Zero-sum Revision Games

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Abstract

In a zero-sum asynchronous revision game, players are equipped with clocks that beat according to independent Poisson processes, and a player is able to revise her action only at a beat of her clock. Payoffs are obtained from the actions implemented at a fixed deadline. The value of this game is called revision value. We characterize it as the unique solution of an ordinary differential equation and provide some comparative statics with respect to the parameters of the model. Then, we study the limit revision value as the duration of the game increases and prove that it does not depend on the initial position. For 2 \times 2 games, we calculate precisely the limit revision value and show that equilibrium paths have a wait-and-wrestle structure: far form the deadline, players stay put at suer-place points, close to the deadline, they take best responses to the action of the opponent.

Key-words: Revision Games, Zero-sum Games, Deadline Effect.
1 Introduction

A revision game models a situation where players prepare the actions they want to see actually implemented at a future predetermined deadline. Players enter the game at an initial time, choose actions and then wait for their actions to be implemented and for payoffs to be distributed according to a known mapping from action profiles into payoffs (the component game). Between the initial time and the deadline, players observe the actions the other players have prepared and may have the opportunity to revise their action and reconsider their choices. As the deadline gets closer and closer, the probability to get another revision opportunity vanishes and the last chosen actions become more and more binding.

Revision games can be used to model a number of economic and social relations. For example, in some stock markets, thirty minutes before the opening hour, traders can submit orders which can be withdrawn and changed until the opening time. Only the orders recorded in the system at the opening time are binding and payoff relevant. Another example are international negotiations, or negotiations about regulations between a government and the industry to be regulated. During an international negotiation process, parties prepare proposals and counterproposals but only the final draft of the agreement signed at the end of the summit matters. Regulation of industries are renewed at known dates and before the expiration of the current regulation, the regulator and the industry representatives negotiate on the next version. At the same time, companies prepare their production technology to complain to the next standards in the regulation.\textsuperscript{1} Sports also provide many applications for revision games. The composition of teams in football or rugby, needs to be made before the match, the choice of the tires and car setting before the care race starts, etc.

In such frameworks, it is common to empirically observe a \textit{deadline effect}. In the stock mar-

\textsuperscript{1}Some example are: the banking sectors regulations Basel 1, 2, and 3 expire and are renewed known basis; Global warming forums set targets for a given period to be reviewed on regular basis; European Union emission standards Euro 4 effective January 1, 2008, Euro 5 effective January 1, 2010 etc.
ket pre-opening, little happens at the beginning of the period whereas most of the activity of submitting and revising orders concentrate in the last five minutes before the official opening hour (Biais et al, 1999). Negotiations often start with little progress in the early phase, whereas most activity is concentrated when the deadline approaches. These phenomena have been explained using war of attrition (Alesina and Drazen, 1991), or bargaining games with incomplete information (Spier, 1992, Vives and Medrano 2001), or optimism (Simsek and Yildiz, 2014)). In a revision game, there is no waiting cost (or war of attrition), no incomplete information, and players are perfectly rational. Nevertheless, we show that if the component game does not have a pure strategy equilibrium, the deadline effect emerges as an equilibrium behavior in a revision game. Thus, we provide a new alternative explanation of the deadline effect in these economic frameworks.

Revision games have been introduced by Kamada and Kandori (2013), and further studied in Calcagno et al. (2014). In particular Calcagno et al. (2014), study revision games where the component game is $2 \times 2$ and has two strict Nash equilibria.

In this paper, we consider the case where the component game is a two player zero-sum game. We are interested in the properties of the revision game equilibrium payoff, which we call the revision value, and its limit when the length of the revision phase goes to infinity, which we call limit revision value.

The revision value exists (from Lovo and Tomala, 2015) and we characterize it as the unique solution of an ordinary differential equation. We prove that players have pure optimal strategies. Then we show that, in contrast with non-zero-sum revision games, the equilibrium payoff (i.e. the value) is a continuous function of the parameters of the model: the component game payoff matrix, the relative frequency with which each player has a revision opportunity, and the duration of the revision game. We show that it is advantageous to be relatively fast, which is not the case for the games studies by Calcagno et al. (2014) where a player might gain from being forced to revise at a relatively low frequency.
Then, we analyze the limit revision value when the length of the duration phase (or equivalently, the frequency of revisions) goes to infinity. We show that the limit revision value does not depend on the action profile that is prepared at the beginning of the game. Moving backward from the deadline, the revision value converges to the limit revision value exponentially fast. When the component game has a pure Nash equilibrium, the limit revision value clearly coincides with the value of the component game. However, when the component game has no pure strategy equilibrium, the limit revision value generically differs from the component game value. In this case, the limit revision value to a player is strictly increasing in the relative frequency with which she can revise her action. The limit revision value converges to the player’s min-max in the component game as this relative frequency goes to infinity.

Finally, we fully characterize the equilibrium strategies for $2 \times 2$ component games. We show that if the component game has a pure Nash equilibrium, when revision phase is sufficiently long, players prepare the component game equilibrium from the start. Then, we consider the case where the component game equilibrium is in mixed strategies. We show that at the beginning of the revision phase, players prepare an action profile that they do not revise until the deadline is close enough. When the end of the revision game approaches, player start changing their action whenever they have a revision opportunity. That is, the equilibrium generate a deadline effect: at the beginning, players settle on a sur-place action profile, that is not an equilibrium of the component game. They wait for the deadline to be close enough and then all the activity of revision is concentrated at the last minute. This is consistent with robust empirical regularity observed in bargaining and negotiations.

The paper is organized as follows: The model is introduced in Section 2. Section 3 analyzes the equilibrium property of a general zero-sum revision game. The finite duration phase is studied in Section 3. Section 4 considers the limit case when the duration goes to infinity. In Section 5, we characterize the equilibrium of the $2 \times 2$ component game. The proof for this case are relegated to the Appendix. Section 6 concludes.
2 Zero-sum revision games

2.1 Component Game

The first ingredient is a two-player zero-sum game that we will call the ‘component game’. Let $X_1$ and $X_2$ be finite and non-empty sets of actions for players 1 and 2 respectively, and denote $X = X_1 \times X_2$ the set of action profiles. Let $U : X \rightarrow \mathbb{R}$ be player 1’s payoff function, the payoff function of player 2 is $-U$. We also denote by $U$ the payoff matrix $(U(x))_{x \in X}$.

For any pure action $x_{-i} \in X_i$ of player $-i$, denote $BR^U_i(x_{-i}) \subseteq X_i$ the set of player $i$’s best responses to $x_{-i}$ in the component game with payoff matrix $U$. Formally, for any $x_1$ in $X_1$ and $x_2$ in $X_2$, we denote the best responses by:

$$BR^U_1(x_2) := \arg \max_{y_1 \in X_1} U(y_1, x_2), \quad BR^U_2(x_1) := \arg \min_{y_2 \in X_2} U(x_1, y_2)$$

We say that $U$ is regular if for each player $i$ and $x_{-i}$, $BR^U_i(x_{-i})$ is a singleton.

Let $V$ be the minmax value of this game in mixed strategies, that is, $V$ is player 1’s unique Nash equilibrium payoff of the game $U$. Let $S_i$ be player $i$’s Stackelberg payoff in the game where the first mover is player $i = 1, 2$. That is, $S_1$ (resp. $S_2$) is the maxmin (resp. minmax) value of the game in pure strategies. One has,

$$\max_{x_1} \min_{x_2} U(x_1, x_2) = S_1 \leq V \leq S_2 = \min_{x_2} \max_{x_1} U(x_1, x_2)$$

and $S_1 = S_2$ if and only if the component game has a Nash equilibrium in pure strategies.

2.2 Revision Game

We describe now the revision game built on the component game. Time is continuous, $t \in [0, T]$ where $T$ is a fixed deadline. At time 0, there is an initial “prepared” action profile
$x(0) \in X$, given exogenously. For each player $i$, there is a Poisson process with intensity $\lambda_i > 0$, the two processes being independent. A player is able to change her prepared action only at an arrival of her Poisson process which we call a revision opportunity. At every time $t < T$, if the current prepared action profile is $x(t)$ and player $i$ receives a revision opportunity, then she chooses $y_i \in X_i$ and the new prepared action profile is $(y_i, x_{-i}(t))$. At time $t = T$, the action profile $x(T)$ is actually played and each player receives the corresponding payoff, $U(x(T))$ for player 1 and $-U(x(t))$ for player 2. This is the only payoff that players receive in the revision game.

Revision opportunities and prepared actions are perfectly and publicly observed, the description and the parameters of the game are common knowledge.

The revision game is called asynchronous, since with probability one, only one player receives a revision opportunity at a time.

We will use the following parametrization. Denote $\lambda = \lambda_1 + \lambda_2$ and $q = \lambda_1/\lambda$. The revision process is equivalently described as follows. There is a common Poisson process with intensity $\lambda$, which we call the clock. At each arrival, or beat of the clock, a coin flip $(q, 1 - q)$ determines whether player 1 or player 2 gets the revision opportunity.

Next, we define strategies. A history of length $n$ is a sequence $(x_0, \tau_1, i_1, x_1, \tau_2, \ldots, \tau_n, i_n, x_n)$ in $H_n = X \times ([0, T] \times \{1, 2\} \times X)^n$, with $H_0 = X$ and where $x_0$ denotes the initial action profile, $\tau_m$ the time of the $m$-th beat of the clock, $i_m$ the player who had the opportunity to revise at time $\tau_m$ and $x_m$ the new action profile after the revision. A strategy $\sigma_i$ of player $i$ is a measurable map from $\bigcup_{n \geq 0}(H_n \times [0, T])$ to $\Delta(X_i)$. A strategy is Markov if it depends only on the time of revision and the other player’s current action. Formally, a Markov strategy $\sigma_i$ for player $i$ is a measurable map from $X_{-i} \times [0, T]$ to $\Delta(X_i)$.

A pair $\sigma = (\sigma_1, \sigma_2)$ induces a unique probability $P_\sigma$ over plays in $\bigcup_{n \geq 0}H_n$ (disjoint union), where plays are simply finite histories. If a play $(x, \tau_1, i_1, x_1, \tau_2, \ldots, \tau_n, i_n, x_n)$ is realized, it means
that \( \tau_n \) was the last ring of the clock and the terminal action profile is \( x(T) = x_n \). If the history \( x_0 \in H_0 = X \) is realized, it means that the clock did not ring and \( x(T) = x_0 \). A play induces a unique right-continuous trajectory for the profile of actions \( (x(t))_{t \in [0,T]} \) which jumps at revision times. This allows to define an expected payoff \( \mathbb{E}_\sigma[U(x(T))] \).

We define the minmax and maxmin value of this game as,

\[
\pi(T, x) = \inf_{\sigma_2} \sup_{\sigma_1} \mathbb{E}_\sigma[U(x(T))], \quad u(T, x) = \sup_{\sigma_1} \inf_{\sigma_2} \mathbb{E}_\sigma[U(x(T))].
\]

If these two quantities are equal, the game is said to have a value \( u(T, x) = \pi(T, x) = u(T, x) \).

In the sequel we denote by \( t \geq 0 \) the remaining time to the deadline, whereas \( T \) denotes the total length of the revision game, so that \( t = T - \tau \) in the above description. The revision game over the interval \( [0, T] \) with initial position \( x \) is denoted \( \Gamma(T, x) \). From Lovo and Tomala (2015), revision games have Markov perfect equilibria, hence for any \( t \geq 0 \) the revision game with length \( t \) and initial position \( x \) has a value denoted by \( u(t, x) \).

Notations: We write \( u(t) \) for the matrix \( (u(t, x))_{x \in X} \), and \( \|u(t)\| := \max_{x \in X} |u(t, x)| \). For \( t \geq 0 \) and \( x = (x_1, x_2) \), we write:

\[
 u^-(t, x) := \min_{y_2 \in X_2} u(t, x_1, y_2) \quad \text{and} \quad u^+(t, x) := \max_{y_1 \in X_1} u(t, y_1, x_2).
\]

Clearly, \( u^- \leq u \leq u^+ \).

3 Revision values

We start by analyzing the equilibrium and value for a revision game where \( T \) and \( \lambda \) are fixed. Note first that existence of equilibrium and value is guaranteed by the results of Lovo and Tomala (2015):
Lemma 1  The revision game admits a Markov perfect equilibrium $\sigma$, the revision game with length $t$ and initial position $x$ has a value denoted by $u(t, x)$. This equilibrium and the value are such that for every $x \in X$ and almost every $t \in [0, T]$,

$$\sigma_i(t, x_{-i}) \in BR_i^{u(t)}(x_i).$$

(1)

Proof. The result follows immediately from Lemma 2.1 and Theorem 4.1 in Lovo and Tomala (2015).

From this result, a zero-sum revision game has an equilibrium and hence a value. Also there exist an equilibrium where each player’s strategy only depends on the last action prepared by the other player and the remaining time to the end of the game. This Markov equilibrium strategies satisfy equation (1). To interpret this equation, suppose that at time $t$ from the end, player $i$ has a revision opportunity whereas player $-i$’s last prepared action is $x_{-i}$. After preparing an action $y_i \in X_i$, the players will be playing a revision game with length $t$ and where the initial action profile is $x(0) = (y_i, x_{-i})$, that is game $\Gamma(t, (y_i, x_{-i}))$. Thus, at a revision time, player $i$ has the opportunity to choose which revision game he prefers to play in the set $\{\Gamma(t, (y_i, x_{-i}))\}_{y_i \in X_i}$. Because the equilibrium payoff associated to $\Gamma(t, (y_i, x_{-i}))$ is $u(t, (y_i, x_{-i}))$, it is optimal for player $i$ to prepare an action $y_i \in BR_i^{u(t)}(x_{-i})$. That is, if a player has a revision opportunity at $t$, then he will prepare an action that is a best response to the other player’s last prepared action in a component game where the player payoff is $u(t)$. Thus, for any remaining time $t \geq 0$ before the deadline, a player continuation payoff only depends on the current state $x$ and $t$ and is $u(t, x)$.

From now on, we call revision value the mapping $(t, x) \mapsto u(t, x)$. The main result of this section is that the revision value is characterized as the solution of an ordinary differential equation. We deduce then regularity properties.

We first prove that $u(t, x)$ is continuous, in fact Lipschitz, with respect to time. When $t$ and $t'$ are close, with high probability there will be no revision between $t$ and $t'$, thus $u(t, x)$ and
u(t',x) are close. In other words, a small increase of the duration of revision game has a small effect in players equilibrium payoff. More formally:

**Lemma 2** For all \( x \in X \), \( u(.,x) \) is \( 2\lambda\|U\|\)-Lipschitz.

**Proof.** Consider two remaining times \( t' \) and \( t \), with \( t' > t \). Since histories in \( \Gamma(t,x) \) are also histories in \( \Gamma(t',x) \), any strategy \( \sigma'_i \) in \( \Gamma(t',x) \) induces naturally (by restriction) a strategy in \( \Gamma(t,x) \) (also denoted \( \sigma'_i \) for simplicity). On the other hand, any strategy \( \sigma_i \) in \( \Gamma(t,x) \) may be seen as the restriction of some strategy in \( \Gamma(t',x) \). For instance, one may define a strategy \( \hat{\sigma}_i \) in \( \Gamma(t',x) \) in which player \( i \) uses \( \sigma_i \) when the remaining time is less than \( t \), and does not revise his action when the remaining time is between \( t \) and \( t' \). As a consequence, \( u(t,x) = \sup_{\sigma'_1} \inf_{\sigma'_2} \mathbb{E}_{\sigma}[u(x(t))] \) where the supremum and infimum range over strategies in \( \Gamma(t',x) \).

For strategy profile \( (\sigma'_1, \sigma'_2) \), the probability that at least one revision time occurs in the interval \( (t,t'] \) is \( (1 - e^{-\lambda|t-t'|}) \), so that \( P_{\sigma'}(x(t) = x(t')) \geq e^{-\lambda|t-t'|} \), which implies,

\[
\mathbb{E}_{\sigma'}[|u(x(t)) - u(x(t'))|] \leq 2\|U\|(1 - e^{-\lambda|t-t'|}) \leq 2\|U\|\lambda|t-t'|.
\]

The result follows. ■

Next, we show that the value function \( u(t,x) \) can be obtained as the solution of a system of differential equations.

**Proposition 1** For all \( t \geq 0 \) and all \( x \in X \), the value \( u(t,x) \) of the revision game \( \Gamma(t,x) \) is the unique solution of the following system of differential equations:

**Evolution equations:** \( \forall x \in X, \forall t \geq 0, \)

\[
\frac{\partial u(t,x)}{\partial t} = \lambda_1(u^+(t,x) - u(t,x)) + \lambda_2(u^-(t,x) - u(t,x)), \ u(0,x) = U(x). \tag{2}
\]

**Proof.**
Let us first observe that the value satisfies the following Dynamic Programming Principle. For \( s \geq 0 \), denote \( u^s(t, x) \) the value of the revision game with length \( t \), initial position \( x \), and component game \( U = u(s) \). Then \( u^0(t, x) = u(t, x) \), \( u^t(0, x) = u(t, x) \). More generally:

**Dynamic Programming Principle:**

\[
\forall x \in X, \forall t \geq 0, \forall s \in [0, t], \; u^s(t - s, x) = u(t, x). \tag{3}
\]

Consider now the game with length \( t \), and let \( s \) be the realization of the waiting time before the first revision opportunity, drawn from the exponential distribution. If \( s \geq t \), no revision occurs and the payoff is \( u(x) \). If \( s \leq t \) and player 1 has the opportunity to revise, the new payoff will be \( u(t - s, \sigma_1(x, s), x_2) = u^+ (t - s, x) \), and if player 2 has the opportunity to revise, the new payoff will be \( u(x_1, \sigma_2(x, s), t - s) = u^- (t - s, x) \). So that:

\[
u(t, x) = U(x) e^{-\lambda t} + \int_{s=0}^{t} \lambda e^{-\lambda s} \left( qu^+ (t - s, x) + (1 - q) u^- (t - s, x) \right) ds. \tag{4}\]

The change of variable \((s \mapsto t - s)\) yields:

**Dynamic programming equation:**

\[
u(t, x) = U(x) e^{-\lambda t} + \int_{s=0}^{t} e^{-\lambda(t-s)} \left( \lambda_1 u^+ (s, x) + \lambda_2 u^- (s, x) \right) ds. \tag{5}\]

Since \( u^+ \) and \( u^- \) are continuous, \( u \) is continuously differentiable in \( t \), and the law of motion of the continuation values is given by the system:

\[
\frac{\partial u(t, x)}{\partial t} = \lambda_1 (u^+(t, x) - u(t, x)) + \lambda_2 (u^-(t, x) - u(t, x)), \; u(0, x) = U(x).
\]

The mapping \((u \mapsto \lambda_1 u^+ + \lambda_2 u^- - \lambda u)\) is Lipschitz continuous, so by the Cauchy-Lipschitz Theorem, this ordinary differential equation has a unique solution. ■
From this characterization, we deduce the following.

**Corollary 1** The revision game has a Markov perfect equilibrium in pure strategies.

**Proof.** Consider a revision game \( \Gamma(t, x) \). As argued before, this game has an equilibrium, or equivalently, players have optimal strategies guaranteeing the value, but these are possibly mixed. We deduce that there are also pure optimal strategies. The main argument is that a player who is randomizing can select any of the pure actions in the mixing support without affecting the continuation payoffs. The zero-sum aspect is crucial here, in a non zero-sum game, it is possible to change the continuation payoff of one player without affecting the continuation payoff of the opponent.

We now detail the construction, note that we have to make sure that the strategy is measurable w.r.t. time. To this end, we identify the action set \( X_i \) with \( \{1, \ldots, |X_i|\} \) according to some fixed ordering. This allows us to break ties in a measurable way.

Define a strategy \( \sigma(1) \) of player 1 in the game \( \Gamma(t, x) \) as follows:

- If the first beat of the clock occurs at time \( s \) and if player 1 is called to play, then she chooses the action,

\[
x'_1 = \min \left( \argmax_{y_1 \in X_1} u^{t-s}(y_1, x_2) \right).
\]

That is, instead of playing the original Markov equilibrium strategy (which is possibly mixed), player 1 plays the action in the support of the mixed strategy which is minimal in the given ordering.

- Otherwise, player 1 plays some (possibly mixed) optimal strategy given by some fixed Markov perfect equilibrium.

Against any strategy of player 2, this strategy \( \sigma(1) \) guarantees the payoff:

\[
u(x)e^{-\lambda t} + \int_{s=0}^{t} \lambda e^{-\lambda s} \left( qu^+(t-s, x) + (1-q)u^-(t-s, x) \right) ds.
\]
From equation (4), this is precisely the value $u(t, x)$, and therefore $\sigma(1)$ is optimal in the game $\Gamma(t, x)$.

So in the game $\Gamma(t, x)$, player 1 has an optimal strategy which is pure at the first beat of the clock. Iteratively, define for every $n \geq 1$, the following strategy $\sigma(n)$ of player 1:

- If for some $m \leq n$, player 1 receives a revision opportunity at the $m$-th beat of the clock occurring at time $s$, player 1 chooses the action $x_1' = \min \left( \arg\max_{y_1 \in X_1} u^{t-s}(y_1, x_2(s)) \right)$, where $x_2(s)$ denotes the current action of player 2 at time $s$.

- Otherwise, player 1 plays some (possibly mixed) optimal strategy given by some fixed Markov perfect equilibrium.

A simple induction argument shows that $\sigma(n)$ is optimal for player 1 in the game $\Gamma(t, x)$. Then, for any length $t$, the clock will almost surely beat finitely many times. Thus, letting $n$ go to infinity, shows that the strategy $\sigma(\infty)$ which plays, at any time $s$ the pure action $x_1' = \min \left( \arg\max_{y_1 \in X_1} u^{t-s}(y_1, x_2(s)) \right)$, is a pure optimal Markov strategy of player 1.

By symmetry, the same holds for player 2, which concludes the proof.

Our next result states that the revision value $u(t, x)$ is continuous with respect to the parameters of the model: the payoff matrix $U$ and the revision parameters $\lambda$ and $q$.

**Lemma 3** The revision value $u(t, x)$ is 1-Lipschitz in $U$, Lipschitz-continuous in $\lambda \in (0, \infty)$, continuous and non-decreasing in $q \in (0, 1)$.

This result is in contrast with properties of non zero-sum revision games, see Calcagno et al. (2014). In a Battle of the Sexes games, a slight change in the relative speed of players, measured by $q$, may drastically change the equilibrium payoff. Further, it might be advantageous to be relatively slower that then opponent. This cannot be the case in a zero-sum game.

**Proof.**
i) **1-Lipschitz in** \( U \). Consider first two payoff matrices \( U \) and \( U' \). For any \( T \geq 0 \) and \( x \) in \( X \), the strategy spaces of the players in the revision games \( \Gamma(T, x) \) and \( \Gamma'(T, x) \) with respective component games \( U \) and \( U' \) are the same. The payoffs of a given strategy profile in the two games can be written \( \mathbb{E}_P(U) \) and \( \mathbb{E}_P(U') \), with \( P \) the same probability distribution over \( X \), hence they can not differ by more than \( \|U - U'\| := \max_x |U(x) - U'(x)| \). Consider now the equilibrium strategy in the revision game with component game \( U \). Player 1 's equilibrium strategy yields a payoff at least \( u(T, x) \), no matter the strategy of player 2. If player 1 uses the same strategy in the revision game with component game \( U' \), she cannot lose more than \( \|U - U'\| \). Thus, we have \( |u(T, x) - u'(T, x)| \leq \|U - U'\| \) for all \( T \) and \( x \).

ii) **Lipschitz-continuity in** \( \lambda \). For fixed \( q \), modifying \( \lambda \) just changes the frequency of rings of the Poisson clock, but not the relative frequency of rings of player 1 and player 2. In other words a change in \( \lambda \) is equivalent to a change in the unit of time. Namely, denoting \( u(t, x)|_\lambda \) the value of \( \Gamma(t, x) \) with parameter \( \lambda \), we have \( u(t, x)|_{\lambda'} = u(\frac{t}{\lambda}, x)|_{\lambda}. \) Thus, \( u(t) \) is Lipschitz in \( \lambda \) since it is Lipschitz in \( t \).

iii) **Continuity in** \( q \). Let us prove the continuity of \( u(t, x) \) in the parameter \( q \in [0, 1] \), \( \lambda \) and \( U \) being fixed. Given a strategy profile \( \sigma \), denote \( \mathbb{E}_\sigma \) the expectation with respect to \( P_\sigma \) when the parameter is \( q \), and \( \mathbb{E}'_\sigma \) the expectation with respect to \( P'_\sigma \) when the parameter is \( q' \). We have the following uniform continuity property:

\[
\forall \varepsilon > 0, \exists \alpha > 0, |q - q'| \leq \alpha \Rightarrow \forall \sigma, |\mathbb{E}_\sigma(U) - \mathbb{E}'_\sigma(U)| \leq \varepsilon
\]

To see this, fix a strategy \( \sigma \) and compare \( \mathbb{E}_\sigma(U) \) with \( \mathbb{E}'_\sigma(U) \). To evaluate these expectations, what matters is the distribution of the total number of beats, and of the alternation between players. This distribution is continuous with respect to \( q \). For any \( \varepsilon > 0 \), there is an integer \( M \) large enough such that, with probability at least \( 1 - \varepsilon \), the total number of beats is no more than \( M \). Then, conditional on a number of beats \( m \leq M \), finitely many configurations of alternations of
players are possible. For given $m$, the conditional probability of these configurations is continuous with respect to $q$ (for each configuration, this can be written $q^k(1-q)^{m-k}$, with $k$ the number of revisions for player 1 in the configuration). So, for $q'$ close enough to $q$, the differences in probabilities are bounded by $\varepsilon$.

The payoff function of the revision game $\Gamma(t,x)$ with parameter $q$, is thus close to the payoff function of the revision game $\Gamma'(t,x)$ with parameter $q'$, uniformly over strategies. The values of the two games are therefore also close to each other.

iv) Monotonicity in $q$. Take $q' < q$ and let $u(t,x)|_q$ be the revision game value for a given $q$. We prove that $u(t,x)|_{q'} \leq u(t,x)|_q$ for all $t$ and $x$. This is a direct consequence of the Gronwall inequality. At $t = 0$, we have $u(0)|_{q'} = u(0)|_q = U$ and from equation (2) we have $\left. \frac{\partial u(0,x)|_{q'}}{\partial t} \right|_{q'} \leq \left. \frac{\partial u(0,x)|_q}{\partial t} \right|_{q}$ for all $x$. From Gronwall, the inequality is maintained over time and $u(t,x)|_{q'} \leq u(t,x)|_q$ for all $t$ and $x$.

4 Limit revision value

In this section we analyse the asymptotic behavior of the revision value, as the duration $T$ of the revision phase goes to infinity, or equivalently, as the intensity of arrivals of revisions $\lambda$ goes to infinity (for $T$ fixed).

We denote by $\underline{R}(x)$ and $\overline{R}(x)$ the lim inf and lim sup of the revision value $u(t,x)$ as $t$ goes to infinity. If $\underline{R}(x) = \overline{R}(x) = R$ is independent of $x$, then we call $R$ the limit revision value. We will prove the existence of the limit revision value and compare it with the static solutions of the game: $V$, $S_1$ and $S_2$.

Proposition 2 For each component game $U$ and revision parameters $\lambda > 0$ and $q \in (0,1)$, there
exists a number $R \in [S_1, S_2]$, such that for all $t \geq 0$ and all $x \in X$,
\begin{equation}
|u(t, x) - R| \leq 2\|U\| \exp(-\lambda(1 - \max\{q, 1 - q\})t).
\end{equation}

It follows directly that the limit revision value $R = \lim_t u(t, x)$ exists and is independent of the initial position $x$. The number $R$ satisfies $S_1 \leq R \leq S_2$, thus if the component game has a pure Nash equilibrium, then $V = S_1 = S_2 = R$.

Remark that the condition $0 < q < 1$ is important. If $q = 1$, only player 1 can change her action and the value depends on the initial action of player 2.

**Proof.** Consider the maximum $M(t) = \max_{x \in X} u(x, t)$ of the revision value in the game of length $t$. If $t > s$, by the dynamic programming principle (3), we have for all $x$, $u(t, x) = u^s(t - s, x) \leq M(s)$ so $M(t) \leq M(s)$. $M(t)$ is non-increasing with time and thus converges to some $M_\infty$. Similarly, $m(t) = \min_{x \in X} u(t, x)$ is non-decreasing and converges to some $m_\infty$, and we have
\[ m = m(0) \leq m(t) \leq m_\infty \leq M_\infty \leq M(t) \leq M = M(0). \]

For a given, $t \geq 0$, let $y$ be such that $m(t) = u(t, y)$ and $x$ be such that $M(t) = u(t, x)$. From the dynamic programming equation in $\Gamma(t, x)$, we have:
\[ M(t) = u(t, x) \leq \|U\| e^{-\lambda t} + \int_{s=0}^{t} \lambda e^{-\lambda(t-s)} (qM(s) + (1-q)u(s, x_1, y_2))ds \]
since $u^+(s, x) \leq M(s)$ and $u^-(s, x) \leq u(s, x_1, y_2)$. Similarly, considering $\Gamma(t, y)$, we have:
\[ m(t) = u(t, y) \geq -e^{-\lambda t} \|U\| + \int_{s=0}^{t} \lambda e^{-\lambda(t-s)} ((1-q)m(s) + qu(s, x_1, y_2))ds. \]

Define $w(t) = M(t) - m(t)$. If $q \geq 1/2$, we write $qM(s) - (1-q)m(s) = qw(s) + (2q - 1)m(s)$
and deduce:

\[ w(t) \leq 2\|U\|e^{-\lambda t} + q \int_0^t \lambda e^{-\lambda(t-s)} w(s) \, ds + \int_0^t \lambda e^{-\lambda(t-s)} (2q - 1)(m(s) - u(s, x_1, y_2)) \, ds \]

\[ \leq 2\|U\|e^{-\lambda t} + q \int_0^t \lambda e^{-\lambda(t-s)} w(s) \, ds, \]

where the second inequality follows from \((2q - 1)(m(s) - u(s, x_1, y_2)) \leq 0\).

Similarly, if \(q \leq 1/2\), we write \(qM(s) - (1 - q)m(s) = (1 - q)w(s) + (2q - 1)M(s)\) and get,

\[ w(t) \leq 2\|U\|e^{-\lambda t} + (1 - q) \int_0^t \lambda e^{-\lambda(t-s)} w(s) \, ds + \int_0^t \lambda e^{-\lambda(t-s)} (2q - 1)(M(s) - u(s, x_1, y_2)) \, ds \]

\[ \leq 2\|U\|e^{-\lambda t} + (1 - q) \int_0^t \lambda e^{-\lambda(t-s)} w(s) \, ds. \]

We deduce that for each \(q\),

\[ w(t) \leq 2\|U\|e^{-\lambda t} + \max\{q, 1 - q\} \int_0^t \lambda e^{-\lambda(t-s)} w(s) \, ds. \]

Denoting \(g(t) = e^{\lambda t}w(t)\), we have,

\[ g(t) \leq 2\|U\| + \max\{q, 1 - q\} \int_0^t \lambda g(s) \, ds. \]

Applying the Gronwall inequality to the function \(g\) we obtain,

\[ g(t) \leq 2\|U\|e^{\lambda t \max\{q, 1 - q\}}. \]

We deduce finally that,

\[ w(t) \leq 2\|U\|e^{\lambda t \max\{q, 1 - q\} - 1}, \]

which concludes the proof: this implies \(M_\infty = m_\infty = R\) and \(|u(t, x) - R| \leq w(t)|.\)
To show $S_1 \leq R \leq S_2$, it is enough to prove that,

$$S_1 \leq \lim \inf_t u(t, x) \leq \lim \sup_t u(t, x) \leq S_2.$$ 

Consider the strategy of player 1 which always prepares a maxmin action, i.e., an optimal action in the Stackelberg game where player 1 moves first. Since the length of the revision phase is arbitrary long and $\lambda q > 0$, the probability that player 1 gets at least one revision opportunity, is arbitrarily close to one. Hence, this strategy guarantees player 1 get approximately $S_1$ when $T$ is large and $\lim \inf_t u(t, x) \geq S_1$. Dually, $\lim \sup_t u(t, x) \leq S_2$. ■

From this result, we can define for the matrix game $U$, the limit revision value $R(U) = \lim_t u(t, x)$, independently of $x$. Notice that the dynamic programming principle (3) gives that:

$$\forall s \geq 0, \quad R(U(s)) = R(U).$$

As for the fixed duration revision value, the limit revision value $R$ is continuous in the parameters of the model, as stated in the following lemma.

**Lemma 4** The revision game value $R$ is $1$-Lipschitz in $U$, independent of $\lambda \in (0, \infty)$, continuous and non-decreasing in $q \in (0, 1)$. It converges to the pure minmax $S_2$ when $q$ goes to 1, and to the pure maxmin $S_1$ when $q$ goes to 0.

**Proof.**

i) $R$ is $1$-Lipschitz in $U$. Take two payoff matrices $U$ and $U'$. From Lemma 3, for all $t$ and $x$,

$$|u(t, x) - u'(t, x)| \leq \|U - U'\|.$$ 

Letting $t \to \infty$ gives $|R(U) - R(U')| \leq \|U - U'\|$.

ii) $R$ does not depend on $\lambda$. We have seen in the proof of Lemma 3 that a change in $\lambda$
corresponds to a change in the unit of time. Taking the limit when $t$ goes to infinity removes the dependency on $\lambda$, and the revision value $R$ depends only on $q$ and $U$.

**iii) $R$ in continuous in $q \in (0,1)$**. Fix $q \in (0,1)$ and take a sequence $q_n \to q$. For any $\varepsilon > 0$, there exists $t$ large enough such that $u(t, x) \in [R - \varepsilon, R + \varepsilon]$, for all $x$ in $X$. From Lemma 3, $u(t, x)$ is continuous in $q$, and for $n$ large enough, the revision value $u_n(t, x)$ of the game with parameter $q_n$ belongs to $[R - 2\varepsilon, R + 2\varepsilon]$, for all $x$. Thus, for $n$ large enough, the limit revision value of the game with parameter $q_n$ belongs to $[R - 2\varepsilon, R + 2\varepsilon]$, concluding the proof.

**iv) $R$ in non-decreasing in $q$**. From Lemma 3, $u(t, x)$ is continuous in $t$ and non-decreasing in $q$. Thus, $R = \lim_{t \to \infty} u(t, x)$ is also non-decreasing in $q$.

**v) $R$ converges to $S_2$ when $q$ goes to 1**. Take $t$ large enough such that in the game with duration $t$, there is at least one revision with probability at least $1 - \varepsilon$. Whenever called to play, Player 1 can choose a pure best reply to the current position in the component game $U$. With probability at least $(1 - \varepsilon)q$, there is at least one revision opportunity and player 1 is called to play at the last ring. Thus for all $x$:

$$u(t, x) \geq (1 - \varepsilon)qS_2 - (1 - (1 - \varepsilon)q)\|U\|.$$

By letting $t$ go to infinity and $\varepsilon$ go to 0, we get that the revision value satisfies $R \geq qS_2 - (1 - q)\|U\|$. Since $R \leq S_2$, $R$ converges to $S_2$ when $q$ goes to 1.

Symmetrically, $R$ converges to $S_1$ when $q$ goes to 0. ■

### 5 2 × 2 zero-sum revision games

In this section, we focus on $2 \times 2$ component games. For this class, we characterize the equilibrium strategies and the limit revision value $R$.

Consider a $2 \times 2$ zero-sum game and denote $X_i = \{\alpha, \beta\}$ the actions available to player $i$.  

We focus on the generic case where \( u \) is regular. First, we remark that, for regular \( 2 \times 2 \) zero-sum games, there are only three possible scenarios, describing the players’ best responses in the component game.

**Definition 1**

1. **Scenario DD**: Each player has a strictly dominant action.

2. **Scenario DN**: One player has a dominant action, the other player’s best response varies with the opponent’s action.

3. **Scenario NN**: Each player’s best response varies with his opponent’s action.

The game has a pure Nash equilibrium for the scenarios DD and DN, and a mixed one for the scenario NN.

Without loss of generality, let us normalize payoffs by setting \( U(\alpha, \alpha) = 0 \). Observe then that for generic payoffs, if \( U(\alpha, \beta) + U(\beta, \alpha) - U(\alpha, \alpha) - U(\beta, \beta) = 0 \), each player has a strictly dominant action: Player 1’s (Player 2’s) dominant action is \( \alpha \) if \( U(\beta, \alpha) > 0 \) (resp. \( U(\alpha, \beta) > 0 \)) and \( \beta \) if \( U(\beta, \alpha) < 0 \) (resp. \( U(\alpha, \beta) < 0 \)).

Now, generically, \( U(\alpha, \beta) + U(\beta, \alpha) - U(\alpha, \alpha) - U(\beta, \beta) \neq 0 \), and it is possible to further normalize payoffs by setting \( U(\alpha, \beta) + U(\beta, \alpha) - U(\alpha, \alpha) - U(\beta, \beta) = 1 \). Thus, without loss of generality, we adopt thereafter the following specification of \( U \):

\[
\begin{array}{cc}
\alpha & \beta \\
\hline
\alpha & 0 & b \\
\beta & c & b + c - 1
\end{array}
\]

This parametrization is compatible with each of the three scenarios as illustrated in the following example.

**Example 1**

1. If \( c, b > 1 \), the component game scenario is DD: the dominant actions are \( \beta \) for Player 1 and \( \alpha \) for Player 2.
2. If $0 < b < 1$ and $c > 1$, the component game scenario is DN: the dominant action for Player 1 is $\beta$, whereas for Player 2, $BR_2^U(\alpha) = \alpha$, $BR_2^U(\beta) = \beta$.

3. If $0 < b < 1$ and $0 < c < 1$, the component game scenario is NN: for Player 1, $BR_1^U(\alpha) = \beta$, $BR_1^U(\beta) = \alpha$, whereas for Player 2, $BR_2^U(\alpha) = \alpha$, $BR_2^U(\beta) = \beta$.

Consider a Markov equilibrium of the revision game. For any $t \geq 0$, we denote the continuation payoff matrix as follows:

\[
\begin{array}{cc}
\alpha & \beta \\
\alpha & u(t, \alpha, \alpha) & u(t, \alpha, \beta) \\
\beta & u(t, \beta, \alpha) & u(t, \beta, \beta)
\end{array}
\]

where $u(0, x) = U(x)$. The next Proposition shows that, studying the equilibrium strategies for each of the three scenarios, fully characterizes the equilibrium play in the revision game.

**Proposition 3** Consider a regular $2 \times 2$ payoff matrix $U$. The revision game equilibrium is as follows.

1. If the component game scenario is DD, then each player prepares her dominant action in the component game, for all $t \geq 0$. It results that $R = V$.

2. If the component game scenario is DN, then there is a player, say Player $i$, who as a dominant action, and prepares it at all $t \geq 0$.

   There exists $t^* > 0$ such that:

   - for $t < t^*$, Player $-i$ prepares her best response in the component game to the current action of the Player $i$;

   - for $t > t^*$, Player $-i$ prepares her component game best response to Player $i$’s dominant action in the component game.

   It results that $R = V$. 

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3. If the component game scenario is NN, then there are $t^*_2 > t^*_1 > 0$ and a player, say Player $i$, such that:

(a) Player $i$ prepares her component game best response to Player $-i$’s current action when $t < t^*_1$, and prepares a fixed action when $t > t^*_1$.

(b) Player $-i$ prepares her component game best response to Player $i$’s current action when $t < t^*_2$. When $t > t^*_2$, Player $-i$ prepares her component game best response to the fixed action that Player $i$ prepares when $t > t^*_1$.

It results that $R \neq V$ for generic NN component games.

We provide here the intuition for the proof, the details are in the appendix. The proof is divided in three steps. First, if at some $t$ from the deadline, the continuation payoff matrix $u(t)$ is in scenario DD, then at any $t' > t$ the payoff matrix $u(t')$ is also in scenario DD, i.e. the players’ dominant actions are the same in $t$ and in $t'$. Thus, for a component game in scenario DD, at all revision opportunities, players prepare the same dominant actions which correspond to the pure Nash equilibrium of the component game.

Second, we consider scenario DN. We show that if at some $t$, the matrix $u(t)$ is in scenario DN, then there is $t^* > t$, such that for $t' < t^*$ the continuation payoff matrix $u(t')$ is in the same scenario DN, whereas for $t' > t^*$ it is in scenario DD. Further, the scenario DD in the early stages of the revision game ($t' > t^*$) is such that players prepare pure Nash equilibrium of the component game. Once the prepared actions reach this action profile, players do not revise anymore.

Third, we analyze scenario NN, the most interesting case. We show that if $U$ is in scenario NN and payoffs are generic, then there is a finite $t^*_1 > 0$ such that for $t < t^*_1$ the continuation payoff matrix $u(t)$ is in the same scenario NN (with the same best responses). For $t > t^*_1$ and close to $t^*_1$, the payoff matrix $u(t)$ is in scenario DN, and one can apply the analysis of the former two steps.
It results that in the early stages of the revision game, players prepare a *sur-place* action profile, and do not revise until $t^*_1$ from the deadline. It is only when the remaining time is less than $t^*_1$ that players start ‘wrestling’ and change their prepared action at any revision opportunity.

This is illustrated in Figure 1.

![Figure 1: Evolution of scenarios](image)

In Figure 1, the component game scenario is NN: player 2 prefers to play the same action as player 1, who prefers to play the opposite. In red is represented the equilibrium behavior, when the deadline is close, $t < t^*_1$: a revising player prepares the best response to the other player current action. In blue is represented the equilibrium behavior when the remaining time is between $t^*_2$ and $t^*_1$. The continuation payoff function $u(t)$ is in scenario DN: $\alpha$ is dominant for Player 1. If the prepared action profile is $(\alpha, \alpha)$, players do not change their actions. Otherwise, at least one player changes his action. Compared to the strategy used for $t < t^*_1$, only Player 1’s strategy changes: she now prepares $\alpha$ when player 2 prepares $\alpha$ (orange vertical arrow). In black is represented the equilibrium behavior for $t > t^*_2$. Both players’ dominant action is $\alpha$. Compared to the equilibrium strategy for $t \in (t^*_1, t^*_2)$, the only difference is that player 2 prefers $\alpha$ also when player 1 is playing $\beta$ (orange horizontal arrow).

The *sur-place* action depends on the specific values of $b$, $c$ and the relative speed of revision.
\((1 - q)/q\). Note that \(U\) being in scenario NN requires that \(0 < b, c < 1\).\(^2\) Without loss of
generality, we assume \(0 < b, c < 1\) from now on.

Denote \(\lambda_1 = \lambda q\), \(\lambda_2 = \lambda (1 - q)\), define \(\sigma := |\lambda_1^2 - 6\lambda_1\lambda_2 + \lambda_2^2|^{\frac{1}{2}}\) and let \(\hat{t}(A, B, \lambda_1, \lambda_2)\) be the
smallest positive \(t\) such that,
\[
e^{-\frac{\lambda_1 + \lambda_2}{2}t} + A \cos\left(\frac{\sigma t}{2}\right) + \frac{(\lambda_2 - \lambda_1)A + 2\lambda_2 B}{\sigma} \sin\left(\frac{\sigma t}{2}\right) = 0. \tag{7}
\]
If this equation has no solution in \(t \geq 0\), then \(\hat{t}(A, B, \lambda_1, \lambda_2)\) is set to infinity. Let,

\[
\begin{align*}
t_{\alpha,\alpha} & = \hat{t}(2c - 1, 2b - 1, \lambda_1, \lambda_1) \\
t_{\alpha,\beta} & = \hat{t}(2b - 1, 1 - 2c, \lambda_2, \lambda_1) \\
t_{\beta,\alpha} & = \hat{t}(1 - 2b, 2c - 1, \lambda_2, \lambda_1) \\
t_{\beta,\beta} & = \hat{t}(1 - 2c, 1 - 2b, \lambda_1, \lambda_1)
\end{align*}
\]

Then we have:

**Lemma 5** Consider a \(2 \times 2\) component game in scenario NN. The sur-place action profile is,

\[
\hat{x} = \arg\min_{y \in X} t_y.
\]

Then \(t^*_1 = t_{\hat{x}}\), \(\hat{x}\) is a pure Nash equilibrium of \(u(t_{\hat{x}})\) and the limit revision value is \(R = u(t_{\hat{x}}, \hat{x})\).

The proof is in the Appendix.

We present now two specific cases. First, we consider the revision game where \(U\) is the
matching pennies game. This is a non-generic case where in equilibrium, each player prepares

\(^2\)Namely player 2 has a dominant action if if \(b > 1\) or \(b < 0\) whereas player 1 has a dominant action if \(c > 1\)
or \(c < 0\).
her best response in the component game at each revision opportunity. The limit revision value $R$ is then a simple function of $q$.

**Example 2** For $b = c = 1/2$, the component game is the matching pennies game:

$$
\begin{array}{cc}
\alpha & \beta \\
\alpha & 0 & 1/2 \\
\beta & 1/2 & 0 \\
\end{array}
$$

In this case, $\hat{t}_x = \infty$ for all $x \in X$ and therefore, no matter the remaining time, at a revision opportunity each player prepares his component game best response to the other player current action. This implies that the prepared action profile will keep cycling counterclockwise. As $t$ goes to infinity, the equilibrium payoff converges to,

$$R = \frac{q}{2}.$$

Note that in this case $R > V = 1/4$ if and only if $q > 1 - q$, i.e., player 1 gains more in a revision game than in the one-shot game, only if he is faster than the other player. When $q$ goes to 1 or to 0, the fastest player gets approximately the Stackelberg second mover’s payoff.

As a second case, we consider games where players are equally fast. We thus set $q = 1/2$ and analyze how $R$ varies with $b$ and $c$. In this case, $R$ is trigonometric function of $b$, $c$ and $t_1^*$. The latter is defined as a solution of an equation of the form of (7).

**Example 3** Consider parameters $b$ and $c$ in $[0, 1]$ and $q = 1/2$. The sur-place action depends on $b$ and $c$. Figure 2 represents the regions for $b$ and $c$ corresponding to different sur-place actions.

Figure 2 outlines an interesting property of zero-sum revision games. For $b \simeq c \simeq 1/2$, a small change in the payoff $b$ can change the sur-place action to any other action. In other words,
Figure 2: Regions in the square $b \times c$ leading to each of the four action profile as the sur-place action. For $b = c$ we get the matching pennies case of Example 2.

there are regions for $b$ and $c$ in which the equilibrium strategy is “very” discontinuous in the parameters. Nevertheless, the resulting equilibrium payoff change smoothly (see Lemma 3).

What would player 1 prefer between playing the one-shot component game, or playing it within an long duration revision game? This is equivalent to comparing the component game value $V$ with the limit revision value $R$. We already know that when the component game equilibrium is in mixed strategies, then player 1 prefers playing the revision game rather than the component game if $q$ is large (Proposition 2). If one keeps $q = 1/2$, it is not a-priori obvious how $R$ compares to $V$. The next lemma provides a more explicit formula for $R$ in the case $q = 1/2$.

**Lemma 6** If $b, c \in [0,1] \times [0,1]$, and $q = 1/2$, the limit revision value is given by:

$$R(b, c) = \frac{1}{4} (2b + 2c - 1) + \frac{1}{2} (b + c - 1)(b - c) \sin(2\mu) + \frac{1}{4} (2c - 1)(2b - 1) \cos(2\mu), \quad (8)$$
where $\mu$ is the smallest $t$ in $\mathbb{R}_+$ satisfying:

\[
e^{-t} = \max\{(1 - 2c) \cos(t) + (1 - 2b) \sin(t), (1 - 2b) \cos(t) - (1 - 2c) \sin(t),
\]

\[
-(1 - 2b) \cos(t) + (1 - 2c) \sin(t), -(1 - 2c) \cos(t) - (1 - 2b) \sin(t)\}\.
\]

The value of the one-shot component game is $V = bc$. It is then possible to represent the values for $b$ and $c$ for which $R(b, c) > V$. This is illustrated panel (a) of Figure 3.

![Figure 3: Panel (a): The shaded area represents the region of parameters $(b, c)$ for which $R > V$. The lines are $c = b$ and $c = 1 - b$. Panel (b): the shaded area represents the region of parameters $(b, c)$ for which $R > V$ and is superposed over the regions corresponding to the four sur-place actions.](image)

From Figures 2 and 3, one might read that Player 1 prefers the revision game to the component game, when the sur-place action is $(\alpha, \alpha)$ or $(\beta, \beta)$. However, this is not necessarily the case as illustrated by panel (b) of Figure 3 that is obtained by superimposing panel (a) over Figure 2: there is no one-to-one relation between the preference over revision versus component game, and the equilibrium strategy played in the revision game.
6 Conclusion

We have analyzed asynchronous zero-sum revision games and we showed that equilibrium payoff and strategies differ from those in non zero-sum revision games studied in the literature. First, in zero-sum revision games the equilibrium payoff is continuous in all the parameters of the model and a players cannot lose from being faster than his opponent in revising. This applies both to the equilibrium payoff for a finite duration as well as for the limit of the equilibrium payoff as the duration goes to infinity. The opposite occurs for non-zero sum revision games. Second, whereas in the games analyzed in Calcagno et al. all the activity occurs at the beginning of the revision phase, this is not necessarily the case for the zero-sum case. If the component game equilibrium is mixed, then the revision game equilibrium displays a deadline effect where all the revision activity is concentrated toward the end of the revision phase.
7 Appendix

7.1 Proof of Proposition 3

We start with a preliminary result which shows the local uniqueness of equilibrium when the matrix of continuation payoff is generic. Let $u(t)$ be some $2 \times 2$ continuation payoff. We have:

**Lemma 7** If $u(t)$ is regular, then there is $\varepsilon > 0$ such that in all equilibria, and for all $t' \in [t+\varepsilon, t],$

$$\sigma_i(x, t') = BR_i^{u(t)}(x_{-i})$$

**Proof.** The result comes from the fact that $u(t)$ is a continuous function in time. Since $u(t)$ is regular, for any $x_{-i}$ there exists $x_i^*$ such that $BR_i^{u(t)}(x_{-i}) = \{x_i^*\}$. Hence, there is $\delta > 0$ such that for all $i$, all $x_i$ and all $y \neq x_i^*$, $u(t, x_i^*, x_{-i}) > u(t, y, x_{-i}) + \delta$. Fix $\varepsilon < \frac{\delta}{4\lambda\|U\|}$. Since $t \to u(t)$ is $2\lambda\|U\|$-Lipschitz,

$$u(t', x_i^*, x_{-i}) \geq u(t, x_i^*, x_{-i}) - 2\lambda\|U\|\varepsilon \geq u(t, y, x_{-i}) - 2\lambda\|U\|\varepsilon + \delta$$

$$\geq u(t', y, x_{-i}) - 4\lambda\|U\|\varepsilon + \delta > u(t', y, x_{-i}),$$

thus, $x^*$ is the unique best response at $t'$. ■

**Scenario DD:** Without loss of generality we can focus on the case $c, b > 1$ implying that the dominant actions in the component game are $\beta$ and $\alpha$ for Player 1 and Player 2, respectively. From Lemma 7, when the remaining time $t$ is close to 0, these are also the actions prepared by the players. Thus, using backward induction, as the remaining time $t$ increases, players continuation
payoffs evolve according to the following differential system:

\[
\begin{align*}
    u'(t, \alpha, \alpha) &= \lambda_1(u(t, \beta, \alpha) - t(t, \alpha, \alpha)) \\
    u'(t, \alpha, \beta) &= \lambda_1(u(t, \alpha, \alpha) - u(t, \alpha, \beta)) + \lambda_2(u(t, \beta, \beta) - u(t, \alpha, \beta)) \\
    u'(t, \beta, \alpha) &= 0 \\
    u'(t, \beta, \beta) &= \lambda_2(u(t, \beta, \alpha) - u(t, \beta, \beta))
\end{align*}
\]

where derivative is taken with respect to the remaining time \( t \). Solving this system with terminal condition \( u(0, x) = U(x) \) for all \( x \in X \), one finds that for any \( t \geq 0 \),

\[
\begin{align*}
    u(t, \beta, \alpha) - u(t, \alpha, \alpha) &= ce^{-\lambda_1 t} > 0 \\
    u(t, \beta, \beta) - u(t, \alpha, \beta) &= e^{-(\lambda_1 + \lambda_2)t}(ce^{\lambda_2 t} - 1) > 0 \\
    u(t, \alpha, \beta) - u(t, \beta, \alpha) &= e^{-(\lambda_1 + \lambda_2)t}(1 + (b-1)e^{\lambda_1 t}) > 0 \\
    u(t, \beta, \beta) - u(t, \beta, \alpha) &= (b-1)e^{-\lambda_1 t} > 0.
\end{align*}
\]

The first two inequalities, together with the equilibrium condition (1), guarantee that at any revision time \( t \) and no matter the last prepared action profile, player 1 prepares \( \beta \). The former two inequalities guarantee that at any revision time \( t \) and no matter the last prepared action profile, player 2 prepares \( \alpha \).

**Scenario DN**: Without loss of generality, we focus on the case \( 0 < b < 1 \) and \( c < 0 \). This implies that for \( t \) close to 0, in equilibrium Player 1 prepares \( \alpha \), independently of the last prepared action, and Player 2 prepares \( \alpha \) if and only if in the last prepared action of Player 1 is \( \alpha \). Thus, for \( t \) close to 0 the differential system describing the evolution of continuation payoffs is,
\[ u'(t, \alpha, \alpha) = 0 \]
\[ u'(t, \alpha, \beta) = \lambda_2(u(t, \alpha, \alpha) - u(t, \alpha, \beta)) \]
\[ u'(t, \beta, \alpha) = \lambda_1(u(t, \alpha, \alpha) - u(t, \beta, \alpha)) + \lambda_2(u(t, \beta, \beta) - u(t, \beta, \alpha)) \]
\[ u'(t, \beta, \beta) = \lambda_1(u(t, \alpha, \beta) - u(t, \beta, \beta)). \]

If \( \lambda_1 \neq \lambda_2 \), solving this system with terminal condition \( u(0, x) = U(x) \), one finds that for any \( t \geq 0 \),

\[ u(t, \alpha, \alpha) - u(t, \beta, \alpha) = -e^{-(\lambda_1+\lambda_2)t} \]
\[ + e^{-\lambda_1t} \left( 1 - c + b \frac{\lambda_2}{\lambda_1} (1 - e^{(\lambda_1-\lambda_2)t}) \right) \]
\[ u(t, \alpha, \beta) - u(t, \beta, \beta) = e^{-\lambda_1t} \left( 1 - c + b \frac{\lambda_2}{\lambda_1} (1 - e^{(\lambda_1-\lambda_2)t}) \right) \]
\[ u(t, \alpha, \beta) - u(t, \alpha, \alpha) = be^{-\lambda_2t} > 0 \]
\[ u(t, \beta, \alpha) - u(t, \beta, \beta) = e^{-(\lambda_1+\lambda_2)t}(1 - be^{\lambda_1t}). \]

All these expressions are strictly positive for \( t \) close enough to 0. We are interested in the smallest \( t \) such that at least one of these expressions is nil. Note first, that \( u(t, \beta, \alpha) - u(t, \beta, \beta) = 0 \) for \( t = t^* := \frac{1}{\lambda_1} \ln \left( \frac{1}{b} \right) > 0 \). Second, \( u(t, \alpha, \beta) - u(t, \alpha, \alpha) > 0 \) for all \( t \). Third, note that \( u(t, \alpha, \alpha) - u(t, \beta, \alpha) = u(t, \alpha, \beta) - u(t, \beta, \beta) - e^{-(\lambda_1+\lambda_2)t} \), which implies \( u(t, \alpha, \alpha) - u(t, \beta, \alpha) < u(t, \alpha, \beta) - u(t, \beta, \beta) \) for all \( t \). Hence, it is sufficient to show that for \( t \leq t^* \) one has \( u(t, \alpha, \alpha) - u(t, \beta, \alpha) > 0 \). This implies that when the remaining time is less than \( t^* \), the revision strategies coincide with the component game best responses.
Define \( f(t) := (u(t, \alpha, \alpha) - u(t, \beta, \alpha))e^{(\lambda_1 + \lambda_2)t} \), which has the same sign as \( u(t, \alpha, \alpha) - u(t, \beta, \alpha) \).

One has,

\[
f(t) = (1 - b - c)e^{\lambda_2 t} + b\frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} - 1.
\]

Note that \( f(0) = -c > 0 \) and \( f'(0) = \lambda_2(1 - b - c) > 0 \). Note also that \( f'(t) \) has the same sign as \( \frac{1}{\lambda_2}f'(t)e^{-\lambda_2 t} = b\frac{\lambda_1}{\lambda_2 - \lambda_1}(1 - e^{(\lambda_1 - \lambda_2)t}) + 1 - b - c \) which is positive for all \( t > 0 \) if \( \lambda_1 < \lambda_2 \). For \( \lambda_1 > \lambda_2, \frac{1}{\lambda_2}f'(t)e^{-\lambda_2 t} \) is positive for \( t < \log\left(\frac{\lambda_1(1-c)+\lambda_2(b+c-1)}{\lambda_1b}\right)/(\lambda_1 - \lambda_2) \) (\( \equiv \tilde{t} > 0 \)) and negative otherwise.

From this, we deduce that \( u(t, \alpha, \alpha) - u(t, \beta, \alpha) \) is positive at \( t = 0 \) and increasing in \( t \) (for \( \lambda_1 \leq \lambda_2 \)) or hump-shaped (for \( \lambda_1 > \lambda_2 \)). Thus, it is sufficient to show that \( u(t^*, \alpha, \alpha) - u(t^*, \beta, \alpha) > 0 \) for \( \lambda_1 > \lambda_2 \). Note that,

\[
u(t^*, \alpha, \alpha) - u(t^*, \beta, \alpha) = b\left(\frac{\lambda_1}{\lambda_2 - \lambda_1}(b^{\lambda_2/\lambda_1} - b) + 1 - b - c\right)
> b\left(\frac{\lambda_1}{\lambda_2 - \lambda_1}(b^{\lambda_2/\lambda_1} - b) + 1 - b\right),
\]

where the first inequality follows from \( c < 0 \). The last expression has the same sign as \( \phi(\lambda_2/\lambda_1) := 1 - b^{\lambda_2/\lambda_1} - \frac{\lambda_2}{\lambda_1}(1 - b) \) which is strictly positive for \( \lambda_2/\lambda_1 \in (0, 1) \). In facts, this function is nil for \( \lambda_2 = 0 \) and \( \lambda_2 = \lambda_1 \) and strictly concave because \( \phi''(\lambda_2/\lambda_1) = -b^{\lambda_2/\lambda_1}\log(b)^2 < 0 \), as \( b > 0 \).

If \( \lambda_1 = \lambda_2 = \lambda/2 \), solving this system with the terminal condition \( u(0, x) = U(x) \), one finds that for any \( t \geq 0 \),

\[
\begin{align*}
u(t, \alpha, \alpha) - u(t, \beta, \alpha) &= -e^{-\lambda t} + e^{-\lambda t/2}(1 - c - b\frac{\lambda t}{2}) \\
u(t, \alpha, \beta) - u(t, \beta, \beta) &= e^{-\lambda t/2}\left(1 - c - b\frac{\lambda t}{2}\right) \\
u(t, \alpha, \beta) - u(t, \alpha, \alpha) &= be^{-\lambda t/2} > 0 \\
u(t, \beta, \alpha) - u(t, \beta, \beta) &= e^{-\lambda t}(1 - be^{\lambda t/2}).
\end{align*}
\]
Applying exactly the same method, define \( f(t) := (u(t, \alpha, \alpha) - u(t, \beta, \alpha))e^{\lambda t}, \) which has the same sign as \( u(t, \alpha, \alpha) - u(t, \beta, \alpha). \) One has,

\[
f(t) = e^{\frac{\lambda t}{2}}(1 - c - b \frac{\lambda t}{2}) - 1.
\]

Note that \( f(0) = -c > 0 \) and,

\[
f'(t) = \frac{\lambda}{2} e^{\frac{\lambda t}{2}}(1 - c - b - b \frac{\lambda t}{2}).
\]

\( f' \) is positive for \( t < \frac{2(1-c-b)}{b\lambda} (\equiv \tilde{t} > 0) \) and negative otherwise. From this, we deduce that \( u(t, \alpha, \alpha) - u(t, \beta, \alpha) \) is hump-shaped. Thus, it is sufficient to show that \( u(t^*, \alpha, \alpha) - u(t^*, \beta, \alpha) > 0. \)

Note that,

\[
 u(t^*, \alpha, \alpha) - u(t^*, \beta, \alpha) = b(1 - c + b \ln(b) - b) > b(1 + b \ln(b) - b) > 0
\]

where the first inequality follows from \( c < 0. \) To prove the last inequality, note that \( 1 + b \ln(b) - b \) is decreasing and equals 0 for \( b = 1. \)

For all pairs \((\lambda_1, \lambda_2), u(\cdot, \cdot) \) solves (2) and is continuously differentiable. One may therefore compute the derivative \( u'(t^*, x) \) and we have:

\[
u(t^*, \beta, \alpha) - u(t^*, \beta, \beta) = 0,
\]

\[
u'(t^*, \beta, \alpha) - u'(t^*, \beta, \beta) = \lambda_1(u(t^*, \alpha, \alpha) - u(t^*, \beta, \alpha) - (u(t^*, \alpha, \beta) - u(t^*, \beta, \beta))) = -\lambda_1 e^{-(\lambda_1 + \lambda_2)t^*} < 0.
\]
This implies that there exists $\delta$ such that for $t \in (t^*, t^* + \delta)$, we have,

\[
\begin{align*}
    u(t, \alpha, \alpha) - u(t, \beta, \alpha) &> 0 \\
    u(t, \alpha, \beta) - u(t, \beta, \beta) &> 0 \\
    u(t, \alpha, \beta) - u(t, \alpha, \alpha) &> 0 \\
    u(t, \beta, \alpha) - u(t, \beta, \beta) &< 0.
\end{align*}
\]

We deduce that $u(t)$ is in scenario DD, and we may applying the preceding analysis to conclude that,

\[
R = u(t, \alpha, \alpha) = u(t^*, \alpha, \alpha) = u(\alpha, \alpha) = V,
\]

where the second equality follows from the continuity of $u$, the third from the differential system satisfied by $u$ on $[0, t^*]$.

**Scenario NN:** Without loss of generality we focus on the case $0 < b, c, < 1$. This implies that for $t$ close to 0, in equilibrium, Player 1 prepares an action different from Player 2’s last prepared action, whereas Player 2 prepares the same action as the last prepared by Player 1. Thus, for $t$ close to 0, the differential system describing the evolution of continuation payoffs is,

\[
\begin{align*}
    u'(t, \alpha, \alpha) &= \lambda_1(u(t, \beta, \alpha) - u(t, \alpha, \alpha)) \\
    u'(t, \alpha, \beta) &= \lambda_2(u(t, \alpha, \alpha) - u(t, \alpha, \beta)) \\
    u'(t, \beta, \alpha) &= \lambda_2(u(t, \beta, \beta) - u(t, \beta, \alpha)) \\
    u'(t, \beta, \beta) &= \lambda_1(u(t, \alpha, \beta) - u(t, \beta, \beta)).
\end{align*}
\]
If $\theta := |\lambda_1^2 - 6\lambda_1\lambda_2 + \lambda_2^2|^{1/2} \neq 0$, solving this system with terminal condition $u(0, x) = U(x)$, one finds that for $t$ close to 0:

$$u(t, \alpha, \alpha) = \frac{1}{2} \left( b + c - \frac{\lambda_1 e^{-\left((\lambda_1 + \lambda_2) t\right)} + \lambda_2}{\lambda_1 + \lambda_2} - \eta(b + c - 1, b - c, \lambda_1, \lambda_2) \right)$$  \hspace{1cm} (9)

$$u(t, \alpha, \beta) = \frac{1}{2} \left( b + c - \frac{\lambda_2 (1 - e^{-\left((\lambda_1 + \lambda_2) t\right)})}{\lambda_1 + \lambda_2} + \eta(b - c, b + c - 1, \lambda_2, \lambda_1) \right)$$

$$u(t, \beta, \alpha) = \frac{1}{2} \left( b + c - \frac{\lambda_2 (1 - e^{-\left((\lambda_1 + \lambda_2) t\right)})}{\lambda_1 + \lambda_2} - \eta(b - c, b + c - 1, \lambda_2, \lambda_1) \right)$$

$$u(t, \beta, \beta) = \frac{1}{2} \left( b + c - \frac{\lambda_1 e^{-\left((\lambda_1 + \lambda_2) t\right)} + \lambda_2}{\lambda_1 + \lambda_2} + \eta(b + c - 1, b - c, \lambda_1, \lambda_2) \right)$$

with,

$$\eta(A, B, \lambda_1, \lambda_2, t) := e^{-\frac{(\lambda_1 + \lambda_2) t}{2}} \left( A \cos \left( \frac{\theta t}{2} \right) + \frac{(\lambda_2 - \lambda_1) A + 2\lambda_1 B}{\theta} \sin \left( \frac{\theta t}{2} \right) \right).$$

This implies that,

$$u(t, \beta, \alpha) - u(t, \alpha, \alpha) = \mu(2c - 1, 2b - 1, \lambda_1, \lambda_2, t)$$  \hspace{1cm} (10)

$$u(t, \alpha, \beta) - u(t, \beta, \beta) = \mu(1 - 2c, 1 - 2b, \lambda_1, \lambda_2, t)$$  \hspace{1cm} (11)

$$u(t, \alpha, \beta) - u(t, \alpha, \alpha) = \mu(2b - 1, 1 - 2c, \lambda_2, \lambda_1, t)$$  \hspace{1cm} (12)

$$u(t, \beta, \alpha) - u(t, \beta, \beta) = \mu(1 - 2b, 2c - 1, \lambda_2, \lambda_1, t)$$  \hspace{1cm} (13)

with,

$$\mu(A, B, \lambda_1, \lambda_2, t) := \frac{e^{-\frac{(\lambda_1 + \lambda_2) t}{2}}}{2} \left( e^{-\frac{(\lambda_1 + \lambda_2) t}{2}} + A \cos \left( \frac{\theta t}{2} \right) + \frac{(\lambda_2 - \lambda_1) A + 2\lambda_2 B}{\theta} \sin \left( \frac{\theta t}{2} \right) \right).$$

If $\theta = 0$ (i.e. $\lambda_1 = (3 \pm \sqrt{8})\lambda_2$), then one has to replace the above expression by,

$$\mu(A, B, \lambda_1, \lambda_2, t) := \frac{e^{-\frac{(\lambda_1 + \lambda_2) t}{2}}}{2} \left( e^{-\frac{(\lambda_1 + \lambda_2) t}{2}} + A + \frac{(\lambda_2 - \lambda_1) A + 2\lambda_2 B}{2} \right).$$
Let $t_1^*$ be the smallest $t > 0$ such that at least one of expressions (10)-(13) is nil.

For generic values of $b, c$, the time $t_1^*$ is finite and only one of expression (10)-(13) is nil at $t_1^*$. Without loss of generality suppose that $u(t^*_1, \beta, \alpha) - u(t^*_1, \beta, \beta) = 0$. Then, for a remaining time $t > t_1^*$ and close to $t_1^*$, the continuation payoff matrix is in the scenario DN analyzed above and the same conclusion apply. That implies that there is $t_2^* > t_1^*$ such that for $t > t_2^*$, Players 1 and 2 prepare $\beta$ and $\alpha$, respectively. For $t_1^* < t < t_2^*$, Player 1 prepares the action that differs from Player 2's current action, whereas Player 2 prepares $\alpha$, no matter Player 1’s current action. When the remaining time is less than $t_1^*$, the revision strategies coincide with the component game best responses. This ends the proof.

7.2 Proof of Lemma 5

The first part of the Lemma follows from the proof for scenario NN in Proposition 3. To see that $R = u(t_1^*, \hat{x})$, observe that for a remaining time $t$ between $t_1^*$ and $t_1^*$, the continuation payoff matrix $u(t)$ is in scenario DN so that, as long as the prepared action profile is $\hat{x}$, players do not revise their actions. For a remaining time $t$ larger then $t_1^*$, $u(t)$ is in the scenario DD where the dominant actions are $\hat{x}$. Hence, for any $t > t_1^*$ one has $u(t, \hat{x}) = u(t_1^*, \hat{x})$. Observe that no matter the starting action profile, as $t$ goes to infinity, the probability that the prepared action profile becomes $\hat{x}$ before $t_2^*$, goes to 1. Hence $\lim_{t \to \infty} u(t, x) = u(t_1^*, \hat{x})$, for all $x \in X$. 

7.3 Proof of Lemma 6

In order to simplify notations, we will use the labels \((1, 2, 3, 4)\) for \(\{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\}\) and we define,

\[
\begin{align*}
    h_1(t) &= (1 - 2c) \cos(t) + (1 - 2b) \sin(t) \\
    h_2(t) &= (1 - 2b) \cos(t) - (1 - 2c) \sin(t) \\
    h_3(t) &= -(1 - 2b) \cos(t) + (1 - 2c) \sin(t) \\
    h_4(t) &= -(1 - 2c) \cos(t) - (1 - 2b) \sin(t).
\end{align*}
\]

Note for later use that \(h'_1(t) = h_2(t), h'_2(t) = h_4(t), h'_3(t) = h_1(t), h'_4(t) = h_3(t).\) By applying Lemma 4, since \(\lambda_1 = \lambda_2 = 1\), we have \(\theta = 2\). For \(i = 1, 2, 3, 4\), let us denote \(t_i\) the smallest \(t\) such that,

\[e^{-t} = h_i(t).\]

It follows that \(\mu := t^*_1 = \min\{t_1, t_2, t_3, t_4\}\) and \(R(b, c) = u(\mu, \hat{x})\) where \(\hat{x} \in A(b, c) := \arg \min_{i=1,2,3,4} t_i\).

Note that \(A(b, c)\) is generically a singleton. The number \(\mu\) is then the smallest \(t\) such that

\[e^{-t} = \max\{(1 - 2c) \cos(t) + (1 - 2b) \sin(t), (1 - 2b) \cos(t) - (1 - 2c) \sin(t),
\]

\[-(1 - 2b) \cos(t) + (1 - 2c) \sin(t), -(1 - 2c) \cos(t) - (1 - 2b) \sin(t)\}.

Let us assume that \(\hat{x} = 1 = (\alpha, \alpha)\), so that \(R = u(t^*_1, \alpha, \alpha)\). Using the ODE determining \(u(s)\) on the interval \([0, \mu]\), we find that,

\[R = u(\mu, \alpha, \alpha) = \frac{1}{2} \left( b + c - \frac{e^{-2\mu} + 1}{2} - e^{-\mu} ((b + c - 1) \cos(\mu) + (b - c) \sin(\mu)) \right).
\]
Substituting $e^{-\mu} = (1 - 2c) \cos(\mu) + (1 - 2b) \sin(\mu)$, we obtain,

$$
R = \frac{1}{2} \left[ b + c - \frac{1}{2} + \frac{1}{2}(1 - 2c)(1 - 2b) \cos^2(\mu) - \frac{1}{2}(1 - 2c)(1 - 2b) \sin^2(\mu).
+ 2(b + c - 1)(b - c) \sin(\mu) \cos(\mu) \right]
= \frac{1}{2} \left[ b + c - \frac{1}{2} + \frac{1}{2}(1 - 2c)(1 - 2b) \cos(2\mu) + (b + c - 1)(b - c) \sin(2\mu) \right].
$$

Doing the same computation for all profiles $\hat{x} = 2, 3, 4$ leads to the same result which proves the formula.

So far, we have proved the formula only for generic payoff matrices. In order to complete the proof, we extend it to all matrices. It is enough to prove that the expression (8) is continuous in $(b, c)$ since we know that $R(b, c)$ is continuous. Denote $\mu(b, c)$ the smallest $t$ in $\mathbb{R}_+$ satisfying:

$$
e^{-t} = \max\{(1 - 2c) \cos(t) + (1 - 2b) \sin(t), (1 - 2b) \cos(t) - (1 - 2c) \sin(t),
-(1 - 2b) \cos(t) + (1 - 2c) \sin(t), -(1 - 2c) \cos(t) - (1 - 2b) \sin(t)\}.
$$

We argue now that $\mu(b, c)$ is continuous in $(b, c)$.

The function $\mu(b, c)$ is lower semi-continuous (as the minimum of finitely many lower semi-continuous functions). Let us prove that it is upper semi-continuous on $[0, 1]^2 \setminus \{(1/2, 1/2)\}$.

**Claim 1** For $(b, c) \neq (1/2, 1/2)$, $\mu < \infty$ and there exists $i \in A(b, c)$ such that $-e^{-\mu} < h'_i(\mu)$.

**Proof.** Take $i \in A(b, c)$. The map $g(t) = e^{-t} - h_i(t)$ is positive on $[0, \mu)$ and equal to zero at $\mu$. Thus, its derivative at $\mu$ is non-negative and $-e^{-\mu} \leq h'_i(\mu)$. If $g'(\mu) = 0$, then $-e^{-\mu} = h'_i(\mu) = -h'_j(\mu)$ for some $j \neq i$. This implies that $j \in A(b, c)$ and $-e^{-\mu} < h'_j(\mu)$ which proves the claim.

Next, if $i \in A(b, c)$ is such that $-e^{-\mu} < h'_i(\mu)$, then using this claim, for all $\varepsilon > 0$ sufficiently

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small, we have $e^{-(\mu-\varepsilon)} - h_i(\mu - \varepsilon) < 0$ and $e^{-(\mu+\varepsilon)} - h_i(\mu + \varepsilon) > 0$. Since $h_i$ is continuous with respect to $(b, c)$, we deduce that

$$\limsup_{(b',c') \to (b,c)} \mu(b',c') \leq \mu + \varepsilon;$$

which proves that $\mu$ is upper semi-continuous, by sending $\varepsilon$ to zero.

Clearly, $\mu(b, c)$ goes to $+\infty$ when $(b, c)$ goes to $(1/2, 1/2)$ and we deduce that (8) is continuous on $[0, 1]^2$ as a function of $(b, c)$.

### 7.4 About Figure 3

We give now analytic expressions for the curves in Figure 3. Let us consider the four sets $K_i = \{(b, c) \in (0, 1)^2 : i \in A(b, c)\}$. We give parameterized equations for the sets $C_1 = K_1 \cap K_4$, $C_2 = K_2 \cap K_1$, $C_3 = K_3 \cap K_4$ and $C_4 = K_4 \cap K_2$.

If $(b, c) \in C_3$ and $(b, c) \neq (1/2, 1/2)$, then $\mu = \mu(b, c)$ is solution of the system:

$$\begin{cases}
    e^{-t} = h_3(t) = (2b - 1) \cos(t) + (1 - 2c) \sin(t) \\
    e^{-t} = h_4(t) = (2c - 1) \cos(t) + (2b - 1) \sin(t)
\end{cases}$$

This system is equivalent to:

$$(b, c) = \frac{1}{2}(1 + e^{-\mu}(\cos(\mu) - \sin(\mu)), 1 + e^{-\mu}(\cos(\mu) + \sin(\mu))) .$$

Conversely, if the triple $(b, c, t)$ solves the above system, then we claim that $t = \mu(b, c)$.

Indeed, if this holds, the map $e^{-s} - h_3(s)$ is equal to zero at $t$, and its derivative $-e^{-s} - h_4(s)$ is also equal to zero at $t$. The second derivative is $e^{-s} + h_3(s)$ which positive at $t$. The map $e^{-s} - h_3(s)$ is convex on the interval $[r, t]$ where $r$ is defined as the largest point in $[0, t]$ such that
$h'_3(r) = 0$. (because $h_3(s)$ is positive on this interval and $h''_3(s) = -h_3(s)$). $h_3$ attains its global maximum $M$ at $r$ and $h_3 \geq h_j$ for any $j$ on $[r, t]$ (needs a proof based on the periodicity of the maps $h_i$, but this is clear from a drawing). Since all the maps $h_i$ have the same global maximum $M$, this proves that $\mu \geq r$ since $e^{-t}$ is decreasing, and we deduce finally that $\mu = t$.

From this equivalence, we deduce that $C_3$ is exactly the curve given by the equation (14) for $\mu \in \mathbb{R}_+$ together with the point $(1/2, 1/2)$.

Moreover, for any point on this curve, we have

$$R(b, c) = \frac{1}{4}(1 + 2e^{-\mu}\cos(\mu)) + \frac{1}{4}e^{-2\mu}$$

to be compared to the value of the one-shot game

$$bc = \frac{1}{4}(1 + 2e^{-\mu}\cos(\mu)) + \frac{1}{4}e^{-2\mu}\cos(2\mu),$$

which suggests that $R$ is always larger than the value of the one-shot game on $C_3$ (computations to be checked).

The same analysis gives a parametric description of $C_1, C_2, C_4$.

References


1218–1248.


