Limit Order Markets

- All market participants have the choice between submitting limit orders and market orders.
- Quotes must belong to a grid with a minimum tick.
- Cumulative depth on the bid side at price $b := \text{maximum amount of share one can sell for at least } b$.
- Cumulative depth on the ask side at price $a := \text{maximum amount of share one can buy for at most } b$. 

![Order Book Example](http://data.inetats.com - INET Java Bo...)

**INET home** system stats help

**INET**

**GOOG**

**LAST MATCH**

<table>
<thead>
<tr>
<th>Price</th>
<th>384.9000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>15:18:56</td>
</tr>
</tbody>
</table>

**TODAY'S ACTIVITY**

<table>
<thead>
<tr>
<th>Orders</th>
<th>1,295,622</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume</td>
<td>2,791,809</td>
</tr>
</tbody>
</table>

**BUY ORDERS**

<table>
<thead>
<tr>
<th>SHARES</th>
<th>PRICE</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>384.8200</td>
</tr>
<tr>
<td>100</td>
<td>384.8200</td>
</tr>
<tr>
<td>100</td>
<td>384.8100</td>
</tr>
<tr>
<td>300</td>
<td>384.8100</td>
</tr>
<tr>
<td>100</td>
<td>384.8000</td>
</tr>
<tr>
<td>500</td>
<td>384.7900</td>
</tr>
<tr>
<td>200</td>
<td>384.7700</td>
</tr>
</tbody>
</table>

**SELL ORDERS**

<table>
<thead>
<tr>
<th>SHARES</th>
<th>PRICE</th>
</tr>
</thead>
<tbody>
<tr>
<td>93</td>
<td>384.9500</td>
</tr>
<tr>
<td>100</td>
<td>385.0300</td>
</tr>
<tr>
<td>100</td>
<td>385.0600</td>
</tr>
<tr>
<td>100</td>
<td>385.0700</td>
</tr>
<tr>
<td>200</td>
<td>385.0900</td>
</tr>
<tr>
<td>100</td>
<td>385.1800</td>
</tr>
<tr>
<td>100</td>
<td>385.2400</td>
</tr>
</tbody>
</table>
A simple model of limit order market

- Impatient traders (IT): submit market orders. 
  \( F(Q) \): (exogenous) probability the quantity \( \tilde{Q} \) demanded by IT is less than \( Q \). (\( Q > 0 \) meaning trader buys)

- Patient risk neutral traders (PT): submit limit orders.

- Limit order quotes must belong to a grid \( P \) with tick \( \epsilon \):
  \[ P = \{ p_k \}_{k=0,1,...}, \ p_k = \epsilon k \]

- Limit order submission has a cost \( c \).

- Symmetric information.
Timing

1. PT submit their limit orders.
2. IT submit their market orders.
3. Limit orders are executed on the basis of price priority (and time priority).
4. The asset value $\tilde{v}$ is realized.
Let $Y_k$ be the cumulative depth of the bid side for bid $b = p_k$

What is the expected marginal profit for the PT offering to buy the $Y_k$-th share?

$$\Pi^{bid}_k(Y_k) = \Pr(\tilde{Q} \leq -Y_k)(E[\tilde{v}|\tilde{Q} \leq -Y_k] - b) - c$$

Let $Y_j$ be the cumulative depth of the ask side for ask $a = p_j$

What is the expected marginal profit for the PT offering to sell the $Y_j$-th share?

$$\Pi^{ask}_j(Y_j) = \Pr(\tilde{Q} \geq Y_j)(a - E[\tilde{v}|\tilde{Q} \geq Y_j]) - c$$
What is the equilibrium depth of the market for a given level of price?

- PT’s marginal utility from offering to buy one extra share at price $p_k$ is nil:

$$\Pr(\tilde{Q} \leq -Y_k) = \frac{c}{E[\tilde{v}] - p_k}$$

- PT’s marginal utility from offering to sell one extra share at price $p_j$ is nil:

$$\Pr(\tilde{Q} \geq Y_j) = \frac{c}{p_j - E[\tilde{v}]}$$
Example: exponential distribution of market orders

Suppose that $F'[Q] = \frac{1}{2} \theta e^{-\theta |Q|}$

average order size $= \frac{1}{\theta}$

$\Pr(\tilde{Q} \leq Y_k) = \frac{1}{2} e^{\theta Y_k}, Y_k \leq 0$

$\Pr(\tilde{Q} \geq Y_j) = \frac{1}{2} e^{-\theta Y_j}, Y_j > 0$

Implying that the equilibrium bid and ask depths are

$Y_k = \frac{1}{\theta} \ln \left( \frac{E[v] - p_k}{2c} \right)$

$Y_j = \frac{1}{\theta} \ln \left( \frac{p_j - E[v]}{2c} \right)$
Example: exponential distribution of market orders

\[ Y_k = \frac{1}{\theta} \ln \left( \frac{E[v] - p_k}{2c} \right) \]

\[ Y_j = \frac{1}{\theta} \ln \left( \frac{p_j - E[v]}{2c} \right) \]

Implications:

- Depth increases with the average order size \(1/\theta\) and decreases with limit order cost \(c\).
- Minimum bid-ask spread is \(4c\) even without asymmetric information and with infinitesimal small unit of trade.
\( \tilde{v} \in \{ v_1, v_2 \} \)

- Price tick size \( \epsilon \rightarrow 0 \).
- Round lot \( q > 0 \) shares.
- A fraction \( 1 - \mu \) of IT are liquidity traders (LIT):
  \[
  \Pr(\text{LIT sells } 2q) = \Pr(\text{LIT sells } 1q) = \frac{1}{4}
  
  \Pr(\text{LIT buys } 1q) = \Pr(\text{LIT buys } 2q) = \frac{1}{4}
  \]

- A fraction \( \mu \) of IT know \( \tilde{v} \) (IIT):
  - IIT buys all that sells for less than \( \tilde{v} \)
  - IIT sells all that trades for more than \( \tilde{v} \)

- PT are risk neutral and not informed.
PT’s zero profit condition

bid side: \( \Pr(\tilde{Q} \leq -Y_k)(E[\tilde{v} | \tilde{Q} \leq -Y_k] - b) - c = 0 \)

ask side: \( \Pr(\tilde{Q} \geq Y_j)(a - E[\tilde{v} | \tilde{Q} \geq Y_j]) - c = 0 \)

IIT equilibrium behaviour:

- IIT buys all that sells for less than \( \tilde{v} \)
- IIT sells all that trades for more than \( \tilde{v} \)
Informed IT: bid side equilibrium

Let $b \in [v_1, v_2]$

$$
\Pi^{bid}(-3) = q \left( \pi (v_2 - b) * 0 + (1 - \pi)(v_1 - b)\mu \right) - c
$$

$$
\Pi^{bid}(-2) = q \left( \pi (v_2 - b) \frac{1 - \mu}{4} + (1 - \pi)(v_1 - b) \left( \frac{1 - \mu}{4} + \mu \right) \right) - c
$$

$$
\Pi^{bid}(-1) = q \left( \pi (v_2 - b) \frac{1 - \mu}{2} + (1 - \pi)(v_1 - b) \left( \frac{1 - \mu}{2} + \mu \right) \right) - c
$$

Let

$$
b(Y_k) = E[\tilde{v} | \tilde{Q} \leq -Y_k] - \frac{c}{q Pr(\tilde{Q} \leq -Y_k)}
$$

Then $E[v] > b(1) > b(2) > v_1$ and the bid side book consists of

- A buy limit order for $q$ share at price $b(1)$
- A buy limit order for $q$ share at price $b(2)$
- Buy limit orders for any amount of one share at price $V_1$
Informed IT: ask side equilibrium

Let

\[ a(Y_k) = E[\tilde{v}|\tilde{Q} \geq Y_k] + \frac{c}{q \Pr(\tilde{Q} \geq Y_k)} \]

Then \( E[v] < a(1) < a(2) < v_2 \) and the ask side book consists of

- A sell limit orders for \( q \) share at price \( a(1) \)
- A sell limit orders for \( q \) share at price \( a(2) \)
- Sell limit orders for any amount of one share at price \( V_2 \)
The depth of the LOB decreases with the informativeness of the order flow:
- \( b(Y) \) is decreasing in \( \mu \)
- \( a(Y) \) is increasing in \( \mu \)

The bid depth of the LOB decreases with the ex-ante uncertainty \( \pi \).

The ask depth of the LOB decreases with the ex-ante uncertainty \( \pi \).

Minimum bid ask speed is positive even when \( c = 0 \) even for arbitrarily very small size of the round lot \( q \).
Choice between limit and market orders

- **Market order**
  - (+) Instantaneous execution. No uncertainty on the price.
  - (-) Have to accept the price present in the LOB.

- **Limit order**
  - (+) Set a better trading price.
  - (-) Execution risk. Risk of being adverse selected.
The choice between limit and market orders a simple model

- Two population of traders
  - High value traders: $v + L$
  - Low value traders: $v - L$
  - Trading quantities $\{-1, 0, 1\}$

- **time $t$**
  1. A trader arrives with probability $\tau$;
  2. Conditionally on arriving, the trader is he ‘high value’ with probability $1/2$;
  3. the trader submit a market or a limit order;
  4. The market order is executed against previous trader’s limit order (if any);
  5. non-executed limit orders are cancelled.
Claim:

- A high value trader submits a buy limit order at $\hat{b}$ if there is no sell limit order or the sell limit order is at price larger than $\hat{a}$.

- A high value type submits a buy market order if there is a sell limit order at price not larger than $\hat{a}$.

- A low value trader submits a sell limit order at $\hat{a}$ if there is no buy limit order or the buy limit order is at price lower than $\hat{b}$.

- A low value type submits a sell market order if there is a buy limit order at price not smaller than $\hat{a}$.
The make or take decision

- High value type facing a sell limit order at price \( a \):
  Profit from a buy market order:
  \[ v + L - a \]
  Profit from a buy limit order:
  \[ (v + L - b) \frac{\tau}{2} \mathbb{1}_{\{b \geq \hat{b}\}} \]

- Low value type facing a buy limit order at price \( b \):
  Profit from a sell market order:
  \[ b - v + L \]
  Profit from a sell limit market order:
  \[ (a - v + L) \frac{\tau}{2} \mathbb{1}_{\{a \leq \hat{b}\}} \]
Equilibrium bid and ask

\[ v + L - \hat{a} = (v + L - \hat{b}) \frac{\tau}{2} \]

\[ \hat{b} - v + L = (\hat{a} - v + L) \frac{\tau}{2} \]

Implying

\[ \hat{b} = v + L - \frac{4L}{2 + \tau} < v \]

\[ \hat{a} = v - L + \frac{4L}{2 + \tau} > v \]

\[ \hat{a} - \hat{b} = 2L \left( \frac{2 - \tau}{2 + \tau} \right) \]

- The higher \( v \) the higher the quotes
- Bid ask spread decreases with the execution risk \( 1 - \frac{\tau}{2} \) and increases with the difference in private values \( 2L \).
Other possible complications.

- Choice between take or make with asymmetrically informed participants.
- Choice between take or make as a function of the status of the book.
- Shocks to the fundamentals.
- Dynamic models.