Financial Economics
1: Time value of Money

Stefano Lovo

HEC, Paris

What is Finance?

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Finance studies how households and firms allocate monetary resources across **time** and **contingencies.**

Three dimensions:

- Return: how much?
- Time: when?
- Uncertainty: in what circumstances? (risk)

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Example

Choose one of the following three investment opportunities:

1. Today you invest *Eu* 100 and in 5 years time you will receive *Eu* 200 ;
2. Today you invest *Eu* 100 and in 4 years time you will receive *Eu* 190 ;
3. Today you invest *Eu* 100 and in 4 years time you will receive *Eu* 400 or nothing with probability 50%.

Overview of the Course

- Time
  - Time value of money: Compounding and Discounting.
  - Capital budgeting: How to choose among different investment projects (NPV).
- Uncertainty
  - How to describe uncertainty.
  - Portfolio management: How to choose between return and risk.
  - Capital Asset Pricing Model.

Definition

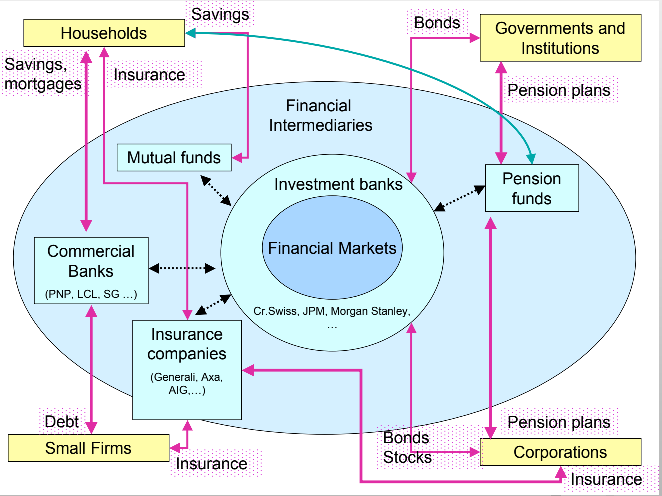
The financial system is a set of markets and intermediaries that are used to carry out financial contracts by allowing demand for different cash flows to meet the supply.

Tasks:

- Transfer resources across time (allow households, firms and governments to borrow and lend).
- Transfer and manage risk (insurance policies, futures contracts . . .)
- Pool resources to finance large scale investments.
- Provide information through prices.

Financial System

→: flows of cash
→: flows of financial assets



Time value of money

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Why?

Time value of money

You can receive either *Eu* $1,000$ today or *Eu* $1,000$ in the future. What do do you prefer?

Why?

- *Uncertainty*: You do not know what will happen tomorrow.
- *Inflation*: Purchase power of *Eu* $1,000$ decreases with time.
- *Opportunity cost*: *Eu* $1,000$ can be invested today and will pay interests in the future.
- Everything you can do with *Eu* $1,000$ received tomorrow can be done if you receive *Eu* $1,000$ today (just save it and spend it tomorrow). The reverse is not true.

FACT: Money received today is better than money received tomorrow.

IMPLICATION: You will lend *Eu* 1 during one year, only if you expect to receive more than *Eu* 1 after one year.

Definition

A **compounding law** is a function of time that tells how many Euros an investor will receive at some future date t for each Euro invested today until t.

Three ways of expressing a compounding law:

- *Effective annual rate,* r_e.
- *Interest rate,* r*, and frequency of compounding,* k.
- *Annual rate* r_a *and frequency of compounding* k.

Compounding law examples: the effective annual rate

Definition

Effective annual rate r_e: how many Euros I will receive after one year in addition to each Euro invested today.

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I invest *Eu* 1 at an effective annual rate $r_e = 2\%$. How much do I have in my bank account...

- After 1 year?

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$$1 + 2\% = 1.02$$

- After 2 years?

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- After 2 years?

$$(1.02)(1.02) = 1.02^2 \simeq 1.0404$$

- After 18 months?

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$$(1.02)^{1.5} \simeq 1.0302$$

- After 1 week?

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- After 2 years?
$$(1.02)(1.02) = 1.02^2 \simeq 1.0404$$
- After 18 months?
$$(1.02)^{1.5} \simeq 1.0302$$
- After 1 week? $(1.02)^{\frac{7}{365}} \simeq 1.00038$

Future Value

Definition

Let r_e be the effective annual rate, then the **future value** of an amount S invested for t years is

$$FV(S, r_e) = S \times (1 + r_e)^t$$

Note that t is in years.

Future Value

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Let r_e be the effective annual rate, then the **future value** of an amount S invested for t years is

$$FV(S, r_e) = S \times (1 + r_e)^t$$

Note that t is in years.

Example

You invest $S = Eu\ 20,000$, the effective annual rate is $r_e = 3\%$ What is the amount of money you will have after $t = 5$ years?

$$FV = 20,000 \times (1 + 0.03)^5 = 23,185.48$$

Compounding laws using interest rate and frequency of compounfing

Definition

A **compounding law** is a function of time that tells how many Euros an investor will receive at some future date t for each Euro invested today until t.

Features:

- *Interest rate* r : how many Euros I will receive after one period in addition to each Euro invested today.
- *Frequency of compounding* k: how often in a year I will receive the interests.

Some examples

Example

1. I invest *Eu* 1 at $r = 2\%$ with frequency $k = 1$ per year.
   - After 2 years:

$$(1.02)(1.02) = 1.02^2 \simeq 1.0404$$

   - After 18 months:

$$(1.02)^{1.5} \simeq 1.0302$$

Some examples

Example

1. I invest *Eu* 1 at $r = 2\%$ with frequency $k = 1$ per year.
   - After 2 years:
   $$(1.02)(1.02) = 1.02^2 \simeq 1.0404$$
   - After 18 months:
   $$(1.02)^{1.5} \simeq 1.0302$$
2. I invest *Eu* 1 at $r = 2\%$ with frequency $k = 12$ times per year.
   - After 1 year:
   $$(1 + 0.02)^{12} \simeq 1.27$$
   - After 2 years:
   $$(1.02)^{24} \simeq 1.61$$
   - After 18 months:
   $$(1.02)^{18} \simeq 1.43$$

Future Value

Definition

Let r be the interest rate and let k be the frequency of compounding, then the **future value** of an amount S invested for t years is

$$FV(S, r, k, t) = S \times (1 + r)^{k \times t}$$

Note that t is in years.

Future Value

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Let r be the interest rate and let k be the frequency of compounding, then the **future value** of an amount S invested for t years is

$$FV(S, r, k, t) = S \times (1 + r)^{k \times t}$$

Note that t is in years.

Example

You invest $S = Eu\ 20,000$, the interest rate is $r = 1.5\%$ paid every 6 months ($k = 2$). What is the amount of money you will have after $t = 5$ years?

$$FV = 20,000 \times (1 + 0.015)^{2 \times 5} = 23,210.8$$

Future Value: Quick-Check Questions

What is the future value of

1. *Eu* $5,000$ invested at $r = 1\%$ frequency $k = 4$ during 3 years? (Ans. *Eu* $5,634.13$)
2. *Eu* $1,000,000$ invested at $r = 2.5\%$ frequency $k = 1$ during 1 day? (Ans. *Eu* $1,000,067.65$)
3. *Eu* 10 invested at $r = 1.5\%$ frequency $k = 1$ during 50 years? (Ans. *Eu* 21.05)
4. *Eu* $30,000$ invested at $r = 3.5\%$ frequency $k = 3$ during 100 days? (Ans. *Eu* $30,860.36$)

Annual Interest Rate

Definition

The **annual interest rate** r_a is the interest rate times the compounding frequency:

$$r_a := r \times k$$

Example

The interest rate is 1.5% paid every 6 months ($k = 2$).
The annual rate is: $r_a = r \times k = 1.5\% \times 2 = 3\%$

Definition

The future value of S invested for t years at annual interest rate r_a with frequency of compounding k is

$$FV = S \times \left(1 + \frac{r_a}{k}\right)^{k \times t}$$

Annual rate: QCQ

You invest *Eu* 100 in a bank account. The annual interest rate is 4%. Interests are compounded every 3 months.

1. What is the interest rate (per quarter)? (Ans. 1%)
2. What is the amount in your bank account after 1 year (Ans. 104.06)

Effective annual rate, annual rate, interest rate

Relation across effective annual rate, interest rate per period and annual rate.

$$r_e = (1+r)^k - 1 = \left(1 + \frac{r_a}{k}\right)^k - 1$$

Example

The interest rate is 1.5% paid every 6 months ($k = 2$).
The effective annual rate is: $(1.015)^2 - 1 = 3.02\%$

Effective annual rate: QCQ

1. The annual interest rate is 4%. Interests are compounded every quarter. What is the effective annual rate? (Ans.

   4.06%)

2. The effective annual rate is 5%. What is the effective monthly rate?

$$(1 + r_{e,month})^{12} = 1 + r_e$$

(Ans. $r_{e,month} = 0.41\%$)

Present Value

Present Value

Definition

The **present value (PV)** of an amount S paid after t years is the amount of money I have to invest today in order to obtain exactly S after t years.

$$S = PV(1 + r_e)^t \Rightarrow PV := \frac{S}{(1 + r_e)^t}$$

Read: "The amount S is discounted for t years at a discount rate r_e."

Remark:

Interest rate: rate used to compute future values.

Discount rate : rate used to compute present values.

Present Value: an example

Example

If $r_e = 2\%$,

1. What is the PV of *Eu* 1,000 received in 20 years?

$$PV = 1,000/(1.02)^{20} = 672.97$$

2. What is the PV of *Eu* 1,000 received in 20 days?

$$PV = 1,000/(1.02)^{\frac{20}{365}} = 998.92$$

Present Value: properties

Remark 1: The present value is decreasing in r_e and in t:

- The higher the interest rate r_e, the lower the amount I have to invest today to reach the target at t.
- The longer is the investment time t, the larger are the interests and hence the lower the amount I have to invest today to reach the target at t.

Present Value: Interpretation

Remark 2: Receiving an amount of money S at a future date t is equivalent to receiving its PV today.

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Example

The discount rate is 5%. The PV of $S = 10,000$ received in 3 years is

$$\frac{10,000}{1.05^3} = 8,638.38$$

	Today	3 years
Receive	$8,638.38$	0
Invest	$-8,638.38$	$8,638.38 \times 1.05^3 = 10,000$
Total	0	$10,000$

Present Value: QCQ

1. The discount rate is 2%.
   - Choose one of the following two investment opportunities:
     1. Today you invest *Eu* 100 and in 5 years time you will receive *Eu* 200 ;
     2. Today you invest *Eu* 100 and in 4 years time you will receive *Eu* 190;

Present Value: QCQ

1. The discount rate is 2%.
   - Choose one of the following two investment opportunities:
     1. Today you invest *Eu* 100 and in 5 years time you will receive *Eu* 200 ;
     2. Today you invest *Eu* 100 and in 4 years time you will receive *Eu* 190;
   - What is the present value of *Eu* $450,000$ received in 3 years time? (Ans. *Eu* $424,045.05$)
   - What is the present value of *Eu* $450,000$ received in 3 months time? (Ans. *Eu* $447,777.78$)

Present Value: QCQ

1. The discount rate is 2%.
   - Choose one of the following two investment opportunities:
     1. Today you invest *Eu* 100 and in 5 years time you will receive *Eu* 200 ;
     2. Today you invest *Eu* 100 and in 4 years time you will receive *Eu* 190;
   - What is the present value of *Eu* 450, 000 received in 3 years time? (Ans. *Eu* 424, 045.05)
   - What is the present value of *Eu* 450, 000 received in 3 months time? (Ans. *Eu* 447, 777.78)
2. The discount rate is 20%.
   - What is the present value of *Eu* 450, 000 received in 3 years time? (Ans. *Eu* 260, 416.67)
   - What is the present value of *Eu* 450, 000 received in 3 days time? (Ans. *Eu* 449.326.17)

Definition

The present value of a stream of future cash flows is equal to the sum of the present values of each cash flow.

Example

The discount rate is 3%. How much do you have to invest today to have exactly *Eu* 103 in 1 year and *Eu* 200 in 2 years?

$$PV = \frac{103}{1.03} + \frac{200}{1.03^2} = 100 + 188.52 = 288.52$$

Present Value of Multiple Cash-flows

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	Today	year 1	year 2
Receive today	288.52	0	0

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	Today	year 1	year 2
Receive today	288.52	0	0
Invest for 1 year	-100	103	

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	Today	year 1	year 2
Receive today	288.52	0	0
Invest for 1 year	-100	103	
Invest for 2 years	-188.52		$188.52 \times 1.03^2 = 200$

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	Today	year 1	year 2
Receive today	288.52	0	0
Invest for 1 year	-100	103	
Invest for 2 years	-188.52		$188.52 \times 1.03^2 = 200$
Total	0	103	200

Ordinary Annuities

Definition

An **ordinary annuity** of length n is a sequence of equal cash flows during n periods, where the cash flows occur at the end of each period.

Example

An ordinary monthly annuity of *Eu* 4,000 lasting 30 months is

today	month 1	month 2	$\cdots$	month 30	month 31
0	4,000	4,000	4,000	4,000	0

PV of Ordinary Annuities

Theorem

If r is the discount rate per period, then the present value of an ordinary annuity with cash flow C and length n periods is

$$A(C, r, n) = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

Proof: Recall that $\sum_{i=0}^{m} \theta^i := 1 + \theta + \theta^2 + \cdots + \theta^m = \frac{1-\theta^{m+1}}{1-\theta}$.
Hence,

$$\begin{aligned} A(C, r, n) &= \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^n} = \frac{C}{(1+r)} \sum_{i=0}^{n-1} \frac{1}{(1+r)^i} \\ &= \frac{C}{(1+r)}\left(\frac{1-\frac{1}{(1+r)^n}}{1-\frac{1}{(1+r)}}\right) = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right) \end{aligned}$$

r is the effective rate corresponding to one period in the annuity.

Definition

An **immediate annuity** of length n is a sequence of equal cash flows during n periods, where the cash flows occur at the <u>beginning</u> of each period.

Example

An immediate monthly annuity of *Eu* 4,000 lasting 30 months is

today	month 1	month 2	$\cdots$	month 29	month 30
$4,000$	$4,000$	$4,000$	$4,000$	$4,000$	0

Theorem

If r is the discount rate per period, then the present value of an immediate annuity with cash flow C and length n periods is

$$A_0(C,r,n) = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right)(1+r)$$

Proof:

$$\begin{aligned} A_0(C,r,n) &= C + \frac{C}{(1+r)} + \frac{C}{(1+r)2} + \ldots + \frac{C}{(1+r)^{n-1}} \\ &= \left(\frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^n}\right)(1+r) \\ &= A(C,r,n)(1+r) \end{aligned}$$

Increasing Ordinary Annuities

Definition

An **increasing ordinary annuity** of length n is a sequence of cash flows increasing at a constant rate g during n periods, where the cash flows occur at the end of each period.

Example

An ordinary monthly annuity of *Eu* C increasing at rate g lasting n months is

today	month 1	month 2	$\cdots$	month n	month $n+1$
0	C	$C(1+g)$	$\ldots$	$C(1+g)^{n-1}$	0

PV of Increasing Ordinary Annuity

Theorem

If r is the discount rate per period, then the present value of an ordinary annuity with cash flows starting from C and increasing at rate g during n periods is

$$IA(C,r,g,n) = \frac{C}{r-g}\left(1 - \left(\frac{1+g}{1+r}\right)^n\right)$$

Proof: Recall that $\sum_{i=0}^{m} \theta^i = \frac{1-\theta^{m+1}}{1-\theta}$. Hence,

$$\begin{aligned} IA(C,r,g,n) &= \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \ldots + \frac{C(1+g)^{n-1}}{(1+r)^n} = \frac{C}{(1+r)} \sum_{i=0}^{n-1} \left(\frac{1+g}{1+r}\right)^i \\ &= \frac{C}{1+r}\left(\frac{1-\left(\frac{1+g}{1+r}\right)^n}{1-\frac{1+g}{1+r}}\right) = \frac{C}{r-g}\left(1 - \left(\frac{1+g}{1+r}\right)^n\right) \end{aligned}$$

Perpetuities

Definition

A **perpetuity** is an ordinary annuity with infinite length.

Example

A perpetuity of C per year is

today	year 1	year 2	$\cdots$	year t	...
0	C	C	C	C	C

Definition

An **increasing perpetuity** is an increasing ordinary annuity with infinite length.

Example

A perpetuity of C per year increasing at rate g is

today	year 1	year 2	$\cdots$	year t	...
0	C	$C(1+g)$	$\cdots$	$C(1+g)^{t-1}$	$\cdots$

Theorem

If r is the discount rate per period, then the present value of a perpetuity increasing at rate $g < r$ and starting from a payment of C is

$$P(C, r, g) = \frac{C}{r - g}$$

Proof: if $g < r$, then

$$P(C, r, g) = \lim_{n \to \infty} \frac{C}{r-g} \left(1 - \left(\frac{1+g}{1+r}\right)^n\right) = \frac{C}{r-g}$$

Annuity QCQ

The effective annual rate is 6%.

1. What is the effective monthly rate? (Ans. 0.487%)
2. What are the cash-flows and the present value of an immediate annuity paying *Eu* $5,000$ every year for the next 35 years? (Ans. *Eu* $76,840.70$)
3. What are the cash-flows and the present value of an ordinary annuity lasting 20 years, with annual payments starting from *Eu* $5,000$ and increasing at annual rate $g = 8\%$? (Ans. *Eu* $113,327.12$)
4. What are the cash-flows and the present value of a monthly perpetuity of *Eu* 100? (Ans. *Eu* $20,544.21$)
5. What are the cash-flows and the present value of an annual perpetuity of *Eu* 100 increasing at rate $g = 8\%$? (Ans. ∞)

Exercise

You borrow *Eu* $205,000$ to buy a house. You will pay your debt in consistent monthly payments C during the next 20 years. The first payment is due in one-month time. The mortgage is at an annual interest of 3.20%, and the frequency of compounding is $k = 12$.

1. What is the effective monthly rate? (Ans. 0.267%)
2. What is the monthly payment C? (Hint: the PV of your payment to the bank equals the amount of money you borrow) (Ans. *Eu* $1,157.56$)