Confidence, aggregation and decision*

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Dedicated to the memory of Philippe Mongin (1950-2020)

Abstract

Integrating experts' judgements into choice is a problem of belief aggregation for the purposes of subsequent decision. As a new approach to this problem, this paper develops an aggregation rule for confidence in beliefs, characterised by a novel Pareto condition that enjoins respecting consensuses borne of compromise. Confidence aggregation generalises standard probability aggregation rules such as linear pooling, whilst avoiding the spurious unanimity issues that have plagued them. It generates the first probability aggregation rule that can faithfully accommodate within-person cross-issue expertise diversity, hence resolving a longstanding challenge. When combined with existing accounts of decision under uncertainty, it naturally generalises Bayesian and non-Bayesian approaches to misspecification of multiple probabilistic models, providing a taxonomy that situates existing approaches along separate aggregation and decision dimensions. It is dynamically rational, insofar as it commutes with update.

Keywords: Belief aggregation, confidence in beliefs, linear pooling, spurious unanimity, expertise, consensus, model averaging, model misspecification, decision under uncertainty.

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1 Introduction

How should policy decisions be taken on the the basis of honest and well-intentioned experts' beliefs? Conceptually, this involves belief aggregation for the purposes of subsequent decision. In practice, institutions at the science-policy frontier often reflect this two-stage structure. Science representatives such as Chief Science Advisor in the UK and the CDC Director in the US are charged with providing policy-relevant assessments that reflect the range of expert opinion; institutions such as the IPCC aim to summarise scientific knowledge on a topic, in order to inform policy. They play the role of 'representatives', collating, evaluating and aggregating expert judgements, providing input to subsequent policy decisions. Much the same goes when the 'experts' are models. Given the informative yet imperfect nature of models in economics, climate science and many other fields, scientists draw on them to form judgements incorporating their uncertainty about model correctness—judgements that feed into subsequent decision making. As such, they play the role of representatives drawing on 'model expertise' and providing input for subsequent decision. These are instances of the *aggregation-for-subsequent-decision* problem, involving: a group of *individuals* each of whom provides input; a *representative* who aggregates them heeding her judgements about the individuals' respective expertise; and a *decision maker* who relies on the aggregate judgement to inform (policy) choice.

The classic Bayesian approach to this problem assumes the individuals' input to come in the form of probability measures; it aggregates them into a 'group' probability measure, which feeds into a Subjective Expected Utility (EU) decision rule. It applies whether the individuals are humans or models: indeed, the popular Bayesian Model Averaging approach to model uncertainty in statistics and economics (Raftery et al., 1997; Steel, 2020) essentially aggregates using a special case of linear pooling (e.g. Stone, 1961; Cooke, 1991; Gilboa et al., 2004), which takes a weighted average of probabilistic beliefs, with subsequent EU-based decision making.

However, this paradigm faces three apparently orthogonal challenges, concerning both the aggregation and the decision dimensions. This paper develops a single approach to the general aggregation-for-subsequent-

	Labour	Real Estate	Both
Laura	0.9	0.1	0.09
Ray	0.1	0.9	0.09
Linear pool	$0.1 + 0.8w^L$	$0.9 - 0.8w^L$	0.09

Table 1: Probability that a certain interest rate has a limited effect on the sector(s) in the top row

Final row gives the results of linear pooling $p(E) = w^L p^L(E) + (1 - w^L)p^R(E)$, with w^L the weight for Laura, and $1 - w^L$ for Ray.

decision problem that resolves them all. We begin by presenting them.

The decision challenge is exemplified by approaches motivated by the fear of *model misspecification*. They take the same input as the Bayesian approach—a probability measure for each individual—but inject them directly into a non-EU decision rule that permits cautious attitudes to model misspecification (Hansen and Sargent, 2001, 2022).

The two aggregation challenges can be illustrated on an example of a (two-member) central bank committee pondering whether to make a given interest rate rise. The committee agree that the determining factor in the choice is whether the rise has a limited (negative) effect on both the labour market and the real estate sector. Table 1 displays the two members' probability judgements for the rise having a limited effect on each of these sectors, and on both. Though both are competent economists, Laura is a specialist in the labour market, whilst Ray's field of expertise is the real estate sector. As is clear from the table, whilst they disagree significantly on the effect of the rise on each sector, they agree on the probability that it will have a limited effect on both sectors.

The first aggregation challenge concerns the treatment of unanimity by linear pooling, the result of which is shown in the final row of Table 1. Irrespective of the weights assigned to the individuals, the linear pool preserves their common judgement on the effect on both sectors—a consequence of the Pareto principle in this context (Mongin, 1995). However, the agreement on this probability is arguably *spurious*, resulting from the fortuitous interplay of two fundamental disagreements. After all, Laura gives a low probability to a limited effect on both sectors because of the low probability she assigns to the real estate event; Ray does so because of the low probability he assigns concerning the labour market; and they disagree on the judgements concerning labour and real estate alone. Several authors have argued that the automatic respect of such *spurious unanimity* is unjustified (Mongin, 2016), and hence a problem for linear pooling (Bradley, 2017b; Mongin and Pivato, 2020; Dietrich, 2021).

The second aggregation challenge involves the way that pooling rules typically incorporate expertise. They do so through the weights in the rule (w^L in Table 1): each individual is allocated a single weight, with larger weights given to individuals with more expertise *overall*. They thus cannot reflect expertise differences across issues. For instance, linear pooling cannot capture the fact that Laura has more expertise on the labour market than the real estate sector (Genest and Zidek, 1986; French, 1985). However, in examples such as this, involving within-person expertise diversity, one might want to respect Laura's opinion more on labour and Ray's more on real estate. Linear pooling, like virtually all probability aggregation rules in the literature, does not allow this. Even model misspecification approaches (Hansen and Sargent, 2022; Cerreia–Vioglio et al., 2025) assign a single weight to each model, hence precluding expertise diversity of the same model across different issues (see Section 5.3).

All three challenges matter for the committee's decision in this example. Integration of model uncertainty or misspecification attitudes can clearly affect the decision. But even sticking to EU, the aggregation issues will impact the subsequent decision. Whilst under linear pooling, the aggregate probability of a limited effect on both sectors is low, following each expert's judgement on their respective sectors of expertise would suggest a significantly higher probability, hence driving a different decision on whether to implement the rise. Moreover, the decision-relevant factor—whether there is a limited effect on both sectors—lies at the intersection of the committee members' fields of expertise, hence posing the problem of how to incorporate their different levels of expertise across issues.

This paper develops a new approach to the general aggregation-for-subsequent-decision problem. By encompassing both Bayesian probability aggregation and recent model misspecification models, it situates the differences between them—hence tackling the decision challenge. It naturally integrates within-person cross-issue expertise diversity, whilst avoiding commitment to spurious unanimity—thus resolving the aggregation challenges.

The heart of the proposal—and first contribution of the paper—is a new aggregation rule for *confidence in beliefs*. Confidence in (experts' or models') beliefs is *prima facie* relevant for reflecting the extent of model uncertainty or misspecification, and several models of decision under uncertainty can be interpreted as involving confidence in beliefs in their representation of belief states (e.g. Marinacci, 2015; Maccheroni et al., 2006; Chateauneuf and Faro, 2009; Hill, 2013, 2019b; Bradley, 2017a). Our *confidence aggregation* rule enjoys independent normative justification, as witnessed by the preference-based axiomatic foundations we provide, showing that it is essentially characterised by a novel Pareto axiom. It preserves consensus, but a more refined notion of consensus than involved in standard Pareto, which leads to problems with spurious unanimity (Mongin, 1995).

Our main contributions show how this single aggregation rule can overcome all the previous challenges, in the context of existing approaches to decision under uncertainty. On the aggregation front, popular probability aggregation rules can be reproduced as special cases of confidence aggregation, corresponding to particular assumptions on confidence in individuals' beliefs. By identifying the point of divergence with classic pooling rules, this analysis sets the stage for the integration of within-person cross-issue expertise diversity. If an individual has more expertise on one issue than another, that justifies more confidence *ceteris paribus* in her beliefs concerning the former issue. Drawing on this insight, we explore the consequences of confidence aggregation when applied in cases involving different degrees of confidence—reflecting differing expertise—across different issues. It yields aggregate judgements that more strongly respect an individual's judgement on the issues on which she has more expertise, as compared to those on which she has less. The approach thus resolves the expertise challenge; moreover, since it does not respect spurious unanimities resulting from ignoring expertise differences, it tackles the other aggregation challenge too.

As an application, we use confidence aggregation to generate a new family of probability aggregation rules that can accommodate within-person expertise diversity. To our knowledge, these *expertise-sensitive pooling rules* are the first of their sort in the literature.

On the decision front, known preference axioms connect the confidence aggregation rule to an existing decision approach incorporating confidence in beliefs (Hill, 2013, 2019b), which generalises variational preferences (Maccheroni et al., 2006) that include in turn the main model misspecification approaches. This provides a general model of aggregation-for-subsequent-decision on the basis of multiple individual inputs. It encompasses all the aforementioned approaches—from linear pooling and Bayesian Model Averaging to model misspecification—situating them along three dimensions. Two of these dimensions—reflecting assumptions about confidence in individual judgements and the only parameter in our aggregation rule—are concerned with the aggregation step; the last—reflecting the importance of the decision, and hence confidence required to take it—concerns decision. As such, our rule situates the divergences and communalities among various existing approaches, as well as suggesting new ones. For instance, combining the techniques used to develop the new expertise-sensitive pooling rule with the settings yielding model misspecification representations, we provide novel model misspecification expertise-sensitive decision rules.

Finally, we briefly consider the issue of dynamic rationality, which is typically invoked to justify geometric pooling of probability measures (Genest and Zidek, 1986; Dietrich, 2021). Drawing on the only existing account of rational update for confidence in belief (Hill, 2022), we show that confidence aggregation fully satisfies dynamically rationality with respect to this update, in the standard sense: the two commute.

The paper is organised as follows. Sections 2 and 3 focus entirely on aggregation: the former sets out the framework and the aggregation rule, whereas the latter shows how it overcomes the aggregation challenges and generates a new family of expertise-sensitive probability aggregation rules. Preferences are introduced in Section 4: it contains a preference-based characterisation of confidence aggregation, while Section 5 expands it to a model encompassing Bayesian as well as model misspecification approaches, and develops a taxonomy of them. Section 6 considers the dynamic rationality of confidence aggregation, and Section 7 discusses remaining related literature. Proofs and supplementary material are contained in the Appendices.

2 Confidence aggregation

The general aggregation-for-subsequent-decision problem involves: a set I of *individuals*, each of which provides input in a specified format; a *representative* who aggregates this input taking into account her evaluations of the individuals' expertise; a *decision maker* who relies on the representative's aggregate judgements to take the subsequent decision(s).

The representative and decision maker will ultimately be modelled by preferences (Sections 4 and 5). By contrast, we make no assumption on the format of individuals' inputs. For instance, they could come in the form of probability measures, as in probability aggregation or model misspecification approaches (Genest and Zidek, 1986; Cerreia–Vioglio et al., 2025). Alternatively, they could be preferences, as in the literature on preference aggregation. As such, all of the development here can support several interpretations, some of which will be highlighted presently.

This section and the subsequent one will focus on the aggregation stage of the aggregation-for-subsequent-decision problem, i.e. that involving the individuals and the representative. This section sets out the heart of our approach: a confidence aggregation rule that takes a representation of confidence in belief for each individual as input, and provides a single confidence in belief representation as output. Section 3 illustrates how it can apply to the general problem (involving alternative input formats).

2.1 Preliminaries

Setup Let Ω be a non-empty set of *states* with a σ-algebra Σ of subsets, called *events*. *Partitions* are sets of mutually disjoint events whose union is Ω . For any partition $\mathcal{P}$ (including Ω itself), $\Delta(\mathcal{P})$ denotes the set of probability measures over $\mathcal{P}$; henceforth, we let $\Delta = \Delta(\Omega)$.[1] For any $p \in \Delta$ and partition $\mathcal{P}_j$, $p|_{\mathcal{P}_j} \in \Delta(\mathcal{P}_j)$ denotes the projection of p into $\Delta(\mathcal{P}_j)$.

$O \subseteq \mathbb{R}$ is an ordered set of confidence levels, endowed with the (strict) order $>$ inherited from $\mathbb{R}$. $\geqslant$ is the corresponding weak order. No general assumptions will be made about the cardinality of O in this paper: we only assume that, if O is not finite, then it is a closed left-bounded interval in

[1] Throughout, we take the weak* topology on Δ and $\Delta(\mathcal{P})$.

$\mathbb{R}$, with the associated topology.[2] We shall use vector notation to denote tuples of confidence levels, i.e. elements of O^n such as $\mathbf{o} = (o_1, \ldots, o_n)$. With slight abuse of notation, we use $\geqslant$ to denote the dominance relation on such tuples: $\mathbf{o} \geqslant \mathbf{o}'$ if and only if $o_i \geqslant o'_i$ for all $i = 1, \ldots, n$.

Beliefs and confidence We work with a general model of confidence in beliefs that, as explained below, underlies many recent models of decision under ambiguity (Hill, 2019b). The belief state of an agent, incorporating confidence, is represented by a *confidence ranking*: a function $c : O \to 2^{\Delta}\backslash\varnothing$ that is increasing in the containment order on sets and is upper semicontinuous.[3] For each confidence level o, $c(o)$ is the set of priors representing the beliefs the agent holds with confidence of at least o. Identifying a probability judgement—such as 'the probability of A is greater than x'—with the set of probability measures where it holds, a judgement $\mathcal{J} \subseteq \Delta$ is held with confidence of at least o if it contains the corresponding set in the agent's confidence ranking, i.e. $\mathcal{J} \supseteq c(o)$. For any $o \in O$ and increasing, upper semicontinuous function $c : \{o' \in O : o' \geqslant o\} \to 2^{\Delta}\backslash\varnothing$, the *natural extension* of c, denoted $\bar{c}$, is the confidence ranking defined by $\bar{c}(o') = c(o')$ for $o' \geqslant o$ and $\bar{c}(o') = c(o)$ otherwise. Π denotes the set of confidence rankings.

The *centre* of confidence ranking c is its smallest element, i.e. $\min_{o \in O} c(o)$. A confidence ranking c is *pointed* if its centre is a singleton. Pointed confidence rankings represent Bayesians with confidence: agents who assign a precise probability to every event (namely, that given by the centre), though may have more confidence in some judgements than others (as represented by the rest of the confidence ranking). A confidence ranking c is *convex* (respectively, *closed*) if, for every $o \in O$, $c(o)$ is a convex (resp. closed) set. For a confidence ranking c, its *convex closure* c^{clconv} is defined in the natural way: for all $o \in O$, $c^{clconv}(o) = clconv(c(o))$, where $clconv(X)$ for a set $X \subseteq \Delta$ is the closure of the convex hull of X.

As an alternative, equivalent representation, each confidence ranking generates a unique *implausibility function* $\iota : \Delta \to O \cup \varnothing$ defined by

[2] It follows that $\geqslant$ is continuous: its upper and lower contour sets are closed.

[3] I.e. for all $o \geqslant o'$, $c(o) \supseteq c(o')$ and for any decreasing sequence $o_i \in O$ with $o_i \to o$, $c(o) = \bigcap_i c(o_i)$.

$\iota(p) = \min\{o \in O : p \in c(o)\}$ whenever the set is non-empty, and $\iota(p) = \varnothing$ otherwise. This yields the 'implausibility' of each probability measure, in terms of the smallest confidence level such that the probability measure doesn't contradict a judgement held with that much confidence.[4]

Related models and distance-based confidence The representation of confidence in beliefs used here includes that in Hill (2013, 2019b) under the calibration developed in Hill (2019a) (see Section 7). Given its equivalence to real-valued implausibility functions on the probability space, it also underlies prominent models of decision under uncertainty which involve such functions in their preference representation. Examples include smooth, variational, multiplier and confidence preferences (Klibanoff et al., 2005; Maccheroni et al., 2006; Hansen and Sargent, 2001; Chateauneuf and Faro, 2009), some of which interpret such functions in terms of confidence.

Moreover, this alternative implausibility-function representation implies that confidence rankings can be generated from single probability measures and (statistical) distances. A *statistical distance* ρ on $\Delta(\mathcal{P})$ is a function $\rho : \Delta(\mathcal{P})^2 \to [0, \infty]$ such that: $\rho(p, q) = 0$ if and only if $p = q$; and $\rho(\bullet, q)$ is a lower semicontinuous function, for all $q \in \Delta(\mathcal{P})$. A distance ρ is *convex* if, for every $q \in \Delta(\mathcal{P})$, the function $\rho(\bullet, q)$ is strictly convex.[5] A (convex) *classical statistical distance* d is the specification, for each partition $\mathcal{P}$ (including Ω itself), of a (convex) statistical distance on $\Delta(\mathcal{P})$. With slight abuse of notation, we use d to refer to the distance for each $\Delta(\mathcal{P})$.

The confidence ranking with centre at probability $p \in \Delta$ and generated by distance ρ on Δ and weight $w \in \mathbb{R}_{>0}$ is defined as follows.

Definition 1. Let $p \in \Delta$ be a probability measure, ρ a statistical distance on Δ and $w \in \mathbb{R}_{>0}$. The w *confidence ranking generated by* p *under* ρ—or simply the w ρ*-confidence ranking generated by* p—is defined by $c(o) = \{q \in \Delta : w\rho(q, p) \leqslant o\}$ for all $o \in O$.

This is clearly the confidence ranking associated with implausibility function $\iota(q) = w\rho(q, p)$. Table 2 lists some well-known classical statistical

[4]Note that c can be defined from ι: $c(o) = \{p \in \Delta : \iota(p) \leqslant o\}$. It follows immediately that the implausibility function ι is lower semicontinuous if c is closed.

[5]That is, for all $p, r \in \Delta$ with $p \neq r$ and $\alpha \in (0, 1)$, $\rho(\alpha p + (1-\alpha)r, q) < \alpha\rho(p, q) + (1-\alpha)\rho(r, q)$.

Generating distance	$\rho(q,p) =$	w ρ-confidence ranking generated by p, $c(o) =$
Euclidean	$\sum_{\omega\in\Omega}(q(\omega)-p(\omega))^2$	$\{q \in \Delta : w\sum_{\omega\in\Omega}(q(\omega)-p(\omega))^2 \leqslant o\}$
Relative entropy	$R(q\|p)$	$\{q \in \Delta : wR(q\|p) \leqslant o\}$
Reverse relative entropy	$R(p\|q)$	$\{q \in \Delta : wR(p\|q) \leqslant o\}$

Table 2: Examples of convex classical statistical distances and distance-generated confidence rankings.
Note: Euclidean distance only well-defined on finite Ω. R is the relative entropy, defined by: $R(p\|q) = -\sum p(\omega)(\log \frac{q(\omega)}{p(\omega)})$.

distances, and the corresponding generated confidence rankings. Several popular models of decision under uncertainty use such distance-generated confidence rankings; for instance, the w relative-entropy-confidence ranking is involved in multiplier preferences (Hansen and Sargent, 2001).

2.2 Consensus-preserving confidence aggregation

Confidence aggregation rules transform profiles of confidence rankings into a confidence ranking, where a *profile* is a tuple $(c^1, \ldots, c^n)$ of confidence rankings. In this section, we refer to the confidence rankings in the profiles as those of the individuals, and the resulting one as that of the group's representative; alternative interpretations will be discussed in Section 3.

The aggregation rule developed here respects consensus, whilst recognising it to be borne of compromise. To introduce the relevant notion of consensus, note that each tuple $\mathbf{o} = (o_1, \ldots, o_n)$ of confidence levels generates a tuple of sets of priors $(c^1(o_1), \ldots, c^n(o_n))$, reflecting the beliefs held by each individual at the confidence level specified for him. If the intersection of these sets is empty, $\bigcap_i c^i(o_i) = \varnothing$, these beliefs are in contradiction. By contrast, if $\bigcap_i c^i(o_i) \neq \varnothing$, they are not: there is a consistent overall *consensus* position, characterised by $\bigcap_i c^i(o_i)$, which incorporates the beliefs of each individual at the assigned confidence level. In this consensus, each individual i only puts beliefs held with confidence o_i or more 'on the

Aggregator	**Definition**
Affine aggregator	$\otimes\mathbf{o} = \sum_{i=1}^{n} \lambda_i o_i + \chi$ for $\lambda_i \in \mathbb{R}_{>0}$, $\chi \in \mathbb{R}$.
Average aggregator	$\otimes\mathbf{o} = \sum_{i=1}^{n} \frac{1}{n} o_i + \chi$ for $\chi \in \mathbb{R}$.
Maximum aggregator	$\otimes\mathbf{o} = \max\{o_i\}$
Minimum aggregator	$\otimes\mathbf{o} = \min\{o_i\}$

Table 3: Examples of confidence-level aggregators.

table', 'setting aside' lower-confidence beliefs. All the former beliefs are retained in the consensus,[6] while the individuals compromise by leaving the latter ones off the table. To that extent, the compromises involved in such a consensus are regulated by the confidence levels determining the beliefs each individual contributes.

There may thus be several such consensuses, involving different compromises—i.e. different tuples of confidence levels. To translate them into levels of confidence deemed relevant for the group, we use a *confidence-level aggregator*: an operator $\otimes : O^n \to O$ that is monotonic in each argument, i.e. such that for every pair of profiles of confidence levels with $\mathbf{o} \geqslant \mathbf{o}'$, $\otimes\mathbf{o} \geqslant \otimes\mathbf{o}'$. For a consensus obtained with individual confidence levels $\mathbf{o}$, the confidence-level aggregator picks out the group confidence warranted in the associated consensus judgements. Monotonicity means that the higher the individual confidence levels $\mathbf{o}$ behind the consensus, the higher the corresponding confidence level for the group's representative. This reflects the fact that higher individual confidence levels translate into more robust consensus, albeit one involving more compromise.

In our preference-based characterisation (Section 4), the relevant confidence-level aggregator will be endogenous; for illustration, Table 3 provides some examples.

Each confidence-level aggregator generates a confidence aggregation rule—i.e. a function from profiles of confidence rankings to confidence

[6]For a probability judgement $\mathcal{J}$, if some individual i holds it at this level of confidence (i.e. $c^i(o_i) \subseteq \mathcal{J}$), then it holds in the consensus ($\bigcap_i c^i(o_i) \subseteq \mathcal{J}$).

rankings—as defined below. Since, in our preference-based characterisation (Section 4), we follow the economic literature and work in a single-profile setup, we also provide the associated definition of consensus-preserving confidence aggregation for a fixed confidence ranking c^{rep} and profile $(c^1, \ldots, c^n)$.

Definition 2. Let $\otimes : O^n \to O$ be a confidence-level aggregator. The *consensus-preserving confidence aggregation rule* $F_\otimes : \Pi^n \to \Pi$ is such that, for every profile $(c'^1, \ldots, c'^n) \in \Pi^n$, $F_\otimes(c'^1, \ldots, c'^n) = \overline{\Phi_\otimes(c'^1, \ldots, c'^n)}$, where, for every $o \in O$ such that $\bigcup_{\mathbf{o}:\otimes\mathbf{o}\leqslant o} \bigcap_i c'^i(o_i) \neq \varnothing$

$$\Phi_\otimes(c'^1, \ldots, c'^n)(o) = \bigcup_{\mathbf{o}:\otimes\mathbf{o}\leqslant o} \bigcap_{i=1}^{n} c'^i(o_i). \tag{1}$$

For a fixed confidence ranking $c^{rep} \in \Pi$ and profile of confidence rankings $(c^1, \ldots, c^n) \in \Pi^n$, c^{rep} is a *consensus-preserving confidence aggregation of* $(c^1, \ldots, c^n)$ if there exists a confidence-level aggregator $\otimes$ such that $c^{rep} = F_\otimes(c^1, \ldots, c^n)$. In this case, we say that c^{rep} is a consensus-preserving confidence aggregation of $(c^1, \ldots, c^n)$ *under* $\otimes$.

Under consensus-preserving confidence aggregation—or *confidence aggregation* for short—the aggregate judgements held with confidence level o are formed by looking at the consensuses considered to warrant confidence level o or less according to $\otimes$.[7] More specifically, the group's representative holds a probability judgement with confidence o if that judgement holds for all such consensuses: this is guaranteed by the union in Eq. (1). Judgements that hold unanimously across the appropriate consensuses are thus preserved. It follows that no judgement held by the representative at confidence level o contradicts the judgements of the corresponding consensuses, though if two consensuses contradict each other on an issue, neither's judgement will be retained by the representative at that confidence level.

Note that this aggregation rule implies that the representative's and individuals' confidence in a judgement co-vary: because of the monotonicity of $\otimes$, the former's confidence in a judgement is higher when the individual beliefs feeding into the relevant consensuses are held with more confidence.

[7] The use of consensuses corresponding to confidence levels less than or equal to o in Eq. (1) ensures that the image of $F_\otimes$ is a well-defined confidence ranking, no matter the $\otimes$. As discussed in Appendix B, 'less than or equal to' can be replaced by 'equal to' for all $\otimes$ in Table 3.

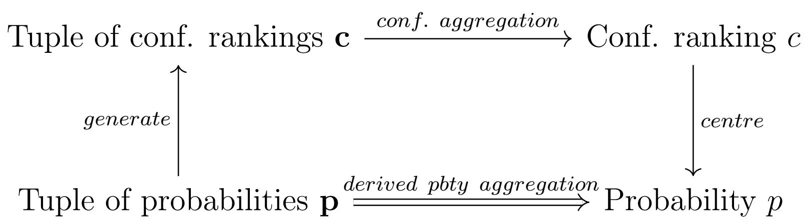


Figure 1: Using confidence aggregation to generate probability aggregation rules

Illustrative applications of confidence aggregation will be provided in the next Section. For reference, note that, as shown in Proposition D.1 (Appendix D.1), this aggregation rule can be formulated in terms of implausibility functions: c^{rep} is a consensus-preserving confidence aggregation of $(c^1, \ldots, c^n)$ under $\otimes$ if and only if, for all $p \in \Delta$

$$\iota^{rep}(p) = \begin{cases} \otimes(\iota^1(p), \ldots, \iota^n(p)) & \text{if } \forall i \ \iota^i(p) \in O \\ \varnothing & \text{otherwise.} \end{cases} \tag{2}$$

3 Confidence, probability aggregation and expertise diversity

Input into aggregation-for-subsequent-decision problems often does not come in the form of confidence rankings: a more common format is as probability measures. In this section, we consider how confidence aggregation can deal with the challenges for existing approaches to aggregating probabilities. We first show that standard pooling rules can be recovered as special cases of confidence aggregation. Analysing the underlying assumptions will naturally reveal how it can resolve both of the challenges mentioned in the Introduction. The discussion also contains several examples illustrating confidence aggregation, and culminates in a new probability aggregation rule integrating within-person cross-issue expertise diversity.

3.1 Recovering probability aggregation from confidence aggregation

Probability aggregation takes as input a profile of probability measures $\mathbf{p} = (p_1, \ldots, p_n) \in \Delta^n$. To connect pooling rules operating on such profiles with confidence aggregation, recall that, once a (statistical) distance and weight are specified, each probability measure generates a unique pointed confidence ranking (Definition 1, Section 2.1). This provides the following possibility for using consensus-preserving confidence aggregation (Section 2.2) to aggregate probability measures. Given a profile of probability measures, take a profile of confidence rankings generated by them under a given distance. Picking a confidence-level aggregator, confidence aggregation can be applied on them, to produce a confidence ranking, call it c. This naturally identifies the 'best-guess' set of probability measures, namely $\min_{o \in O} c(o)$. If this is a singleton (i.e. c is pointed), then the procedure yields a unique probability measure, as required by probability aggregation rules. This schema is summarised in Figure 1.

The following result compares this probability aggregation method to standard pooling rules.

Proposition 1. *Let* $\mathbf{p} = (p_1, \ldots, p_n) \in \Delta^n$ *be a profile of probability measures, and* $(w^1, \ldots, w^n)$ *an* n*-tuple of weights, with* $w^i \geqslant 0$ *for all* i*, with strict inequality for some* i*. For each row in Table 4, the following holds:*

(*) Let c *be the consensus-preserving confidence aggregation under the average confidence-level aggregator of* w^i ρ*-confidence rankings generated by* p_i*, where* ρ *is the distance given in the first column of Table 4. Then its centre is a singleton containing the pool of the* p_i *under the rule specified in the second column of the Table, with weights* $\frac{w^i}{\sum_{i=1}^n w^i}$*. In other words, the centre contains the probability measure satisfying the equation in the third column of the Table.*

Hence the two most prominent pooling rules in the literature (Genest and Zidek, 1986; Mongin, 1995; Dietrich, 2021) correspond to special cases of confidence aggregation. Figure 2b provides a graphical illustration of this result on the example from the Introduction, which will be further analysed below (Example 3.1). Central to it is the use of specific classical

Generating distance ρ	Pooling rule	Centre p satisfies
Euclidean	Linear pooling	$p = \sum_i \frac{w^i}{\sum_{i=1}^n w^i} p_i$
Relative entropy	Geometric pooling	$p(\omega) \propto \prod_i p_i^{\frac{w^i}{\sum_{i=1}^n w^i}}(\omega)$
Reverse relative entropy	Linear pooling	$p = \sum_i \frac{w^i}{\sum_{i=1}^n w^i} p_i$

Table 4: The pooling rules derived from confidence aggregation applied to confidence rankings generated under given classical distances (as in Figure 1). To be read in the context of Proposition 1.

distances (Table 2) to generate confidence rankings from individuals' probability measures. As is clear from the comparison of the cases in Table 4, the 'shape' of the confidence rankings determines the pooling rule reproduced. In this sense, the use of, say, linear pooling, involves an assumption leading to the use of confidence rankings generated by the Euclidean or reverse relative entropy distances. And the evaluation of this pooling rule can thus pass via an appraisal of such assumptions.[8]

The w^i ρ-confidence rankings in Proposition 1—and hence the assumptions underlying them—support two interpretations, which may be relevant in different situations. Under one, they reflect the opinions the representative forms on the basis of each individual's input, taken separately. For instance, if the individuals are models providing probability distributions, each confidence ranking reflects the representative's opinions about the quality of that model (and similarly if they are Bayesian experts). Indeed, the weights in the generated pooling rules, which are often interpreted in terms of the representative's assessment of individuals' expertise (e.g. McConway, 1981), are generated by the confidence rankings. Under this *representative's-confidence* interpretation, using confidence rankings generated by a specific distance amounts to assuming that they properly represent the confidence assessments actually formed by the representative on the basis of each individual's input. If they don't reflect the beliefs she in-

[8]Given that, as noted in Section 2.1, a distance and a probability measure generate a confidence ranking, Proposition 1 is technically related to a literature characterising aggregation rules in terms of distances in probability space (e.g. Abbas, 2009; Kemeny, 1959 initiated a similar approach for preference aggregation). This literature takes the distances as given, whereas we consider them as purported representations of belief states—and, as shall be clear below, evaluate them as such.

fers from the provided probabilities and her evaluation of the individuals' expertise, then this assumption is inappropriate.

Under the other interpretation, the individuals can assess their confidence in their probability judgements, although they have been asked to provide only probabilities. For instance, they may be experts whose belief states are richer than the probability measures requested. Under this *individuals'-confidence* interpretation, consensus-preserving confidence aggregation could be applied on the individuals' full belief representations; it is the setup of the probability aggregation problem that artificially restricts the input to probability measures. Compared to this benchmark, the use of confidence rankings generated by a specific distance involves the assumption that they properly represent individuals' actual confidence in their beliefs. The assumption will be unjustified if individuals' confidence is not well reflected by the distance used.

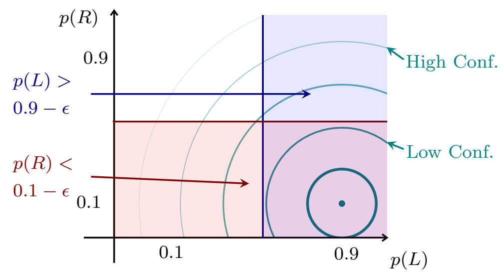
All of the results and points made in this paper will hold under both of these interpretations (and indeed, under hybrid ones where the stipulated confidence reflects individuals' confidence and the representative's assessment). Nevertheless, to lighten the discussion, we shall often conduct it in terms of the latter interpretation.

Proposition 1 suggests a strategy for tackling the aggregation challenges cited in the Introduction. If the weaknesses of traditional pooling rules can be connected to how the confidence rankings are generated, then applying confidence aggregation with different generation methods may avoid them.

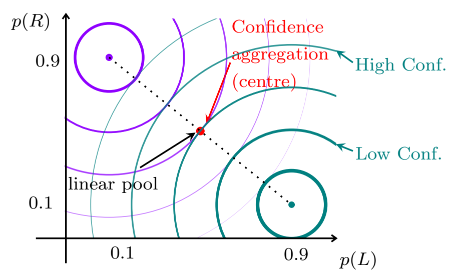
3.2 Classical statistical distances and expertise

One specificity of the classical-distance-based confidence rankings involved in Proposition 1 is a certain 'neutrality' to the identity of the issues involved. All that counts for the confidence with which a probability judgement is held is the classical distance from the centre to the closest probability measure where the judgement doesn't hold—independently of the issue concerned by the judgement. We illustrate this on a running example.

Example 3.1. Formalise the example from the Introduction with a state space $\Omega = \{\omega_{LR}, \omega_L, \omega_R, \omega_N\}$ where ω_{LR} (respectively ω_L, ω_R, ω_N) is the state in which there is a limited effect on both the labour and real es-



(a) Illustration of 'issue-neutrality'.
Note: The blue area represents the probability judgement, $\mathcal{L}_\epsilon$, that $p(L)$ is within ϵ of Laura's best-guess probability $p^L(L) = 0.9$; the red area represents the judgement, $\mathcal{R}_\epsilon$, that $p(R)$ is within ϵ of $p^L(R) = 0.1$. The confidence in these judgements (corresponding to the largest circular set contained in each area; Section 2.1) is the same.



(b) Illustration of Proposition 1.
Note: The red point is the centre of the result of confidence aggregation applied to the two confidence rankings (Proposition 1). Each point on the dotted line is obtained by linear pooling (with some choice of weights). This graph displays the case of $w^L = w^R$; other cases produce centres lying on the dotted line (i.e. coinciding with some linear pool).

Figure 2: Confidence rankings generated as in Proposition 1.
Note: Each graph shows the space of pairs of probability values $(p(L), p(R))$ for the Labour and Real Estate events (L and R; Example 3.1). The areas (sets of probability values) enclosed by the green circles represent the w^L Euclidean confidence ranking generated by Laura's probability p^L (Definition 1): they are the projection of the confidence ranking into this space. Larger, lighter circles correspond to higher confidence levels. The purple circles represent the w^R Euclidean confidence ranking generated by p^R (Ray's probabilities), with $w^R = w^L$.

tate sectors (resp. only the labour market, only the real estate sector, neither). So the event that there is a limited effect on the labour market is $L = \{\omega_{LR}, \omega_L\}$; the corresponding event for real estate is $R = \{\omega_{LR}, \omega_R\}$. Laura's probability judgements (Table 1) define the measure p^L with $p^L(\omega_{LR}) = 0.09$, $p^L(\omega_L) = 0.81$, $p^L(\omega_R) = 0.01, p^L(\omega_N) = 0.09$. Hence, for any $\epsilon \in [0, 0.9]$, she holds both the judgement, $\mathcal{L}_\epsilon$, that the probability of L is greater than $0.9 - \epsilon$, and the judgement, $\mathcal{R}_\epsilon$, that the probability of R is less than $0.1 + \epsilon$.[9] Note that these judgements involve moving the same amount away from her best-guess probability for L (0.9) and R (0.1) respectively. Which of them is she more confident in?

[9] I.e. $\mathcal{L}_\epsilon = \{p \in \Delta : p(L) \geqslant 0.9 - \epsilon\}$ and $\mathcal{R}_\epsilon = \{p \in \Delta : p(R) \leqslant 0.1 + \epsilon\}$.

Proposition D.3 (Appendix D.2) shows that, if her confidence ranking coincides with that resulting from either of the two generating procedures yielding linear pooling (Table 4), the confidence in the two judgements is the same, for all ϵ. Figure 2a illustrates the intuition: given the 'circular' shape of the sets of priors in the confidence ranking, the highest confidence levels at which the judgements hold are the same. Hence the confidence assigned to a judgement that 'deviates' from the best-guess probability by a certain amount depends, in this example, only on the extent of the deviation, but not on the issue concerned by the judgement—labour or real estate.

The confidence rankings generating standard pooling rules thus represent individuals as having or warranting the same confidence in the probability judgements encoded in their probability measure, no matter the issues that these judgements concern. As such, they cannot properly capture an individual with different confidence in judgements pertaining to different issues. The previous example is arguably such a case. Recall that Laura has more expertise on one issue (labour) than another (real estate). But an expertise difference typically translates into a difference in confidence: *ceteris paribus* she will have more confidence in her judgements concerning her issue of expertise than in those that do not. In other words, the confidence rankings involved in Proposition 1, based on classical statistical distances on the probability space, assume that there is no within-person cross-issue difference in expertise.

This observation brings a new perspective on the problem that linear pooling and other standard pooling rules have with within-person expertise diversity. The source of the problem isn't so much the underlying rule in our reconstruction—confidence aggregation—but the use of confidence-ranking-generating procedures which *de facto* assume away within-person cross-issue expertise differences. It thus suggests that confidence aggregation applied to confidence rankings that *do* correctly capture expertise differences could incorporate more faithfully such differences into group beliefs. We now confirm this suggestion, and show how it can produce new expertise-sensitive probability aggregation rules.

3.3 Representing expertise using confidence rankings

For the presentation, we focus on issues that can be related to events in Ω; see Appendix A.2 for a generalisation. Consider a sequence $\mathcal{P}_1, \ldots \mathcal{P}_m$ of partitions of Ω; each partition could be thought of as an *issue*. For instance, a partition could just be an event E and its complement: the issue is whether the event holds. Another partition could have cells corresponding to the event that a parameter takes a given value: the issue is the value of the parameter. We say that a sequence of partitions $\mathcal{P}_1, \ldots \mathcal{P}_m$ is *rich* if, for any $(p_1, \ldots, p_m) \in \prod_{j=1}^{m} \Delta(\mathcal{P}_j)$, there exists at most one $p \in \Delta$ with $p|_{\mathcal{P}_j} = p_j$ for all $j = 1, \ldots, m$. When the sequence of partitions is rich, then each tuple of probability measures, one on each partition, determines at most one probability measure over the whole space.

Example 3.2. In the example from the Introduction, with the state space and events defined in Example 3.1, each of the three issues in Table 1 corresponds to a two-element partition: $\mathcal{P}_L = \{L, L^c\}$ (whether there will be an effect on the labour market), $\mathcal{P}_R = \{R, R^c\}$ (concerning real estate), $\mathcal{P}_B = \{B, B^c\}$, where $B = \{\omega_{LR}\} = L \cap R$ (whether there will be an effect on both). Clearly every specification of a probability on each of these partitions determines at most one probability on Ω, so this sequence of partitions is rich.

Now consider the following family of pointed confidence rankings.

Definition 3. Let $\mathcal{P}_1, \ldots \mathcal{P}_m$ be partitions of Ω and d be a classical statistical distance. For any probability measure $p \in \Delta$, and any vector $\mathbf{w} = (w_1, \ldots, w_m)$ of positive real-valued weights, the $\mathbf{w}$ *d-confidence ranking generated by p* is defined as: for each $o \in O$,

$$c(o) = \left\{ q \in \Delta : \sum_{j=1}^{m} w_j d(q|_{\mathcal{P}_j}, p|_{\mathcal{P}_j}) \leqslant o \right\} \tag{3}$$

For such confidence rankings, at each confidence level, the corresponding set of priors are those for which the weighted sum of the distances from the centre probability, taken over all the partitions (or issues), is less than a certain value. These belong to the family of distance-generated confidence rankings in Definition 1, with implausibility function determined by:

$$\iota(q) = \sum_{j=1}^{m} w_j d(q|_{\mathcal{P}_j}, p|_{\mathcal{P}_j}) \tag{4}$$

Apart from the special case involving a single partition $\mathcal{P} = \Omega$, such ι are not (multiples of) classical statistical distances (Section 2.1).[10]

The issue-specific weights in $\mathbf{w}$ d-confidence rankings can capture relative expertise across issues, with higher weights on a given issue translating more confidence in judgements concerning it. This can be seen on a continuation of the running example.

Example 3.3. Consider p^L as in Example 3.1, and consider the confidence ranking generated by it with Euclidean distance and vector of weights $\mathbf{w^L} = (w^L_L, w^L_R, w^L_B)$, namely:

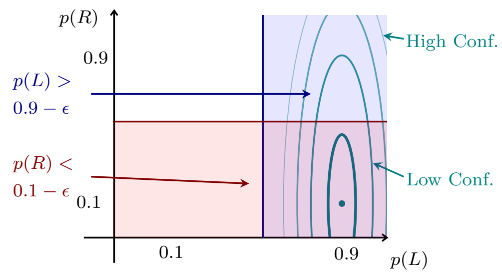
$$c^L(o) = \left\{ q \in \Delta : \sum_{j=\{L,R,B\}} 2w^L_j (q(j) - p^L(j))^2 \leqslant o \right\} \tag{5}$$

The weights reflect the relative confidence in judgements about L, R and B. Larger weights involve a higher 'penalty' for deviating too much on the issue in question, as compared to other issues, so *ceteris paribus*, there is more confidence in judgements concerning issues with higher weights, under this confidence ranking. This is borne out by the following proposition.

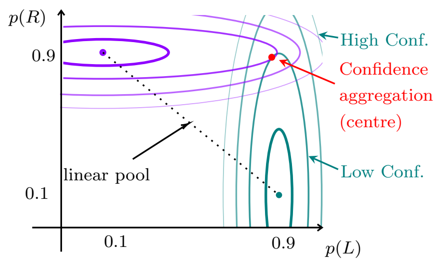
Proposition 2. *Suppose, in (5), that $w^L_L > w^L_R$ and $0.8w^L_B < w^L_L - w^L_R$. Then, for every $\epsilon \in [0, 0.9]$, there exists $o \in O$ with $c^L(o) \subseteq \mathcal{L}_\epsilon$ but $c^L(o) \nsubseteq \mathcal{R}_\epsilon$ (where $\mathcal{L}_\epsilon$, $\mathcal{R}_\epsilon$ are as defined in Example 3.1).*

Whenever w^L_B is not too large, if $w^L_L > w^L_R$, then any judgement $\mathcal{L}_\epsilon$ that the probability of L is higher than a deviation ϵ below its best-guess probability 0.9 is held with more confidence than a judgement about R that involves the same divergence ϵ from its best-guess probability 0.1 ($\mathcal{R}_\epsilon$). Figure 3a illustrates the intuition: when $w^L_L > w^L_R$, the sets in the confidence ranking have an 'elliptical' shape which is thinner along the L dimension, hence translating higher confidence in judgements on this issue.

[10] In particular, they do not automatically provide a distance on any space except Ω.



(a) Illustration of Proposition 2.
Note: As in Figure 2a, the blue area represents the probability judgement, $\mathcal{L}_\epsilon$, that $p(L)$ is within ϵ of Laura's best-guess probability $p^L(L) = 0.9$; the red area represents the judgement, $\mathcal{R}_\epsilon$, that $p(R)$ is within ϵ of $p^L(R) = 0.1$. The confidence in these judgements corresponds to the largest elliptical set contained in each area (Section 2.1): it is higher for the judgement concerning L.



(b) Illustration of expertise-sensitive aggregation (Example 3.4).
Note: The red point is the centre of the result of confidence aggregation applied to the two confidence rankings, which coincides with expertise-sensitive pooling (Definition 4). The aggregate probability of L is closer to Laura's judgement ($p^L(L) = 0.9$), and similarly for R. The dotted line is the set of points obtained by linear pooling (with different weights).

Figure 3: Confidence rankings generated as in Eq. (5).
Note: Each graph shows the space of pairs of probability values $(p(L), p(R))$ for the Labour and Real Estate events (L and R; Example 3.1). The areas (sets of probability values) enclosed by the green ellipses represent the projection into this space of the $\mathbf{w^L}$ Euclidean-confidence ranking generated by p^L—i.e. Eq. (5)—with $w^L_L > w^L_R$ and w^L_B low, representing Laura's confidence in beliefs. Larger, lighter ellipses correspond to higher confidence levels. The purple ellipses represent the $\mathbf{w^R}$ Euclidean-confidence ranking generated by p^R (representing Ray), with $w^R_L < w^R_R$ and w^R_B low.

So $w^L_L > w^L_R$ reflects higher confidence *ceteris paribus* in judgements about the labour market as compared to the real estate sector, and would be a natural assumption for Laura's confidence ranking, given her expertise. The clause concerning w^L_B is related to the constraints that a given value of $p(B)$ places on the possible values of $p(L)$ and $p(R)$, as will be discussed below.

If a confidence ranking of the form (5) with $w^L_L > w^L_R$ is a natural representation for Laura, the corresponding confidence ranking for Ray is centred on p^R as specified in Table 1,[11] with weights $\mathbf{w^R} = (w^R_L, w^R_R, w^R_B)$

[11] I.e. p^R such that $p^R(\omega_{LR}) = 0.09,\ p^R(\omega_L) = 0.01,\ p^R(\omega_R) = 0.81, p^R(\omega_N) = 0.09$.

where $w_R^R > w_L^R$, translating his relative expertise in real estate.

3.4 Confidence aggregation with within-person expertise diversity

Armed with confidence rankings reflecting cross-issue differences in expertise, we now consider aggregation of such rankings. The following Proposition characterises the centre of the confidence ranking obtained by confidence aggregation with an average confidence-level aggregator.

Proposition 3. *Consider a profile of confidence rankings of the form (3), with classical distance d, centre p^i and vector of positive real-valued weights $\mathbf{w^i}$ for each $i = 1, \ldots, n$. Then the centre of the consensus-preserving confidence aggregation under the average confidence-level aggregator is:*

$$\arg\min_{p\in\Delta} \sum_{i=1}^{n}\sum_{j=1}^{m} w_j^i d(p|_{\mathcal{P}_j}, p^i|_{\mathcal{P}_j}) \tag{6}$$

As we now show on the running example, this aggregation naturally incorporates within-person cross-issue expertise diversity.

Example 3.4. Consider, for Laura and Ray, the confidence rankings defined in Example 3.3 with $w_L^L > w_R^L$ and $w_L^R < w_R^R$. As discussed above, these rankings faithfully reflect Laura's higher expertise on the labour issue as compared to the real estate one, and similarly for Ray. Moreover, the example stipulates that Laura has more expertise in the labour market than Ray; in the light of the analysis of confidence rankings of form in Eq. (5), this suggests that $w_L^L > w_L^R$. Similarly, given Ray's higher specialisation in the real estate sector, $w_R^R > w_R^L$. Note that $\frac{w_B^L + w_B^R}{w_L^L + w_L^R}$ reflects the ratio of the overall confidence in the probability judgements on B (across both agents) to the overall confidence in judgements concerning L, and similarly for $\frac{w_B^L + w_B^R}{w_R^L + w_R^R}$.

The following Proposition illustrates some properties of confidence aggregation with such confidence rankings.

Proposition 4. *Suppose that c^L is specified as in Eq. (5) and similarly for c^R, and that $w_L^L > w_L^R$ and $w_R^R > w_R^L$. Then the centre in (6) contains a single probability measure, which we denote p^{rep}. Moreover:*

i. *If* $\frac{w_L^R}{w_L^L} = \frac{w_R^L}{w_R^R}$ *and* $\frac{w_B^L+w_B^R}{w_L^L+w_L^R}, \frac{w_B^L+w_B^R}{w_R^L+w_R^R} \leqslant 1$ *then* $p^L(L) - p^{rep}(L) < p^{rep}(L) - p^R(L)$ *and* $p^R(R) - p^{rep}(R) < p^{rep}(R) - p^L(R)$*;*

ii. *As* $\frac{w_B^L+w_B^R}{w_L^L+w_L^R} \to 0$ *and* $\frac{w_B^L+w_B^R}{w_R^L+w_R^R} \to 0$*:*

$$
\begin{aligned}
p^{rep}(L) \to & \frac{w_L^L}{w_L^L + w_L^R} p^L(L) + \frac{w_L^R}{w_L^L + w_L^R} p^R(L) \\
p^{rep}(R) \to & \frac{w_R^L}{w_R^L + w_R^R} p^L(R) + \frac{w_R^R}{w_R^L + w_R^R} p^R(R) \\
p^{rep}(B) \to & \begin{cases} 0.09 & \textit{if } p^{rep}(L) + p^{rep}(R) - 1 \leqslant 0.09 \\ p^{rep}(L) + p^{rep}(R) - 1 & \textit{otherwise} \end{cases}
\end{aligned}
$$

Recall from the Introduction that, under linear pooling, if the aggregate probability is closer to Laura's judgement than Ray's on one of the issues, then the same holds for the other issue. Proposition 4 shows that this is typically not true under confidence rankings reflecting expertise. Part i. focuses on the case of symmetric expertise—Ray's expertise shortfall as compared to Laura on labour ($\frac{w_L^R}{w_L^L}$) is the same as her expertise shortfall compared to him on real estate. When the overall confidence in the judgements about the issues L and R outweigh the confidence in the judgement concerning the joint issue B, then the resulting probability is closer to that of the individual with more expertise—both for labour and for real estate. Given that, in the examples, Laura's expertise concerns L but not specifically R or $B = L \cap R$, and similarly for Ray, it seems reasonable that the overall confidence will be higher for the issues L and R than for B.[12]

Indeed, in the examples, the overall confidence in the individual issues appears to much higher than in the joint issue. Part ii. considers the limit as this confidence gap increases. It shows that the centre probability for L, $p^{rep}(L)$, tends to the weighted average of Laura's and Ray's judgements on L, where the weights are those in the generation of the confidence rankings that correspond to the issue L. Since Laura has more expertise than Ray on the labour market, $w_L^L > w_L^R$, so the probability for L will be closer to Laura's; and similarly for $p^{rep}(R)$ and Ray's judgement on real estate. So

[12]This result is provided as a simple illustration. Stronger but more complicated versions are derivable from the calculations reported in Appendix A.

again, this part of the result illustrates a typical situation (with potential asymmetric expertise) where, under confidence aggregation, the resulting probability is closer to that of relevant expert, on both issues.

Figure 3b provides a visual illustration: the centre under confidence aggregation belongs to sets with confidence levels that are not too high on either ranking, and this picks out probability measures that are close to both Laura's probability on L and Ray's on R. Confidence aggregation applied to these confidence rankings, which reflect within-person cross-issue expertise differences, thus follows each individual more closely on their area of expertise. As such, it fairs better on this score than linear (or geometric) pooling.

Given that the centres of the confidence rankings generated as in (5) are probability measures assigning probability 0.09 to B, the centre of the resulting ranking will stick as close to this value as possible. If the weights yield issue-wide weighted averages which are consistent with the probability 0.09 for B, then this is the value of $p^{rep}(B)$. If not, as will typically be the case, then $p^{rep}(B)$ takes the value closest to 0.09 that satisfies the constraints, i.e. $p^{rep}(L) + p^{rep}(R) - 1$. Since this is typically not 0.09,[13] this example also demonstrates that the confidence aggregation rule does not respect spurious unanimities (Section 1).[14]

This example illustrates a simple way in which the confidence approach can capture within-person expertise diversity, showing that confidence aggregation faithfully reflects these expertise differences in the resulting beliefs. As such, it resolves the within-person cross-issue expertise diversity challenge. Moreover, it ignores agreements in the individuals' judgements about B when there is little comparative confidence in them—that is, when they are indeed spurious. So the aggregation procedure also tackles the spurious unanimity challenge.

The preceding discussion suggests that confidence aggregation, applied on individuals' actual confidence rankings reflecting their acknowledged differences in expertise across issues, will result in aggregate beliefs that

[13] E.g. when $w^L_L = w^R_R = 0.75$ and $w^L_R = w^L_R = 0.25$, $p^{rep}(B) = 0.4$.

[14] Note that while this example involves 'primitive' views on the joint issue B, an alternative would be to work with opinions on the relationship between L and R. Appendix A.2 shows how the approach set out above extends naturally to such cases.

incorporate within-person cross-issue expertise differences. As argued in Section 3.3, confidence rankings of the sort in Definition 3 are often a closer representation of individuals' beliefs than those underlying standard pooling rules; Propositions 3 and 4 show that, applied to these rankings, confidence aggregation respects expertise differences across issues.

Within-person cross-issue expertise diversity remains relevant under the alternative interpretation of confidence rankings as reflecting the representative's confidence in individuals' judgements (Section 3.1). In the example, the representative is justified in being more confident in Laura's judgement about labour than Ray's—and inversely for real estate. It is thus more reasonable to use the expertise-sensitive generation of confidence rankings (Definition 3) rather than classical distance-based ones (Proposition 1) to aggregate probabilities. As we now show, this generates a new pooling rule.

3.5 Expertise-sensitive pooling

Confidence aggregation using confidence rankings generated by classical distances recoups standard pooling rules (Proposition 1; Figure 1); we now consider what we obtain when applying it with $\mathbf{w^i}$ d-confidence rankings. To this end, consider the correspondence yielding the centre in the result of confidence aggregation applied to $\mathbf{w^i}$ d-confidence rankings.

Definition 4. Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be a set of partitions, and d a classical distance. The correspondence $F^d_{\mathcal{P}_1,\ldots,\mathcal{P}_m} : \Delta^n \rightrightarrows \Delta$ is defined by

$$F^d_{\mathcal{P}_1,\ldots,\mathcal{P}_m}(p^1, \ldots, p^n) = \arg\min_{p\in\Delta} \sum_{i=1}^{n} \sum_{j=1}^{m} w^i_j d(p|_{\mathcal{P}_j}, p^i|_{\mathcal{P}_j}) \tag{7}$$

where $\mathbf{w^i} = (w^i_1, \ldots, w^i_m)$ is a tuple of vectors of positive real-valued weights, one for each individual.

As yet, $F^d_{\mathcal{P}_1,\ldots,\mathcal{P}_m}$ is not a well-defined probability aggregation rule. In particular, since the optimisation problem may have multiple solutions, $F^d_{\mathcal{P}_1,\ldots,\mathcal{P}_m}$ may yield a set of probability measures rather than a unique one. However, under some natural conditions, $F^d_{\mathcal{P}_1,\ldots,\mathcal{P}_m}$ is a well-defined (i.e. single-valued) probability aggregation rule.

Proposition 5. *Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be a rich set of partitions, and d a convex distance. Then $F^d_{\mathcal{P}_1,\ldots,\mathcal{P}_m}$ is a well-defined pooling rule, i.e. a function from Δ^n to Δ.*

Confidence aggregation thus generates a new pooling rule, which we call *expertise-sensitive pooling*. Since this is basically the rule used in Example 3.4, all the conclusions there—including the capacity to naturally respect within-person expertise diversity—apply equally for this pooling rule. Indeed, as discussed in Section 7, the limit expressions concerning L and R in Proposition 4 are reminiscent of early suggestions in the pooling literature; unlike them, however, expertise-sensitive pooling is well-defined.

As an illustration of the technical simplicity of the confidence approach, it is worth setting out the intuition behind Proposition 5. It relies on the observation that the optimisation problem defining $F^d_{\mathcal{P}_1,\ldots,\mathcal{P}_m}$ can typically be reduced to a recognisable, tractable form. More specifically, for a sequence of partitions $\mathcal{P}_1, \ldots \mathcal{P}_m$, let $P_{\mathcal{P}_1,\ldots\mathcal{P}_m} = \{(p|_{\mathcal{P}_1}, \ldots, p|_{\mathcal{P}_m}) \in \prod_{k=1}^m \Delta(\mathcal{P}_k) : p \in \Delta\}$, i.e. the set of sequences of probability measures on the partitions, each of which is derived from some probability measure on Ω. Note that, since projection is a linear map, $P_{\mathcal{P}_1,\ldots\mathcal{P}_m}$ is a convex set. In fact, it is typically defined by a collection of inequalities: for instance, in the case of our running example (Example 3.2), the set is defined by a system of three linear inequalities (see Appendix A.1 for details). The centre of the aggregate confidence ranking (7) can thus be equivalently characterised as the set of probability measures p such that $(p|_{\mathcal{P}_1}, \ldots, p|_{\mathcal{P}_m})$ belongs to:

$$\underset{(p_1,\ldots,p_m)\in P_{\mathcal{P}_1,\ldots,\mathcal{P}_m}}{\arg\min} \sum_{i=1}^{n} \sum_{j=1}^{m} w^i_j d(p_j, p^i|_{\mathcal{P}_j}) \tag{8}$$

For many settings of the parameters, problems of this sort are well-known: for instance, for the $\mathbf{w}$ Euclidean-confidence rankings in Example 3.3, this is a quadratic optimisation problem over a convex set (Appendix A.1). In general, whenever d is convex, (8) is a minimisation of a strictly convex lower semicontinuous function on a convex set, so there is a unique minimum. So whenever $\mathcal{P}_1, \ldots, \mathcal{P}_m$ is rich, (8) defines a unique probability measure in Δ, hence establishing the Proposition.

4 Characterising Confidence Aggregation

In this section we provide a preference-based axiomatisation of consensus-preserving confidence aggregation (Definition 2) in a single-profile variable-population setting. In terms of the general aggregation-for-subsequent-decision outlined at the beginning of Section 2, here we shall thus be concerned with the relationship between the representative's preferences and the individuals' input. We begin by setting out the decision framework and preference representation.

4.1 Representative's preferences and consensus

Preliminaries Let $\mathcal{X}$, the set of *consequences*, be a convex subset of a vector space; for instance it could be the set of lotteries over a set of prizes, as in the version of the Anscombe and Aumann (1963) setup provided by Fishburn (1970). $\mathcal{A}$ is the set of *acts*: simple measurable functions from states Ω to consequences $\mathcal{X}$. Mixtures of acts are defined pointwise as standard: for any $f, g \in \mathcal{A}$ and $\alpha \in [0, 1]$, the α-mixture of f and g, which we denote with $f_\alpha g$, is defined by $f_\alpha g(\omega) = \alpha f(\omega) + (1 - \alpha) g(\omega)$ for all $\omega \in \Omega$. We denote by $\mathcal{A}^c$ the set of a constant acts, i.e. acts yielding the same consequence in all states.

We use $\succ$ (perhaps with superscripts) to denote a strict preference relation on $\mathcal{A}$. $\succeq$ is the derived weak preference, defined as $f \succeq g$ if and only if $g \nsucc f$. Preferences $\succ$ *contradict* $\succ'$ if there exists $f, g \in \mathcal{A}$ with $f \succ g$ and $f \prec' g$. A preference relation $\succ$ is *contradictory* if there exists $f, g \in \mathcal{A}$ with $f \succ g$ and $f \prec g$. As standard, a functional $V : \mathcal{A} \to \mathbb{R}$ is said to represent $\succ$ if, for all acts $f, g \in \mathcal{A}$, $f \succ g$ if and only if $V(f) > V(g)$.

Decision models It is well known that ambiguity-sensitive, complete preferences can be connected to underlying incomplete preferences (Ghirardato et al., 2004; Gilboa et al., 2010). The bridge between the two concerns ambiguity attitudes—a taste factor, which, in the context of the aggregation-for-subsequent-decision problem, typically lies under the competence of the decision or policy maker, not the experts. Hence, whilst the decision maker's (weak) preferences are complete, it is natural to use incomplete preferences to model the representative's relevant preferences.

In so doing, we follow Danan et al. (2016) in studying aggregation in the context of incomplete preferences. As noted in Section 2.1, the representation of confidence in beliefs used here is compatible with several models of decision under uncertainty. A general incomplete preference model it supports (Hill, 2016) is such that for all acts $f, g \in \mathcal{A}$, $f \succ g$ if and only if:

$$\mathbb{E}_p u(f) > \mathbb{E}_p u(g) \qquad \text{for all } p \in c(\max\{D(f), D(g)\}). \tag{9}$$

where $\mathbb{E}_p$ is the expectation with respect to a probability measure $p \in \Delta$,[15] $u : \mathcal{X} \to \mathbb{R}$ is a non-constant affine utility function, c is a closed convex confidence ranking and $D : \mathcal{A} \to O$ is such that $D(x) = \min O$ for all $x \in \mathcal{A}^c$ and it satisfies the following *richness* condition: for every $f, g \in \mathcal{A} \backslash \mathcal{A}^c$ and $o \in D(\mathcal{A} \backslash \mathcal{A}^c)$, there exists $h \in \mathcal{A}$ and $\alpha \in (0, 1]$ such that $\max\{D(f_\alpha h), D(g_\alpha h)\} = o$. The function D, called the *cautiousness coefficient*, picks out the confidence level the decision maker considers relevant for evaluating each act, and hence each decision. As shown in Hill (2013, 2016), it captures the decision maker's attitudes to choosing on the basis of limited confidence—a taste factor.

If (9) holds, we say that (c, D, u) represents $\succ$. The representing u is unique up to positive affine transformation, $c(O)$ is unique up to convex closure, and $c \circ D$ is unique.

General aggregation problem Recall that I is a finite set of individuals. Each individual i provides fixed input $\mathcal{I}^i$. As in single-profile contexts that are common in the relevant literature (e.g. Mongin, 1995; Danan et al., 2016), the sequence of inputs $(\mathcal{I}^i)_{i \in I}$ will be fixed throughout. However, we consider variable-population single-profile aggregation: the representative has access to the inputs from a set $J \subseteq I$ of individuals, which may vary. For each non-empty $J \subseteq I$, $\succ^J$ is the representative's preference formed on the basis of inputs $(\mathcal{I}^i)_{i \in J}$ from individuals in J. As a point of notation, for singleton $J = \{i\}$, we use $\succ^i$ in place of $\succ^{\{i\}}$.

As noted in Section 2, we make no assumption on the format of inputs; accordingly, this framework is rich enough to cover several standard setups in the literature. For instance, each $\mathcal{I}^i$ could be a probability measure

[15] I.e. for any $\phi : \Omega \to \mathbb{R}$, $\mathbb{E}_p \phi = \int \phi(\omega) dp(\omega)$.

$p^i \in \Delta$, as in the case of probability aggregation or model misspecification. In this case, the $\succ^i$ would be preferences based on the confidence rankings generated by these probabilities, as in the representative's-confidence interpretation discussed in Section 3. Alternatively, each $\mathcal{I}^i$ could be a preference satisfying (9). Under the assumption that the representative's preferences $\succ^i$ coincides with the input preferences $\mathcal{I}^i$, this reduces to a classic preference aggregation problem (Crès et al., 2011; Danan et al., 2016) for confidence preferences.

We assume that the representative's preferences are represented according to the incomplete preference model (9), with the same tastes no matter the inputs received. Since the confidence model has two parameters representing tastes—the utility function u and the cautiousness coefficient D—this is expressed by the following assumption.[16]

Assumption 1. *For each non-empty $J \subseteq I$, $\succ^J$ can be represented according to (9) with the same D and u for all J.*

When the inputs are confidence preferences, this corresponds to the standard assumption in the aggregation literature that all individuals and the group share the same tastes (e.g. Crès et al., 2011).

Henceforth, we fix representations (c^J, D, u) of $\succ^J$, for non-empty $J \subseteq I$. As for preferences, we use c^i as shorthand for $c^{\{i\}}$.

Stakes A central idea behind the confidence family is that the beliefs one relies on to decide are held with a level of confidence that is appropriate given the importance of the decision (Hill, 2013, 2016, 2019b; Bradley, 2017a). For instance, (9) represents agents for which determinate preferences held at low stakes—where less confidence is required—may become indeterminate at higher stakes. In the light of this, when higher-confidence beliefs are invoked—i.e. $\max\{D(f), D(g)\} > \max\{D(f'), D(g')\}$—then this is an indication that the agent considers the choice between f and g to be more important than the choice between f' and g': it involves higher

[16]Given the uniqueness of the representation, this is equivalent to the uniqueness of u and D up to appropriate transformations. Indeed, whilst stated on the models for ease, this Assumption can be reformulated in behavioural terms, drawing, for instance, on the choice-based foundations for the weak preference version of (9) provided by Hill (2016) (which can be extended to strict preference using Bewley, 1986; Karni, 2011).

stakes. In the context of (9), this can be formalised by a surjective function $\sigma : \mathcal{A} \times \mathcal{A} \to \mathcal{S} \subseteq \mathbb{R}$, assigning to each binary choice the stakes involved in it. Hill (2016) contains several examples of such (real-valued) notions of stakes.[17] For $(f, g) \in \mathcal{A} \times \mathcal{A}$ and $s \in \mathcal{S}$, we say that (f, g) *has stakes* s if $\sigma(f, g) = s$. Assumption 1 guarantees that the representative's preferences across different J are consistent with a single stakes function σ, in the sense that D and σ are linked by a monotone transformation.[18]

Given a preference relation $\succ$ represented according to (9) and a stakes level $s \in \mathcal{S}$, the derived relation $\succ_s$ is defined as follows: for all $f, g \in \mathcal{A}$, $f \succ_s g$ if and only if there exists $h \in \mathcal{A}$ and $\alpha \in (0, 1]$ such that $(f_\alpha h, g_\alpha h)$ has stakes s and $f_\alpha h \succ g_\alpha h$.[19] As discussed in Hill (2013, 2016), $f \succ_s g$ essentially says that, if the acts were evaluated 'as if' the decision involved stakes s, then f would be preferred. For example, consider two choices. One is between the bet f on the Democrat candidate winning the 2028 US President election, yielding \$1 million if you win and a loss of \$1 million if not, and nothing, g. The other choice is between a similar bet f' on the 2032 election, with stakes (winnings and losses) a million times less in utility terms, and no utility change, g'. An agent with beliefs that are more precise and slightly more favorable for the 2028 bet might nevertheless choose the bet in the 2032 choice but have indeterminate preferences in the 2028 one because of the difference in stakes: with lower stakes, he can rely on low-confidence beliefs when comparing f' and g', but not for the choice between f and g. However, if the 2028 choice were evaluated at the low stakes level, say s, then f would typically be chosen over g: i.e. $f \succ_s g$. When $f \succ_s g$, we say that f *is preferred to* g *at stakes level* s, and we call $\succ_s$ *the preferences at stakes level* s.

Consensus preferences A *profile of stakes levels* is a variable-population tuple of stakes levels $\mathbf{s} = (s_i)_{i \in J} \in \prod_{J \subseteq I} \mathcal{S}^{|J|}$. Under this notation, s_i is understood to be the stakes level for the i^{th} individual under $\mathbf{s}$. The following

[17] Hill (2016) also discusses the relationship between the stakes involved in an act and the stakes in a (binary) choice: in particular, whilst, for simplicity, we adopt one possible relationship in (9) (the stakes in the choice as the highest stakes among available acts), Hill (2016) contains examples of others (for which the results below also hold).

[18] More precisely, for all $f, g \in \mathcal{A}$, $\max\{D(f), D(g)\} = \zeta \circ \sigma(f, g)$ for some strictly increasing real-valued function ζ.

[19] This is well-defined because of the richness of D.

definition shall prove crucial.

Definition 5. For a profile of stakes levels $\mathbf{s} = (s_i)_{i\in J} \in \prod_{J\subseteq I} \mathcal{S}^{|J|}$, define the relation $\succ_{\mathbf{s}}$ on $\mathcal{A}$ by $\succ_{\mathbf{s}} = \bigcup_{i\in J} \succ^i_{s_i}$. $\mathbf{s}$ *exhibits consensus* when $\succ_{\mathbf{s}}$ is not contradictory; it does not exhibit consensus otherwise. Moreover, we say that $\succ^J$ *respects the consensus* $\succ_{\mathbf{s}}$ *at stakes level* s if $\mathbf{s}$ exhibits consensus and $\succ^J_s \subseteq \succ_{\mathbf{s}}$.

The relation $\succ_{\mathbf{s}}$ assembles all the (determinate) preferences corresponding to the individuals in J, at the stakes levels specified by $\mathbf{s}$. This collection of individuals exhibits consensus across $\mathbf{s}$ if none of the assembled preferences contradict each other, in the sense of strictly preferring different acts. In other words, if, for each individual $i \in J$, only their preferences at stakes level s^i were put 'on the table', a coherent consensus position would exist, consisting of all such preferences, across all the individuals in J. In this case, $\succ_{\mathbf{s}}$ represents the preferences under this consensus. The preference $\succ^J$ respects the consensus $\succ_{\mathbf{s}}$ at a given stakes level s if it doesn't decide more than it: all of the preferences decided upon in $\succ^J_s$ appear in the consensus, though some determinate preferences in the consensus may be left open in $\succ^J_s$. In other words, consensus respect at stakes level s means that the representative doesn't adopt stronger preferences than the consensus, at that stakes level.

4.2 Confidence aggregation and Pareto

Axioms The preference-based characterisation of confidence aggregation relies on one main axiom. To introduce it, first consider the Pareto axiom behind linear pooling in a sufficiently rich, single-profile aggregation context (Mongin, 1995). The strict preference version (in our setup) is as follows.

Axiom (Issue-wise Pareto)**.** *For all acts $f, g \in \mathcal{A}$ and non-empty $J \subseteq I$, if $f \succ^i g$ for all $i \in J$, then $f \succ^J g$.*

This principle encodes preservation of unanimity across individual preferences at the level of single issues, or act-comparisons: if everyone holds a certain preference between two acts, then it should be adopted by the group's representative. As noted in the Introduction, this leads to challenges related to spurious unanimity. We consider the following variant.

Axiom (Consensus-wise Pareto)**.** *For all stakes levels $s \in \mathcal{S}$, acts $f, g \in \mathcal{A}$ and non-empty $J \subseteq I$, if $f \succ_{\mathbf{s}} g$ for all* **s** *for which $\succ^J$ respects the consensus at s, then $f \succ^J_s g$.*

Rather than asking the representative to adopt a preference if everyone in the group holds it, Consensus-wise Pareto looks across all relevant consensuses formed from the individuals' preferences. If the preference holds in all consensuses respected at a given stakes level, then the representative adopts that preference at those stakes. Consensus-wise Pareto thus retains the motivation that group-level preferences should respect consensuses among its members, but, unlike Issue-wise Pareto, it does not conceive consensus as simple unanimity in preferences between two acts. Rather, it respects the richer notion of consensus reflected in $\succ_{\mathbf{s}}$, understood as a coherent overall position acceptable to all individuals, insofar as it results from bringing together their preferences whilst compromising by leaving lower-stakes ones 'off the table'. Whenever there are several relevant consensuses of this sort, preferences held in all of them are adopted by the representative at the appropriate stakes level.

Note that more consensuses are respected at higher stakes levels than at lower ones, so fewer preferences hold in all such consensuses. This principle thus applies to fewer preferences at higher stakes levels, in line with the expectation that fewer preferences are held with higher confidence.

Whilst, logically, neither Issue-wise Pareto nor Consensus-wise Pareto imply the other, Proposition 1 shows that linear pooling can be recovered as a special case of confidence aggregation. In this sense, the latter condition could be considered more general.

Our characterisation requires two auxiliary axioms.

Axiom (Consensus-based beliefs)**.** *For all stakes levels $s \in \mathcal{S}$, acts $f, g \in \mathcal{A}$ and non-empty $J \subseteq I$, if $f \not\succ^J_{s'} g$ for every stakes level s' such that some consensus $\succ_{\mathbf{s}}$ is respected at s', then $f \not\succ^J_s g$.*

Axiom (Non-degeneracy)**.** *There exists a tuple of stakes levels $\mathbf{s} = (s_i)_{i \in I}$ exhibiting consensus.*

Aggregate beliefs should come from individuals' beliefs. Under confidence aggregation, the latter translate into the beliefs of the group's representative principally in the context of consensuses. In terms of preferences, this occurs at stakes levels where some consensus is respected. Consensus-based beliefs states that all of the representative's preferences are determined by those formed on the basis of consensuses: in particular, any preferences at a stakes level where no consensus is respected must already be present at a level where some are. Non-degeneracy states that, if individuals leave sufficiently many preferences aside by restricting to high enough stakes levels, they can come to a consensus.

Result We have the following characterisation result.

Theorem 1. *Let $\{\succ^J\}_{J\subseteq I}$ satisfy Assumption 1. They satisfy Consensus-wise Pareto, Consensus-based beliefs and Non-degeneracy if and only if, for each J with $|J| \geqslant 2$, c^J is a consensus-preserving confidence aggregation of $(c^i)_{i\in J}$, up to convex closure.*

Moreover, for each $J \subseteq I$ with $|J| \geqslant 2$, there is a unique minimal confidence-level aggregator $\otimes$ under which c^J is a consensus-preserving confidence aggregation: that is, for all $\otimes'$ such that c^J is a consensus-preserving confidence aggregation of $(c^i)_{i\in J}$ under $\otimes'$ up to convex closure, $\otimes'(\mathbf{o}) \geqslant \otimes(\mathbf{o})$ for all $\mathbf{o}$ such that $\bigcap_{i\in J} c^i(o_i) \neq \varnothing$.

So the central axiom characterising confidence aggregation is Consensus-wise Pareto, which is no more than a reformulation of the standard Pareto condition to apply to a more refined notion of consensus rather than issue-wise unanimity in individual preferences. Indeed, Consensus-based beliefs can be dropped whenever the representative's confidence ranking is pointed. More generally, without it, the representative's confidence ranking is always that obtained by a confidence aggregation, except at confidence levels at the bottom of the ranking.

No particular confidence-level aggregator is assumed in this result; rather, the appropriate aggregator is determined endogenously by the individuals' and representative's preferences. Moreover, there is a unique minimal one: that is, one which always takes the lowest value across all aggregators representing the profile of preferences. Relatedly, no particular confidence-level aggregator is imposed in Theorem 1. Extensions involving specific confidence-level aggregators or relationships between the aggregators for different J lie largely beyond the scope of this paper. To illustrate some possibilities, Appendix B proposes supplementary axioms that characterise some confidence-level aggregators related to those in Table 3 (Section 2.2).

5 Decision making, model misspecification and confidence

Building on the previous developments concerning aggregation, we now bring in decision. This will permit discussion of attitudes to uncertainty, and notably the relationship to recent model misspecification decision rules. In terms of the general aggregation-for-subsequent-decision problem, this requires considering the relationship between the representative's preferences, as provided in the previous section, and the decision maker's. We begin by characterising this link, to obtain a representation of the decision maker's preferences combining the aggregation and decision dimensions.

5.1 Decision maker's Preferences

Suppose that the representative receives input from each individual, and let $\succ^{rep}$ denote her resulting preferences (which coincide with $\succ^{I}$ from Section 4), and $\succ^{dm}$ the decision maker's. We adopt standard 'rationality' assumptions on the decision maker's preferences.

Assumption 2 (Decison maker's preferences). $\succeq^{dm}$ *are complete, transitive and continuous.*

Consider the following axioms relating the representative's and decision maker's preferences.

Axiom (C-Consistency). *For all* $f \in \mathcal{A}$, $x \in \mathcal{A}^c$,

$$f \succ^{rep} x \Rightarrow f \succ^{dm} x$$

Axiom (Caution). *For all* $f \in \mathcal{A}$, $x \in \mathcal{A}^c$

$$f \not\succ^{rep} x \Rightarrow x \succeq^{dm} f$$

These are strict-preference versions of standard axioms relating incomplete and complete preferences (Gilboa et al., 2010; Cerreia–Vioglio et al., 2025), albeit under different interpretations. C-Consistency is the reformulation of the Consistency axiom from Gilboa et al. (2010) in terms of strict preferences and its weakening to comparisons with constant acts. In the current context, it says that, in all such comparisons, the decision maker adopts all of the representative's preferences. Caution is the strict-preference version of Gilboa et al.'s Caution. In this context, it says that the decision maker does not adopt any strict preferences in the previous sort of comparison whenever the representative does not do so.

Proposition 6. *Under Assumptions 1 and 2, the following are equivalent:*

i. $\succ^{rep}$ *and* $\succ^{dm}$ *jointly satisfy C-Consistency and Caution;*

ii. $\succ^{dm}$ *is represented by*

$$\min_{p \in c^{rep}(D(f))} \mathbb{E}_p u(f) \tag{10}$$

where (c^{rep}, D, u) *represent* $\succ^{rep}$ *according to* (9).

Representation (10) encompasses a wide range of popular models in the ambiguity literature. It is the ambiguity averse model in the confidence family (Hill, 2013, 2019b), and as such generalises Gilboa and Schmeidler's (1989) maxmin-EU model. Like maxmin-EU, decision criterion (10) evaluates acts by the worst-case expected utility over a set of priors; unlike it, the set used is determined by the act evaluated, with acts involving higher stakes being evaluated using sets corresponding to higher confidence levels. Moreover, as shown in Appendix C, representation (10) generalises variational preferences (Maccheroni et al., 2006), which underpin the main model misspecification decision rules. In particular, every standardly-used variational preference—and all those used in the model misspecification literature—can be represented according to (10) (Proposition C.1).

5.2 Aggregation for subsequent decision

Combining Proposition 6, relating the decision maker's preferences to the representative's, with Theorem 1, on the formation of latter's preferences from the individuals' input, yields a representation of the decision maker's preferences in terms of the aggregate confidence ranking.

Corollary 1. *Let $\{\succ^J\}_{J\subseteq I}$ satisfy Assumption 1 and Consensus-wise Pareto–Non-degeneracy, and let $\succ^{dm}$ satisfy Assumption 2 and, in conjunction with $\succ^I$, C-Consistency and Caution. Then $\succ^{dm}$ is represented by:*

$$\min_{\substack{p\in\Delta:\\ \otimes((\iota^i)_{i\in I})\leqslant D(f)}} \mathbb{E}_p u(f) \tag{11}$$

for some confidence-level aggregator $\otimes$, where, for each $i \in I$, (c^i, D, u) represents $\succ^i$.[20]

Representation (11) captures decision in the face of multiple information sources that proceeds by aggregation followed by ambiguity-sensitive decision. It admits the two interpretations discussed in Section 3. Under the individuals'-confidence interpretation, the decision maker chooses on the basis of the confidence aggregation of the individuals' beliefs. This result thus connects to the literature on the aggregation of ambiguity preferences (e.g. Crès et al., 2011), providing a characterisation for the confidence approach. Under the representative's-confidence interpretation, the decision maker chooses on the basis of the representative's assessments, formed from the individuals' inputs and her evaluation of their relative expertise. This is relevant in cases where a decision must be taken on the basis of inputs in the form of probability measures, for instance reflecting models.

5.3 Making decisions with models

As discussed in Section 3, when input comes from models, i.e. when $\mathcal{I}^i = p^i \in \Delta$ for each i, confidence rankings generated by the individual probability measure inputs under a distance ρ and weights $w(i)$ can be used in the confidence aggregation rule. Plugging these into (11) yields the representation of the decision maker's preference by:

$$\min_{\substack{q\in\Delta:\\ \otimes_{i\in I} w(i)\rho(q,p^i)\leqslant D(f)}} \mathbb{E}_q u(f) \tag{12}$$

[20]Recall from Section 2.1 that ι^i is an equivalent formulation of c^i, and that confidence aggregation can be formulated in terms of the ι^i (Section 2.2).

In this representation, there are essentially three relevant parameters that determine the evaluation of a given act: ρ (determining confidence in input probabilities); $\otimes$ (regulating aggregation); and the stakes of the decision (determining the relevant $D(f)$ and hence set of priors for the decision).

This case of our aggregate-then-decide decision rule recovers many approaches to choosing on the basis of models in the literature.

Theorem 2. *For each $i \in I$, let $\mathcal{I}^i = p^i \in \Delta$ and $\succ^i$ be represented using a $w(i)$ ρ-confidence ranking generated by p^i, for some statistical distance ρ and individual-specific weights $w(i)$.*

Then, under the assumptions in Corollary 1, for each row of Table 5, the following are equivalent:

i. *ρ, $\otimes$ are as specified in the first two columns of the Table.*

ii. *Preferences $\succ^{dm}$ over acts with stakes specified in the third column of the Table are represented as specified in the fourth column.*

We now discuss the special cases in turn.[21]

Low stakes: Bayesian Model Averaging In evaluating acts involving low stakes, sets of priors held with low confidence levels will be used; if these sets are singletons (i.e. the set in the subscript of the minimisation in (12) is a singleton for small enough $D(f)$), then (12) coincides with subjective expected utility. So, for confidence-level aggregators $\otimes$ and distances ρ yielding the results of, say, linear pooling (Proposition 1), (12) coincides with EU on a linear pool of the models, at low stakes. Confidence aggregation with the confidence decision model (10) thus subsumes Bayesian Model Averaging (Raftery et al., 1997; Steel, 2020), a popular approach to dealing with multiple models that involves the linear pool of the distributions provided by the various models, with weights determined by the posterior probabilities over them (Table 5, row 1). Changing the distance ρ yields EU with geometric pooling (Table 5, row 2).

[21] Although I has been assumed to be finite here—and Table 5 is written for finite I—in the model uncertainty literature the set of models is often infinite. The points made here extend to infinite sets of models, under appropriate technical modifications.

Medium / High stakes: Model misspecification At higher stakes, sets further up the confidence ranking will feature in (12). This is illustrated in rows 3 and 4, featuring representations that generalise Hansen and Sargent's (2001; 2008) constraint preferences, which are the special case of (12) with no aggregation (i.e. singleton I), relative entropy distance and identical $D(f)$ for all f. For instance, row 4 shows that the specification yielding geometric pooling (row 2) generates an 'average' robust control evaluation at higher stakes. Changing the confidence-level aggregator $\otimes$ yields a 'minimum' robust control evaluation (row 3), where the constraint set is determined by the minimum relative entropy distance.

Hansen and Sargent (2001, 2008) also introduce multiplier preferences, which yield the same optima as their constraint preferences on various classes of decision problems. By similar reasoning, for sufficiently convex distances and confidence-level aggregators, (12) will yield the same optima on these problems as:[22]

$$\min_{q \in \Delta} \left(\mathbb{E}_q u(f) + \lambda \otimes_{i \in I} w(i) \rho(q, p^i) \right) \tag{13}$$

for appropriate λ. (13) contains several representations in the recent literature on model misspecification; for instance, the settings of $\otimes$ and ρ in row 3 yield the representation proposed by Hansen and Sargent (2022). It follows from the previous observations that this representation yields the same optima as that in row 3 of the Table on the decision problems studied by Hansen and Sargent (2008).

Model expertise Section 3 demonstrated confidence aggregation's ability to deal with within-person expertise diversity; this remains relevant when the 'experts' are models. In real applications, it is not uncommon for some models to be 'better' on certain issues and 'worse' on others. In climate science, say, one model could have a more detailed representation of cloud formation, whereas another is more accurate on elements of the biosphere: the former might thus be expected to do a better job in predict-

[22] Hansen and Sargent's proof for multiplier preferences relies on the Lagrange multiplier theorem (Luenberger, 1969), and hence on the convexity of the constraint ($R(q\|p)$ in their case) as a function of q. Sufficient convexity conditions ensure the theorem applies to (12) and (13).

ρ	$\otimes$	Stakes	'Overall' Decision rule
Reverse RE	Average	Low	$\mathbb{E}_{\sum_I \frac{w(i)}{\sum_{i\in I} w(i)} i} u(f)$ Bayesian Model Averaging & SEU (Steel, 2020)
RE	Average	Low	$\mathbb{E}_{\chi \prod_I i^{\frac{w(i)}{\sum_I w(i)}}} u(f)$ Geometric pooling & SEU (Dietrich, 2021)[23]
RE	Minimum	Medium / High	$\min_{q\in\Delta:\ \min_{i\in I} R(q\Vert p^i)\leqslant\eta} \mathbb{E}_q u(f)$ 'Minimum' robust control (Hansen and Sargent, 2022; Cerreia–Vioglio et al., 2025) [24]
RE	Average	Medium / High	$\min_{q\in\Delta:\ \sum_I w(i) R(q\Vert p^i)\leqslant\eta} \mathbb{E}_q u(f)$ 'Average' robust control
Exp-sens. RE	Average	Low	$\mathbb{E}_{\arg\min_{q\in\Delta} \sum_I \sum_{j=1}^l w(i,l) R(q\vert_{\mathcal{P}_j} \Vert p^i\vert_{\mathcal{P}_j})} u(f)$ Expertise-sensitive pooling & SEU (Defn 4)
Exp-sens. RE	Minimum	Medium / High	$\min_{q\in\Delta:\ \min_I \sum_{j=1}^l w(i,l) R(q\vert_{\mathcal{P}_j} \Vert p^i\vert_{\mathcal{P}_j})\leqslant\eta} \mathbb{E}_q u(f)$ Expertise-sensitive minimum robust control
Exp-sens. RE	Average	Medium / High	$\min_{q\in\Delta:\ \sum_I \sum_{j=1}^l w(i,l) R(q\vert_{\mathcal{P}_j} \Vert p^i\vert_{\mathcal{P}_j})\leqslant\eta} \mathbb{E}_q u(f)$ Expertise-sensitive average robust control

Table 5: Aggregation for subsequent decision (Eq. (12)): Special cases. See Tables 2 and 3 for definitions of values for ρ and $\otimes$ respectively.

Note: RE stands for 'Relative Entropy'; Exp-sens. RE is short for 'Expertise-sensitive Relative Entropy', i.e. ρ as in Definition 3 (and (4)), with the relative entropy classical distance. In the last three rows, $w : I \times \{1, \ldots, l\} \to \Re_{\geqslant 0}$ is an assignment of weights to models and issues. Other notation is as in Sections 3, 4 and the text.

ing hurricanes; the latter in predicting ground-level temperature (Masson-Delmotte et al., 2021, Section 1.5.3). However, the distances underlying Bayesian Model Averaging and existing misspecification models (Table 5, rows 1–4) encode the assumption that all models are equally 'good'—they have comparable expertise—on all issues (Section 3.3). Like standard pooling rules (Section 3.1), existing model-misspecification decision rules thus cannot cope with intra-model cross-issue expertise diversity.

Using the same expertise-sensitive confidence rankings (Definition 3) in the derivation of our expertise-sensitive pooling rule (Definition 4) yields a expertise-sensitive generalisation of Bayesian Model Averaging (row 5) at low stakes. At higher stakes levels, it yields misspecification-sensitive decision rules that *can* accommodate differing degrees of expertise within models. Rows 6 and 7 provide two examples, obtained by replacing the distance generating existing misspecification models by the expertise-sensitive distance introduced in Section 3.3. For the reasons set out in Section 3, they constitute arguably more pertinent misspecification-sensitive decision rules in situations where models' performance may vary across issues.

6 Dynamic rationality

A common theme in the probability aggregation literature is the interaction between aggregation and update. Dietrich (2021) argues that a 'rational group' requires belief aggregation to be in sync with updating. This is typically formulated in terms of commutivity: aggregation followed by update on some information yields the same group beliefs as updating all individual beliefs on the information and then aggregating. The version of this condition for Bayesian beliefs, where updating is performed on events (or likelihoods) by Bayesian conditionalisation, has been called 'external Bayesianism' in the pooling literature (Genest and Zidek, 1986) or 'Dynamic Rationality' by Dietrich (2021).

However, the natural domain for our aggregation approach (Section 2.2) is not Bayesian beliefs but richer and more refined confidence in beliefs. Here, Bayesian conditionalisation no longer applies, without revision. Hill (2022) proposes a *confidence update* rule for the general representation of confidence in beliefs used here, and argues for its normative validity, suggesting in particular that it deals appropriately with situations where Bayesian update struggles. So the question of dynamic rationality in our context is whether confidence aggregation commutes with confidence up-

[22] As standard, the geometric pool involves a multiplicative constant, denoted χ.

[23] This generalised constraint rule takes $w(i) = 1$ for all $i \in I$. Hansen and Sargent (2022) proposed the multiplier version of this model (i.e. (13) with $\otimes$ and ρ as specified in the Table and $w(i) = 1$ for all $i \in I$); Cerreia–Vioglio et al. (2025, Proposition 6) axiomatise a version with general convex ρ, minimum $\otimes$ and convex, compact set of models I. In such cases, the convexity assumptions needed to run Hansen and Sargent's (2008) argument hold (footnote 22), so the constraint preferences in the Table yield the same optima as the corresponding multiplier version, in the relevant decision problems.

date.

In the framework set out in Section 2.1, the probability-threshold confidence update rule from Hill (2022, Definition 2) can be defined as follows, where, for a set $\mathcal{C} \in 2^{\Delta} \backslash \varnothing$ and event E, $\mathcal{C}_E = \{p(\bullet|E) : p \in \mathcal{C}, p(E) > 0\}$, and a probability-threshold function ρ_E is a decreasing function $O \to [0,1]$.

Definition 6 (Confidence Update (Hill, 2022))**.** For event $E \subseteq 2^{\Delta} \backslash \varnothing$, confidence ranking $c : O \to 2^{\Delta} \backslash \varnothing$ and probability-threshold function $\rho_E : O \to [0,1]$, the confidence update of c by E under ρ_E is the ranking $c|\rho_E = \overline{\Phi}$, where the partial function $\Phi : O \to 2^{\Delta} \backslash \varnothing$ is defined, for all $o \in O$ such that $\{p \in c(o) : p(E) \geqslant \rho_E(o)\} \neq \varnothing$, by:

$$\Phi(o) = \{p \in c(o) : p(E) \geqslant \rho_E(o)\}_E \tag{14}$$

See Hill (2022) for a full discussion and axiomatic characterisation of this confidence update rule.

We have the following result concerning the confidence aggregation rule $F_{\otimes}$ (Definition 2).

Theorem 3. *For every tuple of confidence rankings* $(c^1, \ldots, c^n)$*, every confidence-level aggregator* $\otimes$*, every event* E *and probability-threshold function for it* ρ_E*:*

$$F_{\otimes}(c^1|\rho_E, \ldots, c^n|\rho_E) = F_{\otimes}(c^1, \ldots, c^n)|\rho_E \tag{15}$$

So confidence aggregation commutes with confidence update: it is 'dynamically rational', to use Dietrich's (2021) term. Such coherence has been argued to be an important property of an aggregation rule, so much so that some use it to promote aggregation rules having this property, and to criticise those that don't. Theorem 3 thus provides a reassuring message concerning confidence aggregation's credentials on this score.

7 Discussion

As mentioned above, the approach to the aggregation-for-subsequent-decision proposed here admits several interpretations. Under the representative's confidence interpretation, the individuals' input can come in the form of probabilities; this connects our approach to a related literature on probability aggregation taking probabilities as primitive, rather than working with preferences (Genest and Zidek, 1986). The within-person expertise diversity challenge was first raised in this literature, with some early contributions suggesting averaging with potentially different weights for each event (e.g. Bordley and Wolff, 1981). Such rules turned out not to be well-defined: they fail to yield probability measures unless the weights are the same for all events, in which case one returns to standard linear pooling in the presence of a minimal Pareto-like condition (e.g. McConway, 1981; Genest and Zidek, 1986). This, and in particular the apparent impossibility in capturing within-person expertise diversity, has been argued to be a problem for linear pooling (e.g. French, 1985). The limit case in Example 3.4 (Section 3.4)—involving different weights for the labour and real estate events—shows that confidence aggregation can capture the intuition behind the early proposals. It does so whilst overcoming their limits: as testified by Proposition 5, the expertise-sensitive pooling rule derived from confidence aggregation is always well-defined.

The model misspecification literature also takes probabilities as primitives, with recent work focussing on cases of multiple probability measures (Hansen and Sargent, 2022; Cerreia–Vioglio et al., 2025). Section 5.3 shows that many of these decision rules are essentially special cases of our approach. A central difference, however, concerns separation of the aggregation and decision dimensions of the aggregation-for-subsequent-decision problem. Whilst misspecification-motivated models in the literature bake everything together into the decision rule, the confidence-aggregation perspective proposed here fully separates the *epistemic* issue of identifying the beliefs (and confidence) that can or should be formed on the basis of a set of models from the *pragmatic* question of their role in decision making. This is clear in Table 5: the resulting 'overall' decision rule depends not just on the aggregation parameters (first two columns), but also on the stakes, which under (12) regulate the degree of ambiguity aversion exhibited (Hill, 2013, 2019b). This separation may be particularly relevant in situations, such as those cited in the Introduction, with an actor responsible for aggregating expert opinions about matters of fact, for use by a decision maker who supplies the tastes. Moreover, this separation can bring to light hitherto unrecognised relationships (as in the example of 'average' robust control and geometric pooling; Table 5, rows 2 & 4) and suggest new approaches to model misspecification drawing from advances on aggregation (Table 5, rows 6 & 7). On the decision front, extensions of representation (12) involving different ambiguity attitudes and functionals can be obtained by modifying the Caution axiom in Corollary 1 , in the style of Cerreia–Vioglio et al. (2025, Proposition 7); Lanzani (2025).

Under the individuals'-confidence interpretation, the individuals' input can come in the form of confidence rankings, on which confidence aggregation operates directly. Just as pooling rules tacitly assume interpersonal comparison of probability judgements—one can say when two individuals are assigning the same probability—in direct application, confidence aggregation requires interpersonal comparison of confidence—one can tell when two individuals are talking about the same confidence level. Hill (2019a) discusses the problem of 'calibrating' confidence levels across individuals, providing and theoretically founding a scale for interpersonal confidence comparison which can be used for 'direct' applications of confidence aggregation. Interpersonal comparison can alternatively be provided by preferences, in the context of the appropriate decision models (Sections 2.1 and 4.1).

Indeed, the characterisations in Sections 4 and 5 connect into a literature on belief aggregation working in preference-based frameworks. Spurious unanimity, for instance, first arose as an issue for preference aggregation with potentially differing utilities and subjective probabilities (Mongin, 1995, 2016), and only recently has been recognised as relevant for aggregation of belief *tout court*. For instance, Mongin and Pivato (2020); Dietrich (2021); Pivato (2022) criticise the influential approach of Gilboa et al. (2004)—which characterises utilitarian aggregation of utility and linear pooling of probabilities—on these grounds. Several reactions in this literature work with preferences and consist in restricting the domain of the Pareto condition. Dietrich (2021) restricts it to cases where all agents have identical subjective probabilities, and adds a dynamical rationality condition of the sort discussed in Section 6. Mongin and Pivato (2020); Pivato (2022) restrict Pareto to such an extent that their representations involve "no connection between the social probability and the individual ones". Unlike the aggregation approach developed here, these make no attempt to retain the consensus-preservation intuition behind Pareto. Bommier et al. (2021) present a condition preserving consensuses on prospects yielding identical distributions of outcomes for all individuals, and use it to provide a decision rule involving prospect-dependent aggregate distributions. Alon and Gayer (2016) and Stanca (2021) consider aggregation of EU preferences when group preferences may be non-expected utility, and under identical utilities in the latter case. Both involve versions of Pareto that, were group preferences expected utility, would lead to linear pooling.

To the extent that confidence rankings support both ambiguity averse and incomplete preferences (Sections 4.1 and 5.1), confidence aggregation provides an aggregation rule for both sorts of preferences. Crès et al. (2011) characterises an aggregation rule for maxmin-EU preferences, and Danan et al. (2016) explore aggregation of incomplete preferences, with potentially differing utilities and beliefs. Both adopt conditions comparable with standard, issue-wise Pareto. By contrast, the approach proposed here leverages the non-probabilistic structure of beliefs in aggregation, in concordance with the insight that confidence has a role in consensus formation (Section 2.2). Nau (2002) proposes an aggregation rule for a confidence-based belief representation which is a special case of that used here (see Hill, 2016, Sect. 6). His rule is based on a different intuition, pertaining to the Bayesian risk function of the group, as defined in terms of an opponent's minimum expected loss in a betting game. Neither approach is contained in the other.[24]

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[24]This can be seen from the fact that Nau's rule violates Eqn. (2); see Nau (2002, Figs 2 & 3).

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Appendices: For Online Publication

A Expertise-sensitive pooling: Running Example

In this Appendix, for completeness, we set out the details concerning the running Example in Sections 3.3–3.5, as well as some further comments and extensions to confidence in judgements about probabilistic independence.

A.1 Examples 3.2–3.4

First note that each probability measure p over $\{L, L^c\}$ is determined by $p(L)$, and similarly for the other partitions (Example 3.2). So each tuple $(p_L, p_R, p_B) \in \Delta(\mathcal{P}_L) \times \Delta(\mathcal{P}_R) \times \Delta(\mathcal{P}_B)$ is fully characterised by the vector $(p_L(L), p_R(R), p_B(B)) \in [0,1]^3$. The set $P_{\mathcal{P}_L,\mathcal{P}_R,\mathcal{P}_B} \subseteq \Delta(\mathcal{P}_L) \times \Delta(\mathcal{P}_R) \times \Delta(\mathcal{P}_B)$ of tuples derived from probability measures on the full state space (Section 3.5) is defined by the following linear inequalities imposed by the fact that $B = L \cap R$: $P_{\mathcal{P}_L,\mathcal{P}_R,\mathcal{P}_B}$ is the set of all $(p_L, p_R, p_B) \in \Delta(\mathcal{P}_L) \times \Delta(\mathcal{P}_R) \times \Delta(\mathcal{P}_B)$ satisfying

$$\begin{aligned} p_L(L) &\geqslant p_B(B) \\ p_R(R) &\geqslant p_B(B) \\ 1 &\geqslant p_L(L) + p_R(R) - p_B(B) \end{aligned} \tag{A.1}$$

Writing these inequalities in vector notation, $P_{\mathcal{P}_L,\mathcal{P}_R,\mathcal{P}_B}$ corresponds to the set of vectors $\mathbf{q} \in [0,1]^3$ satisfying the constraint $\mathbf{Aq} \leqslant \mathbf{r}$ where

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Moreover, for each confidence level o in the confidence ranking (5) in Example 3.3, the map of $c^L(o)$ into the space $\Delta(\mathcal{P}_L) \times \Delta(\mathcal{P}_R) \times \Delta(\mathcal{P}_B)$ can be written as:

$$c^L(o) = \{\mathbf{q} \in [0,1]^3 : (\mathbf{q} - \mathbf{p^L})^T \mathbf{D^L} (\mathbf{q} - \mathbf{p^L}) \leqslant o\} \tag{A.2}$$

where

$$\mathbf{p^L} = \begin{pmatrix} 0.9 \\ 0.1 \\ 0.09 \end{pmatrix}, \quad \mathbf{D^L} = \begin{pmatrix} 2w_L^L & 0 & 0 \\ 0 & 2w_R^L & 0 \\ 0 & 0 & 2w_B^L \end{pmatrix}$$

and similarly for Ray, with

$$\mathbf{p^R} = \begin{pmatrix} 0.1 \\ 0.9 \\ 0.09 \end{pmatrix}, \quad \mathbf{D^R} = \begin{pmatrix} 2w_L^R & 0 & 0 \\ 0 & 2w_R^R & 0 \\ 0 & 0 & 2w_B^R \end{pmatrix}$$

It follows that the minimisation problem (8) defining the centre of the confidence aggregation in Example 3.4 becomes the following simple quadratic optimisation problem under constraints:

$$\underset{\mathbf{Aq} \leqslant \mathbf{r}}{\arg\min} \sum_{i=L,R} (\mathbf{q} - \mathbf{p^i})^T \mathbf{D^i} (\mathbf{q} - \mathbf{p^i})$$

If $\left(\frac{w_L^L}{w_L^L + w_L^R} p^L(L) + \frac{w_L^R}{w_L^L + w_L^R} p^R(L)\right) + \left(\frac{w_R^L}{w_R^L + w_R^R} p^L(R) + \frac{w_R^R}{w_R^L + w_R^R} p^R(R)\right) - 1 \leqslant 0.09$, then the constraints are slack, and the unqiue solution is:

$$\begin{aligned} p(L) &= \frac{w_L^L}{w_L^L + w_L^R} p^L(L) + \frac{w_L^R}{w_L^L + w_L^R} p^R(L) \\ p(R) &= \frac{w_R^L}{w_R^L + w_R^R} p^L(R) + \frac{w_R^R}{w_R^L + w_R^R} p^R(R) \\ p(B) &= 0.09 \end{aligned}$$

Henceforth, denote the issue-wide weighted averages by $m_L = \frac{w_L^L p^L(L) + w_L^R p^R(L)}{w_L^L + w_L^R}$

and $m_R = \frac{w_R^L p^L(R) + w_R^R p^R(R)}{w_R^L + w_R^R}$. Otherwise, solving the minimisation problem yields the following unique solution:

$$\begin{aligned} p(L) =& m_L - \frac{1}{w_L^L + w_L^R} \frac{m_L + m_R - 1.09}{\frac{1}{w_B^L + w_B^R} + \frac{1}{w_R^L + w_R^R} + \frac{1}{w_L^L + w_L^R}} \\ p(R) =& m_R - \frac{1}{w_R^L + w_R^R} \frac{m_L + m_R - 1.09}{\frac{1}{w_B^L + w_B^R} + \frac{1}{w_R^L + w_R^R} + \frac{1}{w_L^L + w_L^R}} \\ p(B) =& 0.09 + \frac{1}{w_B^L + w_B^R} \frac{m_L + m_R - 1.09}{\frac{1}{w_B^L + w_B^R} + \frac{1}{w_R^L + w_R^R} + \frac{1}{w_L^L + w_L^R}} \end{aligned}$$

where, as specified, $p^L(L) = p^R(R) = 0.9$, $p^L(R) = p^R(L) = 0.1$, $p^L(B) = p^R(B) = 0.09$. These solutions yield Proposition 4, as follows.

Proof of Proposition 4. The solutions provided above imply that the centre of the aggregate confidence ranking is a singleton. We now consider the other parts of the Proposition.

Part i. If the constraint mentioned above does not bind (i.e. $\left(\frac{w_L^L}{w_L^L + w_L^R} p^L(L) + \frac{w_L^R}{w_L^L + w_L^R} p^R(L)\right) + \left(\frac{w_R^L}{w_R^L + w_R^R} p^L(R) + \frac{w_R^R}{w_R^L + w_R^R} p^R(R)\right) - 1 \leqslant 0.09$), then the result follows immediately from the assumptions (notably $w_L^L > w_L^R$ and $w_R^R > w_R^L$) and the solution provided above. Now consider the case where the constraint is binding. Since $\frac{w_L^R}{w_L^L} = \frac{w_R^L}{w_R^R}$ (and plugging in the values for $p^L(L), p^R(L), p^L(R), p^R(R)$), we have that $m_L = m_R$. Using the solutions provided above, and the fact that the assumption implies that $(w_L^L + w_L^R)\left(\frac{1}{w_B^L + w_B^R} + \frac{1}{w_R^L + w_R^R} + \frac{1}{w_L^L + w_L^R}\right) \geqslant 2$, so

$$m_L - \frac{1}{w_L^L + w_L^R} \frac{m_L + m_R - 1.09}{\frac{1}{w_B^L + w_B^R} + \frac{1}{w_R^L + w_R^R} + \frac{1}{w_L^L + w_L^R}} \geqslant m_L - (m_L - 0.545) = 0.545$$

and similarly for the expression for $p(R)$, yielding the required result.

Part ii. The result follows from taking the limits in the previous solutions as $\frac{w_B^L + w_B^R}{w_L^L + w_L^R} \to 0$ and $\frac{w_B^L + w_B^R}{w_R^L + w_R^R} \to 0$.

□

Note that Proposition 4 considers the case where $\frac{w_B^L + w_B^R}{w_L^L + w_L^R} \to 0$ and $\frac{w_B^L + w_B^R}{w_R^L + w_R^R} \to 0$, which, arguably, is closest to the example described in the Introduction. For completeness, note that, in the opposite case of $\frac{w_B^L + w_B^R}{w_L^L + w_L^R} \to$

∞ and $\frac{w_B^L+w_B^R}{w_R^L+w_R^R} \to \infty$, the confidence in the probability judgements concerning B grows very large comparatively, so these are retained at the expense of others. Hence, we have:

$$\begin{aligned} p(L) \to & m_L - \frac{m_L + m_R - 1.09}{1 + \frac{w_L^L+w_L^R}{w_R^L+w_R^R}} \\ p(R) \to & m_R - \frac{m_L + m_R - 1.09}{1 + \frac{w_R^L+w_R^R}{w_L^L+w_L^R}} \\ p(B) \to & 0.09 \end{aligned}$$

Here the judgement about B is fully preserved, as one would expect given the high confidence postulated in it. This places a strong constraint on $p(L)$ and $p(R)$ (namely, $p(L) + p(R) = 1.09$). The possible probability available is shared between L and R according to the comparison between the issue-wide weighted averages m_L and m_R and the ratio between the overall confidence (i.e. $w_L^L + w_L^R$ v.s. $w_R^L + w_R^R$) in each of these judgements.

A.2 Confidence in independence judgements

A central factor in Example 3.4 is the trade-off between the confidence in the judgements concerning the main two issues—labour and real estate—and what happens to both, considered as a third issue. However, an alternative analysis considers individuals to have opinions on the main issues and their relationship, rather than 'primitive' views on B. We now briefly show that the confidence approach can easily support such perspectives.

Example A.1. Now suppose that Laura and Ray hold beliefs about L and R, and about the independence of L and R: they believe them to be independent, without being maximally confident in this judgement.[25] This can be reflected using a vector of weights $\mathbf{w^L} = (w_L^L, w_R^L, w_I^L)$ (resp.

[25] Independence here refers to the probabilistic sense: $p^i(L \cap R) = p^i(L)p^i(R)$. Note that the belief in independence implies that $p^L(B) = p^R(B) = 0.09$, as per Table 1.

$\mathbf{w^R} = (w_L^R, w_R^R, w_I^R)$) and the following confidence ranking:

$$c_{Ind}^L(o) = \left\{ q \in \Delta : \begin{array}{c} \sum_{j=\{L,R\}} 2w_j^L (q(j) - p^L(j))^2 \\ +2w_I^L \left(q(B) - q(L).q(R)\right)^2 \end{array} \leqslant o \right\} \tag{A.3}$$

and similarly for c_{Ind}^R. These are clearly well-defined confidence rankings. The weighted element corresponding to the event B here is $(q(B) - q(L).q(R))^2$, which reflects the 'distance' from independence of L and R. So, at higher confidence levels, probability measures with larger 'distances' from independence are contained in the set of priors, translating the limited confidence in independence.

The solution of the minimisation problem (8) can be obtained similarly to the analysis in Example 3.4, yielding as centre of the aggregate confidence ranking p with:

$$\begin{aligned} p(L) =& \frac{w_L^L}{w_L^L + w_L^R} p^L(L) + \frac{w_L^R}{w_L^L + w_L^R} p^R(L) \\ p(R) =& \frac{w_R^L}{w_R^L + w_R^R} p^L(R) + \frac{w_R^R}{w_R^L + w_R^R} p^R(R) \\ p(B) =& p(L).p(R) \end{aligned}$$

Here the aggregation on each of the issues L and R uses issue-specific weights, reflecting differing confidence, as in the limit case in Proposition 4. For the issue B, agents' beliefs concerning the independence of L and R generates the probability.

In tandem with the discussion in Section 3, this illustrates that the confidence approach can not only recoup averaging with issue-specific weights whilst retaining consistency, but it can also incorporate varying opinions about independence or more generally the relationship between issues. This is relevant for another recurrent criticism of linear pooling: that it does not preserve independence. As is well known, even if all individuals consider the events L and R to be independent, the linear pool might not (e.g. Genest and Zidek, 1986). This is easy to see on our running example: the linear pool of Laura's and Ray's probabilities with equal weights ($w^L = \frac{1}{2}$) is $p^{LP}(L) = 0.5$, $p^{LP}(R) = 0.5$, $p^{LP}(B) = 0.09$, so L and R are not independent under p^{LP}, though they are under p^L and p^R. The aggregation above based on confidence rankings of the form (A.3) shows how confidence aggregation can respect independence, whilst retaining much of the spirit of linear pooling. For instance, when $w_L^L = w_R^L = w_L^R = w_R^R$, the resulting centre probability is $p^{LP}(L) = 0.5$, $p^{LP}(R) = 0.5$, $p^{LP}(B) = 0.25$: i.e. the same as linear pooling for the issues L and R, but with independence retained (and hence a different B).

The beliefs about the independence of L and R in Example A.1 are considered merely for the purposes of illustration. The point of the example is more general: by incorporating conditional probabilities much in the way proposed in Eq. (A.3), the confidence approach can respect conditional probability judgements (including, but not limited to, judgements about independence) in the aggregate belief. In accordance with the philosophy behind the approach, they are respected to the extent that the individuals are confident in them.

B Characterising confidence aggregation: special cases

In this Appendix, we extend Theorem 1 to characterise, as special cases, confidence aggregation under some of the families of confidence-level aggregators mentioned in Section 2.2 (Table 3) as well as the following generalisations.

Example B.1 (Generalised Maximum aggregator)**.** An aggregator of the form $\otimes\mathbf{o} = \psi(\max\{o_i\})$ for $\psi : O \to O$ an increasing transformation of confidence levels.

Example B.2 (Generalised Minimum aggregator)**.** An aggregator of the form $\otimes\mathbf{o} = \psi(\min\{o_i\})$ for $\psi : O \to O$ an increasing transformation of confidence levels.

Example B.3 (Non-neutral Maximum aggregator)**.** An aggregator of the form $\otimes\mathbf{o} = \max\{\psi_i(o_i)\}$, where $\psi_i : O \to O$ (for $i = 1, \ldots, n$) are increasing transformations of confidence levels.

More specifically, we will provide results for the following stronger representation: $c^{rep} = \dot{F}_{\otimes}(c^1, \ldots, c^n)$, with $\dot{F}_{\otimes}(c'^1, \ldots, c'^n) = \overline{\dot{\Phi}(c'^1, \ldots, c'^n)}$ for every $(c'^1, \ldots, c'^n) \in \Pi^n$, where, for every $o \in O$ such that $\bigcup_{\mathbf{o}:\otimes\mathbf{o}=o} \bigcap_i c'^i(o_i) \neq \varnothing$

$$\dot{\Phi}_{\otimes}(c^1, \ldots, c^n)(o) = \bigcup_{\mathbf{o}:\otimes\mathbf{o}=o} \bigcap_i c'^i(o_i) \tag{B.1}$$

The only difference with respect to Definition 2 is that here the union is taken over all tuples of confidence levels whose confidence-level aggregate equals o, whereas the previous procedure looks at all those with confidence-level aggregate at most o. It follows directly from the fact that confidence rankings are increasing in o that, if $c^{rep} = \dot{F}_{\otimes}(c^1, \ldots, c^n)$, then c^{rep} is a consensus-preserving confidence aggregation in the sense of Definition 2.

Recall that, under Assumption 1, $\max\{D(\bullet), D(\bullet)\}$ is a monotonically increasing transformation of σ. By appropriate choice of normalisation for O and $\mathcal{S}$, it can be assumed that they are identical. Under this assumption, we have the following result, which involves the axioms in Figure 4, and defines clauses according to Table 6.

Theorem B.1. *Suppose that O is infinite, let $\{\succ^J\}_{J \subseteq I}$ satisfy Assumption 1 with $\max\{D(f), D(g)\} = \sigma(f, g)$ for all $f, g \in \mathcal{A}$. For each of the rows in Table 6: $\{\succ^J\}_{J \subseteq I}$ satisfy Consensus-wise Pareto, Consensus-based beliefs, Non-degeneracy and the axiom(s) in the first column of the table if and only if, for each J with $|J| \geqslant 2$, there exists a confidence-level aggregator $\otimes$ of the type specified in the second column such that $c^J = \dot{F}_{\otimes}(c_1, \ldots, c_n)$, up to convex closure.*

We make no particular claim for any of the confidence-level aggregators in Table 6 on normative grounds; we present this result to illustrate the richness of the approach, and exemplify some simple aggregators.

The axiom involved in the characterisation of confidence aggregation with an affine aggregator, Consensus Independence, uses the notion of uncovered consensus. For every tuple of stakes levels $\mathbf{s}$ exhibiting consensus and stakes level s with $\succ^J$ respecting the consensus $\succ_{\mathbf{s}}$ at s, we say that the consensus at s is *covered* when, for all acts f, g, if $f \not\succ_{\mathbf{s}} g$ then there exists a tuple $\mathbf{s}'$ exhibiting consensus with $\mathbf{s}' \not\geqslant \mathbf{s}$ such that $\succ^J$ respects the

Axioms	Aggregator
Consensus Independence	Affine
Consensus Independence, Neutrality	Average
Consensus Join	Non-neutral Maximum
Consensus Join, Neutrality	Generalised Maximum
Consensus Meet, Neutrality	Generalised Minimum

Table 6: Characterisations of special case confidence-level aggregators, to be read in the context of Theorem B.1.

Axiom (Consensus Independence)**.** *For all non-empty $J \subseteq I$, and all tuples of stakes levels $\mathbf{s_1}, \ldots, \mathbf{s_l}, \mathbf{t_1}, \ldots, \mathbf{t_m} \in \mathcal{S}^{|J|}$ exhibiting consensus and $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_m \in [0,1]$ with $\sum_{k=1}^{l} \alpha_k = \sum_{j=1}^{m} \beta_j = 1$ and $\sum_{k=1}^{l} \alpha_k \mathbf{s}_l = \sum_{j=1}^{m} \beta_j \mathbf{t}_j$, if, for some stakes levels $s_1, \ldots, s_l, t_1, \ldots, t_m$, $\succ^J$ does not respect the consensuses $\succ_{\mathbf{s}_i}$ at s_k for each $k = 1, \ldots, l$, and $\succ_{\mathbf{t}_j}$ are uncovered consensuses at t_j for all $j = 1, \ldots, m$, then $\sum_{k=1}^{l} \alpha_k s_k < \sum_{j=1}^{m} \beta_j t_j$.*

Axiom (Consensus Join)**.** *For all non-empty $J \subseteq I$ and any tuples of stakes levels $\mathbf{s}, \mathbf{t}$ exhibiting consensus, if $\succ^J$ respects the consensuses $\succ_{\mathbf{s}}, \succ_{\mathbf{t}}$ at s, then it respects the consensus $\succ_{\mathbf{s} \vee \mathbf{t}}$ at s.*

Axiom (Consensus Meet)**.** *For all non-empty $J \subseteq I$ and any tuples of stakes levels $\mathbf{s}, \mathbf{t}$ exhibiting consensus, if $\succ^J$ respects the consensuses $\succ_{\mathbf{s}}, \succ_{\mathbf{t}}$ at s, then it respects the consensus $\succ_{\mathbf{s} \wedge \mathbf{t}}$ at s.*

Axiom (Neutrality)**.** *For all non-empty $J \subseteq I$ and any stakes levels s, tuple of stakes levels $\mathbf{s}$ and permutation π such that $\mathbf{s}, \pi(\mathbf{s})$ exhibit consensus, $\succ^J$ respects the consensus $\succ_{\mathbf{s}}$ at s if and only if $\succ^J$ respects the consensus $\succ_{\pi(\mathbf{s})}$ at s.*

Where, for any $\mathbf{s}, \mathbf{t} \in \mathcal{S}^n$ and $\alpha \in [0,1]$, $(\alpha\mathbf{s} + (1-\alpha)\mathbf{t})_i = \alpha s_i + (1-\alpha) t_i$, $(\mathbf{s} \vee \mathbf{t})_i = \max\{s_i, t_i\}$ and $(\mathbf{s} \wedge \mathbf{t})_i = \min\{s_i, t_i\}$.

Figure 4: Axioms for special cases

consensus $\succ_{\mathbf{s}'}$ at s and $f \not\succ_{\mathbf{s}'} g$. Otherwise, say that the consensus is *uncovered* at s. When the consensus $\succ_{\mathbf{s}}$ is covered, there is no f, g such that the absence of preference between them according to $\succ_s^J$ can be pinpointed as being due to the respect for consensus $\succ_{\mathbf{s}}$, for there is some other consensus respected at s that does not have any preference either. So, when the consensus is uncovered, it contributes for sure to the construction of group preferences, even in the context of the other relevant consensuses. In particular, it means that the group confidence level assigned to this consensus can't be a lower than that corresponding to stakes level s.

In the light of this, Consensus Independence can be thought of as an Independence-like axiom, adapted to this context. An Independence axiom in this context would imply that if $\succ^J$ does not respect $\succ_{\mathbf{s}_i}$ at s_i, for all i, then it does not respect any mixture $\succ_{\sum_k \alpha_k \mathbf{s}_k}$ exhibiting consensus at $\sum_k \alpha_k s_k$. However, consensus-preserving aggregation with an affine aggregator can violate such a condition when, for instance, the consensus involved is respected 'by accident', because it is covered; so the implication does not hold. Consensus Independence corrects the first-pass independence condition to account for such cases, using the notion of uncovered consensus. It allows that the mixture of uncovered consensuses may not be uncovered, and it allows that a mixture of non-respected consensus may be respected, but doesn't allow the mixture of uncovered consensuses to coincide with a mixture of non-respected ones.

The characterising axiom for a non-neutral maximum confidence-level aggregator, Consensus Join, states that respect for consensus at s is preserved if one takes the consensus corresponding to the largest stakes level for each entry in the tuple (the join). Consensus Meet is the dual axiom, involving the lowest stakes level for each entry. Neutrality is a standard neutrality axiom, adapted to the current context, stating that respect for consensus is preserved under permutation of individuals. Added to the other conditions, it characterises the 'neutral' average, generalised maximum and generalised minimum confidence-level aggregators.

C Connecting Confidence-based and Variational Preferences

We briefly state two results connecting the ambiguity averse confidence-based preferences (10) with variational preferences.

A preference $\succeq$ is *variational* (Maccheroni et al., 2006) if it is represented by

$$V(f) = \min_{p \in \Delta} \left(\mathbb{E}_p u(f) + \gamma(p) \right) \tag{C.1}$$

where u is as in Section 4.1 and $\gamma = \Delta \to [0, \infty]$ is a grounded, convex, lower semicontinuous function on the probability space.[26]

A variational preference $\succeq$ is said to be *strongly grounded* if there is a representing γ which attains it infimum (over Δ). The vast majority of variational preferences used in practice are strongly grounded. For instance, in the main examples of variational preferences generated by divergences (Maccheroni et al., 2006) or drawn from sets of models (Cerreia–Vioglio et al., 2025), the infimum is attained. This is also the case where the probability space is weakly compact, and hence when Ω is finite. The following result shows that this class of variational preferences are in fact confidence preferences, for appropriate c and D (see Appendix D.3 for proofs).

Proposition C.1. *Let $\succeq$ be a strongly grounded variational preference. Then there exists (c, D, u) such that $\succeq$ is represented by (c, D, u) according to (10).*

For a partial converse to this result, let $\succ^{dm}$ be a preference relation represented by (c, D, u) according to (10), and consider the following two assumptions on D.

Assumption C.1. *For all $f \in \mathcal{A} \backslash \mathcal{A}^c$, $x, y \in \mathcal{A}^c$ and $\alpha \in (0, 1]$*

$$D(f_\alpha x) = D(f_\alpha y) \tag{C.2}$$

[26] γ is grounded if its infimum is zero: as elsewhere, Δ is taken to be equipped with the weak* topology.

Assumption C.2. *For all* $f, g \in \mathcal{A}$, $\alpha \in [0,1]$, $x \in \mathcal{A}^c$ *and* $o \in O$, *if* $f \succ^{dm} x$, $g \succ^{dm} x$ *and* $\min_{p \in c(o)} \mathbb{E}_p u(f_\alpha g) \leqslant u(x)$, *then*

$$D(f_\alpha g) < o$$

We have the following equivalence.

Proposition C.2. *Let* $\succeq^{dm}$ *be represented by* (c, D, u) *according to* (10)*. If* D *satisfies Assumptions C.1 and C.2, then* $\succeq^{dm}$ *is a variational preference. Moreover, if* c *is strictly increasing, the converse implication holds.*

So the confidence preferences used in (10) can coincide with variational preferences, under appropriate assumptions on D. Under the philosophy behind confidence preferences, D typically reflect *only* tastes about the confidence level deemed appropriate for evaluating a given act, and hence the stakes involved in it. Such an interpretation is difficult to uphold for Assumption C.2, suggesting that the subfamily of confidence preferences that are variational do not respect taste-belief separation. This may be related to the lack of such separation for variational preferences in general, as suggested by the dependence of γ on u (Maccheroni et al., 2006).

D Proofs

D.1 Proofs of results in Sections 2, 4 and Appendix B

We begin with the following Proposition, mentioned in Section 2.2.

Proposition D.1. c^{rep} *is a consensus-preserving confidence aggregation of* $(c^1, \ldots, c^n)$ *under* $\otimes$ *if and only if*

$$\iota^{rep}(p) = \begin{cases} \otimes(\iota^1(p), \ldots, \iota^n(p)) & \text{if } \forall i \ \iota^i(p) \in O \\ \varnothing & \text{otherwise} \end{cases}$$

where the implausibility function ι^i *for* c^i *is as defined in Section 2.1.*

Proof. By the definition, $p \in c^{rep}(o)$ if and only if, for some $\mathbf{o}$ with $\otimes\mathbf{o} \leqslant o$, $p \in c^i(o_i)$ for all $i = 1, \ldots, n$. First consider p such that $\iota^i(p) \in O$ for all i. For such p, $p \in c^i(\iota^i(p))$ for all i and hence $p \in c^{rep}(\otimes(\iota^1(p), \ldots, \iota^n(p)))$.

Moreover, for any $\mathbf{o}$ with $\otimes\mathbf{o} < \otimes(\iota^1(p), \ldots, \iota^n(p))$, $o_i < \iota^i(p)$ for some i by the monotonicity of $\otimes$; since $\iota^i(p) = \min\{o' \in O : p \in c^i(o')\}$, it follows that $p \notin c^i(o_i)$. Hence, for every $o' < \otimes(\iota^1(p), \ldots, \iota^n(p))$, $p \in c^{rep}(o')$. The first clause of the required formula follows from the definition of ι. As concerns the other case, if there exists i with $\iota^i(p) = \varnothing$, then $p \notin c^i(o)$ for all $o \in O$, so, by the definition of confidence aggregation (notably Eq. (1)), $p \notin c^{rep}(o)$ for all $o \in O$. So $\iota^0(p) = \varnothing$, as required by the second clause. $\square$

We now prove Theorems 1 and B.1. It suffices to prove them for a single $J \subseteq I$, $|J| \geqslant 2$. For the rest of this section, we fix such a J, and use $\succ^0$ to denote $\succ^J$, and c^0 to denote c^J. Recall that, under Assumption 1, $\{(c^i, D, u)\}, (c^0, D, u)$ denote the representations of the $\{\succ^i\}, \succ^0$. Moreover, as noted in Section 4.1 (footnote 18), $\max\{D(f), D(g)\} = \zeta \circ \sigma$ for some strictly increasing $\zeta : \mathbb{R} \to \mathbb{R}$. Throughout the rest of this section, with slight abuse of notation, for any stakes level $s \in \mathcal{S}$, we shall denote $c^i(\zeta(s))$ by $c^i(s)$, for all i.

D.1.1 Proof of Theorem 1

We first show sufficiency of the axioms. Let $X \subseteq \mathcal{S}^n$ be the set of tuples exhibiting consensus. By Non-degeneracy, there is a tuple $(s_i)_{i \in I}$ exhibiting consensus, so there is a tuple $(s_i)_{i \in J}$ exhibiting consensus for any $J \subseteq I$, $|J| \geqslant 2$. Hence $X \neq \varnothing$. Let $\geqslant$ be the dominance ordering on $\mathcal{S}^n$: $\mathbf{s} \geqslant \mathbf{t}$ if and only if $s_i \geqslant t_i$ for all i. X is closed under $\geqslant$: if $\mathbf{s} \in X$ and $\mathbf{t} \geqslant \mathbf{s}$, then $\succ_{\mathbf{s}}$ is not contradictory; but $\succ_{t_i} \subseteq \succ_{s_i}$ for all i by the properties of confidence rankings, so $\succ_{\mathbf{t}}$ is not contradictory and hence $\mathbf{t} \in X$.

The following claim follows immediately from standard arguments (e.g. Ghirardato et al., 2004), for every $\succ_{\mathbf{s}}$ exhibiting consensus.

Claim D.1. $\succ^0$ respects the consensus $\succ_{\mathbf{s}}$ at stakes level s if and only if $c^0(s) \supseteq \bigcap_{i=1}^n c^i(s_i)$.

Claim D.2. For any set $Y \subseteq \mathcal{S}^n$ such that $\succ_{\mathbf{s}}$ exhibits consensus for every $\mathbf{s} \in Y$, $\bigcap_{\mathbf{s} \in Y} \succ_{\mathbf{s}}$ is represented by $\bigcup_{\mathbf{s} \in Y} \bigcap_{i=1}^n c^i(s_i)$ in the following sense: for all $f, g \in \mathcal{A}$, $f \succ_{\mathbf{s}} g$ if and only if

$$\mathbb{E}_p u(f) > \mathbb{E}_p u(g) \quad \text{for all } p \in \bigcup_{\mathbf{s} \in Y} \bigcap_{i=1}^n c^i(s_i) \tag{D.1}$$

Proof of Claim D.2. First consider $\succ_{\mathbf{s}}$ exhibiting consensus, and let $\succ_{\bigcap \mathbf{s}}$ be the 'Bewley' preference such that, for every $f, g \in \mathcal{A}$, $f \succ_{\bigcap \mathbf{s}} g$ if and only if

$$\mathbb{E}_p u(f) > \mathbb{E}_p u(g) \quad \text{for all } p \in \bigcap_{i=1}^{n} c^i(s_i) \tag{D.2}$$

Note that, since the c^i are closed and convex, so is their intersection. For every $f, g \in \mathcal{A}$, $f \succ_{\mathbf{s}} g$ if and only if $f \succ^i_{s_i} g$ for some i and $f \nprec^i_{s_i} g$ for every i. By Assumption 1, this holds if and only if, for some i, $\mathbb{E}_p u(f) > \mathbb{E}_p u(g)$ for all $p \in c^i(s_i)$, and, for every i, it is not the case that $\mathbb{E}_p u(f) < \mathbb{E}_p u(g)$ for all $p \in c^i(s_i)$. Since $\bigcap_{i=1}^n c^i(s_i) \neq \varnothing$, this holds if and only if, for all $p \in \bigcap_{i=1}^n c^i(s_i)$, $\mathbb{E}_p u(f) > \mathbb{E}_p u(g)$. Hence $\succ_{\mathbf{s}} = \succ_{\bigcap \mathbf{s}}$.

Now consider Y as specified. The case in which Y is a singleton has just been treated, so suppose that Y contains several elements. By the previous observation, for every $f, g \in \mathcal{A}$, $f \succ_{\mathbf{s}} g$ for every $\mathbf{s} \in Y$ if and only if $f \succ_{\bigcap \mathbf{s}} g$ for every $\mathbf{s} \in Y$, which holds if and only if $\mathbb{E}_p u(f) > \mathbb{E}_p u(g)$ for all $p \in \bigcap_{i=1}^n c^i(s_i)$ for every $\mathbf{s} \in Y$. This holds if and only if $\mathbb{E}_p u(f) > \mathbb{E}_p u(g)$ for all $p \in \bigcup_{\mathbf{s} \in Y} \bigcap_{i=1}^n c^i(s_i)$, as required.

□

Define the function $G : X \to \mathcal{S}$ as follows:

$$\begin{aligned} G(\mathbf{s}) &= \min \left\{ s : \succ^0_s \subseteq \succ_{\mathbf{s}} \right\} \\ &= \min \left\{ s : c^0(s) \supseteq \bigcap_{i=1}^{n} c^i(s_i) \right\} \end{aligned}$$

where the equality follows from Claim D.1. Note that if $G(X)$ is a finite set, then $\min G(X) \in G(X)$. The following proposition implies that this is the case when $G(X)$, and hence O, is infinite—and hence, given our assumptions, when the confidence rankings are upper semicontinuous.

Proposition D.2. *If the confidence rankings c^i are all upper semicontinuous, then, for any decreasing sequence $\mathbf{s}_j \in X$ with $\mathbf{s}^j \to \mathbf{s}$, $\mathbf{s} \in X$ and $G(\mathbf{s}) \leqslant \lim G(\mathbf{s}^j)$.*

Proof. Consider a decreasing sequence $\mathbf{s}^j \in X$ with $\mathbf{s}^j \to \mathbf{s}$. Since each c^i is upper semicontinuous, $\bigcap_j c^i(s^j_i) = c^i(s_i)$ for each i, so $\bigcap_{i=1}^n c^i(s_i) =$

$\bigcap_{i=1}^n \bigcap_j c^i(s_i^j) = \bigcap_j \bigcap_{i=1}^n c^i(s_i^j) \neq \varnothing$. So $\mathbf{s} \in X$. Moreover, by the definition of G, $c^0(G(\mathbf{s})) \supseteq \bigcap_j \bigcap_{i=1}^n c^i(s_i^j)$, so $G(\mathbf{s}) \leqslant G(\mathbf{s}^j)$ for all j. Hence $G(\mathbf{s}) \leqslant \lim G(\mathbf{s}^j)$, as required. □

Claim D.3. For every $s \geqslant \min G(X)$, $\succ_s^0$ is represented by $\bigcup_{\mathbf{s}\in X: s \geqslant G(\mathbf{s})} \bigcap_i c^i(s_i)$ in the Bewley sense: i.e. for all $f, g \in \mathcal{A}$, $f \succ_s^0 g$ if and only if:

$$\mathbb{E}_p u(f) > \mathbb{E}_p u(g) \quad \text{for all } p \in \bigcup_{\mathbf{s}\in X: s \geqslant G(\mathbf{s})} \bigcap_i c^i(s_i) \tag{D.3}$$

Proof. Fix a stakes level s with $s \geqslant \min G(X)$, and consider any $\mathbf{s}'$ with $G(\mathbf{s}') \leqslant s$. (By the previous observations guaranteeing the existence of a minimum, such $\mathbf{s}'$ exists.) By the definition of G, there exists $s'' \in X$ with $s'' \leqslant s$ and $\succ_{s''}^0 \subseteq \succ_{\mathbf{s}'}$. It follows from the nestedness properties of confidence rankings that $\succ_s^0 \subseteq \succ_{s''}^0 \subseteq \succ_{\mathbf{s}'}$. Since this holds for all $\mathbf{s}'$ with $G(\mathbf{s}') \leqslant s$, it follows that $\succ_s^0 \subseteq \bigcap_{\mathbf{s}\in X: s \geqslant G(\mathbf{s})} \succ_{\mathbf{s}}$.

To establish the opposite containment, consider f, g with $f \succ_{\mathbf{s}} g$ for all $\mathbf{s} \in X$ with $s \geqslant G(\mathbf{s})$. For any $\mathbf{s}'$ such that $\succ^0$ respects the consensus $\succ_{\mathbf{s}'}$ at s, it follows from the definition of G that $s \geqslant G(\mathbf{s}')$, so $f \succ_{\mathbf{s}'} g$ by the assumption specifying f, g. Hence, by Consensus-wise Pareto, $f \succ_s^0 g$. So $\succ_s^0 \supseteq \bigcap_{\mathbf{s}\in X: s \geqslant G(\mathbf{s})} \succ_{\mathbf{s}}$, and hence there is equality. It follows from Claim D.2 that (D.3) holds for all $s \geqslant \min G(X)$.

□

Since $c^0(s)$ represents $\succ_s^0$ by the confidence representation (Hill, 2016), it follows that, up to convex closure, $c^0(s) = \bigcup_{\mathbf{s}\in X: s \geqslant G(\mathbf{s})} \bigcap_i c^i(s_i)$.

By the nestedness of confidence rankings (i.e. the fact that c is increasing in o), we have that, for any $\mathbf{s}, \mathbf{s}'$, if $\mathbf{s}' \geqslant \mathbf{s}$, then $G(\mathbf{s}') \geqslant G(\mathbf{s})$, so G is monotonic. Moreover, if $\succ_{\mathbf{s}} = \succ_{\mathbf{t}}$, then $G(\mathbf{s}) = G(\mathbf{t})$, so G generates a well-defined function on the equivalence classes of $\mathcal{S}^n$ under the relation setting $\mathbf{s}$ and $\mathbf{t}$ equivalent if and only if $\succ_{\mathbf{s}} = \succ_{\mathbf{t}}$, which we also call G. So $\otimes$, defined by

$$\otimes(o_1, \ldots, o_n) = \begin{cases} \zeta \circ G(\zeta^{-1}(o_1), \ldots, \zeta^{-1}(o_n)) & (\zeta^{-1}(o_1), \ldots, \zeta^{-1}(o_n)) \in X \\ \zeta(\min(G(X))) & \text{otherwise} \end{cases}$$

is well-defined; i.e. even if $(\zeta^{-1}(o_1), \ldots, \zeta^{-1}(o_n))$ is multi-valued, for any $\mathbf{s}, \mathbf{t} \in (\zeta^{-1}(o_1), \ldots, \zeta^{-1}(o_n))$, $\succ_{\mathbf{s}} = \succ_{\mathbf{t}}$ by the confidence decision model, and so $G((\zeta^{-1}(o_1), \ldots, \zeta^{-1}(o_n)))$ is well-defined ($G(\mathbf{s}) = G(\mathbf{t})$). Moreover, $\otimes$ is monotonic, and thus a confidence level aggregator. It follows from Claim D.3 that (1) holds up to convex closure for all o with $\bigcup_{\mathbf{o}:\otimes\mathbf{o}\leqslant o} \bigcap_i c^i(o_i) \neq \varnothing$. For any $s < \min G(X)$, by the nestedness of confidence rankings, $\succ_s^0 \subseteq \bigcup_{s' \in G(X)} \succ_{s'}^0$. However, by Consensus-based beliefs, if $f \succ_s^0 g$, then $f \succ_{s'}^0 g$ for some $s' \in G(X)$, so $\succ_s^0 = \bigcup_{s' \in G(X)} \succ_{s'}^0$. Hence, for any o with $\bigcup_{\mathbf{o}:\otimes\mathbf{o}\leqslant o} \bigcap_i c^i(o_i) = \varnothing$, $c(o) = \bigcap_{s' \in G(X)} c^0(s') = c^0(\min G(X))$ (by the upper semicontinuity of confidence rankings), up to convex closure, so c^0 is consensus preserving, as required. This establishes the Theorem.

Moreover, note that since $\otimes$ is monotonic on the domain where $\zeta^{-1}(\mathbf{o}) \in X$, any monotonic operator coinciding with $\otimes$ on this domain is also a confidence level aggregator, and represents aggregated preferences according to (1), hence establishing the 'only if' direction.

The 'if' direction is a direct consequence of (1) and Claims D.1 and D.2.

Finally, suppose that $\otimes' \neq \otimes$ is another confidence level aggregator such that, up to convex closure, c^0 is a consensus-preserving aggregation of $(c^1, \ldots, c^n)$ under $\otimes'$. Let $G'(\mathbf{s}) = \otimes'(\mathbf{s})$. By the confidence representation and the fact that c^0 is a consensus-preserving aggregation of $(c^1, \ldots, c^n)$ under $\otimes'$, for every $\mathbf{s} \in X$, $\succ_{G'(\mathbf{s})}^0 \subseteq \succ_{\mathbf{s}}$. It follows from the definition of G that $G(\mathbf{s}) \leqslant G'(\mathbf{s})$ for all $\mathbf{s} \in X$. So, either $\otimes'$ coincides with $\otimes$ on X, or there $\mathbf{s} \in X$ with $\otimes'\mathbf{o} \neq \otimes\mathbf{o}$, so $\otimes'\mathbf{o} > \otimes\mathbf{o}$. Hence $\otimes$ is the unique $\otimes$ taking minimal values on all consensuses, as required.

Remark 1. Note that the use of profiles of confidence levels with $\otimes\mathbf{o}$ less than or equal to o, rather than just equal, as in (B.1), is a result of the general framework adopted for this result. More specifically, it is clear to see that one can prove, using arguments along the lines above, that one can replace the less than or equal with equality under the condition that, if $\otimes\mathbf{o} < o$, then there exists $\mathbf{o}' \geqslant \mathbf{o}$ with $\otimes\mathbf{o}' = o$. For an example where this condition is not satisfied, consider $O = \{a, b, c\}$ with $a > b > c$, and two agents $1, 2$. Consider $\otimes$ giving the value c on (c, c) and the value a otherwise. Clearly, the condition is not satisfied for b—in fact, there is no $\mathbf{o}$ with $\otimes\mathbf{o} = b$. So there is no $\otimes$ such that (2) holds with equality in the place of the inequality.

D.1.2 Proof of Theorem B.1

Proof for Table 6, row 1 (affine aggregation). Let X be as defined in the proof of Theorem 1. Let

$$C = \{(\mathbf{s}, s) \in \mathbb{R}^{n+1} : \mathbf{s} \in X, \ \succ^0_s \subseteq \succ_{\mathbf{s}}\}$$
$$K = \{(\mathbf{s}, s) \in C : \succ_{\mathbf{s}} \supseteq \bigcap_{\mathbf{s}':(\mathbf{s}',s)\in C, \ \mathbf{s}' \not\geq \mathbf{s}} \succ_{\mathbf{s}'}\}$$

C is the set of consensuses and K is the set of 'covered' consensuses—i.e. where there is consensus because the other consensuses at this s 'cover' this one. For a tuple of stakes levels $\mathbf{s}$ and a stakes level s', $s'_i\mathbf{s}$ is the tuple obtained by replacing the ith stakes level in $\mathbf{s}$ by s'. An individual i is *non-null* if there exist $\mathbf{s}, s'_i\mathbf{s} \in X$ and $t \in \mathcal{S}$ with $(\mathbf{s}, t) \in C\backslash K$ but $(s'_i\mathbf{s}, t) \notin C$. Let $NN = \{i \in \{1, \ldots, n\} : i \ \text{ non-null}\}$ and $Y = \mathcal{S}^{NN} \subseteq \mathbb{R}^n$ be the subspace of $\mathcal{S}^n$ containing the stakes levels for non-null individuals only; we use X_Y, C_Y, K_Y etc to refer to the projection of X, C, K etc onto Y, $Y \times \mathbb{R}$ etc.

Define

$$L = \{(\mathbf{s}, s) \in Y \times \mathbb{R} : \mathbf{s} \in X_Y, \ \succ^0_s \not\subseteq \succ_{\mathbf{s}}\} = (X_Y \times \mathbb{R})\backslash C_Y$$
$$U = \{(\mathbf{s}, s) \in C_Y : \exists s' \leqslant s, (\mathbf{s}, s') \in C_Y\backslash K_Y\}$$

For a set Z, let $conv(Z)$ be the convex hull of Z. Note that $L, U \subseteq X_Y \times \mathbb{R}$, so $conv(L), conv(U) \subseteq conv(X_Y) \times \mathbb{R}$.

Claim D.4. $conv(L) \cap conv(U) = \varnothing$.

Proof. For reductio, suppose that there exist $(\mathbf{s}_1, s_1), \ldots, (\mathbf{s}_l, s_l) \in L$, $(\mathbf{t}_1, t_1), \ldots, (\mathbf{t}_m, t_m) \in U$, $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_m \in [0, 1]$ with $\sum_{i=1}^l \alpha_i = \sum_{i=1}^m \beta_i = 1$, $\sum_{i=1}^l \alpha_i \mathbf{s}_i = \sum_{i=1}^m \beta_i \mathbf{t}_i$ and $\sum_{i=1}^l \alpha_i s_i = \sum_{i=1}^m \beta_i t_i$. Without loss of generality, the t_i can be chosen to be minimal such that $(\mathbf{t}_i, t_i) \in U$. It follows from Consensus Independence (extending to tuples to take any value off NN for which there is consensus, if necessary) that $\sum_{i=1}^{l} \alpha_i s_i < \sum_{i=1}^{m} \beta_i t_i$, which is a contradiction.

□

Claim D.5. $conv(L)$ is open in the subspace topology on $conv(X_Y) \times \mathbb{R}$.

Proof. Note that $L^c \cap (X_Y \times \mathbb{R}) = C_Y = \{(\mathbf{s}, s) \in Y \times \mathbb{R} : \mathbf{s} \in X,\ G(\mathbf{s}) \geqslant s\}$, where G is as defined in the proof of Theorem 1. By Proposition D.2 and the nestedness of the preferences orders at different stakes levels, $L^c \cap (X_Y \times \mathbb{R})$ is closed. Hence L is open in the subspace topology on $X_Y \times \mathbb{R}$. It follows that the convex hull $conv(L)$ is open in the subspace topology on $conv(X_Y) \times \mathbb{R}$. □

By the previous claims and a separating hyperplane theorem (Rockafellar, 1970, Thm 11.3), there exists an linear function $\phi : \mathbb{R}^{NN} \rightarrow \mathbb{R}$ and $\chi \in \mathbb{R}$ with $\phi((\mathbf{s}, s)) < \chi$ for all $(\mathbf{s}, s) \in conv(L)$, and $\phi((\mathbf{s}, s)) \geqslant \chi$ for all $(\mathbf{s}, s) \in conv(U)$. Since it is linear, and without loss of generality, ϕ, χ can be chosen so there exist w_i, $i \in NN$ such that $\phi((\mathbf{s}, s)) = s - \sum_i w_i s_i$. Define $G_{aff} : \mathbb{R}^n \rightarrow S$ by $G_{aff}(\mathbf{s}) = \sum_{i \in NN} w_i s_i + \chi$. Note that G_{aff} is an affine function on $\mathbb{R}^n$, with zero weights on $i \notin NN$. By construction, $s < G_{aff}(\mathbf{s})$ for all $(\mathbf{s}, s) \in conv(L)$, and $s \geqslant G_{aff}(\mathbf{s})$ for all $(\mathbf{s}, s) \in conv(U)$.

We first show that $w_i > 0$ for all $i \in NN$. By the nestedness of confidence rankings, for any $\mathbf{s}', \mathbf{s} \in Y$, $\mathbf{s}' \geqslant \mathbf{s}$, if $(\mathbf{s}, s) \in L$, then $(\mathbf{s}', s) \in L$. For reductio, suppose, for some k, that $w_k < 0$, and consider $(\mathbf{s}, s') \in L$. By construction, s, with $s - \sum_i w_i s_i = \chi$, is such that $(\mathbf{s}, s) \notin L$. Consider $\mathbf{s}' = (s_1, \ldots, s_k - \frac{s - s'}{w_k}, \ldots, s_n)$. $\mathbf{s}' \geqslant \mathbf{s}$ since $w_k < 0$, so $(\mathbf{s}', s') \in L$. However, $s' - \sum_i w_i s'_i = \chi$, contradicting the established properties of ϕ. Hence $w_i \geqslant 0$ for all $i \in NN$. Suppose now that for some $i \in NN$, $w_i = 0$. By the nestedness of the confidence representation and the definition of NN, there exists $\mathbf{s} \in X$, s', t such that $(\mathbf{s}, t) \in U$ and $(s'_i \mathbf{s}, t) \in L$; however, since $w_i = 0$, $G_{aff}(\mathbf{s}) = G_{aff}(s'_i \mathbf{s})$, which contradicts the definition of G_{aff}. So, for all $i \in NN$, $w_i \neq 0$. Hence $w_i > 0$ for all $i \in NN$, and G_{aff} is monotonic.

Claim D.6. For all $s \geqslant \inf G_{aff}(X)$, $\succ_s^0$ is represented by $\bigcup_{\mathbf{s} \in X : s = G_{aff}(\mathbf{s})} \bigcap_i c^i(s_i)$ in the Bewley sense: for all $f, g \in \mathcal{A}$, $f \succ_s^0 g$ if and only if:

$$\mathbb{E}_p u(f) > \mathbb{E}_p u(g) \quad \text{for all } p \in \bigcup_{\mathbf{s} \in X : s = G_{aff}(\mathbf{s})} \bigcap_i c^i(s_i) \tag{D.4}$$

Proof. Fix a stakes level s, with $s \geqslant \inf G_{aff}(X)$. For any $\mathbf{s} \in X$ with $G_{aff}(\mathbf{s}) = s$, by the construction of ϕ and the definition of NN, $\succ_s^0 \subseteq \succ_{\mathbf{s}}$. So $\succ_s^0 \subseteq \bigcap_{\mathbf{s} \in X : s = G_{aff}(\mathbf{s})} \succ_{\mathbf{s}}$.

We now establish the opposite containment. By Consensus-wise Pareto, $\succ_s^0 \supseteq \bigcap_{\mathbf{s}:(\mathbf{s},s) \in C} \succ_{\mathbf{s}}$. Consider any $\mathbf{s}'$ such that $\succ^0$ respects the consensus $\succ_{\mathbf{s}'}$ at s—so $(\mathbf{s}', s) \in C$—and $G_{aff}(\mathbf{s}') < s$. Then by the fact that the $w_i \geqslant 0$ for all i, there exists $\mathbf{s} \geqslant \mathbf{s}'$ with $G_{aff}(\mathbf{s}) = s$; by the nestedness of confidence rankings and the preference representation, $\succ_{\mathbf{s}'} \supseteq \succ_{\mathbf{s}} \supseteq \bigcap_{\mathbf{s}:(\mathbf{s},s) \in C,\ G_{aff}(\mathbf{s}) \geqslant s} \succ_{\mathbf{s}}$. Since this holds for all such $\mathbf{s}'$, $\succ_s^0 \supseteq \bigcap_{\mathbf{s}:(\mathbf{s},s) \in C,\ G_{aff}(\mathbf{s}) \geqslant s} \succ_{\mathbf{s}} = \bigcap_{\mathbf{s}:G_{aff}(\mathbf{s}) = s} \succ_{\mathbf{s}} \cap \bigcap_{\mathbf{s}:(\mathbf{s},s) \in C,\ G_{aff}(\mathbf{s}) > s} \succ_{\mathbf{s}}$, where the equality is due to the construction of G_{aff}. Now consider any $\mathbf{s}'$ with $(\mathbf{s}', s) \in C$ and $G_{aff}(\mathbf{s}') > s$. If $(\mathbf{s}', s) \notin K$, then $(\mathbf{s}', s) \in U$, contradicting the fact that $G_{aff}(\mathbf{s}') > s$ and the construction of G_{aff}. Hence $(\mathbf{s}', s) \in K$, so $\succ_{\mathbf{s}'} \supseteq \bigcap_{\mathbf{s}'':(\mathbf{s}'',s) \in C,\ \mathbf{s}'' \ngeqslant \mathbf{s}'} \succ_{\mathbf{s}''}$. So $\bigcap_{\mathbf{s}:G_{aff}(\mathbf{s}) = s} \succ_{\mathbf{s}} \cap \bigcap_{\mathbf{s}:(\mathbf{s},s) \in C,\ G_{aff}(\mathbf{s}) > s} \succ_{\mathbf{s}} = \bigcap_{\mathbf{s}:G_{aff}(\mathbf{s}) = s} \succ_{\mathbf{s}} \cap \bigcap_{\mathbf{s}:(\mathbf{s},s) \in C,\ G_{aff}(\mathbf{s}) > s, \mathbf{s} \ngeqslant \mathbf{s}'} \succ_{\mathbf{s}}$. Since this holds for all such $\mathbf{s}'$, it follows that $\bigcap_{\mathbf{s}:G_{aff}(\mathbf{s}) = s} \succ_{\mathbf{s}} \cap \bigcap_{\mathbf{s}:(\mathbf{s},s) \in C,\ G_{aff}(\mathbf{s}) > s} \succ_{\mathbf{s}} = \bigcap_{\mathbf{s}:G_{aff}(\mathbf{s}) = s} \succ_{\mathbf{s}}$, so $\succ_s^0 \supseteq \bigcap_{\mathbf{s}:G_{aff}(\mathbf{s}) = s} \succ_{\mathbf{s}}$.

So $\succ_s^0 = \bigcap_{\mathbf{s}:G_{aff}(\mathbf{s}) = s} \succ_{\mathbf{s}}$; it follows from Claim D.2 that (D.4) holds for all $s \geqslant \inf G_{aff}(X)$.

□

Since $c^0(s)$ represents $\succ_s^0$ by the confidence representation (Hill, 2016), it follows that, up to convex closure, $c^0(s) = \bigcup_{\mathbf{s} \in X : G_{aff}(\mathbf{s}) = s} \bigcap_i c^i(s_i)$.

Define $\otimes$ by

$$\otimes \mathbf{o} = \sum w_i o_i + \chi$$

Clearly, this is an affine confidence level aggregator. Moreover, by Claim D.6 and the fact that ζ is the identity, (B.1) holds up to convex closure for every o with $\bigcup_{\mathbf{o}:\otimes o_i = o} \bigcap_i c^i(o_i) \neq \varnothing$. By a similar argument to that used in the proof of Theorem 1, the representation extends to other $o \in O$ as required. Hence, up to convex closure, c^0 is a consensus preserving with affine aggregator $\otimes$ as required.

For the necessity of the Consensus Independence axiom, suppose that there is an affine aggregator $\otimes$ representing preferences. Consider any $\mathbf{s_1}, \ldots, \mathbf{s_l}, \mathbf{t_1}, \ldots, \mathbf{t_m}$ exhibiting consensus, and $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_m \in [0, 1]$ with $\sum_{k=1}^{l} \alpha_k = \sum_{k=1}^{m} \beta_k = 1$ and $\sum_{k=1}^{l} \alpha_k \mathbf{s_k} = \sum_{k=1}^{m} \beta_k \mathbf{t_k}$. If $\succ^0$ does not respect the consensuses $\succ_{\mathbf{s_k}}$ at s_k, then $c^0(s_k) \not\supseteq \bigcap_i c^i((\mathbf{s_k})_i)$, whereas, by the aggregation rule $c^0(\sum_i w_i(\mathbf{s_k})_i + \chi) \supseteq \bigcap_i c^i((\mathbf{s_k})_i)$, so, by the nestedness of confidence rankings, $s_k < w_i(\mathbf{s_k})_i + \chi$. If this holds for all k, then $\sum_{k=1}^{l} \alpha_k s_k < \sum_i w_i \sum_{k=1}^{l} \alpha_k (\mathbf{s_k})_i + \chi$. Similarly, if $\succ_{\mathbf{t_k}}$ is an uncovered consensus at t_k then, by the confidence representation of preferences, $c^0(t_k) \supseteq \bigcap_i c^i((\mathbf{t_k})_i)$ and $\bigcap_i c^i((\mathbf{t_k})_i) \nsubseteq \bigcup_{\mathbf{s} \not\geqslant \mathbf{t_k},\ (\mathbf{s},t_k)\in C} \bigcap_i c^i(s_i) \subseteq c^0(t_k)$. By the aggregation representation, it follows that $c^0(\sum_i w_i(\mathbf{t_k})_i + \chi) = \bigcup_{\mathbf{s}:\ \sum_i w_i s_i = \sum_i w_i (\mathbf{t_k})_i} \bigcap_i c^i(s_i) \subseteq \bigcap_i c^i((\mathbf{t_k})_i) \cup \bigcup_{\mathbf{s} \not\geqslant \mathbf{t_k},\ (\mathbf{s},t_k)\in C} \bigcap_i c^i(s_i) \subseteq c^0(t_k)$, so, by the nestedness of confidence rankings, $\sum_i w_i(\mathbf{t_k})_i + \chi \leqslant t_k$. So if $\succ_{\mathbf{t_k}}$ is an uncovered consensus at t_k for each k, it follows that $\sum_{k=1}^{m} \beta_k t_k \geqslant \sum_i w_i \sum_{k=1}^{m} \beta_k (\mathbf{t_k})_i + \chi$. Since, $\sum_i w_i \sum_{k=1}^{l} \alpha_k (\mathbf{s_k})_i = \sum_i w_i \sum_{k=1}^{m} \beta_k (\mathbf{t_k})_i$, it follows that $\sum_{k=1}^{m} \beta_k t_k > \sum_{k=1}^{l} \alpha_k s_k$, as required.

□

Proof for Table 6, row 2 (averaging aggregation). We show that there exists a representation of the sort obtained in the proof of part i. where the weights are equal. For reductio, suppose not, and consider a representation with an affine aggregator with $w_j > w_k$ for some j, k. First, by Neutrality and Non-degeneracy, $NN = \{1, \ldots, n\}$, so $w_j, w_k \neq 0$.

First consider the case where there exists s and $\mathbf{s}$ such that $(\mathbf{s}, s) \in C$, $\mathbf{s}$ is a maximum, under $\geqslant$, of $\{\mathbf{s}' : (\mathbf{s}', s) \in C\}$, and $s_j \neq s_k$; take any such s and $\mathbf{s}$. By the upper semicontinuity of confidence rankings, for any strictly decreasing sequences $t_l \to s_j$ and $t'_l \to s_k$, $\bigcap_{i\neq j} c^i(s_i) \cap c^k(t'_l) \to \bigcap_i c^i(s_i)$ and $\bigcap_{i\neq k} c^i(s_i) \cap c^j(t_l) \to \bigcap_i c^i(s_i)$ as $l \to \infty$. By the fact that $\mathbf{s}$ is a maximum, $((t_l)_j\mathbf{s}, s) \notin C$, $((t'_l)_k\mathbf{s}, s) \notin C$ for all l. By the affine aggregator representation and the upper semicontinuity of confidence rankings, for each $s'' > s$, there exist $m_t, m_{t'}$ with $((t_l)_j\mathbf{s}, s'') \in C$ and $((t'_l)_k\mathbf{s}, s'') \in C$ for all $l > m_t$ and $l > m_{t'}$. In particular $G_{aff}(t_j\mathbf{s}) > s$ and $G_{aff}(t'_k\mathbf{s}) > s$ for all $t > s_j$, $t' > s_k$, where G_{aff} is as in the proof of part i., though $G_{aff}((t_l)_j\mathbf{s}) \to G_{aff}(\mathbf{s})$ and $G_{aff}((t'_l)_k\mathbf{s}) \to G_{aff}(\mathbf{s})$ as $l \to \infty$, so by the continuity of the affine representation, $G_{aff}(\mathbf{s}) = s$.

If $s_j > s_k$, then $G_{aff}((s_k)_j(s_j)_k\mathbf{s}) < s$, by the form of G_{aff}, the fact that $w_j > w_k$ and the rearrangement inequality. Hence, by the continuity of the representation, for some $t > s_k$, $G_{aff}(t_j(s_j)_k\mathbf{s}) < s$, from which it follows that $(t_j(s_j)_k\mathbf{s}, s) \in C$. Since $t_j(s_j)_k\mathbf{s}$ is a permutation of $t_k\mathbf{s}$, it follows from Neutrality that $(t_k\mathbf{s}, s) \in C$, contradicting the maximality of $\mathbf{s}$. If $s_j < s_k$, then $G_{aff}((s_k)_j(s_j)_k\mathbf{s}) > s$. By the construction of $\mathbf{s}$ there exists $s'' < G_{aff}((s_k)_j(s_j)_k\mathbf{s})$ and $t > s_k$ with $(t_k\mathbf{s}, s'') \in C$. Moreover, since t can be chosen such that there exists $t' > t$ with $(t'_k\mathbf{s}, s'') \notin C$, t can be chosen so that $(t_k\mathbf{s}, s'') \notin K$. By Neutrality, it follows that $(t_j(s_j)_k\mathbf{s}, s'') \in C\backslash K$, so $(t_j(s_j)_k\mathbf{s}, s'') \in U$, contradicting the construction of G_{aff} and the fact that $G_{aff}(t_j(s_j)_k\mathbf{s}') \geqslant G_{aff}((s_k)_j(s_j)_k\mathbf{s}') > s''$.

Now consider the case where there does not exist s and $\mathbf{s}$ such that $(\mathbf{s}, s) \in C$, $\mathbf{s}$ is a maximum, under $\geqslant$, of $\{\mathbf{s}' : (\mathbf{s}', s) \in C\}$, and $s_j \neq s_k$. Hence, for all s and $\mathbf{s}$ such that $(\mathbf{s}, s) \in C$ and $\mathbf{s}$ is a maximum, under $\geqslant$, of $\{\mathbf{s}' : (\mathbf{s}', s) \in C\}$, $s_j = s_k$. For any $\mathbf{s}$, let $\hat{\mathbf{s}}$ be such that: $\hat{s}_i = s_i$ when $i \neq j, k$, $\hat{s}_j = \hat{s}_k = \max\{s_j, s_k\}$. So, in the case under consideration, for every $\mathbf{s}$ with $s_j \neq s_k$ and every stakes level s, $(\mathbf{s}, s) \in C$ if and only if $(\hat{\mathbf{s}}, s) \in C$.

Hence the map $\psi : \mathcal{S}^n \to \mathcal{S}^{n-1}$, defined by $\psi(\mathbf{s})_i = s_i$ for $i \neq j, k$ and $\psi(\mathbf{s})_j = \max\{s_j, s_k\}$, is a well-defined map sending C to $\psi(C) = \hat{C}$ which is such that $\psi^{-1}(\hat{C}) = C$. Hence images of other sets in the proof of part i., which are defined in terms of C, can be defined in terms of $\hat{C}$ and have the same pull-back property. It follows that the argument in the proof of part i. goes through, yielding a representation of c^0 in terms of an affine function $\widehat{G_{aff}} : \mathbb{R}^{n-1} \to \mathbb{R}$ of the following form: for all $s \geqslant \inf \widehat{G_{aff}}(\hat{X})$:

$$c^0(s) = \bigcup_{\mathbf{s} \in \hat{X} : s = \widehat{G_{aff}}(\mathbf{s})} \bigcap_{i \neq k} c^i(s_i) \cap c^k(s_j)$$

up to convex closure. Letting $\widehat{G_{aff}}(\mathbf{s}) = \sum_{i \neq k} w_i s_i + \chi$, define $G'_{aff} : \mathbb{R}^n \to \mathbb{R}$ by $\widehat{G_{aff}}(\mathbf{s}) = \sum_{i \neq j,k} w_i s_i + \frac{w_j}{2} s_j + \frac{w_j}{2} s_k + \chi$. Noting that, for all $\mathbf{s} \in \mathcal{S}^n$ with $s_j = s_k$, $G'_{aff}(\mathbf{s}) = \widehat{G_{aff}}(\mathbf{s}|_{\{1,\ldots n\}\backslash\{k\}})$, we have that, for all $s \geqslant \inf G'_{aff}(X)$,

$$c^0(s) = \bigcup_{\mathbf{s} \in X : s = G'_{aff}(\mathbf{s}),\ s_j = s_k} \bigcap_{i=1}^{n} c^i(s_i)$$

up to convex closure.

For any $\mathbf{s}$ with $s_j \neq s_k$ and $G'_{aff}(\mathbf{s}) = s$, since $(\mathbf{s}, s) \in C$, it follows that $(\hat{\mathbf{s}}, s) \in C$ by the specification of the case. So $\bigcap_i c^i(s_i) \subseteq \bigcap_i c^i(\hat{s}_i) \subseteq \bigcup_{\mathbf{s}' \in X : s = G'_{aff}(\mathbf{s}'),\ s_j = s_k} \bigcap_i c^i(s_i)$. Hence, for all $s \geqslant \inf G'_{aff}(X)$,

$c^0(s) = \bigcup_{\mathbf{s}\in X:s=G'_{aff}(\mathbf{s})} \bigcap_{i=1}^n c^i(s_i)$, up to convex closure. So there is an affine aggregator representation with equal weights for j and k, as required.

Necessity of Neutrality is straightforward.

□

Proof for Table 6, row 3 (non-neutral maximum aggregator). Consider G as defined in the proof of Theorem 1. By Consensus Join, for any $\mathbf{s}, \mathbf{t}$, $G(\mathbf{s} \vee \mathbf{t}) \leqslant \max\{G(\mathbf{s}), G(\mathbf{t})\}$. However, by the monotonicity of G, since $\mathbf{s} \vee \mathbf{t} \geqslant \mathbf{s}, \mathbf{t}$, $G(\mathbf{s} \vee \mathbf{t}) \geqslant \max\{G(\mathbf{s}), G(\mathbf{t})\}$, so $G(\mathbf{s} \vee \mathbf{t}) = \max\{G(\mathbf{s}), G(\mathbf{t})\}$. For each $s \geqslant \min G(X)$, consider $\mathbf{t}^s = \bigvee_{\mathbf{s}:G(\mathbf{s})\leqslant s} \mathbf{s}$. By the previous observation, $G(\mathbf{t}^s) = s$ and for any $\mathbf{s}$ with $s_i > t_i^s$ for some i, $G(\mathbf{s}) > s$. Since, for any $\mathbf{s}$, if $\mathbf{s} \leqslant \mathbf{t}^s$, then $G(\mathbf{s}) \leqslant s$ by the monotonicity of G, we have that, for all $\mathbf{s}$, $G(\mathbf{s}) > s$ if and only if there exists i with $s_i > t_i^s$. Hence $G(\mathbf{s}) < s$ if and only if there exists $s' < s$ with $s_i \leqslant t_i^{s'}$ for all i. Hence $G(\mathbf{s}) = s$ if and only if $\mathbf{s} \leqslant \mathbf{t}^s$ and there is no $s' < s$ with $\mathbf{s} \leqslant \mathbf{t}^{s'}$.

Moreover, since, by the nestedness of the confidence representation, $\bigcap_i c^i(s_i) \subseteq \bigcap_i c^i(t_i^s)$ for all $\mathbf{s}$ with $G(\mathbf{s}) \leqslant s$, it follows that $\bigcap_i c^i(t_i^s) = \bigcup_{\mathbf{s}\in X:s=G(\mathbf{s})} \bigcap_i c^i(s_i) = \bigcup_{\mathbf{s}\in X:s\geqslant G(\mathbf{s})} \bigcap_i c^i(s_i)$. So, up to convex closure, $c^0(s) = \bigcap_i c^i(t_i^s)$.

For $i = 1, \ldots, n$, define $\psi_i : O \to O$ by $\psi_i(o) = \zeta(\min\{s \in \mathcal{S} : \zeta(t_i^s) \geqslant o\})$. Since ζ is strictly increasing and, by the confidence representation, t_i^s is increasing in s for all i, ψ_i is increasing for all i. For any $\mathbf{o} \in O^n$, $s \in \mathcal{S}$ and $\mathbf{s} \in \zeta^{-1}(\mathbf{o})$, $G(\mathbf{s}) = s$ if and only if $\mathbf{s} \leqslant \mathbf{t}^{\mathbf{s}}$ and $\mathbf{s} \nleqslant \mathbf{t}^{\mathbf{s}'}$ for all $s' < s$, which is the case if and only if $\max_i \psi_i(o_i) = s$. Hence $G(\zeta^{-1}(\mathbf{o}))$ is well-defined, and $G(\zeta^{-1}(\mathbf{o})) = \max_i \psi_i(o_i)$. Defining $\otimes$ by $\otimes\mathbf{o} = \max_i \psi_i(o_i)$, we thus have that, for every o with $\bigcup_{\mathbf{o}:\otimes o_i=o} \bigcap_i c^i(o_i) \neq \varnothing$, (B.1) holds with $\otimes$, up to convex closure. By a similar argument to that used in the proof of Theorem 1, the representation extends to other $o \in O$ as required. Since the ψ_i are increasing, $\otimes$ is monotonic, and hence a generalized maximum aggregator. Hence, up to convex closure, c^0 is a consensus preserving with generalised maximum aggregator $\otimes$ as required.

The proof of necessity of Consensus Join is straightforward.

□

Proof for Table 6, row 4 & 5 (generalised maximum & minimum aggregators). We present the proof for the maximum aggregator; the case of the minimum aggregator is similar. Consider $\mathbf{t^s}$, as defined in the proof of part iii; we show that $t^s_j = t^s_k$ for all j, k. For reductio, suppose that this is not the case for some j, k, and suppose without loss of generality that $t^s_j > t^s_k$. By Neutrality, $G((t^s_k)_j(t^s_j)_k\mathbf{t^s}) = G(\mathbf{t^s}) = s$; but since $t^s_j > t^s_k$, it follows by the properties of G established in the proof of part iii. that $G((t^s_k)_j(t^s_j)_k\mathbf{t^s}) > G(\mathbf{t^s}) = s$, which is a contradiction. So $t^s_j = t^s_k$ for all j, k and s. Hence, for ψ_i as defined in the proof of part iv., $\psi_j(o) = \psi_k(o) = \psi(o)$ for all j, k and $o \in O$, whence $\otimes$ as defined in that proof of the proof can be written as $\otimes\mathbf{o} = \max_i \psi(o_i) = \psi(\max_i o_i)$. Hence it is a maximum aggregator, as required.

□

D.2 Proofs of results in Section 3

Proof of Proposition 1. By (2), the centre of c is:

$$\begin{aligned}\arg\min_{p\in\Delta} \otimes(\iota^1(p), \ldots, \iota^n(p)) &= \arg\min_{p\in\Delta}(\sum_{i=1}^{n} \frac{1}{n}\iota^i(p) + \chi) \\ &= \arg\min_{p\in\Delta} \sum_{i=1}^{n} \frac{1}{n}\iota^i(p)\end{aligned}$$

For the first row of Table 4, $\iota^i(p) = w^i \sum_{\omega\in\Omega}(p(\omega) - p_i(\omega))^2$, so the centre of c is $p = \arg\min_{p\in\Delta} \sum_{i=1}^n w^i \sum_{\omega\in\Omega}(p(\omega) - p_i(\omega))^2$. It is well-known that this is the mean of the distributions: the FOC is $\frac{d}{dp(\omega)} = 2\sum_{i=1}^n w^i(p(\omega) - p_i(\omega)) = 0$ for each $\omega \in \Omega$, yielding $p(\omega) = \sum_{i=1}^n \frac{w^i}{\sum_{i=1}^n w^i} p_i(\omega)$ for every $\omega \in \Omega$, which belongs to Δ.

For the second row of the Table, $\iota^i(p) = w^i R(p\|p_i)$, so the centre of c is $p = \arg\min \sum_{i=1}^n w^i R(p\|p_i)$. Yet:

$$\begin{aligned}
\sum_{i=1}^{n} w^i R(p\|p_i) &= -\sum_{i=1}^{n} w^i \sum_{\omega\in\Omega} p(\omega)(\log\frac{p_i(\omega)}{p(\omega)}) \\
&= -\sum_{\omega\in\Omega} p(\omega)\log\left(\prod_{i=1}^{n}\frac{p_i(\omega)^{w^i}}{p(\omega)^{w^i}}\right) \\
&= -\left(\sum_{i=1}^{n} w^i\right)\sum_{\omega\in\Omega} p(\omega)\log\left(\frac{\prod_{i=1}^{n} p_i(\omega)^{\frac{w^i}{\sum_{i=1}^{n} w^i}}}{p(\omega)}\right) \\
&= \left(\sum_{i=1}^{n} w^i\right)\left[-\sum_{\omega\in\Omega} p(\omega)\log\left(\frac{\prod_{i=1}^{n} p_i(\omega)^{\frac{w^i}{\sum_{i=1}^{n} w^i}}}{p(\omega)}.\frac{1}{\sum_{\omega\in\Omega}\prod_{i=1}^{n} p_i(\omega)^{\frac{w^i}{\sum_{i=1}^{n} w^i}}}\right)\right. \\
&\quad \left. +\log\left(\frac{1}{\sum_{\omega\in\Omega}\prod_{i=1}^{n} p_i(\omega)^{\frac{w^i}{\sum_{i=1}^{n} w^i}}}\right)\right] \\
&= \left(\sum_{i=1}^{n} w^i\right)\left[-\sum_{\omega\in\Omega} p(\omega)\log\left(\frac{GM(p_i)(\omega)}{p(\omega)}\right)+\log\left(\frac{1}{\sum_{\omega\in\Omega}\prod_{i=1}^{n} p_i(\omega)^{\frac{w^i}{\sum_{i=1}^{n} w^i}}}\right)\right] \\
&= \left(\sum_{i=1}^{n} w^i\right)\left[R(p\|GM(p_i))+\log\left(\frac{1}{\sum_{\omega\in\Omega}\prod_{i=1}^{n} p_i(\omega)^{\frac{w^i}{\sum_{i=1}^{n} w^i}}}\right)\right]
\end{aligned}$$

where $GM(p_i)(\omega) = \frac{\prod_{i=1}^{n} p_i(\omega)^{\frac{w^i}{\sum_{i=1}^{n} w^i}}}{\sum_{\omega\in\Omega}\prod_{i=1}^{n} p_i(\omega)^{\frac{w^i}{\sum_{i=1}^{n} w^i}}}$. This expression is clearly minimised at $p = GM(p_i) \in \Delta$, so the centre of c is $GM(p_i)$, as required.

For the third row, $\iota^i(p) = w^i R(p_i\|p)$, so the centre of c is $p = \arg\min \sum_{i=1}^{n} w^i R(p_i\|p)$. Yet:

$$\begin{aligned}
\sum_{i=1}^{n} w^i R(p_i \| p) &= -\sum_{i=1}^{n} w^i \sum_{\omega\in\Omega} p_i(\omega)(\log \frac{p(\omega)}{p_i(\omega)}) \\
&= \sum_{i=1}^{n} w^i \sum_{\omega\in\Omega} p_i(\omega)\log p_i(\omega) - \sum_{\omega\in\Omega} \log p(\omega) \sum_{i=1}^{n} w^i p_i(\omega) \\
&= \sum_{i=1}^{n} w^i \sum_{\omega} p_i(\omega)\log p_i(\omega) - \left(\sum_{i=1}^{n} w^i\right)\left(\sum_{\omega\in\Omega} AM(p_i)(\omega)\log AM(p_i)(\omega)\right) \\
&+ \left(\sum_{i=1}^{n} w^i\right)\left(\sum_{\omega\in\Omega} (\log AM(p_i)(\omega) - \log p(\omega))AM(p_i)(\omega)\right) \\
&= \sum_{i=1}^{n} w^i \sum_{\omega} p_i(\omega)\log p_i(\omega) - \left(\sum_{i=1}^{n} w^i\right)\left(\sum_{\omega} AM(p_i)(\omega)\log AM(p_i)(\omega)\right) \\
&+ \left(\sum_{i=1}^{n} w^i\right) R(AM(p_i)\|p)
\end{aligned}$$

where $AM(p_i) = \sum_{i=1}^{n} \frac{w^i}{\sum_{i=1}^{n} w^i} p_i$. This expression is clearly minimised at $p = AM(p_i) \in \Delta$, so the centre of c is $AM(p_i)$, as required.

□

The next two results and proofs adopt the notation from Example 3.1.

Proposition D.3. *Under the conditions and setup of Example 3.1, let c^{Eucl} be the w^L Euclidean confidence ranking generated by p^L (with $\omega' = \omega_R$), c^{RE} be the w^L reverse relative entropy confidence ranking generated by p^L. Then, for all $o \in O$, $c^{Eucl}(o) \subseteq \mathcal{L}_\epsilon$ if and only if $c^{Eucl}(o) \subseteq \mathcal{R}_\epsilon$, and $c^{RE}(o) \subseteq \mathcal{L}_\epsilon$ if and only if $c^{RE}(o) \subseteq \mathcal{R}_\epsilon$.*

Proof. It suffices to show that the appropriate distance (or, equivalently ι-value) between p and the closest q with $q(L) = 0.9 - \epsilon$ is the same as the distance to the closest q' with $q'(R) = 0.1 + \epsilon$.

Both the distance functions involved (Euclidean distance, relative entropy) are functions of $p^L(\omega_{LR}), p^L(\omega_L), p^L(\omega_N), p(\omega_{LR}), p(\omega_L), p(\omega_N)$; write this function as $\phi(p^L(\omega_{LR}), p^L(\omega_L), p^L(\omega_N), p(\omega_{LR}), p(\omega_L), p(\omega_N))$. More specifically, in the Euclidean case,

$$\begin{aligned}
&\phi(p^L(\omega_{LR}), p^L(\omega_L), p^L(\omega_N), p(\omega_{LR}), p(\omega_L), p(\omega_N)) \\
=&(p(\omega_{LR}) - p^L(\omega_{LR}))^2 + (p(\omega_L) - p^L(\omega_L))^2 + ((p(\omega_N) - p^L(\omega_N))^2 \\
&+ ((1 - p(\omega_{LR}) - p(\omega_L) - p(\omega_N)) - (1 - p^L(\omega_{LR}) - p^L(\omega_L) - p^L(\omega_N)))^2
\end{aligned}$$

In the relative entropy case,

$$\begin{aligned}
&\phi(p^L(\omega_{LR}), p^L(\omega_L), p^L(\omega_N), p(\omega_{LR}), p(\omega_L), p(\omega_N)) \\
=&- p^L(\omega_{LR}) \log\left(\frac{p(\omega_{LR})}{p^L(\omega_{LR})}\right) - p^L(\omega_L) \log\left(\frac{p(\omega_L)}{p^L(\omega_L)}\right) - p^L(\omega_N) \log\left(\frac{p(\omega_N)}{p^L(\omega_N)}\right) \\
&- (1 - p^L(\omega_{LR}) - p^L(\omega_L) - p^L(\omega_N)) \log\left(\frac{(1 - p(\omega_{LR}) - p(\omega_L) - p(\omega_N))}{(1 - p^L(\omega_{LR}) - p^L(\omega_L) - p^L(\omega_N))}\right)
\end{aligned}$$

Note that, since $p^L(\omega_{LR}) = p^L(\omega_N)$, $\phi(p^L(\omega_{LR}), p^L(\omega_L), p^L(\omega_N), p(\omega_{LR}), p(\omega_L), p(\omega_N)) = \phi(p^L(\omega_{LR}), p^L(\omega_L), p^L(\omega_N), p(\omega_N), p(\omega_L), p(\omega_{LR}))$ for all p.

Let q minimise the distance from p^L among all p with $p(L) = 0.9 - \epsilon$. I.e. q minimises $\phi(p^L(\omega_{LR}), p^L(\omega_L), p^L(\omega_N), q(\omega_{LR}), q(\omega_L), q(\omega_N))$ among all p with $p(L) = 0.9 - \epsilon$. Hence, by the previous observation, q minimises $\phi(p^L(\omega_{LR}), p^L(\omega_L), p^L(\omega_N), q(\omega_N), q(\omega_L), q(\omega_{LR}))$ among all p with $p(L) = p(\omega_L) + p(\omega_{LR}) = 0.9 - \epsilon$. Define q' by $q'(\omega_{LR}) = q(\omega_N)$, $q'(\omega_L) = q(\omega_L)$, $q'(\omega_N) = q(\omega_{LR})$. By the previous observation, q' minimises $\phi(p^L(\omega_{LR}), p^L(\omega_L), p^L(\omega_N), q'(\omega_{LR}), q$ among all p with $p(R^c) = p(\omega_L) + p(\omega_N) = 0.9 - \epsilon$. So q' minimises the distance from p^L among all p with $p(R) = 0.1 + \epsilon$. By the previous observation, the distance between q and p^L is the same as the distance between q' and p^L, as required. □

Proof of Proposition 2. Take $o = 2\left(w_L^L \epsilon^2 + w_B^L (\max\{\epsilon - 0.81, 0\})^2\right)$. q, defined by $q(\omega_{LR}) = p^L(\omega_{LR}) - \max\{\epsilon - 0.81, 0\} = 0.09 - \max\{\epsilon - 0.81, 0\}$, $q(\omega_R) = p^L(\omega_R) + \max\{\epsilon - 0.81, 0\} = 0.01 + \max\{\epsilon - 0.81, 0\}$, $q(\omega_L) = p^L(\omega_L) - \epsilon = 0.81 - \min\{\epsilon, 0.81\}$ and $q(\omega_N) = p^L(\omega_N) + \min\{\epsilon, 0.81\} = 0.09 + \min\{\epsilon, 0.81\}$ is a probability measure over Ω. Moreover, $\sum_{j=\{L,R,B\}} 2w_j^L (q(j) - p^L(j))^2 = 2\left(w_L^L \epsilon^2 + w_B^L (\max\{\epsilon - 0.81, 0\})^2\right)$, so $q \in c^L(o)$. Since, for any q' with $q'(L) < 0.9 - \epsilon$, $\sum_{j=\{L,R,B\}} 2w_j^L (q'(j) -$

$p^L(j))^2 > 2\left(w_L^L\epsilon^2 + w_B^L\left(\max\{\epsilon - 0.81, 0\}\right)^2\right)$, such $q' \notin c^L(o)$, so $c^L(o) \subseteq \mathcal{L}_\epsilon$. For any $\delta \in [0, 0.9]$, consider q_δ defined by $q_\delta(\omega_{LR}) = p^L(\omega_{LR}) + \max\{0, \delta - 0.09\} = 0.09 + \max\{0, \delta - 0.09\}$, $q(\omega_R) = p^L(\omega_R) + \min\{\delta, 0.09\} = 0.01 + \min\{\delta, 0.09\}$, $q_\delta(\omega_L) = p^L(\omega_L) - \max\{0, \delta - 0.09\} = 0.81 - \max\{0, \delta - 0.09\}$ and $q_\delta(\omega_N) = p^L(\omega_N) - \min\{0.09, \delta\} = 0.09 - \min\{0.09, \delta\}$; this is clearly a probability measure. $\sum_{j=\{L,R,B\}} 2w_j^L(q_\delta(j) - p^L(j))^2 = 2\left(w_R^L\delta^2 + w_B^L\left(\max\{0, \delta - 0.09\}\right)^2\right)$. Noting that $w_R^L\epsilon^2 + w_B^L\left(\max\{0, \epsilon - 0.09\}\right)^2 < w_L^L\epsilon^2 + w_B^L\left(\max\{\epsilon - 0.81, 0\}\right)^2$ if and only if $w_B^L\frac{1}{\epsilon^2}\left(\left(\max\{0, \epsilon - 0.09\}\right)^2 - \left(\max\{\epsilon - 0.81, 0\}\right)^2\right) < w_L^L - w_R^L$, it is straightforward to check that this is the case for all $\epsilon \in [0, 0.9]$ whenever $0.8w_B^L = w_B^L\frac{0.81^2 - 0.09^2}{0.9^2} < w_L^L - w_R^L$. It follows that there exists $\delta > \epsilon$ with $\sum_{j=\{L,R,B\}} 2w_j^L(q_\delta(j) - p^L(j))^2 \leqslant 2\left(w_L^L\epsilon^2 + w_B^L\left(\max\{\epsilon - 0.81, 0\}\right)^2\right) = o$, so $c^L(o) \nsubseteq \mathcal{R}_\epsilon$, as required.

□

Proposition 3 is an immediate corollary of the characterisation of confidence aggregation in Eq. (2) (Proposition D.1) and the observation that the confidence ranking defined in Eq. (3) can equivalently expressed by the implausibility function in Eq. (4). The proof of Proposition 4 is provided in Appendix A.

D.3 Proofs of results in Section 5 and Appendix C

Proof of Proposition 6. The proof essentially follows that of Hill (2016, Proposition 3); given the differences in setup (e.g. using strict rather than weak preference as primitive), we include a full proof for completeness.

The implication from the representation to the axioms is straightforward. Consider the implication from the axioms to the representation, and let (c^{rep}, u, D) represent $\succ^{rep}$. For each $f \in \mathcal{A}$, let $\underline{x_f} \in \mathcal{A}^c$ be any $\succ^{rep}$-minimal element of the set of all $y \in \mathcal{A}^c$ such that, for all $x \in \mathcal{A}^c$, if $f \succ^{rep} x$, then $y \succ^{rep} x$. Consider any $f \in \mathcal{A}$. Noting, since $D(x) \leqslant D(f)$ for all $x \in \mathcal{A}^c$ and $f \in \mathcal{A}$, that $\max\{D(f), D(x)\} = D(f)$, it follows from the representation of $\succ^{rep}$ that $u(\underline{x_f}) = \min_{p \in c^{rep}(D(f))} \mathbb{E}_p u(f)$. By representation (9), for any $y \in \mathcal{A}^c$ with $y \prec^{rep} \underline{x_f}$, $f \succ^{rep} y$, whence $f \succ^{dm} y$ by C-Consistency. By the representation, $f \not\succ^{rep} \underline{x_f}$, whence $\underline{x_f} \succeq^{dm} f$ by Caution. Hence, by the continuity of $\succeq^{dm}$, $f \sim^{dm} \underline{x_f}$. Since this holds for all $f \in \mathcal{A}$, $\succeq^{dm}$ is represented by $V(f) = u(\underline{x_f}) = \min_{p \in c^{rep}(D(f))} \mathbb{E}_p u(f)$ as required. □

Proof of Proposition C.1. Let $\succeq$ be a strongly grounded variational preference represented by the variational functional V with u and γ (Eq. (C.1)). For each $f \in \mathcal{A} \backslash \mathcal{A}^c$, let

$$\mathcal{P}_f = \{p \in \Delta : V(f) = \mathbb{E}_p u(f) + \gamma(p)\}$$

i.e. the set of probability measures achieving the minimum in the variational representation for f. Let $\gamma_f = \min\{\gamma(p) : p \in \mathcal{P}_f\}$. Let $\mathcal{F}_f = \{g \in \mathcal{A} : \gamma_g \geqslant \gamma_f\}$.

Consider the set $\mathcal{C}_f \subseteq \Delta$ defined by:

$$\mathcal{C}_f = \bigcap_{g \in \mathcal{F}_f} \{p \in \Delta : \mathbb{E}_p u(g) \geqslant V(g)\}$$

Since it is an intersection of half-spaces, this is convex, closed set. We now establish several properties of it.

Claim D.7. For each $f \in \mathcal{A} \backslash \mathcal{A}^c$, $\mathcal{C}_f \neq \varnothing$.

Proof. Since $\succeq$ is strongly grounded, $\{p \in \Delta : \gamma(p) = 0\} \neq \varnothing$. For each $f \in \mathcal{A} \backslash \mathcal{A}^c$, by the variational representation, for each $g \in \mathcal{F}_f$, $\mathbb{E}_p u(g) \geqslant V(g)$ for each $p \in \Delta$ with $\gamma(p) = 0$. Hence $\mathcal{C}_f \supseteq \{p \in \Delta : \gamma(p) = 0\} \neq \varnothing$, so $\mathcal{C}_f$ is non-empty. □

Claim D.8. For each $f, g \in \mathcal{A} \backslash \mathcal{A}^c$, if $\gamma_f \geqslant \gamma_g$, then $\mathcal{C}_f \supseteq \mathcal{C}_g$.

Proof. Immediate from the definition of $\mathcal{C}_f$. □

Claim D.9. For each $f \in \mathcal{A} \backslash \mathcal{A}^c$, $\min_{p \in \mathcal{C}_f} \mathbb{E}_p u(f) = V(f)$.

Proof. As a point of notation, let $B(\Omega)$ be the space of real-valued Σ-measurable simple functions on Ω under the sup norm; $ba(\Omega)$ is the dual space of bounded and finitely additive set functions, under the total variation norm. $\geqslant$ is the standard ordering on $B(\Omega)$, and $\mathbf{1}$ the constant function taking value 1. Each act $f \in \mathcal{A}$ generates a function $u \circ f \in B(\Omega)$, as standard; let $B(\Omega)^{\mathcal{A}}$ be the image of $\mathcal{A}$ under u. For any $a \in B(\Omega)^{\mathcal{A}}$, we use $u^{-1}(a) \in \mathcal{A}$ as shorthand for an act mapped to a under u.[27] We reason using Fan's Theorem (Fan, 1956, Theorem 13; see also the proof of Chateauneuf (1991, Theorem 2)). It implies that, for a family $\{a_j\}_{j\in J}$ of elements in $B(\Omega)$ and corresponding real numbers $\{\alpha_j\}_{j\in J}$, the system:

$$\phi(a_j) \geqslant \alpha_j \qquad \forall j \in J \tag{D.5}$$

has a solution in $ba(\Omega)$ if and only if

$$\sup \sum_{j=1}^{n} \lambda_j \alpha_{k_j} < +\infty \tag{D.6}$$

when $n = 1, 2, 3, \ldots$, k_j, λ_j vary under the conditions $\lambda_j > 0$ for all $1 \leqslant j \leqslant n$ and $\| \sum_{j=1}^{n} \lambda_j a_{k_j} \| = 1$.

To establish the claim, it suffices to show that, for $f \in \mathcal{A}\backslash\mathcal{A}^c$, with $a_f = u \circ f$, there exists $\phi \in ba(\Omega)$ such that:[28]

$$\begin{aligned} \phi(a_f) \geqslant V(f), &\qquad \phi(-a_f) \geqslant -V(f), \\ \phi(\mathbf{1}) \geqslant 1, &\qquad \phi(-\mathbf{1}) \geqslant -1, \\ \phi(u \circ g) \geqslant V(g) &\quad \forall g \in \mathcal{F}_f, \\ \phi(a) \geqslant 0 &\quad \forall a \in B(\Omega)^{\mathcal{A}}, a \geqslant \mathbf{0} \end{aligned} \tag{D.7}$$

Note first, by Claim D.7, $\mathcal{C}_f$ is non-empty, so there is a solution for the set of inequalities (D.7'), obtained from (D.7) by removing the inequality $\phi(-a_f) \geqslant -V(f)$.

Clearly, the supremum in (D.6) for (D.7) is equal to the maximum of the supremum across all vectors $(\lambda_1, \ldots \lambda_n)$, $(a_f, -a_f, \mathbf{1}, -\mathbf{1}, u \circ g_{k_5}, \ldots u \circ g_{k_m}, a_{m+1}, \ldots, a_n)$ satisfying the conditions stated after (D.6) such that $\lambda_1 - \lambda_2 \geqslant 0$, and the supremum across all vectors such that $\lambda_1 - \lambda_2 < 0$. It suffices thus to show that each of these suprema is finite.

Consider a pair of vectors $(\lambda_1, \ldots \lambda_n)$, $(a_f, -a_f, \mathbf{1}, -\mathbf{1}, u \circ g_{k_5}, \ldots u \circ g_{k_m}, a_{m+1}, \ldots, a_n)$ involved in the supremum (D.6) for the system (D.7),

[27] All statements in this proof using this notation will hold for all members of the relevant inverse image.

[28] As a point of notation, $\mathbf{0}$ is the constant function taking value 0.

i.e. such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geqslant 0$, $\lambda_j > 0$ for all $5 \leqslant \lambda \leqslant n$, and

$$\left\| \lambda_1 a_f + \lambda_2(-a_f) + \lambda_3 \mathbf{1} + \lambda_4(-\mathbf{1}) + \sum_{j=5}^{m} \lambda_j (u \circ g_{k_j}) + \sum_{k=m+1}^{n} \lambda_k a_k \right\| = 1,$$

There is clearly a one-to-one map between the subset of such vectors with $\lambda_1 - \lambda_2 \geqslant 0$ and pairs $(\lambda_1 - \lambda_2, \lambda_3 \ldots \lambda_n)$, $(a_f, \mathbf{1}, -\mathbf{1}, u \circ g_{k_5}, \ldots u \circ g_{k_m}, a_{m+1}, \ldots, a_n)$ involved in the supremum (D.6) for (D.7'); moreover, the corresponding expressions in (D.6) are identical. Since, as noted above, this set of inequalities has a solution, by Fan's Theorem this supremum is finite, so the supremum (D.6) for the system (D.7) over all vectors with $\lambda_1 - \lambda_2 \geqslant 0$ is finite.

Now consider pairs of vectors $(\lambda_1, \ldots \lambda_n)$, $(a_f, -a_f, \mathbf{1}, -\mathbf{1}, u \circ g_{k_5}, \ldots u \circ g_{k_m}, a_{m+1}, \ldots, a_n)$ involved in the supremum (D.6) for the system (D.7), with $\lambda_1 - \lambda_2 < 0$. It follows from

$$\left\| \lambda_1 a_f + \lambda_2(-a_f) + \lambda_3 \mathbf{1} + \lambda_4(-\mathbf{1}) + \sum_{j=5}^{m} \lambda_j (u \circ g_{k_j}) + \sum_{k=m+1}^{n} \lambda_k a_k \right\| = 1,$$

that

$$(\lambda_1 - \lambda_2) a_f + (\lambda_3 - \lambda_4) \mathbf{1} + \sum_{j=5}^{m} \lambda_j (u \circ g_{k_j}) + \sum_{k=m+1}^{n} \lambda_k a_k \leqslant \mathbf{1}$$

This holds if and only if:

$$\frac{\lambda_3 - \lambda_4 - 1}{\lambda_2 - \lambda_1} \mathbf{1} + \sum_{j=5}^{m} \frac{\lambda_j}{\lambda_2 - \lambda_1} (u \circ g_{k_j}) + \sum_{k=m+1}^{n} \frac{\lambda_k}{\lambda_2 - \lambda_1} a_k \leqslant a_f$$

By the concavity and monotonicity (and normalisation) of the functional V, this implies that:

$$\frac{\lambda_3 - \lambda_4 - 1}{\lambda_2 - \lambda_1} + \sum_{j=5}^{m} \frac{\lambda_j}{\lambda_2 - \lambda_1} V(g_{k_j}) + \sum_{k=m+1}^{n} \frac{\lambda_k}{\lambda_2 - \lambda_1} V(u^{-1}(a_k)) \leqslant V(f)$$

Whence, since $a_k \geqslant 0$,

$$\frac{\lambda_3 - \lambda_4 - 1}{\lambda_2 - \lambda_1} + \sum_{j=5}^{m} \frac{\lambda_j}{\lambda_2 - \lambda_1} V(g_{k_j}) \leqslant V(f)$$

So

$$\lambda_1 V(f) + \lambda_2(-V(f)) + \lambda_3 - \lambda_4 + \sum_{j=5}^{m} \lambda_j V(g_{k_j}) \leqslant 1,$$

and hence the supremum (D.6) for (D.7) over all vectors with $\lambda_1 - \lambda_2 < 0$ is finite.

Hence, the supremum in (D.6) for (D.7) is finite. Thus, by Fan's Theorem, the set of equations (D.7) has a solution, i.e. there exists $p \in \mathcal{C}_f$ with $\mathbb{E}_p u(f) = V(f)$, as required. □

Let $O = [0, \sup_{p\in\Delta} \gamma(p))$, define $c(o) = \bigcap_{f:\gamma_f \geqslant o} \mathcal{C}_f$. By the definition and the previously noted facts, c is a convex, closed confidence ranking. Define $D : \mathcal{A} \to O$ by $D(f) = \gamma_f$. It remains to show that (c, D, u) represent $\succeq$ according to (10). This is immediate for constant acts, so consider $f \in \mathcal{A}\backslash\mathcal{A}^c$. By the definitions of c and D, and Claims D.8 and D.9:

$$\begin{aligned} \min_{p\in c(D(f))} \mathbb{E}_p u(f) &= \min_{\bigcap_{g\in\mathcal{F}_f}\{p\in\Delta:\mathbb{E}_p u(g)\geqslant V(g)\}} \mathbb{E}_p u(f) \\ &= V(f) \end{aligned}$$

as required. □

Proof of Proposition C.2. For the first clause, by the axiomatisation in Maccheroni et al. (2006), it clearly suffices to show that $\succeq^{dm}$ satisfies Weak C-Independence and Uncertainty Aversion (see Maccheroni et al., 2006, Section 3 for definitions).

Weak Certainty Independence Suppose that $f_\alpha x \succeq^{dm} g_\alpha x$, for $f, g \in \mathcal{A}$, $x, y \in \mathcal{A}^c$ and $\alpha \in (0, 1]$. Then, by the representation:

$$\min_{c(D(f_\alpha x))} \mathbb{E}_p u(f) \geqslant \min_{c(D(g_\alpha x))} \mathbb{E}_p u(g)$$

If $f, g \notin \mathcal{A}^c$, then by Assumption C.1, $D(f_\alpha y) = D(f_\alpha x)$ and $D(g_\alpha y) = D(g_\alpha x)$ whence the previous inequality holds for $D(f_\alpha y)$ and $D(g_\alpha y)$, so $f_\alpha y \succeq^{dm} g_\alpha y$, as required. Similar arguments establish the other cases.

Uncertainty Aversion Suppose that $f \sim^{dm} g \sim^{dm} x$ for $x \in \mathcal{A}^c$. By the representation, for every $y \in \mathcal{A}^c$ with $y \prec^{dm} x$, $f \succ^{dm} y$ and $g \succ^{dm} y$, so by Assumption C.2, $D(f_\alpha g)$ is such that $\min_{p \in c(D(f_\alpha g))} \mathbb{E}_p u(f_\alpha g) > u(y)$, whence $f_\alpha g \succ^{dm} y$. It follows from continuity of $\succ^{dm}$ that $f_\alpha g \succeq^{dm} x$, as required.

For converse implication, suppose that c is strictly increasing.

Assumption C.1 Consider $f \in \mathcal{A} \backslash \mathcal{A}^c$, $x, y \in \mathcal{A}^c$ and $\alpha \in (0, 1]$, and let $z \in \mathcal{A}^c$ be such that $f_\alpha x \sim^{dm} z_\alpha x$ (such z exists by standard arguments). Since $\succeq^{dm}$ satisfies Weak C-Independence, $f_\alpha y \sim^{dm} z_\alpha y$. Plugging this into representation (10), it follows that:

$$\min_{c(D(f_\alpha x))} \mathbb{E}_p u(f) = \min_{c(D(f_\alpha y))} \mathbb{E}_p u(f)$$

which, given that c is strictly increasing and $f \notin \mathcal{A}^c$, can only hold if $D(f_\alpha x) = D(f_\alpha y)$ as required.

Assumption C.2 Consider $f, g \in \mathcal{A}$, $\alpha \in [0, 1]$, $x \in \mathcal{A}^c$ and $o \in O$, and suppose that $f \succ^{dm} x$, $g \succ^{dm} x$ and $\min_{p \in c^J(o)} \mathbb{E}_p u(f_\alpha g) \leqslant u(x)$. Since variational preferences are concave, it follows that $f_\alpha g \succ^{dm} x$, whence, by the representation (10), $\min_{p \in c(D(f_\alpha g))} \mathbb{E}_p u(f_\alpha g) > u(x)$. Since c is increasing, it follows that $o > D(f_\alpha g)$, as required. □

D.4 Proofs of results in Section 6

Proof of Theorem 3. Fix E and ρ_E, and define $c^\rho : O \to 2^\Delta \backslash \varnothing$ by $c^\rho(o) = \{p \in \Delta : p(E) \geqslant \rho_E(o)\}$. Clearly, for any confidence ranking c, $c|\rho_E = \overline{\Phi}$ for $\Phi(o) = (c(o) \cap c^\rho(o))_E$, whenever $c(o) \cap c^\rho(o) \neq \varnothing$ (and it is undefined otherwise).

By Definition 6 and the definition of $F_\otimes$, for every $o \in O$ such that $\left(\bigcup_{\mathbf{o}:\otimes\mathbf{o} \leqslant o} \bigcap_i c^i(o_i)\right) \cap c^\rho(o) \neq \varnothing$

$$
\begin{aligned}
&F_{\otimes}(c_1, \ldots, c_n)|\rho_E(o) \\
&= \left(\left(\bigcup_{\mathbf{o}:\otimes\mathbf{o}\leqslant o} \bigcap_i c^i(o_i) \right) \cap c^{\rho}(o) \right)_E \\
&= \left(\bigcup_{\mathbf{o}:\otimes\mathbf{o}\leqslant o} \bigcap_i \big(c^i(o_i) \cap c^{\rho}(o) \big) \right)_E \\
&= \left(\bigcup_{\mathbf{o}:\otimes\mathbf{o}\leqslant o} \bigcap_i \big(c^i(o_i) \cap c^{\rho}(o) \big)_E \right) \\
&= F_{\otimes}(c_1|\rho_E, \ldots, c_1|\rho_E)(o)
\end{aligned}
$$

as required. □