Economic Theory **28**, 331–355 (2006)
DOI: 10.1007/s00199-005-0634-5

Economic Theory

Ascending auctions for multiple objects: the case for the Japanese design*

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Received: September 2, 2004; revised version: April 20, 2005

Summary. We consider two ascending auctions for multiple objects, namely, an English and a Japanese auction, and derive a perfect Bayesian equilibrium of the Japanese auction by exploiting its strategic equivalence with the survival auction, which consists of a finite sequence of sealed-bid auctions. Thus an equilibrium of a continuous time game is derived by means of backward induction in finitely many steps. We then show that all equilibria of the Japanese auction induce equilibria of the English auction, but that many collusive or signaling equilibria of the English auction do not have a counterpart in the Japanese auction.

Keywords and Phrases: Multi-unit auctions, Ascending auctions, FCC auctions, Complementarities, Collusion, Signaling.

JEL Classification Numbers: C72, D44.

* We are indebted to Philippe Jehiel for useful discussions and to Nicolas Vieille for suggestions in the proof of Proposition 2; we also thank seminar audiences in Athens, Basel, Beer-Sheva, Berlin, Brussels, Cambridge, Edinburgh, Exeter, Lausanne, Lisbon, London, Louvain-la-Neuve, Namur and Tel Aviv.
Germano acknowledges financial support from Euopean Commission, TMR Network Grant ERBFM-RXCT0055, "Cooperation and Information" as well from the Spanish Ministry of Science and Technology, Grants SEC2001-0792, SEJ2004-06319, and in form of a Ramon y Cajal Fellowship. Lovo is member of GREGHEC, unité CNRS, FRE-2810. The support of the Economic and Social Research Council (ESRC) is also gratefully acknowledged. The work was part of the programme of the ESRC Research for Economic Learning and Social Evolution.
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1 Introduction

Since the first series of spectrum auctions held by the Federal Communications Commission (FCC) in the United States, academics and policymakers alike have pointed out at least three advantages of the FCC auction rules: they ensure a transparent bidding process, they enable extensive information revelation of bidders' valuations, and they allow bidders to build fairly efficient aggregations of licenses.[1] At the same time, an important drawback that has been documented in the literature is the large potential for signaling and collusion, with corresponding negative consequences for efficiency and revenue.[2]

In this paper we consider two basic auction mechanisms: the Japanese Auction for Multiple Objects (JAMO) and the Simultaneous English Auction for Multiple Objects (SEAMO), which is very close to the actual FCC auctions. We show that the JAMO is much more immune to collusive and signaling equilibria than the SEAMO. Both auctions are simultaneous ascending auctions, thus both ensure a transparent bidding process, enabling extensive information revelation, and efficient aggregation of objects. However, unlike the SEAMO and the FCC auctions, in the JAMO, prices are raised directly by the auctioneer, and closing is not simultaneous but rather license-by-license. These two basic differences eliminate many unwanted collusive equilibria of the SEAMO.

More specifically, Brusco and Lopomo (2002) construct multiple collusive equilibria for the SEAMO, where each bidder signals her most preferred item to competitors during early phases of the auction. The goal of the signaling strategy is to "split the market" and keep prices low. Thus the openness and simultaneity of the SEAMO, while allowing for transparent bidding, also provides bidders with effective communication devices that can be exploited to achieve collusive outcomes. The JAMO, on the other hand, by imposing a strong activity rule (bidders have no control over the pace at which prices rise) and license-by-license closing, rules out communication devices among bidders that might be used to achieve tacit collusion, while preserving the positive features of the ascending bid process. Indeed Albano et al. (2001), within an example with 2 objects and 4 bidders, show that the JAMO obtains close to ex-post efficiency with higher revenues than the revenue-maximizing ex-post efficient mechanism (Vickrey-Clarke-Groves mechanism), and that it dominates both the sequential and the one-shot simultaneous auctions in terms of ex-ante efficiency. Branco (1997, 2001) obtains related results in a somewhat different framework.

In order to capture some of the most salient features of actual FCC auction environments, we consider a framework in which two licenses are auctioned by the seller to two different sets of participants: unit and bundle bidders. Unit bidders are interested in one license only, bundle bidders are interested in both. We also

[1] See e.g., McAfee and McMillan (1996), Cramton (1997, 1998), Milgrom (1998), Cramton and Schwartz (1999, 2000), Klemperer (2001).

[2] Cramton and Schwartz (1999, 2000) report on bidding phases of the FCC which illustrate many of the communication and coordination devices tacitly used in practice by bidders; Klemperer (2001) provides further evidence and discussion, also relating to the recent European UMTS auctions; Salmon (2004) contains a survey of collusive equilibria in ascending auctions.

assume that there exist positive synergies or complementarities between licenses. These synergies may arise, for instance, from the saving of infrastructure costs whenever the two licenses correspond to two neighboring regions.

We show that every perfect Bayesian equilibrium (PBE) of the JAMO induces a PBE in the SEAMO, while the reverse is not true. This result implies that the set of equilibria of the JAMO is strictly smaller than that of the SEAMO. The rules of the JAMO eliminate many (unwanted) collusive or signaling equilibria that are equilibria of the SEAMO. In particular, jump bid equilibria constructed in Gunderson and Wang (1998) and collusive equilibria constructed in Engelbrecht-Wiggans and Kahn (1998) and Brusco and Lopomo (2002) are not equilibria of the JAMO.

Knowing that most signaling and collusive equilibria of the SEAMO do not have a counterpart in the JAMO, leaves the issue of the optimal bidding behavior in the latter mechanism open. This paper also provides a novel approach to characterize a "competitive" (symmetric) perfect Bayesian equilibrium (PBE) of the JAMO. While unit bidders' optimal behavior simply corresponds to that in a standard one-object second-price auction, the derivation of the bundle bidders' optimal strategies in a framework in which they have both *private* stand-alone values for the objects and a *common* value given by the synergy is by far less trivial.

Specifically, we adopt an indirect but constructive approach. We first extend to the two-object case the strategic equivalence between the JAMO and the Survival Auction (SA). The SA was used by Fujishima et al. (1999) who proved the strategic equivalence with the Japanese Auction in the one-object case. The SA consists in a finite sequence of sealed-bid auctions. At each round the auctioneer announces only the lowest bid on both licenses, and the identity of the "losing" bidder(s), that is, the one(s) who submitted the lowest bid(s). In the following round, the losing bidder(s) of the previous round are not allowed to bid on the object(s) on which they had submitted the losing bid(s), and the surviving bidders are allowed to submit new sealed-bid offers provided that they are not less than the losing bid(s) of the previous round. Bundle bidders' optimal behavior in the SA can be characterized by using backward induction. Starting from the terminal nodes of the game tree, we can reconstruct the bidder's continuation payoffs at each decision node of the game. Thus we derive optimal bids in the SA which in turn translate to optimal exiting times in the JAMO.

The remainder of the paper is organized as follows. In Section 2, we describe the actual auction rules and in Section 3 we construct our indirect approach to characterize a PBE of the JAMO. In Section 4, we consider a series of collusive and signaling equilibria and show that many equilibria of the SEAMO have no counterpart in the JAMO. Section 5 concludes with directions for future research. The proof of Proposition 2 is contained in the Appendix.

2 Three ascending auctions

2.1 The framework

Throughout the paper we work with a framework close to the ones of Krishna and Rosenthal (1996) and Brusco and Lopomo (2002). Two objects are auctioned to a set of participants of two types: M bundle bidders who are interested in both objects and N_k unit bidders who are interested in only one of the two objects, $k = 1, 2$. Both bundle and unit bidders draw their values independently from some smooth distribution F with positive density f, both defined over $[0, 1]$. Let v_k and u_k denote the value of object $k = 1, 2$ to a bundle and to a unit bidder respectively. The value of the bundle v_B to a bundle bidder is greater or equal than the sum of stand-alone values, that is,

$$v_B = v_1 + v_2 + \alpha,$$

where $\alpha \geq 0$ is commonly known and coincides across all bundle bidders. The nature of bidders, bundle and unit, is also commonly known.[3]

We further restrict the analysis to the following cases: (i) $v_1 = v_2 \in [0, 1]$ and $\alpha \geq 0$; (ii) $v_1, v_2 \in [0, 1]$ and $\alpha = 0$; (iii) $v_1, v_2 \in [0, 1]$ and $\alpha > 1$; Krishna and Rosenthal (1996) consider case (i); Brusco and Lopomo (2002) consider cases (ii) and (iii). We will refer to these three cases throughout the paper.

Finally, we introduce some notation that will be used later. Let $F(z \mid t) = \frac{F(z)-F(t)}{1-F(t)}$ be the distribution function of a unit bidder's valuation given that his valuation is at least t and $f(z|t) = F'(z|t)$. Let $F_N(\cdot \mid t)$ denote the distribution function of the *lowest* valuation among N unit bidders given that their valuations for the object are at least t, and let $f_N(\cdot \mid t) = F'_N(\cdot \mid t)$. Since unit bidders' valuations are i.i.d. random variables with distribution function $F(\cdot)$, we can write $F_N(z \mid t) = 1 - (1 - F(z \mid t))^N$.

2.2 Auction rules

The two main auction mechanisms we consider (JAMO and SEAMO) are more or less simplified versions of the simultaneous ascending auctions used by the FCC for the sale of spectrum licenses in the US. The third mechanism (SA) is equivalent to the first (JAMO) and is used mainly to simplify some of the analysis. We briefly describe the rules. All auctions have in common a tie-breaking rule that assigns the object with equal probability; also, we assume it is specified before the auction begins on which objects the different bidders are going to bid. In particular, we assume throughout the paper that bidders bid only on objects they value, that is, we assume local bidders bid on one object only. This always happens if bidders have to pay a participation fee proportional to the number of objects they want to bid for.

[3] The fact that a bidder with $v_1 = 0$ and $v_2 > 0$ qualifies as a bundle bidder when $\alpha = 0$ is a degenerate case; it may be worth stressing that what distinguishes bundle from unit bidders is that bundle bidders have a potential (in the eyes of other bidders) for obtaining a positive value from each object, besides the typically positive complementarity.

JAMO: Prices start from zero for all objects and are simultaneously and continuously increased on all objects until only one agent is left on a given object, in which case prices on that object stop and continue to rise on the remaining auctions. Once an agent has dropped from a given object, the exit is irrevocable. The last agent receives the object at the price at which the auction stopped. Whenever an agent exits one object, the clock (price) temporarily stops on both objects giving the opportunity to other bidders to exit at the same price. If all active bidders exit simultaneously on one object, then the object will be allocated randomly among those bidders that exit after the price has stopped. The number and the identity of agents active on any object is publicly known at any given time. The overall auction ends when all agents but one have dropped out from all objects. We refer to this mechanism as the Japanese auction for multiple objects (JAMO); some also refer to it as the English clock auction.

SEAMO: The auction proceeds in rounds. At each round, $n = 1, 2, ..$, each bidder submits a vector of bids where bids for single objects are taken from the set $\{\emptyset\} \cup (b^k(n-1), +\infty)$, where $\emptyset$ denotes "no bid", and $b^k(n-1)$ is the "current outstanding bid", that is, the highest submitted bid for object k up to round $n-1$. Thus for each object k a bidder can either remain silent or raise the high bid of the previous round of at least $\nu > 0$, (ν arbitrarily close to zero). All objects close simultaneously. The auction ends if all bidders remain silent on all objects, and the winners are the "standing high bidders" determined at round $n-1$ and they pay their last bids. If there is more than one standing high bidder on one object, then that object will be allocated randomly among these bidders. Given the simultaneity of closing, we refer to this mechanism as the simultaneous English auction for multiple objects (SEAMO).

Two basic differences distinguish the two mechanisms. First, the JAMO does not allow for rounds of bidding; bidders press buttons corresponding to the objects on which they wish to bid; by releasing a button, a bidder quits that auction irrevocably; thus, bidders have "smaller" strategy spaces than in SEAMO; in particular they have no influence on the pace at which prices rise. Second, closing is not simultaneous in the JAMO but rather object-by-object. We shall highlight the role of these distinguishing features in the emergence of collusive and signaling equilibria.

SA: The auction proceeds in rounds. At round $n = 1, 2, ..$, each bidder submits a vector of sealed bids for objects on which they are allowed to bid. Bids for a single object are taken from the set $[b_{\min}(n-1), +\infty)$, where $b_{\min}(0) = 0$ and for $n > 1$, $b_{\min}(n-1)$ is the lowest among *all* bids submitted during the previous round on *all* objects. In the following rounds, all the bidders who offered $b_{\min}(n)$ in the current round, are not allowed to bid again on the object on which they submitted $b_{\min}(n)$. At the end of each round the auctioneer only announces $b_{\min}(n)$, the object for which $b_{\min}(n)$ was submitted and the identity of the bidder(s) that submitted that bid. An object is attributed to the last bidder having the right to bid on that object and the winner will pay an amount of money equal to the last lowest bid on that object. If all active bidders bid the minimum admissible bid $b_{\min}(n)$ on one object, then that object will be randomly allocated among those bidders. Note that since at least one bidder exits some object in any given round, the two objects will be attributed after at most $2(M + N_1 + N_2) - 2$ rounds. Following Fujishima et al. (1999), we refer to this auction as the survival auction (SA).

Example of the SA: Suppose there are three bidders: A, G and L. Suppose A only bids on object 1, whereas both G and L bid on both objects. At round 1, bids are as follows: $b_1^A = .4$, $(b_1^G, b_2^G) = (.1, .6)$, $(b_1^L, b_2^L) = (.2, .2)$. Having observed all bids, the auctioneer only announces that the lowest bid on all objects is .1 (i.e., $b_{\min}(n = 1) = .1$), and that this bid has been submitted by bidder G on object 1. Thus bidder G is excluded from object 1, and, in the following round, bidder A will bid on object 1, bidder L will bid on both objects, while bidder G will bid on object 2 only. The lowest admissible bid is .1. Suppose now that at round 2, the bids are as follows: $b_1^A = .4$, $b_2^G = .6$, $(b_1^L, b_2^L) = (.3, .3)$. Then $b_{\min}(2) = .3$, bidder L exits both objects simultaneously, and only one bidder is left on each object. The auction ends at this point, (i.e., after two rounds), and bidders A and G are awarded objects 1 and 2 respectively at price .3.

3 Main results

In this section, we derive the main results on the three mechanisms just described. The main result (Proposition 2) characterizes a symmetric perfect Bayesian equilibrium (PBE) of the SA, which in turn induces a corresponding equilibrium in the other two mechanisms.

In the JAMO or the SA, the information available to a bidder at any time t, is described by H_t. In the JAMO, t coincides with the current level reached by prices, while, in the SA, t represents the current minimum admissible bid on any of the objects; in both mechanisms, H_t contains, for each object, the set of active bidders on that object, as well as the price at which the other bidders dropped out. The following proposition, which extends Fujishima et al. (1999), is useful in characterizing equilibria of the JAMO; it is used in the proof of Proposition 2.

Proposition 1. *The JAMO and the SA are strategically equivalent.*

Proof. Fujishima et al. (1999) cover the case of a single object. Given our rules for the SA and the JAMO, their proof can easily be adapted to the multi-object case. In order to show strategic equivalence between the JAMO and the SA, one needs show that: (a) An isomorphism exists between each bidder's set of decision nodes in the two auctions such that the precedence relation is preserved; (b) there exists an isomorphism between feasible action sets at the same decision nodes, which is consistent with the precedence relation; (c) bidders' payoffs are always the same at the same terminal nodes. We follow Fujishima et al. (1999) to prove points (a)-(c) for the present two-object case.

(a) In the SA, the new information that a bidder obtains between round n and $n+1$, provided that he does not submit the lowest bid, is the identity of the loser(s) in round n and the losing bid in that round. Note that the losing bid might well be a vector of identical bids submitted by a bundle bidder on the two objects. Thus each (surviving) bidder's decision node in round $n+1$ can be described by the $2n$-dimensional vector of losing bids and losing bidders in the first n rounds. (There may well be more than one losing bidder submitting the same losing bid at a given round, possibly even on two different objects.) In the JAMO, the new information that a bidder receives after the n-th drop-out, provided he has not yet exited himself at n, is again the $2n$-dimensional vector of exiting times with corresponding bidders who have exited. Thus the isomorphism of the bidders' set of decision nodes is the identity mapping. Suppose for instance that bidder h who is active on objects 1 and 2 observes that bidders i and j active only on object 2 have both submitted the lowest bid of $b(1)$ on object 2 in round 1 of the SA. Then bidder h's decision of how much to bid on objects 1 and 2 in round 2 of the SA is equivalent to the decision of when to exit before being the next to exit in the JAMO after observing that bidders i and j exited object 2 at $b(1)$. Notice that this holds regardless of whether bidder are active on two (or more) objects rather than one object.

(b) In the $(n+1)$st round of the SA, both a bundle and a unit bidder can submit bid(s) greater or equal to the lowest bid submitted on the two (or more) objects in round n. In the JAMO, a bidder can decide either to exit exactly at the same time of the n-th drop-out or to wait longer until being the next one to exit. Thus the feasible action sets at any given decision node are identical in the two auctions, and consistency of the precedence relation with the decision node isomorphism clearly also holds.

(c) If play in the two auctions reaches the same terminal node, then all actions must have been the same at same decision nodes. This implies that the same bidder(s) will win, the bidder(s) will pay the same amount(s), and the information available to all bidders at that terminal point will be the same. □

The proposition implies that the JAMO and the SA are outcome equivalent and that their equilibria coincide. This allows us to use the easier SA to analyze the JAMO. Since at each round of the SA there is at least one bidder that "exits" from at least one object, with finitely many bidders (and objects), the SA ends in a finite number of rounds; hence we can use a backward induction argument to construct equilibria of a game that, as the JAMO, is actually in continuous time.

Specifically, at each round of the SA, unit bidders bid their valuation for that object. This is simply a bidder's weakly dominant strategy in a one-object, second-price auction. The same happens for bundle bidders in the absence of synergies (case (ii)). The bundle bidders' equilibrium strategies in the presence of synergies (cases (i) and (iii)) are less trivial. Still they display some simple features. A bundle bidder always makes the same bid on the two objects unless she has already won one of them. The level of this (double) bid depends on three factors: the bundle bidder's overall valuation for the bundle, the numbers of unit bidders active on objects 1 and 2, and whether or not other bundle bidders are still active. That is, the optimal bid does not depend on the number of other bundle bidders active provided that there is at least one other active bundle bidder, it does however depend on the numbers of unit bidders. Moreover, a bundle bidder's optimal bid increases as unit bidders quit the auction.

We characterize now a symmetric PBE of the SA (hence also of the JAMO). As a general rule, equilibrium bids can be determined as the smallest bids at which each bidder's expected marginal payoff of submitting the bid, computed with appropriate beliefs, is equal to zero. In particular, a unit bidder bids his value u_k on object k, the only object on which he is active, and submits such a bid until he either wins the object or is eliminated from the auction; this behavior mimics the weakly dominant strategy in a standard single-object, second-price auction. The same logic applies to bundle bidders in two instances; first, a bundle bidder, say of type (v_1, v_2) who has already won object 1, will submit a bid equal to $v_2 + \alpha$ at each round; second, in the absence of synergies, i.e., in case (ii), a bundle bidder will submit v_1 and v_2 on object 1 and 2 respectively, starting from the first round. More difficult to determine, are the bundle bidders' equilibrium bids in cases (i) and (iii), when they are active on two objects. In any given round H_t, the bids will depend on the bundle bidder's type (v_1, v_2), on the minimum admissible bid t, and on the set of other bidders that are still allowed to bid in that round, as well as beliefs about their behavior. Equilibrium strategies are determined by backward induction. We distinguish two cases: (1) $M = 1$ and (2) $M \geq 2$ bundle bidders.

(1) $M = 1$: Suppose first the SA has reached a stage H_t in which only one bundle bidder is active on both objects, and $N_1, N_2 \geq 1$ unit bidders are active on objects 1 and 2 respectively. What are the bundle bidder's optimal bids on the two objects? We first show that the bundle bidder optimally submits the same bid on both objects; the bid must maximize his current expected payoff given the information at H_t and given the competitors' optimal strategies in the current and following rounds. If in the current round the bundle bidder submits the same bid $p \geq t$ on both objects, then only three outcomes can occur with positive probability: (a) p is the lowest among all bids submitted in the current round, and the bundle bidder exits the auction; (b) a unit bidder on object 1 sets the lowest bid, exits from that object, and all other bidders move to the following round; (c) a unit bidder on object 2 sets the lowest bid, exits from that object, and all other bidders move to the following round.

In order to define the bundle bidder's overall expected payoff at H_t, let $\pi_{L_k}(s, H_t)$ denote his continuation payoff on both objects if the lowest of all submitted bids in round H_t is equal to s and it has been submitted by a unit bidder on object $k \in \{1, 2\}$, (we drop arguments v_1, v_2, and α from π_{L_k} to simplify notation). The overall expected payoff can be written as the sum of the the expected (continuation) payoffs in cases (a)–(c). We derive each of them in turn. In case (a), the bundle bidder's expected payoff is zero since he quits the auction without buying any object. In case (b), if the lowest of all submitted bids is $s \in [t, p)$, and has been submitted by a unit bidder on object 1, then the bundle bidder's expected continuation payoff will be exactly $\pi_{L_1}(s, H_t)$ and this event has a density $(1 - F_{N_2}(s \mid t)) f_{N_1}(s \mid t)$, i.e., the probability that the lowest of the unit bidders' bid on object 2 is greater than s times the density of having the lowest bid on object 1 equal to s, (this assumes unit bidders bid their valuations). The overall expected (continuation) payoff for (b) is then $\int_t^p \pi_{L_1}(s, H_t)(1 - F_{N_2}(s \mid t)) f_{N_1}(s \mid t) ds$. Similarly, the expected (continuation) payoff in (c) is $\int_t^p \pi_{L_2}(s, H_t)(1 - F_{N_1}(s \mid t)) f_{N_2}(s \mid t) ds$.

Let $\Pi(p; H_t)$ denote the bundle bidder's overall expected payoff from bidding p on the two objects at information set H_t, then we have,

$$\Pi(p; H_t) = 0 + \int_t^p \pi_{L_1}(s, H_t)(1 - F_{N_2}(s|t)) f_{N_1}(s|t) ds \tag{1}$$

$$+ \quad \int_t^p \pi_{L_2}(s, H_t)(1 - F_{N_1}(s \mid t)) f_{N_2}(s \mid t) ds.$$

where 0 is the expected payoff in case (a), and the two integrals represent cases (b) and (c) respectively. Assume now that $\Pi(p; H_t)$ is differentiable, (differentiability of $\Pi(p, H_t)$ and second-order conditions are shown in the Appendix). The necessary condition for optimality of the bundle bidder's bid is obtained from the first derivative of $\Pi(p; H_t)$ with respect to p,

$$\begin{aligned} \Pi'(p; H_t) &= \pi_{L_1}(p, H_t)(1 - F_{N_2}(p \mid t)) f_{N_1}(p \mid t) \\ &\quad + \pi_{L_2}(p, H_t)(1 - F_{N_1}(p \mid t)) f_{N_2}(p \mid t) \\ &= \left(\frac{N_1}{N_1 + N_2} \pi_{L_1}(p, H_t) + \frac{N_2}{N_1 + N_2} \pi_{L_2}(p, H_t) \right) f_{N_1 + N_2}(p \mid t), \end{aligned} \tag{2}$$

since $(1 - F_{N_k}(p \mid t)) f_{N_{3-k}}(p \mid t) = N_{3-k}(1 - F(p \mid t))^{N_1 + N_2 - 1} f(p \mid t)$ for $k = 1, 2$. The optimal bid is the smallest $p \geq t$ such that $\Pi'(p; H_t) \leq 0$. Denote the corresponding bid by $\tau(H_t)$.[4]

(2) $M \geq 2$: Suppose next that the SA is at a stage H_t where $M \geq 2$ bundle bidders are active on both objects and there are $N_1, N_2 \geq 0$ unit bidders active on objects 1 and 2 respectively. Again, it is possible to show that bundle bidders will exit both objects simultaneously and that optimal bids can be determined as the smallest bids at which corresponding expected marginal payoffs are zero. However, it is also shown in the Appendix that the equations determining the optimal bids reduce to simpler equations that are independent of the number of bundle bidders active at H_t. Let $\pi_{BB}(p, H_t)$ denote a bundle bidder's continuation payoff on both objects at H_t, where the minimum admissible bid is $p \geq t$, all unit bidders are present, but all other bundle bidders have already left both objects with a bid of p.[5] We distinguish three possible cases: (a) there are no unit bidders active (N_1, $N_2 = 0$);(b) there are unit bidders active on only one of the two objects ($N_1 = 0$ or $N_2 = 0$); and (c) there are unit bidders active on both objects ($N_1, N_2 \geq 1$). Hence we can write more explicitly,

$$\pi_{BB}(p, H_t) = \begin{cases} v_1 + v_2 + \alpha - 2p, & \text{if } N_1 = 0,\ N_2 = 0 \\ v_{3-k} - p + \int_p^{v_k + \alpha} (v_k + \alpha - z) N_k F^{N_k - 1}(z|s) f(z|p) dz, & \text{if } N_k \geq 1,\ N_{3-k} = 0 \\ \int_p^{\tau(v_B, H_p^{'})} \left(\frac{N_1}{N_1 + N_2} \pi_{L_1}(s, H_p^{'}) + \frac{N_2}{N_1 + N_2} \pi_{L_2}(s, H_p^{'}) \right) f_{N_1 + N_2}(s|H_p^{'}) ds, & \text{else,} \end{cases}$$

where $N_k F^{N_k - 1}(z \mid s) f(z \mid p) dz$ is the density of the highest among the $N_k \geq 1$ unit bidders' valuations for object k, $k \in \{1, 2\}$; $H_p^{'}$ denotes the information set H_t updated by the fact that all other bundle bidders exited with a bid p, and $\tau(v_B, H_p^{'})$ is the optimal bid previously computed for the case $M = 1$ and $N_1, N_2 \geq 1$, which are

[4] We show in the Appendix that the bundle bidder's optimal bid only depends on $v_B = v_1 + v_2 + \alpha$ and not on v_1 and v_2 separately.

[5] This event does not arise with positive probability at equilibrium. It is merely instrumental to the characterization of the equilibrium bid when $M \geq 2$.

now the remaining bidders. It is shown in the Appendix that, in cases where $M \geq 2$, optimal bids are determined as the smallest $p \geq t$ such that $\pi_{BB}(p, H_t) \leq 0$. For example, it can be verified that in case (a) where $N_1, N_2 = 0$, the optimal bid is equal to $v_B/2$; and also in case (c) where $N_1, N_2 \geq 1$, the optimal bid is given by $\tau(v_B, H_t^{'})$ the same bid that one obtains without other bundle bidders. Finally, for a bundle bidder active on both objects, define

$$\psi(p; H_t) = \begin{cases} \left(\frac{N_1}{N_1+N_2}\pi_{L_1}(p, H_t) + \frac{N_2}{N_1+N_2}\pi_{L_2}(p, H_t)\right), \text{if} & M = 1 \\ \pi_{BB}(p, H_t), & \text{if} \;\; M \geq 2. \end{cases} \quad (3)$$

We are now in a position to describe our PBE of the SA (or JAMO).

Proposition 2. *The following constitutes a symmetric PBE of the SA:*

(E1) A unit bidder on object k bids u_k, $k = 1, 2$;
(E2) A bundle bidder active only on object k bids $v_k + \alpha$ if he has won the other object and bids v_k otherwise, $k = 1, 2$;
(E3) In case (ii), a bundle bidder active on both objects bids v_k on object k, $k = 1, 2$;
(E4) In cases (i) and (iii), a bundle bidder that is active on both objects at H_t submits the same bid $\tau(v_B, H_t)$ on the two objects, which is determined as the smallest $p \geq t$ that solves $\psi(p; H_t) \leq 0$ if $\tau(v_B, H_t) \leq 1$ and is equal to $v_B/2$ otherwise.
(E5) Out-of-equilibrium-path beliefs: In case (i) and (iii), if at price t, bundle bidder h has quit object $3 - k$ only and continues on object k, then all other bidders hold that $\Pr[v_k^h \leq t + \delta \mid \tau_{3-k}^h = t] = 1$, δ positive and small, i.e., they believe that with probability 1 bundle bidder h will bid at most $t + \delta$ on object k.

Proof. See the Appendix. □

Note that by Proposition 1 the optimal bids described in Proposition 2 correspond to optimal exiting times in the JAMO. As mentioned above, in cases (i) and (iii), in equilibrium, it never happens that a bundle bidder quits one object to continue on the other one. This greatly simplifies the analysis when there are synergies. In fact, for intermediate values of $\alpha \in (0, 1)$, and if v_1 and v_2 are different, submitting the same bid on the two objects is not always optimal; a bundle bidder's optimal bids will depend on how exiting on one object will affect the other bundle bidders' behavior on the other object; showing existence of a PBE is already problematic in this case.[6] Next, we relate equilibria of the SEAMO and the JAMO.

Proposition 3. *Every PBE of the JAMO induces a PBE of the SEAMO.*

Proof. Consider the exiting times constituting a PBE for the JAMO. Then all bidders bidding the standing high bid plus an arbitrarily small bid increment in each round

[6] For example, Athey's (2001) theorem does not apply due to the presence of signals which are not uni-dimensional; McAdams' (2003) theorem does not apply because of a modularity condition on the payoffs.

and stopping to bid according to these exiting times (also along out of equilibrium paths) constitutes an (arbitrarily close) PBE of the SEAMO. Because winning bidders pay their own last high bids and because of the simultaneity of the closing, there are no profitable deviations from the above strategies. □

This also implies that the set of outcomes induced by PBE of the JAMO is contained in the set of outcomes induced by PBE of the SEAMO. The converse of this as well as of Proposition 3 is not true; the SEAMO has many more equilibria. In what follows, we will see examples of equilibria that are PBE of the SEAMO but not otherwise.

4 Collusive and signaling equilibria

In this section, we consider certain collusive and signaling equilibria that have been studied in the literature, typically in the framework of the SEAMO, and show that they are not viable in the JAMO, due to the more restrictive nature of the strategy spaces.

4.1 Some signaling devices

Bidders in the FCC auctions attempted to communicate in a variety of ways. Since there is no way of proving any private exchange of information among bidders, we are bound to analyze communication arising through the exploitation of the auction rules themselves. This section analyzes some common communication devices also apparently used in the actual FCC auctions, namely code and jump bidding, and withdrawal bids, from the viewpoint of the JAMO.

Code bidding: Code bidding is one of the more obvious forms of signaling. Since bids are expressed in dollars and since, at least in the FCC auctions, most objects displayed six-digit prices, bidders could use the last three digits to encode messages. Code bids had different natures. Some bidders used the last three digits to "disclose" their identities. For example, in the AB auction (Auction 4), GTE frequently used "483" as the last three digits; this number corresponds to "GTE" on the telephone keypad. In other circumstances code bidding had a *reflexive* nature. The last three digits were used by a bidder both to signal an object of special interest to her and the object on which the same bidder was punishing competitors for not bumping the first market.[7]

In the JAMO, code bidding would take a simple form. Bidders have to stop bidding on one object as soon as the price encodes "meaningful" digits. However, this strategy would irrevocably exclude that bidder from competing for that object, and with two objects, would therefore also exclude her from bidding for the bundle; moreover it would also exclude her from performing any retaliation since exit is irrevocable. It then follows that code bidding is ineffective in any PBE of the JAMO

[7] See Cramton (1997) and Cramton and Schwartz (1999, 2000) for detailed accounts of collusive behavior in the actual FCC auctions.

(with two objects). The result, however, may not extend to more than two objects if bidders can bid for objects they do not value. For example, suppose three objects are being auctioned, suppose a bidder is interested in purchasing only one object, say 1, and that she is active on all objects at an early stage of the auction. Then she can stop bidding on, say, object 3 at a price whose digits encode a message similar to the one used by GTE, while remaining active on the other two objects. This allows her to use object 2 as a potential threat for retaliation. Nevertheless this code bidding might be too costly to implement if bidders participation fee increases with the number of objects for which they can bid. Moreover, such a signaling device becomes more difficult and costly to use if prices are raised not continuously but in predetermined finite amounts.

Jump bidding: It need not always be in the interest of the bidders to increase prices at the minimum pace required by the auction rules. In fact, Gunderson and Wang (1998) show how a bidder in a SEAMO can benefit by using jump bids as a signal of a high valuation, possibly causing other bidders to drop out earlier; this may lead to lower revenues for the seller.[8] While jump bids are possible in the SEAMO they are obviously not in the JAMO. The FCC's recent decision to limit the amount by which bids can be raised, e.g., in the LMDS auction (Auction 17), may suggest a change in this direction.[9]

Bid withdrawals: While the FCC had originally allowed unlimited number of bid withdrawals in order to allow bidders to make more efficient aggregations of objects, it was soon noticed that they could be used as signaling devices. As Cramton and Schwartz (2000) report, withdrawal bids were apparently used in FCC auctions as part of a warning or of retaliatory strategies, as well as part of cooperative strategies, where bidders attempted to split objects among themselves. Neither the JAMO nor the SEAMO versions described above allow for withdrawal bids. Again, the FCC's recent decision to limit their number to two, e.g., in the LMDS auction (Auction 17), suggests another change in this direction.

4.2 Closing rules

Milgrom (2000) contains a description of the tâtonnement logic that inspired many of the FCC auction rules. In particular, the rules specified that bidding would remain open on all objects until there were no new bids on *any* object. This simultaneous closing rule allows each losing bidder to switch at any time from the lost object to a substitute or to stop bidding on a complement. However, as Milgrom points out, it is also vulnerable to collusion.

Milgrom's example: Consider the following example from Milgrom (2000). Two bidders bid for two objects 1 and 2, which are each worth 1 to both bidders. Milgrom shows that there exists a sequential equilibrium of the SEAMO (with complete information) such that the selling price for both objects is ν, i.e., the smallest

[8] See also Avery (1998) for further equilibria involving jump bids in the context of one-object English auctions with affiliated values.

[9] See also Cramton and Schwartz (2000).

possible bid, and the bidders realize the highest collusive payoff of $2 \cdot (1 - \nu)$, (see Theorem 8, p. 264).

The logic of the equilibrium is that both players buy one object each at the lowest possible price by using a simple threatening strategy: bidder 1 bids ν on auction 1 if bidder 2 has never bid on 1; otherwise he does not bid. If bidder 2 has bid on 1, then bidder 1 reverts to a "competitive" bidding strategy, that is to keep bidding on each object until a price of 1 is reached; bidder 2 plays symmetrically.

As Milgrom suggests, such a low revenue equilibrium is avoided if closing is not simultaneous but rather object-by-object. According to such closing rule, bidding would stop on an object if at any round there is no new bid on that object. The JAMO provides an example of object-by-object closing. Indeed, once all bidders but one drop from one object and remain active on the other, the first object is awarded irrevocably. The result of Theorem 9 in Milgrom (2000), which states that at each (trembling-hand) perfect equilibrium with object-by-object closing the price of each object is at least $1 - \nu$ carries over to the JAMO (also with complete information), where in fact the price of each object is exactly 1. By applying Proposition 2 to the example described above where, as in our usual framework, the bidders' values are private information, it is easy to establish the following result:

Corollary 1. *If bidders 1 and 2 have (private) values of 1 for both objects, and* $\alpha = 0$*, then, in the PBE of the JAMO, the selling price is 1 for each object.*

Such a selling price of 1 (or $1 - \nu$) is also not guaranteed in the SEAMO with incomplete information as the equilibria constructed in Engelbrecht-Wiggans and Kahn (1998) and Brusco and Lopomo (2002) show.

The collusive equilibria of Brusco and Lopomo: Brusco and Lopomo (2002) (henceforth BL) construct several kinds of PBE in undominated strategies of the SEAMO (in our usual framework), some of which are very similar to the ones constructed by Milgrom under complete information. Kwasnica and Sherstyuk (2002) provide some experimental evidence for such equilibria when there are few players and with small complementarities. We shall see that none of BL's equilibria are possible in the JAMO.

The logic of their collusive equilibria is as follows: Consider two bundle bidders and, for simplicity, take $\alpha = 0$. The bidders use the first round to signal to each other which of the two objects they value the most. If they rank the objects differently, bidders confirm their initial bids in all subsequent rounds and obtain their most preferred object at the minimum price; otherwise they revert to the "competitive" strategy of raising prices on both objects up to their private values. BL then go on to refine this type of collusive equilibrium by allowing bidders to signal more than just the identity of the higher valued object. This allows them to obtain collusive equilibria even more favorable to the bidders. In particular, they show that a collusive equilibrium may also arise when bidders have the same ranking for the objects, also if there are more than two bidders as well as if there are positive complementarities ($\alpha > 1$); they also show, however, that the scope for collusion diminishes as the number of bidders increases and the number of objects is fixed at two; and the possibility of collusion is lowered if the complementarities are large and variable.

Again, the rule driving the presence of such equilibria is simultaneous closing. The JAMO mechanism instead is built around the irrevocable exit and induces object-by-object closing, which makes the rounds of signaling necessary in the above equilibria impossible. In these examples bidders always have an incentive to bid for any object for which they have positive value. In particular, it follows:

Corollary 2. *The collusive equilibria constructed as PBE of the SEAMO in Brusco and Lopomo (2002) are never PBE of the JAMO.*

Note also that these collusive equilibria are not PBE of the JAMO even if one allows for rounds of cheap talk between the bidders prior to the auction.

As it has often been pointed out, simultaneous closing has the advantage of being more flexible in allowing bidders to revise and update their bidding behavior in forming aggregates, (see e.g., Cramton, 1997, 1998; Milgrom, 1998, 2000; Cramton and Schwartz, 2000, 2002). Moreover, Kagel and Levin (2004) point out that, especially for intermediate values of the complementarities, ascending auctions may suffer from the *exposure problem* by which bundle bidders may drop out too early from individual objects thus reducing efficiency. Although their comparison is with one-shot sealed bid auctions, it seems plausible that the exposure problem would be even more pronounced in auctions with object-by-object closing than in ones with simultaneous closing. This is something that needs to be further investigated, also in connection with the rules for withdrawing bids.

5 Conclusion

Recent research on multi-unit ascending auctions has highlighted the existence of two potentially conflicting features of the auction rules adopted by the FCC and subsequently in some of the European UMTS auctions. On one hand, the transparency and flexibility of the bidding process eases an efficient aggregation of objects; on the other, the amount of information available to bidders together with the strategic possibilities allowed by the rules may be used to implement tacit collusive agreements, see Cramton and Schwartz (2000, 2002) and Klemperer (2002).

By not allowing bidders to set the pace at which prices rise on individual objects, the auctioneer can make bidders' signaling devices blunt without losing the information revelation feature of the ascending mechanism. In this sense we maintain that the SEAMO facilitates tacit collusion relative to the JAMO and showed that several collusive equilibria, which appear in the SEAMO, do not have a counterpart in the JAMO. A more complete assessment of the relative performance of the two auction formats certainly requires further study. We outline some directions for future research.

First, the framework, while in line with the existing literature, is admittedly restrictive. For example, if the number of objects is greater than two, the set of equilibria is likely to depend on the composition of the bundles that bundle bidders are interested in acquiring. That is, with more than two objects there are several ways preferences over bundles can overlap, (Krishna and Rosenthal, 1996, mention some examples). It is also possible that code-bidding may reappear even in the JAMO. But even with only two objects, the case of small synergies may already pose non-trivial existence problems.

Second, an issue that has not been addressed is the rationale of having prices on the two objects rise simultaneously in the JAMO. We have imposed the same "speed" on both objects, being aware that there is no theoretical or empirical justification for this assumption.

Third, other aspects of the FCC auctions such as activity rules, the number of allowable bid withdrawals, and the simultaneity of closing deserve further investigation. Although some modifications of the standard SEAMO undertaken by the FCC may be seen as changes in direction of the JAMO, there seems to be no general agreement on e.g., whether closing should be simultaneous or not. Albano et al. (2001) and Branco (1997, 2001) show that under certain conditions, object-by-object closing may perform rather well theoretically. Kagel and Levin (2004) on the other hand provide experimental evidence indicating that, at least within certain ranges of bidders' valuations, inefficiencies may arise due to what they call the "exposure" problem. Clearly, more needs to be done to better assess the theoretical and empirical performance of the "Japanese" vs. "English" design of the auction and the simultaneous vs. object-by-object closing, as well as of other rules mentioned. Also, while the JAMO and the SA are theoretically equivalent it would be useful to obtain further experimental evidence contrasting their relative performance.

Finally, motivated by considerations of market structure and bidder asymmetries, Klemperer (1998, 2002) suggests an auction format he calls"Anglo-Dutch" that combines an ascending or "English" auction with a first-price sealed-bid or "Dutch" auction. Our results suggest that an alternative that may be worth considering in similar environments is a combination of a "Japanese" with a first-price sealed-bid auction. Similarly, Ausubel and Milgrom (2002) suggest an English ascending auction that allows for package bidding in order to improve efficiency while avoiding some of the problems arising for example from Vickrey-Clarke-Groves mechanisms. Also here it may be worthwhile to consider a "Japanese" design while keeping the remaining features that allow bidders to bid on packages; of course, here the question of how to increase the prices of the items for sale becomes even more pressing.

Appendix

Proof of Proposition 2. We argue using both the SA and the JAMO; as mentioned above, bids in the SA correspond to exiting times in the JAMO. We first prove points (E1)–(E3). If bidders only bid on objects they value, then unit bidders' strategies in (E1) are clearly optimal. For, a unit bidder with value u_k for object k and who is active only on object k, it is optimal to always submit a bid of u_k in the SA, and to remain active until the prices reach u_k in the JAMO. Similarly, bundle bidders currently bidding on one object behave *as if* they were unit bidders, with valuations for the object depending upon whether or not they won the other object; hence strategy (E2). For bundle bidders that are still bidding on two objects in case (ii), i.e. $v_1, v_2 \in [0, 1]$ and $\alpha = 0$, we have that, given the other bidders' strategies, no bundle bidder has any incentive to exit from object k after v_k; on the other hand, there is no incentive to exit before v_k either , $k = 1, 2$; hence strategy (E3). It remains to show (E4).

Given the strategies (E1) and (E2), we use Lemmas 1 to 4, to show that strategy (E4) is globally optimal for a bundle bidder bidding on two objects in cases (i), $v_1 = v_2 = x \in [0, 1]$ and $\alpha \geq 0$, and (iii), $v_1, v_2 \in [0, 1]$ and $\alpha > 1$. First, we show that, if, by exiting only one object a bundle bidder does not induce a sensitive increase in the probability of winning the other object, then he will prefer submitting the same bid for both objects; this is shown in Lemma 1. Second, Lemma 2 characterizes a bundle bidder's optimal bid when he faces only unit bidders; from Lemma 1, this bid is the same on the two objects since unit bidders' strategies under (E1) are independent of the bundle bidder's actions. Third, in Lemma 3, we show that a bundle bidders' bids defined in (E4) are increasing in their valuation for the bundle. Finally, we use the latter result to characterize bundle bidders' equilibrium bids when facing both unit and bundle bidders, Lemma 4; for this, we also show that the assumption in Lemma 1 concerning the effect on other bidders' strategies of exiting only one object, is satisfied under the out-of-equilibrium-path beliefs specified under (E5), so that Lemma 1 still applies.

All the following lemmas refer to the cases (i) $v_1 = v_2 = x \in [0, 1]$, $\alpha \geq 0$, and (iii) $v_1, v_2 \in [0, 1]$, $\alpha > 1$, and to information sets H_t, where a bundle bidder is active on both objects and hence faces at least one other bidder active on object 1 and at least one other bidder active on object 2. Consider the SA and fix such a bundle bidder h that is active on the two objects and let y_k denote the highest bid on object k among all other bidders in the current and following rounds. For any given strategy profile of all other bidders, denote by $G_k(\cdot \mid H_t)$ the cumulative distribution function of y_k given the information set H_t. Let further H_t^k denote the information node that is reached starting from H_t if bundle bidder h exits object k at t. Then we can state:

Lemma 1. *For any H_t, there exists $\varepsilon > 0$ such that if $G_k(\cdot \mid H_t^{3-k}) \leq G_k(\cdot \mid H_t) + \varepsilon$, $k = 1, 2$, then it is optimal for bundle bidders to submit the same bid $\tau \geq \max\{v_1, v_2\}$ on both objects.*

Proof. We will argue using the JAMO. The conditions on G_k and ε imply that if bundle bidder h exits one object only, the probability of winning the other object will not increase too much. (We will see later that the distribution function $G_k(\cdot \mid H_t^1)$ of y_2 given that the bundle bidder exits object 1 only at t is always well-defined, given the out-of-equilibrium-path beliefs (E5)). The lemma states that in this case a bundle bidder who is active on two objects has nothing to lose if he stays on the two objects at least until the price reaches $\max\{v_1, v_2\}$, or, in the terminology of the SA, if he submits the same bid $\tau \geq \max\{v_1, v_2\}$ for both objects. The intuition is that when ε is small, if he exits one object before $\max\{v_1, v_2\}$, he does not increase too much the probability of winning the other object while he loses the opportunity of winning the bundle and of gaining the synergy α. More formally, assume without loss of generality that $v_2 \geq v_1$. We need to show that exiting object 1 at $t < v_2$ is worse than waiting at least until v_2 before exiting any object at all. At $t < v_2$, the bundle bidder's expected profit from exiting object 1 is

$$E[(v_2 - y_2)\mathbf{1}_{\{y_2<v_2\}}|H_t^1] = \int_t^{v_2} (v_2 - y_2)dG_2(y_2 \mid H_t^1). \tag{4}$$

Indeed, after exiting object 1 at $t < v_2$, it becomes optimal for the bundle bidder to stay on object 2 until v_2. If y_2 happens to be less than v_2, then the bundle bidder will win object 2 and his payoff will be $(v_2 - y_2)$. We want to show that bundle bidder h prefers to wait until v_2 before exiting any object at all.

Consider case (i) and take $\varepsilon \leq \frac{1}{2}E[\alpha\mathbf{1}_{\{y_1,y_2<v_2\}} \mid H_t]$. Suppose that the bundle bidder stays on the two objects until v_2 and then exits at v_2 the objects he has not won. In this case four outcomes are possible: (a) $y_2 < v_2 < y_1$, the bundle bidder wins only object 2 at y_2 and exits object 1 at v_2, (b) $y_1 < v_2 < y_2$, he wins only object 1 at y_1 and exits object 2 at v_2; (c) $y_1, y_2 < v_2$, he wins both objects at prices smaller than v_2, (d) $v_2 < y_1, y_2$, he exits both objects at v_2; (again, we implicitly assume that the probability that $y_k = v_2$ is zero and that G_2 has a positive density, which we will see later is verified in equilibrium since bidders' strategies are strictly increasing in their valuations that are continuously distributed). Therefore the expected payoff from this strategy is:

$$\begin{aligned}
&E[(v_2 - y_2)\mathbf{1}_{\{y_2<v_2<y_1\}} \mid H_t] + E[(v_1 - y_1)\mathbf{1}_{\{y_1<v_2<y_2\}} \mid H_t] \\
&\quad + E[(v_1 + v_2 + \alpha - y_1 - y_2)\mathbf{1}_{\{y_1,y_2<v_2\}} \mid H_t] + 0 \\
&= E[(v_2 - y_2)\mathbf{1}_{\{y_2<v_2\}} \mid H_t] + E[(v_2 - y_1)\mathbf{1}_{\{y_1<v_2\}} \mid H_t] \\
&\quad + E[\alpha\mathbf{1}_{\{y_1,y_2<v_2\}} \mid H_t] \\
&> E[(v_2 - y_2)\mathbf{1}_{\{y_2<v_2\}} \mid H_t] + (v_2 - t)\varepsilon \\
&\geq E[(v_2 - y_2)\mathbf{1}_{\{y_2<v_2\}} \mid H_t^1],
\end{aligned}$$

where we used $v_1 = v_2$ for the first equality and $v_2 > t$ and $\varepsilon \leq \frac{1}{2}E[\alpha\mathbf{1}_{\{y_1,y_2<v_2\}} \mid H_t] \leq \frac{E[\alpha\mathbf{1}_{\{y_1,y_2<v_2\}}|H_t]}{2(v_2-t)}$ for the strict inequality. Since $G_2(\cdot \mid H_t^1) \leq G_2(\cdot \mid H_t) + \varepsilon$ implies that $E[(v_2 - y_2)\mathbf{1}_{\{y_2\leq v_2\}} \mid H_t^1] \leq E[(v_2 - y_2)\mathbf{1}_{\{y_2\leq v_2\}} \mid H_t] + (v_2 - t)\varepsilon$, we have the last inequality. Thus the bundle bidder prefers waiting at least until v_2 before exiting object 1.

Now consider case (iii) and take $\varepsilon \leq \frac{1}{2}E\left[(v_1 + \alpha - 1)\mathbf{1}_{\{\min\{y_1,y_2\}<v_2\}} \mid H_t\right]$. Suppose that bundle bidder h stays on the two objects until v_2, exits the two objects only if he wins no object before v_2, and continues optimally otherwise. Then, if a bundle bidder h wins one object, he will necessarily win both objects. This is because, if all remaining bidders follow strategies (E1),(E2), after winning object k, bundle bidder h will remain active on the other object until $v_{3-k} + \alpha > 1$, whereas his competitors will remain active until their valuations for that single object which are at most 1. Thus, the strategy mentioned above leads to an expected payoff equal to

$$\begin{aligned}
&E[(v_1 + v_2 + \alpha - (y_1 + y_2))\mathbf{1}_{\{\min\{y_1,\, y_2\}\leq v_2\}} \mid H_t] \\
&\qquad = E[(v_2 - \min\{y_1,\, y_2\})\mathbf{1}_{\{\min\{y_1,\, y_2\}\leq v_2\}} \mid H_t] \\
&\qquad\quad + E[(v_1 + \alpha - \max\{y_1,\, y_2\})\mathbf{1}_{\{\min\{y_1,\, y_2\}\leq v_2\}} \mid H_t]
\end{aligned}$$

$$
\begin{aligned}
&> E[(v_2 - \min\{y_1,\, y_2\})\mathbf{1}_{\{\min\{y_1,\, y_2\} \leq v_2\}} \mid H_t] + \varepsilon(v_2 - t) \\
&\geq E[(v_2 - \min\{y_1,\, y_2\})\mathbf{1}_{\{y_2 \leq v_2\}} \mid H_t] + \varepsilon(v_2 - t) \\
&\geq [(v_2 - y_2)\mathbf{1}_{\{y_2 \leq v_2\}} \mid H_t] + \varepsilon(v_2 - t) \\
&\geq E[(v_2 - y_2)\mathbf{1}_{\{y_2 < v_2\}} \mid H_t^1],
\end{aligned}
$$

where the first inequality follows again from

$$
\varepsilon \leq \frac{E\left[(v_1+\alpha-1)\mathbf{1}_{\{\min\{y_1,y_2\}<v_2\}}|H_t\right]}{2} \leq \frac{E\left[(v_1+\alpha-1)\mathbf{1}_{\{\min\{y_1,y_2\}<v_2\}}|H_t\right]}{2(v_2-t)}
$$

and the second inequality is a consequence of the fact that $\{y_2 < v_2\}$ is a subset of $\{\min\{y_1,\, y_2\} \leq v_2\}$ and that we are taking expectation of positive random variables. Thus for ε small the bundle bidder strictly prefers waiting at least until v_2 before exiting object 1. This proves that in case (i) and (iii) a bundle bidder does not exit object 1 before v_2. As it cannot be optimal for a bundle bidder to exit object 2 before v_2 or to stay only on object 2 when $t > v_2$, exiting simultaneously the two objects at a price no smaller than v_2 is an optimal strategy. □

Lemma 1 implies that in the SA, assuming the condition on the distribution functions G_k is verified, a bundle bidder will submit the same bid on both objects and that in the JAMO a bundle bidder will exit simultaneously from both objects. We use this result in the following Lemma that proves strategy (E4) for the case where a bundle bidder faces only unit bidders.

Lemma 2. *The strategy (E4) in Proposition 2 is globally optimal whenever* $M = 1$ *and* $N_1, N_2 \geq 1$.

Proof. Consider bundle bidder h that has reached an information node H_t in the SA, where the minimum admissible bid is t, and where he is active on the two objects and faces $N_1, N_2 \geq 1$ unit bidders and no other bundle bidders. According to (E1) in Proposition 2, a unit bidder will bid his valuation for the object at each round of the SA as long as the minimum admissible bid is below his valuation; this regardless of the bundle bidder's strategy. Hence, when a bundle bidder faces only unit bidders, by exiting only one object the bundle bidder does not modify the probability of winning the other object simply because he does not affect unit bidders' strategies. Therefore, we have $G_k(\cdot \mid H_t) = G_k(\cdot \mid H_t^{3-k})$ and we can apply Lemma 1 and deduce that it is optimal for the bundle bidder to submit the same bid on the two objects. If in round H_t the bundle bidder submits the same bid $p \geq t$ on the two objects, then his expected payoff is given by expression (1). Suppose for the time being that Π is differentiable, then the first order condition is $\Pi'(p, H_t) = 0$, where $\Pi'(p, H_t)$ is given by (3). Note first that $\pi_{L_k}(p, H_t)$ is non-increasing in p, for $k \in \{1, 2\}$. Indeed, $\pi_{L_k}(p, H_t)$ is the bundle bidder's continuation payoff given that all remaining unit bidders' valuations for the objects are at least p, which also represents the minimum admissible bid in the next round. Clearly, an increase in p cannot raise $\pi_{L_k}(p, H_t)$. Note also that as unit bidders valuations are not greater than 1, we have $f_{N_1+N_2}(p \mid t) > 0$ if and only if $p \in [t, 1]$.

We can now prove that the bid given by (E4) represents a global maximum when $M = 1$. Suppose first that $\tau(v_B, H_t) \leq 1$. Given the definition of $\tau(v_B, H_t)$

and that $\pi_{L_k}(p, H_t)$ is non-increasing in p, it must be that $\Pi'(p; H_t) > 0$ for $p \in [t, \tau(v_B, H_t))$ and $\Pi'(p; H_t) \leq 0$ for $p \geq \tau(v_B, H_t)$. In other words, this says that $\Pi(p; H_t)$ reaches its global maximum at $p = \tau(v_B, H_t)$. Second, suppose that

$$\left(\frac{N_1}{N_1 + N_2}\pi_{L_1}(p, H_t) + \frac{N_2}{N_1 + N_2}\pi_{L_2}(p, H_t)\right) > 0 \;\; \forall p \leq 1. \tag{5}$$

In particular, suppose the bundle bidder's continuation payoff is positive even at $p = 1$; this last case arises only if $v_B > 2$. Since $f_{N_1+N_2}(p \mid t) > 0$ if and only if $p \in [t, 1]$, condition (5) implies that $\Pi'(p; H_t) > 0$ for all $p < 1$ and $\Pi'(p; H_t) = 0$ for all $p \geq 1$. Thus $\tau(v_B, H_t) = v_B/2 > 1$ is a maximizer of $\Pi(p; H_t)$ in this case.

In order to complete the proof, we need to show that $\Pi(p; H_t)$ is differentiable for $p \leq 1$. With expression (1), we argue that, if π_{L_k} is differentiable, then $\Pi(p; H_t)$ must also be differentiable; hence, it is sufficient to show that π_{L_k} is differentiable. The proof is by induction. First, suppose the SA has reached a stage in which the bundle bidder faces only one unit bidder on each of the two objects. Then $\pi_{L_k}(\cdot, H_t)$ is obviously differentiable in p as it will be equal to[10]

$$\pi_{L_k}(p, H_t) = v_k - p + \int_p^{v_{3-k}+\alpha} (v_{3-k} + \alpha - s) f(s \mid p) ds. \tag{6}$$

Second, suppose that $\pi_{L_1}(\cdot, H_t)$ and $\pi_{L_2}(\cdot, H_t)$ are differentiable and take an information node $H_{t'}$ that immediately precedes H_t. We want to prove that $\pi_{L_k}(\cdot, H_t)$ is differentiable, $k = 1, 2$. Information node $H_{t'}$ is such that if the lowest among all bids in the round $H_{t'}$ is t and it is submitted by a unit bidder on object k, then node H_t is reached. Thus, $\pi_{L_k}(t, H_{t'})$ represents the bundle bidder's equilibrium payoff at information node H_t. Denoting with $\tau(v_B; H_t)$ the bundle bidder's optimal bid in round H_t we have

$$\begin{aligned} \pi_{L_k}(t, H_{t'}) &= \Pi(p; H_t)\big|_{p=\tau(v_B;H_t)} = \\ &= \int_t^{\tau(v_B;H_t)} \left(\frac{N_1}{N_1{+}N_2}\pi_{L_1}(s, H_t){+}\frac{N_2}{N_1{+}N_2}\pi_{L_2}(s, H_t)\right) f_{N_1+N_2}(s \mid t) ds \end{aligned} \tag{7}$$

where N_1 and N_2 represent the number of unit bidders at node H_t that are active on object 1 and 2 respectively. Thus, since $\pi_{L_k}(\cdot, H_t)$ is differentiable, then $\pi_{L_k}(\cdot, H_{t'})$ is twice differentiable. Finally notice that starting form expression (6) and using expression (7) and the definition of $\tau(\cdot)$, it is possible to derive recursively $\pi_{L_k}(s, H_t)$ and the bundle bidder's optimal bid for all H_t such that the bundle bidder only faces local bidders. This completes the proof of the Lemma. □

Lemma 3. *The bids defined by strategy (E4) of Proposition 2 are strictly increasing functions of v_B.*

[10] Note that, if the unit bidder on object k sets the lowest bid p, then the bundle bidder wins object k, he pays p. In the following round the bundle bidder will bid $v_{3-k} + \alpha$ on object $3 - k$ whereas the remaining unit bidder will bid his valuation for the object.

Proof. Notice first that a bundle bidder's bid depends on $v_B = v_1 + v_2 + \alpha$ rather than the individual values of his type (v_1, v_2). This is clearly true since $v_1 = v_2 = (v_B - \alpha)/2$. By Lemma 1, the bundle bidder submits the same bid for both objects. Therefore, in case (iii), because $\alpha > 1$, with probability one, either he wins both objects or none; this further implies that his expected payoff and hence his bid will also depend only on the value of the bundle v_B.

If $\tau(v_B, H_t) = v_B/2$, then the exiting time is clearly increasing in v_B. Suppose then that $\tau(v_B, H_t) \leq 1$. In this case, the probability of winning any object at any given price depends only on the other bidders' valuations. Moreover, the bundle bidder's ex-post payoff is non-decreasing in v_B if he wins just one object and is strictly increasing if he wins both. Thus, if $\bar{v}_B > v_B$ is the bundle bidder's value for the bundle, then the corresponding expressions $\bar{\pi}_{L_1}(p, H_t)$, $\bar{\pi}_{L_2}(p, H_t)$ and $\bar{\pi}_{BB}(p, H_t)$, computed with the higher value $\bar{v}_B$, are strictly greater than the respective counterparts $\pi_{L_1}(p, H_t)$, $\pi_{L_2}(p, H_t)$ and $\pi_{BB}(p, H_t)$, computed with v_B, since there is always a strictly positive probability that he may win both objects, as long as he is active on both objects. From the definition of $\Psi(\cdot)$, we deduce that $\overline{\Psi}(p, H_t)$ computed with the higher value $\bar{v}_B$ is strictly greater than $\Psi(p, H_t)$ computed with v_B. Moreover, as $\pi_{L_1}(p, H_t)$, $\pi_{L_2}(p, H_t)$ and $\pi_{BB}(p, H_t)$ are non-increasing in p, it results that also Ψ is so. Thus the definition of τ implies $\tau(\overline{v}_B, H_t) > \tau(v_B, H_t)$, and hence the claim. □

Lemma 4. *The strategy (E4) in Proposition 2 is globally optimal for M, N_1, and N_2 arbitrary.*

Proof. The proof is by backward induction. Consider a bundle bidder who after some sufficiently large number of rounds of the SA is still active on both objects. If there is no other bundle bidder active on both objects, Lemma 2 applies; if all active bidders are bundle bidders, then $\pi_{BB}(p, H_t) = v_1 + v_2 + \alpha - 2p = v_B - 2p$. Thus, the optimal bid $\tau(v_B, H_t)$ defined in point (E4) reduces to $v_B/2$ for a bundle bidder with value for the bundle equal to v_B, and it is easy to check that bidding $v_B/2$ on both objects is indeed a best reply in this case.

Given the above and Lemma 2, in what follows, we consider a round H_t where there are at least two bundle bidders active on the two objects and at least one unit bidder who is active on some object, i.e., $M \geq 2$ and $N_1 + N_2 \geq 1$. Note that the latter requirement implies that $t \leq 1$. Fix bundle bidder h with valuations v_1 and v_2. Suppose unit bidders' strategies are those defined by (E1) and all bundle bidders different from h use strategies defined by (E4). Moreover, suppose that in the following rounds all bundle bidders' strategies, including bundle bidder h, are *symmetric and strictly increasing* in the value of the bundle. We show that the strategy defined by (E4) is a best reply for bundle bidder h in the current round. To begin with, we first restrict to strategies where bundle bidder h submits the same bid for both objects and show optimality under this assumption; later, we will show that Lemma 1 applies and it is indeed optimal to submit the same bid on both objects.

Suppose then that bundle bidder h submits bid p for both objects. We distinguish three possible outcomes that can occur with positive probability at H_t:[11] (a) bundle

[11] From (E1) in Proposition 2 and Lemma 3, bidders' strategies are strictly increasing in their valuations so that ties occur with probability zero.

bidder h's bid p is the lowest among all bids submitted in the current round, and bundle bidder h quits the auction without buying any object; (b) a unit bidder sets the lowest bid $s \in [t, p)$ on one of the two objects, quits the auction, and the other bidders move to the following round; (c) a bundle bidder different from h sets the lowest bid $s \in [t, p)$ on both objects, quits the auction (on both objects), and the remaining bidders move to the following round.

We can therefore write bundle bidder h's expected payoff from bidding p on both objects at an information set H_t with $M \geq 2$ bundle bidders and N_1, N_2 unit bidders, where $N_1 + N_2 \geq 1$ as

$$\begin{aligned} \Pi(p; H_t) = & \int_t^p \pi_L(s, H_t)(1 - F_B(s \mid H_t)) f_{N_1+N_2}(s \mid H_t) ds + \\ & + \int_t^p \pi_B(s, H_t)(1 - F_{N_1+N_2}(s \mid H_t)) f_B(s \mid H_t) ds, \end{aligned} \tag{8}$$

where $\pi_L(s, H_t)$ is bundle bidder h's continuation payoff if in the current round the lowest bid is $s \geq t$ and it has been submitted by a unit bidder (outcome (b)), $\pi_B(s, H_t)$ is bundle bidder h's continuation payoffs if another bundle bidder sets the lowest bid $s \geq t$ on the two objects (outcome (c)), $F_B(\cdot \mid H_t)$ is the cumulative distribution of the *lowest* bid among all bundle bidders different from h, and $f_B(\cdot \mid H_t) = F_B'(\cdot \mid H_t)$. Note that from Lemma 3, the strategies of bundle bidders different from h are strictly increasing in their valuations for the bundle, and individual valuations are independently and continuously distributed random variables; therefore, from bundle bidder h's perspective, the lowest bid among all other bundle bidders is a continuously distributed random variable with a well defined density function $f_B(\cdot \mid H_t)$.

Differentiability of $\Pi(\cdot; H_t)$ follows from the differentiability of π_L and π_B. In order to prove differentiability of the latter, it is sufficient to check that, in the terminal rounds of the auction, π_L and π_B are both differentiable; it can then be checked that, if in a given round, π_L and π_B are differentiable, then they will also be differentiable in the preceding round; the induction argument is the same as the one used in Lemma 2.

Differentiating (8) with respect to p, we have

$$\begin{aligned} \Pi'(p; H_t) = & \pi_L(p, H_t)(1 - F_B(p \mid H_t)) f_{N_1+N_2}(p \mid H_t) \\ & + \pi_B(p, H_t)(1 - F_{N_1+N_2}(p \mid H_t)) f_B(p \mid H_t). \end{aligned} \tag{9}$$

We need to show that $\tau(v_B, H_t)$ defined by (E4) is globally optimal for bidder h. We do this by showing that $\Pi'(p, H_t) \geq 0$ for $p < \tau(v_B, H_t)$ and $\Pi'(p, H_t) \leq 0$ for $p \geq \tau(v_B, H_t)$. This is done by showing that for $p < \tau(v_B, H_t)$ both the first and the second term in (9) are non-negative, whereas for $p \geq \tau(v_B, H_t)$ they are both non-positive.

Recall that we are considering information sets where $M \geq 2$ and $N_1 + N_2 \geq 1$ so that it is sufficient to distinguish the following four cases: (1) $M = 2$, $N_1 + N_2 = 1$, (2) $M = 2$, N_1 or $N_2 = 0$, (3) $M = 2$, $N_1, N_2 \geq 1$, (4) $M \geq 3$, $N_1, N_2 \geq 0$.

(1) Suppose that $M = 2$, $N_1 = 0$, and $N_2 = 1$, (the case $M = 2$, $N_1 = 1$, and $N_2 = 0$ is analogous). Let (v_1, v_2) be bundle bidder h's type, denote by i the other

bundle bidder and let (v_1^i, v_2^i) be his type. We have

$$\pi_B(p, H_t) = \pi_{BB}(p, H_t) = v_1 - p + \int_p^{v_2+\alpha} (v_2 + \alpha - s) f(s \mid p) ds, \qquad (10)$$

since if bundle bidder i (the only other bundle bidder active at H_t) exits the two objects at p, bundle bidder h wins object 1, pays p, and in the following round will bid $v_2 + \alpha$ against the unit bidder on object 2. Note that $\pi_B(p, H_t)$ is strictly decreasing in p and increasing v_B.[12] Moreover, if $v_B = 2$ then $\pi_B(p, H_t)|_{p=1} = 0$. This implies that whenever $v_B > 2$, then $v_1 - p + \int_p^{v_2+\alpha} (v_2 + \alpha - s) f(s|p) ds > 0$ for all $p \in [t, 1]$ and consequently $\tau(v_B, H_t) = v_B/2 > 1$. If $v_B \leq 2$, then $\tau(v_B, H_t)$ will be the minimum $p \in [t, 1]$ that solves $v_1 - p + \int_p^{v_2+\alpha} (v_2 + \alpha - s) f(s|p) ds \leq 0$. In both cases the second term of (9) is not negative for $p < \tau(v_B, H_t)$ and not positive for $p \geq \tau(v_B, H_t)$.[13]

Denote by $f_B(v_B^i | \tau(v_B^i, H_t) > p)$ the density of bundle bidder i's valuation for the bundle conditional on the fact that in the current round he has submitted a bid larger than p. Then we have

$$\pi_L(p, H_t) = \int_{\tau^{-1}(p, H_t)}^{v_B} (v_B - s) f_B(s | \tau(s, H_t) > p) ds, \qquad (11)$$

where $\tau^{-1}(p; H_t)$ is the v_B^i such that $\tau(v_B^i, H_t) = p$. Indeed, if in the current round the unit bidder quits the auction at p, then bundle bidder h understands that bundle bidder i has bid more than p and that v_B^i is at least $\tau^{-1}(p, H_t)$. Moreover in the following round the only active bidders will be the two bundle bidders still active on both objects, and each one of them will bid half his valuation for the bundle on both objects. Thus, bundle bidder h can win only if $v_B^i \leq v_B$. Note that as $\tau(\cdot; H_t)$ is increasing in the value of the bundle, then $\Pr(v_B^i \leq v_B | \tau(v_B^i, H_t) > p) > 0$ for $p < \tau(v_B, H_t)$ and $\Pr(v_B^i \leq v_B | \tau(v_B^i, H_t) > p) = 0$ for $p \geq \tau(v_B, H_t)$. Consequently, $\pi_L(s, H_t) > 0$ for $p < \tau(v_B, H_t)$ and $\pi_L(p, H_t) = 0$ for $p \geq \tau(v_B, H_t)$.

Summing up, and taking in to account expression (9), we deduce that $\Pi'(p; H_t) \geq 0$ for $p < \tau(v_B, H_t)$ and $\Pi'(p; H_t) \leq 0$ for $p \geq \tau(v_B, H_t)$, which is what we needed to show global optimality of $\tau(v_B, H_t)$ at H_t. In particular, note that the level of $\tau(v_B, H_t)$ can be determined only by means of the expression $\pi_{BB}(p, H_t)$ given in (10), and that it is strictly increasing in v_B for Lemma 3.

(2) Suppose now $M = 2$, $N_1 = 0$ and $N_2 \geq 1$, (the case $M = 2$, $N_1 \geq 1$ and $N_2 = 0$ is analogous). Again, since $M = 2$, if the other bundle bidder exits both objects at p, then h automatically buys object 1 at p, and in the following round will bid $v_2 + \alpha$ against N_2 unit bidders on object 2; hence we have

$$\pi_B(p, H_t) = \pi_{BB}(p, H_t) = v_1 - p + \int_p^{v_2+\alpha} (v_2 + \alpha - s) N_2 F^{N_2 - 1}(s|p) f(s|p) ds, \qquad (12)$$

[12] To see this point recall that we are considering the cases where either $v_1 = v_2$ and $\alpha \geq 0$, or $v_1 \neq v_2$ and $\alpha > 1$.

[13] Note that as unit bidders bid at most 1, we have $(1 - F_{N_1+N_2}(p|H_t)) = 0$ for $p > 1$ and so the second term in expression (9) vanishes for $p > 1$.

where $N_1 F^{N_1-1}(s|p)f(s|p)$ is the density of the highest among $N_2 \geq 1$ unit bidders' valuations for object 2; h will win object 2 only if the highest valuation among the unit bidders is below $v_2+\alpha$. Applying the same arguments used for case (1), we deduce that the second term of (9) is not negative for $p < \tau(v_B, H_t)$ and not positive for $p \geq \tau(v_B, H_t)$. Note now that for the first term of (9), $\pi_L(p, H_t) > 0$ for $p < \tau(v_B, H_t)$ and $\pi_L(p, H_t) = 0$ for all $p \geq \tau(v_B, H_t)$. In order to see this, recall that $\pi_L(p, H_t)$ is bundle bidder h's continuation payoff given that in the current round the lowest bid p is set by a unit bidder; this implies that bundle bidder i submitted a bid greater than p, i.e., $\tau(v_B^i, H_t) > p$. Thus, if $p \geq \tau(v_B, H_t)$, then $\tau(v_B^i, H_t) > \tau(v_B, H_t)$. The latter implies that $v_B^i > v_B$ since Lemma 3 tells us that $\tau(\cdot, H_t)$ is strictly increasing in its first argument. As in the following rounds bundle bidders' strategies are supposed to be symmetric and increasing in their valuations for the bundle, bundle bidder h cannot expect to win any object as he will surely quit the auction before bundle bidder i. Therefore his continuation payoff $\pi_L(p, H_t)$ will be zero. By contrast, if $p < \tau(v_B, H_t)$, then the probability that $v_B^i < v_B$ remains positive, hence bundle bidder h wins with positive probability and so $\pi_L(p, H_t) > 0$.

Summing up, and taking into account expression (9), we can deduce that $\Pi'(p; H_t) \geq 0$ for $p < \tau(v_B, H_t)$ and $\Pi'(p; H_t) \leq 0$ for $p \geq \tau(v_B, H_t)$. Finally note again that also in this case, $\tau(v_B, H_t)$ can be determined directly from the expression $\pi_{BB}(p, H_t)$ given in (12), and that it is strictly increasing in v_B by Lemma 3.

(3) Suppose now $M = 2$, and $N_1, N_2 \geq 1$. With the exit of the other bundle bidder, the only remaining bundle bidder is h who faces N_1 and $N_2 \geq 1$ unit bidders on the two objects, in which case we can apply Lemma 2 to get

$$\pi_B(p, H_t) = \pi_{BB}(p, H_t) = \tag{13}$$
$$= \int_p^{\tau(v_B, H_p')} \left(\frac{N_1}{N_1+N_2} \pi_{L_1}(s, H_p') + \frac{N_2}{N_1+N_2} \pi_{L_2}(s, H_p') \right) f_{N_1+N_2}(s|H_p')ds,$$

where H_p' is the information set H_t updated by the fact that the other bundle bidder exited the auction with a bid of p, and $\tau(v_B, H_p')$ is the bundle bidder h optimal bid in the round corresponding to information node H_p'. Since in node H_p' there are no other bundle bidders besides h, we can use Lemma 2. From the definition of $\tau(v_B, H_p')$ we know that the argument of the integral in (13) is non-negative for $p < \tau(v_B, H_p')$ and non-positive for $p \geq \tau(v_B, H_p')$ and from the definition of $\tau(v_B, H_t)$ given in point (E4), we have $\tau(v_B|H_t) = \tau(v_B|H_p')$. Moreover, $\pi_L(p, H_t) > 0$ for $p < \tau(v_B, H_t)$ and $\pi_L(p, H_t) = 0$ for all $p \geq \tau(v_B, H_t)$ by the same argument used in case (2). Thus, using the expression for $\Pi'(p; H_t)$ given in (9), we deduce that $\Pi'(p; H_t) \geq 0$ for $p < \tau(v_B, H_t)$ and $\Pi'(p; H_t) \leq 0$ for $p \geq \tau(v_B, H_t)$, which again proves the claim in this case. Note once again that $\tau(v_B, H_t)$ can be determined directly from the expression $\pi_{BB}(p, H_t)$ given in (13), and that it is strictly increasing in v_B by Lemma 3.

(4) Consider now the remaining case, where H_t is such that there are several unit bidders and several bundle bidders still active besides bundle bidder h, i.e., $M \geq 3$ and $N_1, N_2 \geq 1$. As shown in the previous cases, all the bundle bidders' strategies

in the following stages of the game, including bidder h's strategies, are symmetric and strictly increasing in v_B. Hence, bundle bidder h's continuation payoff will be non-positive if in the current round there is at least one other bundle bidder i whose bid $\tau(v_B^i, H_t)$ is larger than $\tau(v_B, H_t)$. Indeed, if $M \geq 3$ the continuation payoffs $\pi_L(p, H_t)$ and $\pi_B(p, H_t)$ both correspond to situations where there is at least one other bundle bidder, say i, who in the current round has bid more than p; hence, as in the discussion for π_L in cases (1)-(3), if $p \geq \tau(v_B, H_t)$, then there exists a bundle bidder i with $\tau(v_B^i, H_t) > \tau(v_B, H_t)$, which implies $v_B^i > v_B$ and therefore bundle bidder h cannot expect to win any object and his expected continuation payoff will be zero. In particular, $\pi_L(p, H_t), \pi_B(p, H_t) \leq 0$ if $p \geq \tau(v_B, H_t)$. On the other hand, if $p < \tau(v_B, H_t)$, then the probability that the remaining bundle bidders' valuations are less than v_B remains positive, and so bundle bidder h's expected continuation payoff will be non-negative, i.e., $\pi_L(p, H_t), \pi_B(p, H_t) \geq 0$ if $p < \tau(v_B, H_t)$. Finally, notice that $\tau(v_B, H_t)$ is determined from expression $\pi_{BB}(p, H_t)$ by construction, and that by symmetry and monotonicity of $\tau(\cdot, H_t)$, it followed that $\Pi'(p; H_t) \geq 0$ for $p < \tau(v_B, H_t)$ and $\Pi'(p; H_t) \leq 0$ for $p \geq \tau(v_B, H_t)$.

So far we have shown that, if all bidders but bundle bidder h follow strategies (E1)-(E4) and bundle bidder h is restricted to submitting the same bid on the two objects, then the latter's optimal bid is given by (E4) as well. What remains to be proven is that it is not optimal for bundle bidder h to submit two different bids. By Lemma 1, bundle bidder h is not willing to submit two different bids if, by exiting only one object, he does not induce a sensitive increase in the probability of winning the other object. In the presence of other bundle bidders, the effect that exiting only one object has on the probability of winning the other object depends on how the other bundle bidders behave after observing this (out-of-equilibrium-path) event, and this will depend on their out-of-equilibrium-path beliefs. We now show that the out-of-equilibrium-path beliefs specified in (E5) do the job.

Suppose that the SA has reached node H_t corresponding to the event that the lowest bid t has been submitted by bundle bidder h on object 1 only. That is, contrarily to the symmetric equilibrium strategy, bundle bidder h does not exit the two objects simultaneously. Notice first the the unit bidders' optimal bids are not affected by this event. Suppose next that all surviving bundle bidders $i \neq h$ hold the (out-of-equilibrium-path) beliefs under (E5), i.e., $\Pr[v_2 \leq t + \delta \mid \tau_1^h = t] = 1$, for δ positive and small. That is, after observing bundle bidder h exiting object 1 at t, bundle bidders $i \neq h$ believe that v_2 is not greater than $t + \delta$ and, therefore, that bidder h will remain active at most until $t + \delta$. Choosing δ sufficiently small will guarantee that the continuation payoffs of bundle bidders $i \neq h$ are arbitrarily close to those arising in a situation where bundle bidder h exits both objects at the price t. Thus their optimal bids will be arbitrarily close to the ones computed along the equilibrium path. Therefore Lemma 1 applies. Since by exiting only one object a bundle bidder cannot induce a significant increase in the probability of winning the other object, in equilibrium a bundle bidder never exits one object and continues on the second object. $\square$

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