = \widehat{C^r}'(E_r - \delta_r)$$

otherwise. As noted above, δ_r is well-defined and it satisfies (A.4) for all $r \in (0,1]$ except a set of measure zero.

Let $\hat{\theta}_r$ be any cost function satisfying all the constraints for probability threshold r in the definition of $\mathcal{C}$ such that $C_1(E_r - \delta_r, \hat{\theta}_r) = \widehat{C^r}'(E_r - \delta_r)$ and $C(E_r - \delta_r, \hat{\theta}_r) = \widehat{C^r}(E_r - \delta_r)$; such a function exists by the definition of $\widehat{C^r}$ and the fact that it is differentiable at the points involved. Consider the function $G : [0, E_r + \varepsilon_\tau] \to \mathbb{R}$ defined by

$$G(E) = \begin{cases} C(E, \hat{\theta}_r) & \text{if } E \leq E_r - \delta_r \\ C(E_r - \delta_r, \hat{\theta}_r) + \widehat{C^r}'(E_r - \delta_r)(E - E_r + \delta_r) \\ + \frac{\underline{C''^r}}{2}(E - E_r + \delta_r)^2 & \text{if } E_r - \delta_r < E \leq E_r + \varepsilon_\tau \end{cases}$$

This hits the $\widehat{C^r}$ curve at a point, and then increases with the r-lowest second derivative until the marginal abatement cost reaches τ, which occurs at $E_r + \varepsilon_\tau$. Note that G is increasing, differentiable, strictly convex and satisfies all the constraints for probability threshold r in the definition of $\mathcal{C}$ on its domain. Hence there exists $\underline{\theta_r} \in \operatorname{supp}\mathcal{C}$ with $C(E, \underline{\theta_r}) = G(E)$ for all $E \in [0, E_r + \varepsilon_\tau]$. Take any such $\underline{\theta_r}$.

Note that, since

$$\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r} = \frac{2}{\delta_r^2}\left(\widehat{C^r}(E_r - \delta_r) - \widehat{C^r}(E_r) + \tau\delta_r\right) \tag{A.5}$$

is the average second-order derivative of $\widehat{C^r}$ in this range (Eq. (2)), we have

$$\widehat{C^r}(E_r - \delta_r) = \widehat{C^r}(E_r) - \tau\delta_r + \frac{\delta_r^2}{2}\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r}$$

from which it follows that

$$\widehat{C^r}'(E_r - \delta_r) = \tau - \delta_r\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r} \tag{A.6}$$

Plugging this into (A.4) yields:

$$\delta_r = \underline{C''^r}\frac{\varepsilon_\tau}{\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r} - \underline{C''^r}} \tag{A.7}$$

for all $r \in (0, 1]$ except a set of measure zero. $\delta_r = (\underline{C''^r} + \rho_r)\frac{\varepsilon_\tau}{\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r} - (\underline{C''^r} + \rho_r)}$ otherwise.

It follows, substituting these equations in appropriately, that

$$\begin{aligned}
C(E_r + \varepsilon_\tau, \underline{\theta_r}) = {} & C(E_r - \delta_r, \underline{\theta_r}) + \widehat{C^r}'(E_r - \delta_r)(\varepsilon_\tau + \delta_r) + \frac{\underline{C''^r}}{2}(\varepsilon_\tau + \delta_r)^2 \\
= {} & C(E_r, \theta_r) - \tau\delta_r + \frac{\delta_r^2}{2}\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r} \\
& + (\tau - \delta_r\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r})(\varepsilon_\tau + \delta_r) + \frac{\underline{C''^r}}{2}(\varepsilon_\tau + \delta_r)^2 \\
= {} & C(E_r, \theta_r) - \frac{\delta_r^2}{2}\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r} + \tau\varepsilon_\tau - \delta_r\varepsilon_\tau\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r} + \frac{\underline{C''^r}}{2}(\varepsilon_\tau + \delta_r)^2 \\
= {} & C(E_r, \theta_r) + \tau\varepsilon_\tau + \frac{\varepsilon_\tau^2}{2(\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r} - \underline{C''^r})^2}\Big[-\underline{C''^r}^2\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r} \\
& - 2\underline{C''^r}\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r}(\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r} - \underline{C''^r}) + \underline{C''^r}\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r}{}^2\Big] \\
= {} & C(E_r, \theta_r) + \tau\varepsilon_\tau - \frac{\varepsilon_\tau^2}{2}\frac{\underline{C''^r}\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r}}{\widehat{C^r}''^{\tau}_{E_r, E_r-\delta_r} - \underline{C''^r}}
\end{aligned} \tag{A.8}$$

for all $r \in [0, 1]$ except a set of measure zero.

For a uniform distribution u over $[0, 1]$, let $\underline{p}$ over $\{\underline{\theta_r}\}$ be the distribution over cost functions that it generates; i.e. $\underline{p}(\{\underline{\theta_r} : r \leq r'\}) = u([0, r'])$ for all r'. $\underline{p} \in \mathcal{C}$ because by construction it satisfies all the probabilistic constraints. Note moreover that, by construction, $\epsilon_{\tau,\underline{p}} = \epsilon_\tau^{+\varepsilon_\tau}$. So

$$
\begin{aligned}
\mathbb{E}_{\underline{p}} C(C^{-1}(\tau,\theta),\theta) = \mathbb{E}_u C(E_r+\varepsilon_\tau,\underline{\theta_r}) \\
=& \mathbb{E}_u C(E_r,\theta_r) + \tau\varepsilon_\tau - \frac{\varepsilon_\tau^2}{2}\mathbb{E}_u \frac{\underline{C''r}\,\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_r}}{\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_r} - \underline{C''r}} \\
=& \hat{C}(E_\tau) - \frac{1}{2}\left(\mathbb{E}_u \widehat{C^r}''^{\tau}_{E_r,E_\tau}\sigma_\tau^2 + cov(\widehat{C^r}''^{\tau}_{E_r,E_\tau},(E_r-E_\tau)^2)\right) + \tau\varepsilon_\tau \\
&- \frac{\varepsilon_\tau^2}{2}\mathbb{E}_u \frac{\underline{C''r}\,\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_r}}{\widehat{C^r}''^{\tau}_{E_r,E_r-\delta_r} - \underline{C''r}}
\end{aligned}
\tag{A.9}
$$

by (A.2) and the fact that (A.8) holds for all $r \in (0,1]$ except a set of measure zero.

Take any $\hat{D}' \in \partial\hat{D}(E_\tau)$ and recall that:

$$
\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau} = \frac{2}{(\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E - E_\tau)^2}\left(\hat{D}(\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E) - \hat{D}(E_\tau) - \hat{D}'(\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E - E_\tau)\right)
$$

is the average second order derivative of $\hat{D}$ between E_τ and $\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E = E_\tau + \varepsilon_\tau$. Hence:

$$
\hat{D}(\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E) = \hat{D}(E_\tau) + \hat{D}'\varepsilon_\tau + \frac{\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau}}{2}\varepsilon_\tau^2 \tag{A.10}
$$

by the specification of $\underline{p}$.

Moreover, for any $q \in \mathcal{D}$ with $\mathbb{E}_q D(\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E,\xi) = \hat{D}(\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E)$,

$$
\begin{aligned}
\mathbb{E}_q\mathbb{E}_{\epsilon_{\tau,\underline{p}}} D(E,\xi) =& \mathbb{E}_q D(\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E,\xi) + \mathbb{E}_q\mathbb{E}_{\epsilon_{\tau,\underline{p}}} D_1(\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E,\xi)(E - \mathbb{E}_{\epsilon_{\tau,\underline{p}}}E) \\
&+ \mathbb{E}_q\mathbb{E}_{\epsilon_{\tau,\underline{p}}}\frac{D''(\xi)}{2}(E - \mathbb{E}_{\epsilon_{\tau,\underline{p}}}E)^2 \\
=& \hat{D}(\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E) + \frac{\mathbb{E}_q D''(\xi)}{2}\sigma_\tau^2
\end{aligned}
\tag{A.11}
$$

since $\epsilon_{\tau,\underline{p}} = \epsilon_\tau^{+\varepsilon}$ and so $\mathbb{E}_{\epsilon_{\tau,\underline{p}}}(E - \mathbb{E}_{\epsilon_{\tau,\underline{p}}}E)^2 = \sigma_\tau^2$, and where, for each $\xi \in \Xi$

$$
D''(\xi) = \frac{2}{\mathbb{E}_{\epsilon_{\tau,\underline{p}}}(E - \mathbb{E}_{\epsilon_{\tau,\underline{p}}}E)^2}\left(\mathbb{E}_{\epsilon_{\tau,\underline{p}}}D(E,\xi) - D(\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E,\xi) - \mathbb{E}_{\epsilon_{\tau,\underline{p}}}D_1(\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E,\xi)(E - \mathbb{E}_{\epsilon_{\tau,\underline{p}}}E)\right)
$$

Note that

$$
D''(\xi) = \frac{\mathbb{E}_{\epsilon_{\tau,\underline{p}}}\left(D(\bullet,\xi)''^{D_1(\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E,\xi)}_{\mathbb{E}_{\epsilon_{\tau,\underline{p}}}E,E}(E - \mathbb{E}_{\epsilon_{\tau,\underline{p}}}E)^2\right)}{\mathbb{E}_{\epsilon_{\tau,\underline{p}}}(E - \mathbb{E}_{\epsilon_{\tau,\underline{p}}}E)^2} \tag{A.12}
$$

so $D''(\xi)$ is a (weighted) expectation, over $\epsilon_{\tau,\underline{p}}$, of the average second derivative of the damage function $D(\bullet,\xi)$ between $\epsilon_{\tau,\underline{p}}$ and each point in the support of $\epsilon_{\tau,\underline{p}}$. So $\mathbb{E}_q D''(\xi)$ is an expected average second derivative over $\epsilon_{\tau,\underline{p}}$, under the distribution q.

Combining (A.11) and (A.10), and denoting $\mathbb{E}_q D''(\xi)$ by $\mathbb{E}\hat{D}''$, yields:

$$\begin{aligned}\max_{q'\in\mathcal{D}} \mathbb{E}_{q'}\mathbb{E}_{\epsilon_\tau,\underline{p}} D(E,\xi) \geq & \mathbb{E}_q\mathbb{E}_{\epsilon_\tau,\underline{p}} D(E,\xi) \\ &= \hat{D}(E_\tau) + \hat{D}'\varepsilon_\tau + \frac{\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau}}{2}\varepsilon_\tau^2 + \frac{\mathbb{E}\hat{D}''}{2}\sigma_\tau^2\end{aligned} \tag{A.13}$$

Combining (A.9) and (A.13), we obtain:

$$\begin{aligned} T_\tau &= \max_{p\in\mathcal{C}} \left(\max_{q\in\mathcal{D}} \mathbb{E}_q\mathbb{E}_{\epsilon_\tau,p} D(E,\xi) + \mathbb{E}_p C(C_1^{-1}(\tau,\theta),\theta) \right) \\ &\geq \max_{q\in\mathcal{D}} \mathbb{E}_q\mathbb{E}_{\epsilon_\tau,\underline{p}} D(E,\xi) + \mathbb{E}_{\underline{p}} C(C_1^{-1}(\tau,\theta),\theta) \\ &\geq \hat{C}(E_\tau) + \hat{D}(E_\tau) + \varepsilon_\tau(\tau+\hat{D}') \\ &\quad + \frac{\sigma_\tau^2}{2}\left(\mathbb{E}\hat{D}'' - \left(\mathbb{E}_u \widehat{C^r}''^\tau_{E_r,E_\tau} + \frac{1}{\sigma_\tau^2} cov(\widehat{C^r}''^\tau_{E_r,E_\tau}, (E_r-E_\tau)^2)\right)\right) \\ &\quad + \frac{\varepsilon_\tau^2}{2}\left[\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau} - \mathbb{E}_u \frac{\underline{C''^r}\widehat{C^r}''^\tau_{E_r,E_r-\delta_r}}{\widehat{C^r}''^\tau_{E_r,E_r-\delta_r} - \underline{C''^r}}\right] \end{aligned} \tag{A.14}$$

as required. □

Let L^* be the optimal quantity policy. Under the tax policy τ,

$$\begin{aligned} & T_\tau - T_{quantity} \\ &\geq \underbrace{\left(\hat{C}(E_\tau) + \hat{D}(E_\tau)\right) - \left(\hat{C}(L^*) + \hat{D}(L^*)\right)}_{\text{Misspecification}} + \underbrace{\varepsilon_\tau(\tau+\hat{D}')}_{\text{1st order}} \\ &\quad + \underbrace{\frac{\sigma_\tau^2}{2}\left(\mathbb{E}\hat{D}'' - \left(\mathbb{E}_u \widehat{C^r}''^\tau_{E_r,E_\tau} + \frac{1}{\sigma_\tau^2} cov(\widehat{C^r}''^\tau_{E_r,E_\tau}, (E_r-E_\tau)^2)\right)\right)}_{\text{Risk}} \\ &\quad + \underbrace{\frac{\varepsilon_\tau^2}{2}\left[\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau} - \mathbb{E}_u \frac{\underline{C''^r}\widehat{C^r}''^\tau_{E_r,E_r-\delta_r}}{\widehat{C^r}''^\tau_{E_r,E_r-\delta_r} - \underline{C''^r}}\right]}_{\text{Uncertainty}} \end{aligned} \tag{A.15}$$

At this level of generality, the optimal tax policy τ^* does not necessarily yield the optimal quota for emissions reduction in expectation: i.e. E_{τ^*} may differ from L^*. The first 'Misspecification' term is the difference between the expected total costs under the expected emissions reduction level generated by the pricing and quantity policies. Since the quantity policy provides the emissions reductions level minimising expected total costs, this term is always positive. Relatedly, the marginal damages at the expected emissions reduction level E_{τ^*} need not match (the negation of) the optimal tax level τ^*; whence the second, '1st-order' term. Whenever the expected worst-case emissions reduction under tax τ^* is equal to the optimal emissions reduction level L^*, these two terms reduce to zero. By specifying this in Assumption 1, we concentrate on this case in the text, hence focusing on the 'Risk' and 'Uncertainty' terms.

A.2 Proofs of Theorems 1 and 2

Proof of Theorem 1. Let τ^* be an optimal tax level. Since neither 0 nor E^{max} are optimal quantities, $\inf_{(0,E^{max})} \partial\hat{C}(E) < \tau < \sup_{(0,E^{max})} \partial\hat{C}(E)$. Since $\hat{C}$ is strictly convex, there exists E with $\tau^* \in \partial\hat{C}(E)$. Suppose that $\tau^* \in \partial\hat{C}(E)$ for some $E \neq L^*$. By Lemma A.1 (Appendix A.7), there exists $\theta \in \operatorname{supp}\mathcal{C}$ with $C_1(E,\theta) = \tau^*$ and $C(E,\theta) = \hat{C}(E)$, so:

$$\begin{aligned}\hat{D}(C_1^{-1}(\tau^*,\theta)) + C(C_1^{-1}(\tau^*,\theta),\theta) =& \hat{D}(E) + \hat{C}(E) \\ >& \hat{D}(L^*) + \hat{C}(L^*) = T_{quant}\end{aligned}$$

since, by the strict convexity of $\hat{D}$ and $\hat{C}$ (Section 2.3), L^* is the unique quantity optimum. If $\tau^* \in \partial\hat{C}(L^*)$, then by a similar argument there exists $\theta \in \operatorname{supp}\mathcal{C}$ with $\hat{D}(C_1^{-1}(\tau^*,\theta)) + C(C_1^{-1}(\tau^*,\theta),\theta) = \hat{D}(L^*) + \hat{C}(L^*) = T_{quant}$. So $T_{price} \geq T_{quant}$.

As concerns the second clause of the theorem, note first that the reasoning in the proof of Theorem A.1 applies for any $\varepsilon \leq \varepsilon_{\tau^*}$. Noting that when $\mathcal{C}$ is generated by constraints $\sigma^2_{\tau^*} = 0$ and that, since L^* is the quantity optimum, $\tau^* \in \partial\hat{D}(L^*)$, the reasoning in the proof of Theorem A.1 implies that, for $\tau^* \in \partial\hat{C}(L^*)$ and any $\varepsilon \in [0, \varepsilon_{\tau^*}]$:

$$T_{\tau^*} \geq \hat{C}(L^*) + \hat{D}(L^*) + \frac{\varepsilon^2}{2}\left[\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon} - \frac{\underline{C''}\hat{C}''^{\tau^*}_{L^*,L^*-\delta}}{\hat{C}''^{\tau^*}_{L^*,L^*-\delta} - \underline{C''}}\right] \tag{A.16}$$

where δ satisfies $\delta = \underline{C''}\frac{\varepsilon}{\hat{C}''^{\tau^*}_{L^*,L^*-\delta} - \underline{C''}}$. Since, for sufficiently small ε, $\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon} \approx \hat{D}''(L^*)$ and $\hat{C}''^{\tau^*}_{L^*,L^*+\varepsilon} \approx \hat{C}''(L^*)$ whenever these functions are second-order differentiable, $T_{\tau^*} > T_{quantity}$ whenever $\hat{D}''(L^*) > \frac{\hat{C}''(L^*)\underline{C''}}{\hat{C}''(L^*) - \underline{C''}}$, as required. □

Proof of Theorem 2. Under Assumption 1, the 'Misspecification' and '1st order' terms reduce to zero. Moreover $\widehat{C^r}''^{\tau^*}_{E_r,E_{\tau^*}} \approx \widehat{C^r}''(E_{\tau^*})$ so $\mathbb{E}_u\widehat{C^r}''^{\tau^*}_{E_r,E_{\tau^*}} \approx \mathbb{E}_u\widehat{C^r}''(E_{\tau^*}) = \hat{C}''(E_{\tau^*})$, noting that $\hat{C}''(E_{\tau^*})$ exists because $\widehat{C^r}''(E_{\tau^*})$ does for all r. Substituting this in, proceeding similarly for $\mathbb{E}\hat{D}''$ and incorporating the assumption on the covariance yields the 'Risk' term. The substitution yielding the 'Uncertainty' term follows similarly, noting that the Theorem involves a single lower bound on the second order derivative for all probability thresholds. □

A.3 Proofs: Higher derivatives

Proof of Theorem 3. The proof of the first clause is essentially identical to the proof of Theorem 1. The proof of the second clause is similar to that of Theorems 1 and A.1, with the passage in the latter theorem concerning the function G (from Eq. (A.3) to Eq. (A.8)) replaced by the following reasoning. (For notational simplicity, we provide the version for probabilistic constraints with a single lower bound $\underline{C'''}$ and a single upper bound $\overline{C'''}$ on the third derivative. The rest of the notation is as in the proof of Theorem A.1.)

For each $r \in (0,1]$, let δ_r be such that

$$\tau - (\varepsilon_\tau + \delta_r)\widehat{C^r}''(E_r - \delta_r) - \frac{1}{2}(\varepsilon_\tau + \delta_r)^2\underline{C'''} = \widehat{C^r}'(E_r - \delta_r) \tag{A.17}$$

if such δ_r exists (i.e. when $\widehat{C^r}$ is twice differentiable at $E_r - \delta_r$), and the solution to

$$\tau - (\varepsilon_\tau + \delta_r)\widehat{C^r}''(E_r - \delta_r) - \frac{1}{2}(\varepsilon_\tau + \delta_r)^2(\underline{C'''} + \rho_r) = \widehat{C^r}'(E_r - \delta_r)$$

for some small $\rho_r > 0$ such that such a solution exists, otherwise. By the reasoning in the proof of Theorem A.1 (and the fact that, by the third-derivative constraint, the marginal abatement curve is convex, so $\widehat{C^r''}$ exists except on a set of measure zero, by Rademacher's Theorem), δ_r is well-defined, and for each r, it satisfies (A.17) except a set of measure zero.

Let $\hat{\theta}_r$ be any cost function satisfying all the constraints for probability threshold r in the definition of $\mathcal{C}$ such that $C_1(E_r - \delta_r, \hat{\theta}_r) = \widehat{C^r}'(E_r - \delta_r)$, $C(E_r - \delta_r, \hat{\theta}_r) = \widehat{C^r}(E_r - \delta_r)$ and $C_{11}(E_r - \delta_r, \hat{\theta}_r) = \widehat{C^r}''(E_r - \delta_r)$; such a function exists by the definition of $\widehat{C^r}$ (and the fact that it is twice differentiable at the points involved). Consider the function $G : [0, E_r + \varepsilon_\tau] \to \mathbb{R}$ defined by

$$G(E) = \begin{cases} C(E, \theta_r) & \text{if } E \leq E_r - \delta_r \\ C(E_r - \delta_r, \hat{\theta}_r) + \widehat{C^r}'(E_r - \delta_r)(E - E_r + \delta_r) & \\ +\frac{\widehat{C^r}''(E_r - \delta_{r,t})}{2}(E - E_r + \delta_r)^2 + \frac{\underline{C'''}}{6}(E - E_r + \delta_r)^3 & \text{if } E_r - \delta_r < E \leq E_r + \varepsilon_\tau \end{cases}$$

This hits the $\widehat{C^r}$ curve at a point, and then increases with the r-lowest third derivative until the marginal abatement cost reaches τ, which occurs at $E_r + \varepsilon_\tau$. Note that G is increasing, twice differentiable, strictly convex and satisfies all the constraints for the probability threshold r in the definition of $\mathcal{C}$ on its domain. Hence there exists $\underline{\theta_r} \in \mathcal{C}$ with $C(E, \underline{\theta_r}) = G(E)$ for all $E \in [0, E_r + \varepsilon_\tau]$. Take any such $\underline{\theta_r}$.

Analogous to the definition of average second derivative (Section 2.5), define the average third derivative $\widehat{C^r}'''^{\tau}_{E_r, E_r - \delta_r}$ to be such that:

$$\widehat{C^r}(E_r - \delta_r) = \widehat{C^r}(E_r) - \widehat{C^r}'(E_r)\delta_r + \frac{\delta_r^2}{2}\widehat{C^r}''(E_r) - \frac{\delta_r^3}{6}\widehat{C^r}'''^{\tau}_{E_r, E_r - \delta_r}$$

from which it follows that

$$\widehat{C^r}'(E_r - \delta_r) = \tau - \delta_r\widehat{C^r}''(E_r) + \frac{\delta_r^2}{2}\widehat{C^r}'''^{\tau}_{E_r, E_r - \delta_r} \tag{A.18}$$

and

$$\widehat{C^r}''(E_r-\delta_r)=\widehat{C^r}''(E_r)-\delta_r\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r} \tag{A.19}$$

Plugging this into (A.17) yields:

$$(\delta_r+\varepsilon_\tau)^2=2\frac{\varepsilon_\tau\widehat{C^r}''(E_r)+\frac{\varepsilon_\tau^2}{2}\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}}{\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}-\underline{C'''}} \tag{A.20}$$

for all $r\in[0,1]$ except a set of measure zero. A similar expression holds, with $\underline{C'''}$ replaced by $\underline{C''}+\rho_r$, otherwise. Hence

$$\delta_r=(X-1)\varepsilon_\tau \tag{A.21}$$

for $X=\left(\frac{\frac{2}{\varepsilon_\tau}\widehat{C^r}''(E_r)+\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}}{\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}-\underline{C'''}}\right)^{\frac{1}{2}}>0$.

It follows, substituting these equations in appropriately, that

$$\begin{aligned}
C(E_r+\varepsilon_\tau,\underline{\theta_r})=&\,C(E_r-\delta_r,\underline{\theta_r})+\widehat{C^r}'(E_r-\delta_{r,t})(\varepsilon_\tau+\delta_r)\\
&+\frac{\widehat{C^r}''(E_r-\delta_r)}{2}(\varepsilon_\tau+\delta_r)^2+\frac{\underline{C'''}}{6}(\varepsilon_\tau+\delta_r)^3\\
=&C(E_r,\theta_r)-\tau\delta_r+\frac{\delta_r^2}{2}\widehat{C^r}''(E_r)-\frac{\delta_r^3}{6}\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}\\
&+(\tau-\delta_r\widehat{C^r}''(E_r)+\frac{\delta_r^2}{2}\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r})(\varepsilon_\tau+\delta_r)\\
&+\frac{\widehat{C^r}''(E_r)-\delta_r\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}}{2}(\varepsilon_\tau+\delta_r)^2+\frac{\underline{C'''}}{6}(\varepsilon_\tau+\delta_r)^3\\
=&C(E_r,\theta_r)+\tau\varepsilon_\tau+\frac{\varepsilon_\tau^2}{2}\widehat{C^r}''(E_r)\\
&+\frac{\varepsilon_\tau^3}{2}\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}\left(-\frac{1}{3}(X-1)^3+(X-1)^2X-X^2(X-1)\right)+\frac{\varepsilon_\tau^3}{2}\frac{\underline{C'''}}{3}X^3\\
=&C(E_r,\theta_r)+\tau\varepsilon_\tau+\frac{\varepsilon_\tau^2}{2}\widehat{C^r}''(E_r)\\
&-\frac{\varepsilon_\tau^3}{6}\left[X^3\left(\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}-\underline{C'''}\right)-\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}\right]\\
=&C(E_r,\theta_r)+\tau\varepsilon_\tau+\frac{\varepsilon_\tau^2}{2}\left[\widehat{C^r}''(E_r)+\frac{1}{3}\varepsilon_\tau\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}-\frac{\varepsilon_\tau^{-\frac{1}{2}}\left(2\widehat{C^r}''(E_r)+\varepsilon_\tau\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}\right)^{\frac{3}{2}}}{3\left(\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}-\underline{C'''}\right)^{\frac{1}{2}}}\right]
\end{aligned} \tag{A.22}$$

for all $r\in[0,1]$ except a set of measure zero. Continuing as in the proof of Theorem A.1 yields an expression like Eq. (A.1) except that the uncertainty term (see Eq. (A.15)) is replaced by:

$$\frac{\varepsilon_{\tau}^{2}}{2}\left[\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau}+\mathbb{E}_u\left(\widehat{C^r}''(E_r)+\frac{1}{3}\varepsilon_\tau\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}-\frac{\varepsilon_\tau^{-\frac{1}{2}}\left(2\widehat{C^r}''(E_r)+\varepsilon_\tau\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}\right)^{\frac{3}{2}}}{3\left(\widehat{C^r}'''^{\tau}_{E_r,E_r-\delta_r}-\underline{C'''}\right)^{\frac{1}{2}}}\right)\right] \tag{A.23}$$

Noting, as in the proof of Theorem 1, the simplifications implied by the problem under constraints, we obtain that $T_{\tau^*}>T_{quantity}$ whenever

$$\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon_{\tau^*}}+\hat{C}''(L^*)+\frac{1}{3}\varepsilon_{\tau^*}\hat{C}'''(L^*)-\frac{\left(2\hat{C}''(L^*)+\varepsilon_{\tau^*}\hat{C}'''(L^*)\right)^{\frac{3}{2}}}{3\varepsilon_{\tau^*}^{\frac{1}{2}}\left(\hat{C}'''(L^*)-\underline{C'''}\right)^{\frac{1}{2}}}>0$$

The expression in the Theorem follows immediately by basic algebra.

We note finally that this proof also establishes a generalisation of Theorem A.1 to incorporate uncertainty in the third derivative of abatement costs. □

A.4 Constraints on parameters

We first provide a general result applying to any form of uncertainty concerning abatement costs. Proposition 1 concerning uncertainty generated by parametric constraints is a corollary, as shown below.

Theorem A.2. *Consider any $\mathcal{D}\subseteq\Delta(\Xi)$ characterising uncertainty about damages, and any $\mathcal{C}\subseteq\Delta(\Theta)$ characterising uncertainty about abatement costs. For any τ such that $\tau\in\partial\hat{C}(E)$ for some $E\in[0,E^{max}]$, any $\hat{D}'\in\partial\hat{D}(E_\tau)$, any $\underline{p}\in\mathcal{C}$ such that $\epsilon_{\tau,\underline{p}}E=\epsilon_\tau^{+\varepsilon_\tau}$ and any family $\theta_r,\underline{\theta_r}\in\operatorname{supp}\mathcal{C}$ for $r\in(0,1]$ such that $C_1(E_r,\theta_r)=\tau$, $C(E_r,\theta_r)=\widehat{C^r}(E_r)$, $C_1(E_r+\varepsilon_\tau,\underline{\theta_r})=\tau$ and $\underline{\theta_r}\in\operatorname{supp}\underline{p}$ for every $r\in(0,1]$:*

$$T_\tau\geq\hat{C}(E_\tau)+\hat{D}(E_\tau)+\varepsilon_\tau(\tau+\hat{D}') \tag{A.24}$$

$$+\frac{\sigma_\tau^2}{2}\left(\mathbb{E}\hat{D}''-\left(\mathbb{E}_u\widehat{C^r}''^{\tau}_{E_r,E_\tau}+\frac{1}{\sigma_\tau^2}cov(\widehat{C^r}''^{\tau}_{E_r,E_\tau},(E_r-E_\tau)^2)\right)\right) \tag{A.25}$$

$$+\frac{\varepsilon_\tau^2}{2}\left[\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon_\tau}+\frac{2}{\varepsilon_\tau^2}\mathbb{E}_u\left(\int_0^{E^r}e\left(C_{11}(e,\theta_r)-C_{11}(e,\underline{\theta_r})\right)de-\int_{E_r}^{E_r+\varepsilon_\tau}eC_{11}(e,\underline{\theta_r})de\right)\right] \tag{A.26}$$

where all the terms are as in Theorem A.1.

Proof. The proof is identical to the proof of Theorem A.1, apart from the part introducing δ_r (from Eq. A.3 to Eq. (A.9)), which should be replaced by the following.

Now consider $\underline{p}\in\mathcal{C}$ with $\epsilon_{\tau,\underline{p}}E=\epsilon_\tau^{+\varepsilon_\tau}$ (by the definition of ε_τ, such $\underline{p}$ exists), and $\underline{\theta_r}\in$

$\operatorname{supp} \underline{p}$ as specified in the statement of the Theorem. For each such $\underline{\theta_r}$, we have:

$$\begin{aligned}
C(E_r+\varepsilon_\tau,\underline{\theta_r}) &= C(E_r,\theta_r)+\left(\int_0^{E_r+\varepsilon_\tau} C_1(e,\underline{\theta_r})de-\int_0^{E_r} C_1(e,\theta_r)de\right)\\
&= C(E_r,\theta_r)+\left(\tau(E_r+\varepsilon_\tau)-\int_0^{E_r+\varepsilon_\tau} eC_{11}(e,\underline{\theta_r})de-\tau E_r+\int_0^{E_r} eC_{11}(e,\theta_r)de\right)\\
&= C(E_r,\theta_r)+\tau\varepsilon_\tau+\int_0^{E_r} e\left(C_{11}(e,\theta_r)-C_{11}(e,\underline{\theta_r})\right)de-\int_{E_r}^{E_r+\varepsilon_\tau} eC_{11}(e,\underline{\theta_r})de
\end{aligned}$$

where the second inequality follows by integration by parts, and θ_r is as specified in the statement of the Theorem (as well as in the proof of Theorem A.1). Whence

$$\begin{aligned}
\mathbb{E}_{\underline{p}}C(C_1^{-1}(\tau,\theta),\theta) &= \mathbb{E}_u C(E_r+\varepsilon_\tau,\underline{\theta_r})\\
&= \mathbb{E}_u C(E_r,\theta_r)+\mathbb{E}_u\tau\varepsilon_\tau\\
&+\mathbb{E}_u\left(\int_0^{E_r} e\left(C_{11}(e,\theta_r)-C_{11}(e,\underline{\theta_r})\right)de-\int_{E_r}^{E_r+\varepsilon_\tau} eC_{11}(e,\underline{\theta_r})de\right)\\
&= \hat{C}(E_\tau)+\tau\varepsilon_\tau-\frac{1}{2}\left(\mathbb{E}\widehat{C}''\sigma_\tau^2+cov(\widehat{C}'',\sigma_\tau^2)\right)\\
&+\mathbb{E}_u\left(\int_0^{E_r} e\left(C_{11}(e,\theta_r)-C_{11}(e,\underline{\theta_r})\right)de-\int_{E_r}^{E_r+\varepsilon_\tau} eC_{11}(e,\underline{\theta_r})de\right)
\end{aligned} \tag{A.27}$$

where the third equality comes from substituting in Eq. (A.2).

□

Proof of Proposition 1. First note that, by simple algebra, the parametric form f can be shown to have the following single-crossing property: for every $(a,b),(a',b')\in A$, there exists at most one E such that $f(E,a,b)=f(E,a',b')$. It follows that $(\overline{a},\overline{b}),(\underline{a},\underline{b})$ as specified in the statement of the Proposition exist.

For the first clause of the Proposition, note that, if $\tau^*\in\partial\hat{C}'(E)$, then there exists $(a,b)\in A$ such $C(E,\theta_{f,(a,b)})=\hat{C}(E)$ and $f(E,a,b)\geq\tau^*$ and $(a',b')\in A$ such $C(E,\theta_{f,(a',b')})=\hat{C}(E)$ and $f(E,a',b')\leq\tau^*$. By simple algebra, it is clear that there exists $(a''b'')\in\mathbb{R}^2_{\geq 0}$ with $C(E,\theta_{f,(a'',b'')})=\hat{C}(E)$ and $f(E,a'',b'')=\tau^*$. By the aforementioned single-crossing property, it follows that there must be $E'\leq E$ such that $f(F,a'',b'')\geq f(F,a,b)$ for all $F\leq E'$ and $f(F,a'',b'')\leq f(F,a,b)$ for all $F\geq E'$ (if not, that would contradict $C(E,\theta_{f,(a,b)})=C(E,\theta_{f,(a'',b'')})$ and $C_1(E,\theta_{f,(a,b)})\geq C_1(E,\theta_{f,(a'',b'')})$). So $C(F,\theta_{f,(a,b)})\leq C(F,\theta_{f,(a'',b'')})$ for all $F\leq E$ and $C(F,\theta_{f,(a,b)})\geq C(F,\theta_{f,(a'',b'')})$ for all $F\geq E$. By similar reasoning, there must exist $E''\leq E$ such that $f(F,a'',b'')\leq f(F,a,b)$ for all $F\leq E''$ and $f(F,a'',b'')\geq f(F,a,b)$ for all $F\geq E''$. So $C(F,\theta_{f,(a,b)})\geq C(F,\theta_{f,(a'',b'')})$ for all $F\leq E$ and $C(F,\theta_{f,(a,b)})\leq C(F,\theta_{f,(a'',b'')})$ for all $F\geq E$. It follows, since $\mathcal{C}_{f,A}$ is parameter-convex, that $\theta_{f,(a'',b'')}\in\mathcal{C}_{f,A}$. The rest of the proof of the first clause is identical to the proof of Theorem 1, using $\theta_{f,(a'',b'')}$ in the place of θ there.

For the second clause, taking Theorem A.2, applying the reductions employed in the proof of Theorem 1 and substituting the parametric expressions for $C(\bullet,\theta)$ yields, for L^* the quantity optimum:

$$\begin{aligned} T_{\tau^*} \geq & \hat{C}(L^*) + \hat{D}(L^*) + \frac{\varepsilon^2}{2}\left[\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon} + \frac{2}{\varepsilon^2}\left((\bar{a}\left(\frac{L^*}{E^{max}}\right)^{\bar{b}})\frac{\bar{b}L^*}{\bar{b}+1} - (a\left(\frac{L^*+\varepsilon}{E^{max}}\right)^{b})\frac{b(L^*+\varepsilon)}{b+1}\right)\right] \\ = & \hat{C}(L^*) + \hat{D}(L^*) + \frac{\varepsilon^2}{2}\left[\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon} + \frac{2\tau^*}{\varepsilon^2}\left(\frac{\bar{b}L^*}{\bar{b}+1} - \frac{b(L^*+\varepsilon)}{b+1}\right)\right] \\ = & \hat{C}(L^*) + \hat{D}(L^*) + \frac{\varepsilon^2}{2}\left[\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon} + \frac{2\tau^*}{\varepsilon^2}\left(\frac{\bar{b}}{\bar{b}+1}\left(\frac{\tau^*(E^{max})^{\bar{b}}}{\bar{a}}\right)^{\frac{1}{\bar{b}}} - \frac{b}{b+1}\left(\frac{\tau^*(E^{max})^{b}}{a}\right)^{\frac{1}{b}}\right)\right] \end{aligned} \tag{A.28}$$

using the fact that $(\bar{a}\left(\frac{L^*}{E^{max}}\right)^{\bar{b}}) = (a\left(\frac{L^*+\varepsilon_{\tau^*}}{E^{max}}\right)^{b}) = \tau^*$. For (a,b) taken such that $\epsilon > 0$ is sufficiently small, as in the proof of Theorem 1, this establishes the required result.

Remark A.1. The proof of the first clause of Proposition 1 only relies on the single-crossing property of the parametric family mentioned at that point, and the fact that, if $\tau^* \in \partial\hat{C}(E)$, then a member of the family can be found with abatement cost $\hat{C}(E)$ and marginal abatement cost τ^* at E. Since this holds for many parametric families, this establishes the non-optimality of taxes for a wide range of parametric forms used in the literature.

For instance, it also holds for the logarithmic marginal abatement cost family used by Nordhaus (1991), which is defined as in Section 5.2 with $f(E,a,b) = a - b\ln(E^{max} - E)$ for $(a,b) \in \mathbb{R}^2_{>0}$. In the case of this family, the following sufficient condition for $T_{price} > T_{quant}$ can be obtained, by a reasoning analogous to that in the proof of Proposition 1:

$$\hat{D}''^{\tau^*}_{L^*,L^*+\varepsilon} > \frac{2\left[-\left(b(L^*+\varepsilon) - \bar{b}L^*\right) + E^{max}(\bar{a} - \bar{b}\ln E^{max} - (a - b\ln E^{max}))\right]}{\varepsilon^2} \tag{A.29}$$

where L^*, τ^*, $(\bar{a},\bar{b})$ and (a,b) are as in Proposition 1.

The second term in this expression, $\bar{a} - \bar{b}\ln E^{max} - (a - b\ln E^{max})$, is the difference in the marginal cost of the first unit of emissions under this parametric family; as suggested by Figure 1), it is typically fairly small. Moreover, substituting in the expressions for L^* and $L^*+\varepsilon$ derived from f, it is straightforward to check that the first term is negative for $b < \bar{b}$ and decreasing in the ratio between two. Hence, for a range of typical parameter sets for this family, the condition will be satisfied.

A.5 Market imperfections

In the main text, we have followed Weitzman (1974) in considering the case of competitive markets, where the emission reductions resulting from a carbon price τ are determined by the effective aggregate marginal abatement cost function, i.e. as $C_1^{-1}(\tau,\theta)$ for abatement cost function θ (see Section 3.2). A simple exercise for bringing out the extent to which the results extend beyond this assumption involves retaining the abatement cost functions as specified in

Section 2 for the policy maker's objective, but allowing reactions to policies to be determined by a function that does not coincide with the aggregate marginal abatement cost.

To this end, let Ψ be the space of possible *reaction functions*: for each emissions reduction level E, actual abatement cost function $\theta \in \Theta$ and $\psi \in \Psi$, $R(E, \theta, \psi)$ is the carbon price that results in emissions reductions level E under the reaction function ψ when the real abatement function is θ. Each reaction function reflects a possible market reaction (e.g. reaction of all firms and consumers) to a carbon price; for instance, under the standard assumptions adopted in the main text, $R^{-1}(\tau, \theta, \psi) = C_1^{-1}(\tau, \theta)$ for all τ. In the presence of market imperfections, or if firms do not fully know their emissions or abatement costs when deciding on production, the market reaction function may differ. For instance, there are several cases of monopolistic or oligopolistic markets where the optimal tax is less than it would be under perfect competition (Barnett, 1980; Levin, 1985), suggesting that $R^{-1}(\tau, \theta, \psi) > C_1^{-1}(\tau, \theta)$ for such markets ψ.

As defined, reaction functions are compared to underlying effective marginal abatement cost functions. Without such a benchmark, the comparison in Theorem 1 extends almost trivially: whenever there is uncertainty, the optimal quantity policy strictly outperforms the optimal pricing one.[21] Rather, we allow any correlation between reaction function values and actual (marginal) abatement costs as long as they are *local*: for all $\psi \in \Psi$ and every $\theta, \theta' \in \Theta$, if $C(E, \theta) = C(E, \theta')$, $C_1(E, \theta) = C_1(E, \theta')$, $C_{11}(E, \theta) = C_{11}(E, \theta')$ (and similarly for higher order derivatives where defined), then $R(E, \theta, \psi) = R(E, \theta', \psi)$. This reflects the idea that a market reaction to prices corresponding to a certain emissions reduction level can be connected to (marginal) abatement costs for this level, but not for levels too far away from it. Ψ contains all non-negative-valued, strictly increasing, differentiable functions that are local in this sense.

The statements of the quantity and pricing policy optimisation problems are as in Section 3, with $R^{-1}(\tau, \theta, \psi)$ replacing $C_1^{-1}(\tau, \theta)$ in Eq. (4) (and the addition of the expectation and maximisation for Ψ). We establish a version of Theorem 1 which extends beyond the standard market assumptions, involving categorical uncertainty (on the part of the policy maker) in the form of constraints on the abatement costs and on the reaction function, where the sets of priors are defined similarly to Eq. (6). In other words, uncertainty about abatement costs is characterised by $\mathcal{C}_{[m,M],[m^1,M^1],[\underline{C''},\overline{C''}]}$ as in Section 4.1, and uncertainty about the reaction function is characterised by, for each $\theta \in \Theta$:

$$\mathcal{E}_{[m_\theta,M_\theta],[\underline{R'},\overline{R'}]} = \left\{ p \in \Delta(\Psi) : \forall \psi \in \operatorname{supp} p, \quad \begin{array}{l} R(E,\theta,\psi) \in [m_\theta(E), M_\theta(E)] \\ R_1(E,\theta,\psi) \in [\underline{R'}, \overline{R'}] \end{array} \right\} \tag{A.30}$$

where m_θ, M_θ and increasing functions of emissions reductions, for each θ, and $\underline{R'} < \overline{R'}$ are non-negative real numbers. Note that since R gives the aggregate marginal abatement cost representing market behaviour, its first derivative indicates its slope.

[21] To see this, take any tax τ with $R(L^*, \theta, \psi) = \tau$ for some θ, ψ and optimal quantity policy L^*; since there is uncertainty, $R(L, \theta', \psi') = \tau$ for some other L, θ' and ψ'. If the R value is independent of θ these two equalities hold for all $\theta \in \Theta$. So the worst-case total cost of this tax is greater than or equal to $\hat{D}(L) + \hat{C}(L) > \hat{D}(L^*) + \hat{C}(L^*)$, since L^* is the unique optimal quantity.

Proposition A.1. *Consider any $\mathcal{D} \subseteq \Delta(\Xi)$ characterising uncertainty about damages. Suppose that uncertainty about abatement costs is characterised by $\mathcal{C}_{[m,M],[m^1,M^1],[\underline{C''},\overline{C''}]}$ for some $[m, M]$, $[m^1, M^1]$, $\underline{C''} < \overline{C''}$, and uncertainty about reactions by $\mathcal{E}_{[m_\theta,M_\theta],[\underline{R'},\overline{R'}]}$ for each $\theta \in \operatorname{supp} \mathcal{C}_{[m,M],[m^1,M^1],[\underline{C''},\overline{C''}]}$, for some $[m_\theta, M_\theta]$ and $\underline{R'} < \overline{R'}$. Then $T_{price} \geq T_{quant}$.*

Moreover, the inequality is strict whenever $\varepsilon_{\tau^*} > 0$ for optimal tax τ^*, $\hat{D}$ and $\hat{C}$ are second-order differentiable at the optimal quantity level $L^* \in (0, E^{max})$, $\hat{C}'(L^*) \geq \hat{R}(L^*)$ and

$$\hat{D}''(L^*) > \frac{\underline{R'}\hat{C}''(L^*)}{\hat{R}'(L^*) - \underline{R'}} - \frac{\hat{R}'(L^*)\left(\underline{C''}\hat{R}'(L^*) - \underline{R'}\hat{C}''(L^*)\right)}{(\hat{R}'(L^*) - \underline{R'})^2}$$

As is clear from its proof (Appendix A.8), the conditions in the Proposition are sufficient, but far from necessary. Nevertheless, the first condition $\hat{C}'(L^*) \geq \hat{R}(L^*)$ is, as noted above, typically satisfied for a range of monopolistic and oligopolistic models; as is clear from the proof (see Eq. (A.44)) this is often sufficient for the optimal quantity policy to outperform the optimal pricing one. The other condition is reminiscent of that in Theorem 1: again, for sufficiently small lower bounds on the slope of the effective marginal abatement cost function and the reaction function, which arguably correspond to the sort of scientific uncertainty present today (Section 4.1), it is satisfied.

A detailed discussion of this result is beyond the scope of this Appendix. There is no need to enter into details, however, to see that it shows that the main message of the paper – that uncertainty justifies a re-evaluation of the purported superiority of pricing policies – holds even in the absence of standard market assumptions.

A.6 Uncertainty Attitudes

The maxmin EU evaluation rule (Eq. (1)) used in the main text encapsulates *aversion to uncertainty* (or *ambiguity*). In order to ascertain the extent to which this uncertainty aversion drives our results, we consider a simple generalisation that admits a wider range of ambiguity attitudes, namely the α-maxmin EU rule (Ghirardato et al., 2004; Gul and Pesendorfer, 2015). In the present context, it evaluates policy P, leading to a total cost $T(P, \theta, \xi)$ under abatement cost function θ and damage function ξ, by the negation of

$$\alpha \max_{p\in\mathcal{C}} \max_{q\in\mathcal{D}} \mathbb{E}_p\mathbb{E}_q T(P, \theta, \xi) + (1 - \alpha) \min_{p\in\mathcal{C}} \min_{q\in\mathcal{D}} \mathbb{E}_p\mathbb{E}_q T(P, \theta, \xi) \tag{A.31}$$

for $\alpha \in [0, 1]$ an index of uncertainty aversion – higher α translates more aversion to uncertainty. Maxmin-EU corresponds to the case where $\alpha = 1$; in the case of categorical uncertainty, (A.31) reduces to a version of the Hurwicz criterion (Hurwicz, 1951), which evaluates a policy by a mixture of its best- and worst-case values across all potential abatement cost and damage functions. For each $\alpha \in [0, 1]$, let T^α_{quant} and T^α_{price} be the evaluations of the optimal quantity and pricing policies under (A.31) with uncertainty aversion index α, defined analogously to Eqs

(3) and (4).[22]

As a simple indication of the role of uncertainty aversion, the following result extends Theorem 1 for continuous strictly increasing bounds on the marginal abatement cost and fixed lower and upper bounds on its slope.[23] To state it, we define, analogously to $\hat{C}$ and $\hat{D}$, the following best-case cost and damage functions:

$$\underset{\vee}{C}(E) = \min_{p \in \mathcal{C}} \mathbb{E}_p C(E, \theta)$$

$$\underset{\vee}{D}(E) = \min_{q \in \mathcal{D}} \mathbb{E}_q D(E, \xi)$$

Proposition A.2. *Consider any $\mathcal{D} \subseteq \Delta(\Xi)$ characterising uncertainty about damages, and suppose that uncertainty about abatement costs is characterised by $\mathcal{C}_{[m^1, M^1], [\underline{C''}, \overline{C''}]}$ for some non-negative, differentiable, strictly increasing functions m^1, M^1 and real numbers $\underline{C''}, \overline{C''}$ with $\underline{C''} \leq \frac{d}{dE} m^1(E), \frac{d}{dE} M^1(E) \leq \overline{C''}$ for all $E \in [0, E^{max}]$. Then*

$$\begin{aligned} T^{\alpha}_{price} - T^{\alpha}_{quant} \geq & \alpha(\tau^*_\alpha + \hat{D}'(E_{\tau^*_\alpha}))(\underset{\vee}{C}^{-1}(\tau^*_\alpha) - L^*_\alpha) \\ & + \alpha \frac{\varepsilon^2_{\tau^*_\alpha}}{2} \left[\hat{D}''^{\tau^*_\alpha}_{E_{\tau^*_\alpha}, E_{\tau^*_\alpha} + \varepsilon_{\tau^*_\alpha}} - \frac{\underline{C''} \hat{C}''_{E_{\tau^*_\alpha}, E_{\tau^*_\alpha} - \delta}}{\hat{C}''_{E_{\tau^*_\alpha}, E_{\tau^*_\alpha} - \delta} - \underline{C''}} - \frac{\hat{C}''_{E_{\tau^*_\alpha}, L^*_\alpha} + \hat{D}''_{E_{\tau^*_\alpha}, L^*_\alpha}}{2} \frac{(E_{\tau^*_\alpha} - L^*_\alpha)^2}{\varepsilon^2_{\tau^*_\alpha}} \right] \\ & + (1 - \alpha) \left[\frac{(\tau^*_\alpha - \underset{\vee}{C}'(\underline{L}))^2}{2(\overline{C''} - \underset{\vee}{C}''_{\underline{L}, \underline{L} - \gamma})} - \frac{\underset{\vee}{C}''_{\underline{L}, L^*_\alpha} + \underset{\vee}{D}''_{\underline{L}, L^*_\alpha}}{2} (\underline{L} - L^*_\alpha)^2 \right] \end{aligned} \tag{A.34}$$

where τ^*_α is an optimal tax and L^*_α an optimal quantity for uncertainty aversion index α, $E_{\tau^*_\alpha} = \hat{C}'^{-1}(\tau^*_\alpha)$, δ is as in Theorem A.1, γ is as in Lemma A.2 below, and $\underline{L}$ is a solution to:

$$\min_L \min_{p \in \mathcal{C}} \min_{q \in \mathcal{D}} \mathbb{E}_p \mathbb{E}_q \left(D(L, \xi) + C(L, \theta) \right)$$

Noting that $L^*_\alpha \to E_{\tau^*_\alpha}$ and $\hat{D}'(E_{\tau^*_\alpha}) \to -\tau^*_\alpha$ as $\alpha \to 1$, (A.34) reduces to the expression in Theorems A.1 and 1 for the $\alpha = 1$ maxmin-EU case considered in the main text. It also suggests that these are not knife-edge results. For large $\alpha < 1$, L^*_α is close to $E_{\tau^*_\alpha}$, so $\frac{(E_{\tau^*_\alpha} - L^*_\alpha)^2}{\varepsilon^2_{\tau^*_\alpha}}$

[22] More precisely:

$$T^{\alpha}_{quant} = \min_L \begin{pmatrix} \alpha \max_{p \in \mathcal{C}} \max_{q \in \mathcal{D}} \mathbb{E}_p \mathbb{E}_q \left(D(L, \xi) + C(L, \theta) \right) \\ +(1 - \alpha) \min_{p \in \mathcal{C}} \min_{q \in \mathcal{D}} \mathbb{E}_p \mathbb{E}_q \left(D(L, \xi) + C(L, \theta) \right) \end{pmatrix} \tag{A.32}$$

and

$$T^{\alpha}_{price} = \min_\tau \begin{pmatrix} \alpha \max_{p \in \mathcal{C}} \max_{q \in \mathcal{D}} \mathbb{E}_p \mathbb{E}_q \left(D(C_1^{-1}(\tau, \theta), \xi) + C(C_1^{-1}(\tau, \theta), \theta) \right) \\ +(1 - \alpha) \min_{p \in \mathcal{C}} \min_{q \in \mathcal{D}} \mathbb{E}_p \mathbb{E}_q \left(D(C_1^{-1}(\tau, \theta), \xi) + C(C_1^{-1}(\tau, \theta), \theta) \right) \end{pmatrix} \tag{A.33}$$

[23] Extending the notation introduced in Section 4.1, we use $\mathcal{C}_{[m^1, M^1], [\underline{C''}, \overline{C''}]}$ to denote the set of priors generated by constraints on the marginal abatement costs and their slope, with no specific constraints on abatement costs. (So, in terms of Eq. (6), $\mathcal{C}_{[m^1, M^1], [\underline{C''}, \overline{C''}]}$ is shorthand for $\mathcal{C}_{(-\infty, \infty), [m^1, M^1], [\underline{C''}, \overline{C''}]}$.) Note that an extension of this result to probabilistic constraints can be established combining the proof of this Proposition with that of Theorem A.1.

is small and the sign of the second 'α' term will be dictated by the sign of $\hat{D}''^{\tau^*_\alpha}_{E_{\tau^*_\alpha}, E_{\tau^*_\alpha}+\varepsilon_{\tau^*_\alpha}} - \frac{\underline{C}''\hat{C}''_{E_{\tau^*_\alpha}, E_{\tau^*_\alpha}-\delta}}{\hat{C}''_{E_{\tau^*_\alpha}, E_{\tau^*_\alpha}-\delta} - \underline{C}''}$. As discussed in Section 4.1, this can plausibly be taken to be positive for the current state of scientific uncertainty about abatement costs. Similarly, for α close to 1, $\hat{D}'(E_{\tau^*_\alpha})$ will be close to $-\tau^*_\alpha$, suggesting that the first term, though potentially negative, will be small in absolute value. Since $\varepsilon_{\tau^*_\alpha} > \underset{\smile}{C}^{-1}(\tau^*_\alpha) - L^*_\alpha$, this suggests that the combined contribution of the first two terms is positive for large $\varepsilon_{\tau^*_\alpha}$ and α close enough to 1. The sign of the final '$1-\alpha$' term will depend on the comparison of the distance between the tax and a lowest marginal abatement cost ($\tau^*_\alpha - \underset{\smile}{C}'(\underline{L})$) and how far the $\underline{L}$ minimising the best-case total cost is from the optimal quantity policy ($\underline{L} - L^*_\alpha$). This will depend on the details of the worst- and best-case marginal costs and damages; for instance if $m^1(E) = M^1(E) - \mu$ for some constant μ, and similarly for $\mathcal{D}$, then $\underline{L} - L^*_\alpha = 0$ and the whole '$1 - \alpha$' term is positive. Summing up, (A.34) thus shows that there will typically be $\alpha < 1$ large enough for which optimal quantity policies outperform optimal pricing ones.

Proposition A.2 also suggests that this may not hold when the policy maker displays a sufficient degree of uncertainty seeking, i.e. when α is far away enough from 1. However, simple systematic comparison is hampered by the dependence on the details of the marginal abatement costs and damages. For instance, in the $m^1(E) = M^1(E) - \mu$ case discussed above, where $\underline{L} - L^*_\alpha = 0$, a fully uncertainty-seeking evaluation, with $\alpha = 0$, will still rank quantity policies higher than pricing ones whenever there is uncertainty about the marginal abatement costs (i.e. $\tau^*_\alpha - \underset{\smile}{C}'(\underline{L}) > 0$). So, whilst (A.34) reveals that permits cannot be guaranteed to outperform taxes under sufficiently uncertainty-seeking evaluations, it does not ensure that taxes systematically outperform quantities, at least in the absence of further details.

A.7 Other results and remarks

Proposition A.3. *Consider $\mathcal{C}$ generated by probabilistic constraints (be they on costs or parametric). For all $r \in (0, 1]$, $\widehat{C^r}$ is strictly convex.*

Proof. We reason for the case of probabilistic constraints on costs; the case of probabilistic parametric constraints is analogous. Let $\mathcal{C} = \mathcal{C}_{\{m^{(\rho)}, M^{(\rho)}, m^{(\rho)1}, M^{(\rho)1}, m^{(\rho)2}, M^{(\rho)2}\}}$ and for every $\rho \in (0, 1]$, let $\Theta_\rho \subseteq \Theta$ be $\left\{ \theta \in \Theta : \begin{array}{c} C(E,\theta) \in [m^{(\rho)}(E), M^{(\rho)}(E)] \\ C_1(E,\theta) \in [m^{(\rho)1}(E), M^{(\rho)1}(E)] \\ C_{11}(E,\theta) \in [m^{(\rho)2}(E), M^{(\rho)2}(E)] \end{array} \right\}$. Hence, by the definition of $\mathcal{C}$, for every $p \in \mathcal{C}$, $p(\Theta_\rho) \geq \rho$.

Suppose that there exists x and $p \in \mathcal{C}$ with $p(\{\theta : C(E,\theta) \geq x\}) \geq r$. Since, by the previous observation, $p(\Theta_{1-r}) \geq 1-r$, it follows that there exists $\theta \in \Theta_{1-r}$ with $C(E,\theta) \geq x$. So $\{x : \exists p \in \mathcal{C},\ p(\{\theta : C(E,\theta) \geq x\}) \geq r\} \subseteq \{x : \exists \theta \in \Theta_{1-r},\ C(E,\theta) \geq x\}$. Conversely, consider x such that $C(E,\theta) \geq x$ for some $\theta \in \Theta_{1-r}$. By the nestedness of the family of constraints defining $\mathcal{C}$, any p satisfying the constraints for $s < 1 - r$ and putting weight r on θ satisfies all of the constraints (including those for $t \geq 1 - r$); so there exists such $p \in \mathcal{C}$. Since, by construction, $p(\{\theta : C(E,\theta) \geq x\}) \geq r$, we have the converse inclusion. Hence

$$\{x : \exists p \in \mathcal{C},\ p(\{\theta : C(E,\theta) \geq x\}) \geq r\} = \{x : \exists \theta \in \Theta_{1-r},\ C(E,\theta) \geq x\}$$

It thus follows that

$$\begin{aligned}
\widehat{C^r}(E) &= \max_{p\in\mathcal{C}} \sup\left\{x : p(\{\theta : C(E,\theta) \geq x\}) \geq r\right\} \\
&= \sup\{x : \exists p \in \mathcal{C},\ p(\{\theta : C(E,\theta) \geq x\}) \geq r\} \\
&= \sup\{x : \exists \theta \in \Theta_{1-r},\ C(E,\theta) \geq x\} \\
&= \sup\{C(E,\theta) : \theta \in \Theta_{1-r}\}
\end{aligned}$$

So $\widehat{C^r}$ is the pointwise supremum of strictly convex functions, and hence strictly convex. □

Remark A.2 (Average second-order derivatives). Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable, and consider $f''_{x,y}$ for $x, y, \in \mathbb{R}$, defined as in Eq. (2). Then:

$$\begin{aligned}
f(y) &= f(x) + \int_x^y f'(z)dz \\
&= f(x) + \int_x^y \left(f'(x) + \int_x^z f''(u)du\right) dz
\end{aligned}$$

So

$$f''_{x,y} = \frac{2}{(y-x)^2}\left(f(y) - f(x) - f'(x)(y-x)\right) \tag{A.35}$$

$$= \frac{\int_x^y \left(\int_x^z f''(u)du\right) dz}{\int_x^y \left(\int_x^z du\right) dz} \tag{A.36}$$

which is a (normalised) expectation over the values of f'' between x and y.

Lemma A.1. *Under the conditions of Theorem 1, if $y \in \partial\hat{C}(E)$ for some E, then there exists $\theta \in \Theta$ with $\delta_\theta \in \operatorname{supp}\mathcal{C}$ such that $C_1(E,\theta) = y$ and $C(E,\theta) = \hat{C}(E)$.*

Proof. Let $\partial\hat{C}(E) = [x, z]$. By standard results in convex analysis, there exists $\delta_{\underline{\theta}}, \delta_{\overline{\theta}} \in \mathcal{C}$ with $C(E,\underline{\theta}) = C(E,\overline{\theta}) = \hat{C}(E)$, $C_1(E,\underline{\theta}) = x$ and $C_1(E,\underline{\theta}) = z$. Since $y \in \partial\hat{C}(E)$, there exists $\alpha \in [0,1]$ with $y = \alpha x + (1-\alpha)z$. Consider the function $C' : [0, E^{max}] \to \mathbb{R}$ such that $C'(e) = \alpha C(e,\underline{\theta}) + (1-\alpha)C(e,\overline{\theta})$. This function is clearly increasing, differentiable and strictly convex since $C(\bullet,\underline{\theta}), C(\bullet,\overline{\theta})$ are; hence there exists $\theta \in \Theta$ such that $C' = C(\bullet,\theta)$. Moreover, for constraints m, M and every e, $m(e) \leq C(e,\underline{\theta}), C(e,\overline{\theta}) \leq M(e)$, the same holds for $C(E,\theta)$. Since the same holds for constraints on marginal cost and its slope, it follows that θ satisfies all the constraints, so $\delta_\theta \in \operatorname{supp}\mathcal{C}$, establishing the result. □

A.8 Proofs of Results in Sections A.6 and A.5

Proof of Proposition A.1. Similarly to the definition of $\hat{C}$ (Section 3), define, for every $E \in [0, E^{max})$ and $\theta \in \Theta$

$$\bar{R}(E,\theta) = \max_{r \in \mathcal{E}_\theta} \mathbb{E}_r R(E,\theta,\psi)$$

where $\mathcal{E}_\theta$ is the set of priors over Ψ associated with θ. Since each reaction function is strictly increasing and differentiable, $\bar{R}$ is strictly increasing and continuous in the first argument, for all $\theta \in \Theta$. Moreover, define

$$\hat{R}(E) = \max_{\theta \in \operatorname{supp} \mathcal{C} : C(E,\theta) = \hat{C}(E),\ C'(E,\theta) \in \partial \hat{C}(E)} \bar{R}(E,\theta)$$

where the set over which the maximum is taken is non-empty, due to Lemma A.1 (Appendix A.7). (It is closed, by the definition of $\mathcal{C}$.) Again, $\hat{R}$ is strictly increasing and continuous in the first argument.

Let τ^* be an optimal tax level. Since neither 0 nor E^{max} are optimal and $\hat{R}$ is continuous and strictly increasing, there exists E with $\tau^* = \hat{R}(E)$. Suppose that $\tau^* = \hat{R}(E)$ for some $E \neq L^*$. So there exists $\theta \in \operatorname{supp} \mathcal{C}$ and $\psi \in \operatorname{supp} \mathcal{E}_\theta$ with $R(E,\theta,\psi) = \tau^*$ and $C(E,\theta) = \hat{C}(E)$, so:

$$\begin{aligned}\hat{D}(R(\tau^*,\theta,\psi)) + C(R(\tau^*,\theta,\psi),\theta) =& \hat{D}(E) + \hat{C}(E) \\ &> \hat{D}(L^*) + \hat{C}(L^*) = T_{quant}\end{aligned}$$

since, by the strict convexity of $\hat{D}$ and $\hat{C}$ (Section 2.3), L^* is the unique quantity optimum. If $\tau^* = \hat{R}(L^*)$, then by a similar argument there exists $\theta \in \operatorname{supp} \mathcal{C}$ and and $\psi \in \operatorname{supp} \mathcal{E}_\theta$ with $\hat{D}(R(\tau^*,\theta,\psi)) + C(R(\tau^*,\theta,\psi),\theta) = \hat{D}(L^*) + \hat{C}(L^*) = T_{quant}$. So $T_{price} \geq T_{quant}$.

As concerns the second clause, consider any tax level τ such that $\varepsilon_\tau > 0$, and pick any ε such that $0 < \varepsilon \leq \varepsilon_\tau$. Since $\hat{R}$ is strictly increasing and continuous, there exists a unique E such that $\hat{R}(E) = \tau$; call this value E_τ. Moreover, it also follows that there exists δ such that:

$$\tau - (\varepsilon + \delta)\underline{R'} = \hat{R}(E_\tau - \delta) \tag{A.37}$$

By the definition of $\hat{R}$, there exist $\hat{\theta} \in \Theta, \hat{\psi} \in \Psi$ such that $C_1(E_\tau - \delta, \hat{\theta}) = \hat{C}'(E_\tau - \delta)$, $C(E_\tau - \delta, \hat{\theta}) = \hat{C}(E_\tau - \delta)$, and $R^{-1}(E_\tau - \delta, \hat{\theta}, \hat{\psi}) = \hat{R}(E_\tau - \delta)$; take any such functions and consider the function $G : [0, E_\tau + \varepsilon] \to \mathbb{R}$ defined by

$$G(E) = \begin{cases} C(E,\hat{\theta}) & \text{if } E \leq E_\tau - \delta \\ C(E_\tau - \delta, \hat{\theta}) + \hat{C}'(E_\tau - \delta)(E - E_\tau + \delta) \\ + \frac{C''}{2}(E - E_\tau + \delta)^2 & \text{if } E_\tau - \delta < E \leq E_\tau + \varepsilon \end{cases}$$

Note that G is increasing, differentiable, strictly convex and satisfies all the constraints in the definition of $\mathcal{C}$. Hence there exists $\underline{\theta} \in \operatorname{supp} \mathcal{C}$ with $C(E, \underline{\theta}) = G(E)$ for all $E \in [0, E_\tau + \varepsilon]$. Take any such $\underline{\theta}$.

By the locality of the reaction functions in Ψ, $\bar{R}(E_\tau - \delta, \underline{\theta}) = \hat{R}(E_\tau - \delta)$, so there exists $\hat{\psi}$ with $R(E_\tau - \delta, \underline{\theta}, \hat{\psi}) = \hat{R}(E_\tau - \delta)$. By a similar argument to that above, there thus exists $\underline{\psi}$ with $R(E_\tau - \delta, \underline{\theta}, \underline{\psi}) = \hat{R}(E_\tau - \delta)$ and $R(E, \underline{\theta}, \underline{\psi}) = \hat{R}(E_\tau - \delta) + \underline{R}'(E - E_\tau + \delta)$ for all $E \in [E_\tau - \delta, E_\tau + \varepsilon]$. It follows in particular that $R(E_\tau + \varepsilon, \underline{\theta}, \underline{\psi}) = \tau$.

Note that, since

$$\hat{C}''_{E_\tau, E_\tau - \delta} = \frac{2}{\delta_\tau^2} \left(\hat{C}(E_\tau - \delta) - \hat{C}(E_\tau) + \hat{C}'(E_\tau)\delta \right) \tag{A.38}$$

is the average second-order derivative of $\hat{C}$ in this range (Eq. (2)), we have

$$\hat{C}(E_\tau - \delta) = \hat{C}(E_\tau) - \hat{C}'(E_\tau)\delta + \frac{\delta^2}{2} \widehat{C^\tau}''_{E_\tau, E_\tau - \delta}$$

from which it follows that

$$\hat{C}'(E_\tau - \delta) = \hat{C}'(E_\tau) - \delta \hat{C}''_{E_\tau, E_\tau - \delta} \tag{A.39}$$

Similarly,

$$\hat{R}(E_\tau - \delta) = \hat{R}(E_\tau) - \delta \hat{R}'_{E_\tau, E_\tau - \delta} \tag{A.40}$$

where $\hat{R}_{E_\tau, E_\tau - \delta}$ is the average first-order derivative in this range. Plugging this into (A.37) and using the fact that $\hat{R}(E_\tau) = \tau$ yields:

$$\delta = \underline{R}' \frac{\varepsilon}{\hat{R}'_{E_\tau, E_\tau - \delta} - \underline{R}'} \tag{A.41}$$

It follows, substituting these equations in appropriately, that

$$
\begin{aligned}
C(E_\tau+\varepsilon,\underline{\theta}) =& C(E_\tau-\delta,\underline{\theta}) + \hat{C}'(E_\tau-\delta)(\varepsilon+\delta) + \frac{\underline{C''}}{2}(\varepsilon+\delta)^2 \\
=& \hat{C}(E_\tau) - \hat{C}'(E_\tau)\delta + \frac{\delta^2}{2}\hat{C}''_{E_\tau,E_\tau-\delta} \\
&+ (\hat{C}'(E_\tau) - \delta\hat{C}''_{E_\tau,E_\tau-\delta})(\varepsilon+\delta) + \frac{\underline{C''}}{2}(\varepsilon+\delta)^2 \\
=& \hat{C}(E_\tau) - \frac{\delta^2}{2}\hat{C}''_{E_\tau,E_\tau-\delta} + \hat{C}'(E_\tau)\varepsilon - \delta\varepsilon\hat{C}''_{E_\tau,E_\tau-\delta} + \frac{\underline{C''}}{2}(\varepsilon+\delta)^2 \\
=& \hat{C}(E_\tau) + \hat{C}'(E_\tau)\varepsilon + \frac{\varepsilon^2}{2(\hat{R}'_{E_\tau,E_\tau-\delta} - \underline{R'})^2}\Big[-\underline{R'}^2\hat{C}''_{E_\tau,E_\tau-\delta} \\
&-2\underline{R'}\hat{C}''_{E_\tau,E_\tau-\delta}\left(\hat{R}'_{E_\tau,E_\tau-\delta} - \underline{R'}\right) + \underline{C''}\hat{R}'_{E_\tau,E_\tau-\delta}{}^2\Big] \\
=& \hat{C}(E_\tau) + \hat{C}'(E_\tau)\varepsilon + \frac{\varepsilon^2}{2}\left[-\frac{\underline{R'}\hat{C}''_{E_\tau,E_\tau-\delta}}{\hat{R}'_{E_\tau,E_\tau-\delta} - \underline{R'}} + \frac{\hat{R}'_{E_\tau,E_\tau-\delta}\left(\underline{C''}\hat{R}'_{E_\tau,E_\tau-\delta} - \underline{R'}\hat{C}''_{E_\tau,E_\tau-\delta}\right)}{(\hat{R}'_{E_\tau,E_\tau-\delta} - \underline{R'})^2}\right]
\end{aligned} \tag{A.42}
$$

Take any $\hat{D}' \in \partial\hat{D}(E_\tau)$ and recall that:

$$
\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon} = \frac{2}{\varepsilon^2}\left(\hat{D}(E_\tau) + \hat{D}'\varepsilon - \hat{D}(E_\tau+\varepsilon)\right)
$$

is the average second-order derivative of $\hat{D}$ between E_τ and $E_\tau+\varepsilon$. Hence:

$$
\hat{D}(E_\tau+\varepsilon) = \hat{D}(E_\tau) + \hat{D}'\varepsilon + \frac{\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon}}{2}\varepsilon_\tau^2 \tag{A.43}
$$

Combining (A.42) and (A.43), we obtain:

$$
\begin{aligned}
T_{price} =& \max_{p\in\mathcal{C},r\in\mathcal{E}}\left(\max_{q\in\mathcal{D}}\mathbb{E}_q D(R(\tau,\theta,\psi),\xi) + \mathbb{E}_p\mathbb{E}_r C(R(\tau,\theta,\psi),\theta)\right) \\
\geq& \max_{q\in\mathcal{D}}\mathbb{E}_q D(R(\tau,\underline{\theta},\underline{\psi}),\xi) + \mathbb{E}_{\underline{p}} C(R(\tau,\underline{\theta},\underline{\psi}),\underline{\theta}) \\
\geq& \hat{C}(E_\tau) + \hat{D}(E_\tau) + \varepsilon(\hat{C}'(E_\tau) + \hat{D}') \\
&+ \frac{\varepsilon^2}{2}\left[\hat{D}''^{\hat{D}'}_{E_\tau,E_\tau+\varepsilon} - \frac{\underline{R'}\hat{C}''_{E_\tau,E_\tau-\delta}}{\hat{R}'_{E_\tau,E_\tau-\delta} - \underline{R'}} + \frac{\hat{R}'_{E_\tau,E_\tau-\delta}\left(\underline{C''}\hat{R}'_{E_\tau,E_\tau-\delta} - \underline{R'}\hat{C}''_{E_\tau,E_\tau-\delta}\right)}{(\hat{R}'_{E_\tau,E_\tau-\delta} - \underline{R'})^2}\right]
\end{aligned} \tag{A.44}
$$

Since $\tau^* \in \partial\hat{D}(L^*)$ and $\tau^* = \hat{R}'(L^*)$ for quantity optimum L^* and this inequality holds for every ε in the specified range, the result follows. □

Proof of Proposition A.2. Note firstly that, by the specification of $\mathcal{C}_{[m^1,M^1],[\underline{C''},\overline{C''}]}$, there exists $\hat{\theta} \in \operatorname{supp}\mathcal{C}_{[m^1,M^1],[\underline{C''},\overline{C''}]}$ with $C(E,\hat{\theta}) = \hat{C}(E)$ for all $E \in [0, E^{max}]$, and $C_1(E,\hat{\theta}) =$

$M^1(E)$ for all $E \in [O, E^{max}]$. It follows, in particular, that, for every τ, $C_1^{-1}(\tau, \hat{\theta}) \leq C_1^{-1}(\tau, \theta)$ for all $\theta \in \operatorname{supp} \mathcal{C}_{[m^1,M^1],[\underline{C''},\overline{C''}]}$. Similarly, by the specification of $\mathcal{C}_{[m^1,M^1],[\underline{C''},\overline{C''}]}$, there exists $\underset{\smile}{\theta} \in \operatorname{supp} \mathcal{C}_{[m^1,M^1],[\underline{C''},\overline{C''}]}$ with $C(E, \underset{\smile}{\theta}) = \underset{\smile}{C}(E)$ for all $E \in [0, E^{max}]$, and $C_1(E, \underset{\smile}{\theta}) = m^1(E)$ for all $E \in [O, E^{max}]$. Moreover, it follows from these observations that $\hat{C}$ and $\underset{\smile}{C}$ are differentiable.

For any tax level τ, let $E_\tau = \hat{C}'^{-1}(\tau)$. We start with the following Lemma.

Lemma A.2. *For any tax level τ such that $m^1(\underline{L}) \leq \tau \leq M^1(\underline{L})$:*

$$\begin{aligned} &\min_{p \in \mathcal{C}} \mathbb{E}_p \left(\underset{\smile}{D}(C_1^{-1}(\tau, \theta)) + C(C_1^{-1}(\tau, \theta), \theta) \right) \\ \geq &\underset{\smile}{D}(\underline{L}) + \underset{\smile}{C}(\underline{L}) + \frac{(\tau - \underset{\smile}{C}'(\underline{L}))^2}{2(\overline{C''} - \underset{\smile}{C}''_{\underline{L},\underline{L}-\gamma})} \end{aligned}$$

for γ satisfying $\underset{\smile}{C}'(E^* - \gamma) = \tau - \overline{C''}\gamma$.

Proof. Analogous to the proof of Theorem A.1, consider $\overline{\theta} \in \operatorname{supp} \mathcal{C}_{[m^1,M^1],[\underline{C''},\overline{C''}]}$ such that:

$$C(E, \overline{\theta}) = \begin{cases} \underset{\smile}{C}(E) & \text{if } E \leq \underline{L} - \gamma \\ \underset{\smile}{C}(\underline{L} - \gamma) + \underset{\smile}{C}'(\underline{L} - \gamma)(E - \underline{L} + \gamma) & \\ + \frac{\overline{C''}}{2}(E - \underline{L} + \gamma)^2 & \text{if} \underline{L} - \gamma < E \leq \underline{L} \end{cases}$$

where γ satisfies

$$\underset{\smile}{C}'(\underline{L} - \gamma) = \tau - \overline{C''}\gamma$$

By reasoning analogous to that used in the proof of Theorem A.1, such $\overline{\theta}$ (and γ) exists. By construction, $C_1(\underline{L}, \overline{\theta}) = \tau$. Moreover, by the definition of $\underline{L}$ (in particular the fact that $\underline{L}$ minimises $\underset{\smile}{D}(L) + \underset{\smile}{C}(L)$) and by construction (in particular the fact that $C(E, \overline{\theta}) = \underset{\smile}{C}(E)$ for $E \leq \underline{L} - \gamma$), for all $\theta \in \operatorname{supp} \mathcal{C}_{[m^1,M^1],[\underline{C''},\overline{C''}]}$, $\underset{\smile}{D}(C_1^{-1}(\tau, \theta)) + C(C_1^{-1}(\tau, \theta), \theta) \geq \underset{\smile}{D}(C_1^{-1}(\tau, \overline{\theta})) + C(C_1^{-1}(\tau, \overline{\theta}), \overline{\theta})$.

Since

$$\underset{\smile}{C}(\underline{L} - \gamma) = \underset{\smile}{C}(\underline{L}) - \gamma \underset{\smile}{C}(\underline{L}) + \frac{\gamma^2}{2} \underset{\smile}{C}''_{\underline{L},\underline{L}-\gamma}$$

it follows from similar algebra to that in the proof of Theorem A.1 that

$$\gamma = \frac{\tau - \underset{\smile}{C}'(\underline{L})}{\overline{C''} - \underset{\smile}{C}''_{\underline{L},\underline{L}-\gamma}}$$

Hence

$$
\begin{aligned}
C(E,\overline{\theta}) =& \underset{\smile}{C}(\underline{L}-\gamma) + \underset{\smile}{C}'(\underline{L}-\gamma)\gamma + \frac{\overline{C''}}{2}\gamma^2 \\
=& \underset{\smile}{C}(\underline{L}) - \gamma \underset{\smile}{C}'(\underline{L}) + \frac{\gamma^2}{2}\underset{\smile}{C}''_{\underline{L},\underline{L}-\gamma} + \underset{\smile}{C}'(\underline{L})\gamma - \gamma^2 \underset{\smile}{C}''_{\underline{L},\underline{L}-\gamma} + \frac{\overline{C''}}{2}\gamma^2 \\
=& \underset{\smile}{C}(\underline{L}) + \frac{(\tau - \underset{\smile}{C}'(\underline{L}))^2}{2(\overline{C''} - \underset{\smile}{C}''_{\underline{L},\underline{L}-\gamma})}
\end{aligned}
$$

establishing the result. □

We consider taxes such that $m^1(\underline{L}) \leq \tau^* \leq M^1(\underline{L})$. (If not, a similar reasoning applies, leading to a weaker condition than that in the Proposition.)

By the definition of average second-order derivatives:

$$
\begin{aligned}
\underset{\smile}{C}(\underline{L}) =& \underset{\smile}{C}(L^*_\alpha) + \underset{\smile}{C}'(\underline{L})(\underline{L} - L^*_\alpha) - \frac{\underset{\smile}{C}''_{\underline{L},L^*_\alpha}}{2}(\underline{L} - L^*_\alpha)^2 \\
\underset{\smile}{D}(\underline{L}) =& \underset{\smile}{D}(L^*_\alpha) + \underset{\smile}{D}'(\underline{L})(\underline{L} - L^*_\alpha) - \frac{\underset{\smile}{D}''_{\underline{L},L^*_\alpha}}{2}(\underline{L} - L^*_\alpha)^2
\end{aligned}
$$

Hence, by Lemma A.2:

$$
\begin{aligned}
&\min_{p\in\mathcal{C}} \mathbb{E}_p \left(\underset{\smile}{D}(C_1^{-1}(\tau,\theta)) + C(C_1^{-1}(\tau,\theta),\theta)\right) \\
\geq& \underset{\smile}{D}(\underline{L}) + \underset{\smile}{C}(\underline{L}) + \frac{(\tau - \underset{\smile}{C}'(\underline{L}))^2}{2(\overline{C''} - \underset{\smile}{C}''_{\underline{L},\underline{L}-\gamma})} \\
=& \underset{\smile}{D}(L^*_\alpha) + \underset{\smile}{C}(L^*_\alpha) - \frac{\underset{\smile}{D}''_{\underline{L},L^*_\alpha}}{2}(\underline{L} - L^*_\alpha)^2 - \frac{\underset{\smile}{C}''_{\underline{L},L^*_\alpha}}{2}(\underline{L} - L^*_\alpha)^2 + \frac{(\tau - \underset{\smile}{C}'(\underline{L}))^2}{2(\overline{C''} - \underset{\smile}{C}''_{\underline{L},\underline{L}-\gamma})}
\end{aligned}
$$

for any τ with $m^1(\underline{L}) \leq \tau \leq M^1(\underline{L})$, since $\underset{\smile}{C}'(\underline{L}) = -\underset{\smile}{D}'(\underline{L})$ because $\underline{L}$ is a minimum.

Let $\overline{L}$ be a solution to:

$$
\min_L \max_{p\in\mathcal{C}} \max_{q\in\mathcal{D}} \mathbb{E}_p\mathbb{E}_q \left(D(L,\xi) + C(L,\theta)\right)
$$

Theorem A.1 implies that:

$$
\begin{aligned}
&\max_{p\in\mathcal{C}} \max_{q\in\mathcal{D}} \mathbb{E}_p\mathbb{E}_q \left(D(C_1^{-1}(\tau,\theta),\xi) + C(C_1^{-1}(\tau,\theta),\theta)\right) \\
\geq& \hat{C}(E_\tau) + \hat{D}(E_\tau) + \varepsilon_\tau(\tau + \hat{D}'(E_\tau)) + \frac{\varepsilon_\tau^2}{2}\left[\hat{D}''^{\tau^*}_{E_\tau,E_\tau+\varepsilon_\tau} - \frac{\underline{C''}\hat{C}''_{E_\tau,E_\tau-\delta}}{\hat{C}''_{E_\tau,E_\tau-\delta} - \underline{C''}}\right] \\
=& \hat{C}(L^*_\alpha) + \hat{D}(L^*_\alpha) + (\tau + \hat{D}'(E_\tau))(E_\tau - L^*_\alpha) - \frac{\hat{C}''_{E_\tau,L^*_\alpha}}{2}(E_\tau - L^*_\alpha)^2 - \frac{\hat{D}''_{E_\tau,L^*_\alpha}}{2}(E_\tau - L^*_\alpha)^2 + \varepsilon_\tau(\tau + \hat{D}'(E_\tau)) \\
&+ \frac{\varepsilon_\tau^2}{2}\left[\hat{D}''^{\tau^*}_{E_\tau,E_\tau+\varepsilon_\tau} - \frac{\underline{C''}\hat{C}''_{E_\tau,E_\tau-\delta}}{\hat{C}''_{E_\tau,E_\tau-\delta} - \underline{C''}}\right]
\end{aligned}
$$

where $\varepsilon_\tau = \underset{\smile}{C}'^{-1}(\tau) - \hat{C}'^{-1}(\tau)$, δ is as defined in the Theorem and the final equation follows from the definition of average second derivatives (Section 2.5).

Combining the previous two inequalities yields that, for any τ with $\hat{C}^{-1}(\tau) \leq L^*_\alpha \leq \underline{L} \leq \underset{\smile}{C}^{-1}(\tau)$:

$$
\begin{aligned}
&\alpha \max_{p\in\mathcal{C}} \max_{q\in\mathcal{D}} \mathbb{E}_p\mathbb{E}_q \left(D(C_1^{-1}(\tau,\theta),\xi) + C(C_1^{-1}(\tau,\theta),\theta)\right) \\
&\quad + (1-\alpha) \min_{p\in\mathcal{C}} \min_{q\in\mathcal{D}} \mathbb{E}_p\mathbb{E}_q \left(D(C_1^{-1}(\tau,\theta),\xi) + C(C_1^{-1}(\tau,\theta),\theta)\right) \\
\geq&\alpha \Bigg(\hat{C}(L^*_\alpha) + \hat{D}(L^*_\alpha) + (\tau + \hat{D}'(E_\tau))(\underset{\smile}{C}^{-1}(\tau) - L^*_\alpha) - \frac{\hat{C}''_{E_\tau,L^*_\alpha} + \hat{D}''_{E_\tau,L^*_\alpha}}{2}(E_\tau - L^*_\alpha)^2 \\
&\quad + \frac{\varepsilon_\tau^2}{2}\left[\hat{D}''^{\tau^*}_{E_\tau,E_\tau+\varepsilon_\tau} - \frac{\underline{C''}\hat{C}''_{E_\tau,E_\tau-\delta}}{\hat{C}''_{E_\tau,E_\tau-\delta} - \underline{C''}}\right]\Bigg) + (1-\alpha)\Bigg(\underset{\smile}{D}(L^*_\alpha) + \underset{\smile}{C}(L^*_\alpha) - \frac{\underset{\smile}{C}''_{\underline{L},L^*_\alpha} + \underset{\smile}{D}''_{\underline{L},L^*_\alpha}}{2}(\underline{L} - L^*_\alpha)^2 \\
&\quad + \frac{(\tau - \underset{\smile}{C}'(\underline{L}))^2}{2(\overline{C''} - \underset{\smile}{C}''_{\underline{L},\underline{L}-\gamma})}\Bigg) \\
=&\alpha\left(\hat{C}(L^*_\alpha) + \hat{D}(L^*_\alpha)\right) + (1-\alpha)\left(\underset{\smile}{C}(L^*_\alpha) + \underset{\smile}{D}(L^*_\alpha)\right) + \alpha(\tau + \hat{D}'(E_\tau))(\underset{\smile}{C}^{-1}(\tau) - L^*_\alpha) \\
&\quad + \alpha\frac{\varepsilon_\tau^2}{2}\left[\hat{D}''^{\tau^*}_{E_\tau,E_\tau+\varepsilon_\tau} - \frac{\underline{C''}\hat{C}''_{E_\tau,E_\tau-\delta}}{\hat{C}''_{E_\tau,E_\tau-\delta} - \underline{C''}} - \frac{\hat{C}''_{E_\tau,L^*_\alpha} + \hat{D}''_{E_\tau,L^*_\alpha}}{2}\frac{(E_\tau - L^*_\alpha)^2}{\varepsilon_\tau^2}\right] \\
&\quad + (1-\alpha)\left[\frac{(\tau - \underset{\smile}{C}'(\underline{L}))^2}{2(\overline{C''} - \underset{\smile}{C}''_{\underline{L},\underline{L}-\gamma})} - \frac{\underset{\smile}{C}''_{\underline{L},L^*_\alpha} + \underset{\smile}{D}''_{\underline{L},L^*_\alpha}}{2}(\underline{L} - L^*_\alpha)^2\right]
\end{aligned}
$$

which establishes the result whenever τ^*_α satisfies the specified condition. The other case follows from a similar argument.

□