Problem 1 Consider an auction in a private value framework: the $n > 1$ bidders valuation are independently and identically distributed on $[0, 1]$ with cumulative distribution F. The auction rules are as follows: Bids are in virtual coins. Virtual coins must be purchased before the auction at price of $c > 0$ Euros per coin. The highest bidder wins and pays an amount of euros equal to $e > 0$ euros times his bid in coins. Formally,

1. Each bidder i choses the number of coins x_i he wants to purchase and immediately pays $x_i c$ to the auctioneer.
2. Each bidder i submit a bid $b_i \leq x_i$
3. The highest bidder wins the object and pays $b_i e$ euros to the seller.

Consider a symmetric equilibrium where the hypothesis of the Revenue Equivalence Theorem are satisfied. Let v_i denote bidder i's valuation for the object.

a) Determine the equilibrium level of the coin-purchase function $x(v_i)$ and of the bidding function $b(v_i)$.

b) Suppose F is uniform on $[0, 1]$. If the number n of bidders increases, then which of the following statement is correct?

- All bidders will bid more aggressively.
- All bidders will bid less aggressively.
- The way a bidder reacts to an increase in n depends on his valuation v.

Explain.

Problem 2

Consider a quote driven centralized market (à la Glosten and Milgrom (1985)). A risky assets is sequentially exchanged for a risk-less asset. In every period risk neutral market makers set their bid and ask quotes so that their expected profit from buying or selling is equal to 0. Then a trader comes to the market, decides whether to buy, sell or not trade and leaves the market. Trading quantities are normalized to one unit of the asset per trade.

Let $\tilde{W}$ be the fundamental value of the asset. Assume that

- $\tilde{W} = \tilde{V} + \tilde{e}$;
- $\tilde{V} \in \{V_1, V_2\}$, $V_1 < V_2$, and $\Pr(\tilde{V} = V_2) = \pi$;
- $E[\tilde{e} = 0]$ and $Var[\tilde{e}] = \sigma^2 > 0$;
- $\tilde{V}$ and $\tilde{e}$ are independently distributed.

Market participants are:

- A number $n > 1$ of market-makers: Market makers are risk neutral, they have no private information about the true values of $\tilde{V}$ and $\tilde{e}$.
- A mass of traders: They are privately informed about the true value of $\tilde{V}$ but have no private information about $\tilde{e}$. They are mean-variance investors with risk aversion γ. For example, a trader's expected utility from buying 1 unit of the assets at price a is

$$U(I+1, a, \tilde{V}) := E[\tilde{W}(I+1) - a|\tilde{V}] - \frac{\gamma}{2} Var[\tilde{W}(I+1) - a|\tilde{V}]$$

$$= \tilde{V}(I+1) - a - \frac{\gamma}{2}(I+1)^2\sigma^2,$$

where $\tilde{V} \in \{V_1, V_2\}$ is known by the trader, I is that trader's initial initial inventory of the asset and γ is the trader's risk aversion.

Let $IB(\tilde{V}, a)$ denote the set of levels of inventory I such that a trader would prefer buying at price a rather than not trading:

$$IB(\tilde{V}, a) := \{I \in \mathbf{R} \text{ such that } U(I+1, a, \tilde{V}) > U(I, 0, \tilde{V})\}$$

Similarly let $IS(\tilde{V}, b)$ denote the set of level of inventory I such that a trader would prefer selling at price b rather than not trading:

$$IS(\tilde{V}, b) := \{I \in \mathbf{R} \text{ such that } U(I-1, -b, \tilde{V}) > U(I, 0, \tilde{V})\}$$

NB: Only questions 1.A) and 1.E) require some calculations.

Question 1.A: Determine $IB(V_1, a)$, $IB(V_2, a)$, $IS(V_1, a)$, $IS(V_2, a)$.

Question 1.B: Is it possible that in equilibrium market makers set quotes bigger than V_2 or smaller than V_1? Briefly explain.

Question 1.C: Is it possible that in equilibrium a trader decides to buy from market makers even if he knows that $\tilde{V} = V_1$? Briefly explain.

Let $F(x)$ be the probability that the trader coming to the market has an inventory smaller than x. Fix $A > 1$ and assume that $F(x) = 0$ for $x < -A$, $F(x) = \frac{x+A}{2A}$ for $x \in [-A, A]$, and $F(x) = 1$ for $x > A$.

Let $\rho := (A - 1/2)\gamma\sigma^2$, $E[\tilde{V}] = \pi V_2 + (1-\pi)V_1$ and $Var[\tilde{V}] = \pi(1-\pi)(V_2 - V_1)^2$.

Question 1.D: How would you show that the it is an equilibrium for market makers to set ask and bid quotes equal to :

$$a(\pi) := E[\tilde{V}] + \frac{\rho}{2} - \sqrt{\frac{\rho^2}{4} - Var[\tilde{V}]}$$

$$b(\pi) := E[\tilde{V}] - \frac{\rho}{2} + \sqrt{\frac{\rho^2}{4} - Var[\tilde{V}]}$$

You need to provide the exact equations that would lead to these ask and bid quotes but you do not have to algebraically solve these equations.

Question 1.E: How is the bid ask spread affected by an increase of γ or an increase of σ^2? Please provide both a formal proof and the economic intuition behind your findings.

Question 1.F: In the long run, would you expect the trading quotes to converge to $\tilde{V}$, to $\tilde{W}$, or to neither $\tilde{V}$nor $\tilde{W}$? Briefly exaplin.

Problem 3

The purpose of this exercise is to see how a simple extension of the Glosten Milgrom (1985) model can lead market makers to set simultaneously different bid and ask quotes.

Consider a quote driven centralized market (à la Glosten and Milgrom (1985)). A risky assets is sequentially exchanged for a risk-less asset.

Market participants are: A number $n > 3$ of risk neutral market-makers, a mass of liquidity traders and a mass of risk neutral informed traders. Let $\tilde{v} \in \{0, 1\}$ be the fundamental value of the risky asset. Market makers have no private information regarding $\tilde{v}$ and let $\pi = 1/2$ denote their initial belief that $\tilde{v} = 1$. Informed traders know the value of $\tilde{v}$.

In every period t, first each market maker sets his bid and ask quotes at which he stands ready to buy or sell exactly one unit. Second, a new trader comes to the market observes the quotes and submits a market order for the desired quantity $x \in \{-3, -2, -1, 0, 1, 2, 3\}$. Third, the trader's market order is instantaneously executed against the best quotes of market makers. For example, a trader willing to sell 2 units will sell his first unit to the market maker proposing the highest bid price and the second unit to the market maker proposing the second highest bid. Fourth, the trader leaves the market so that a trader can trade at most once.

There are four groups of liquidity traders: those who wont to buy 1 unit of the asset, those who want to buy 2 units of the asset, those who wont to sell 1 unit of the asset, those who want to sell 2 units of the asset. The mass of liquidity trader for each group is $\mu_1/2$, $\mu_2/2$, $\mu_1/2$ and $\mu_2/2$, respectively, with $0 < \mu_1 + \mu_2 < 1$.

An informed trader will trade the quantity $x_I \in \{-3, -2, -1, 0, 1, 2, 3\}$ maximizing his trading profit given his information and market makers' quotes. The mass of informed traders is $1 - \mu_1 - \mu_2$.

Let consider an equilibrium satisfying the following properties:

1. Each market maker sets a bid and an ask quote. For each quote he is willing to trade exactly 1 unit of the asset.

2. Each market maker sets quotes so that his expected profit conditional on trading is nil.

3. The resulting quotes-size book is as follows:

Bid Quotes	Size	Ask Quotes	Size
b_1	1	a_1	1
b_2	1	a_2	1
b_3	$n-2$	a_3	$n-2$

with $0 \leq b_3 < b_2 < b_1 < \pi < a_1 < a_2 < a_3 \leq 1$. Where (b_1, a_1) are the bid and ask quotes set by market maker 1, (b_2, a_2) are the bid and ask quotes set by market maker 2 and all other market makers set their bid and ask at (b_3, a_3).

Note that because $n > 3$, an informed trader can actually choose the traded quantity in $\{-3, -2, -1, 0, 1, 2, 3\}$.

Questions

1. What are the optimal market orders of an informed trader in this equilibrium?

2. Consider market makers 1, 2 and 3.

   (a) For each one of these market makers compute the equilibrium ex-ante probability that he has to buy one asset. What are these probability for selling?

   (b) For each one of these market makers compute the equilibrium probability that he has to buy one asset conditional on $\tilde{v} = 1$. What are these probability when conditioning on $\tilde{v} = 0$?

3. What are (b_1, a_1), (b_2, a_2) and (b_3, a_3)?

4. Let $\pi(x_1)$ be the expected value of the asset conditional on the time 1 trader market order x_1. What can you say about the relation between $\pi(x_1) - \pi$ and x_1?