Financial Economics
3: Uncertainty

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A simple but risky investment

Today the price of one share of Total is $P_0 = Eu$ 39. You buy 1000 share of Total. After one year

- you receive the dividends $\widetilde{D}$ payed by Total.
- you resell the shares of Total at price $\widetilde{P}_1$.

What will the annual rate of return of your investment be?

$$\widetilde{r} := \frac{\widetilde{P}_1 + \widetilde{D} - P_0}{P_0}$$

- Capital gain/loss $:= \frac{\widetilde{P}_1 - P_0}{P_0}$
- Yield component $:= \frac{\widetilde{D}}{P_0}$

A simple but risky investment

Example

Today the price of one share of Total is $P_0 = Eu\ 39$. You buy 1000 share of Total. After one year

- you receive the dividends *Eu* 4 per-sahre.
- you resell the shares of Total at price *Eu* 38.

What is the annual rate of return of your investment?

$$r = \frac{38 + 4 - 39}{39} = 7.7\% = -2.6\% + 10.3\%$$

- Capital gain/loss : $\frac{38-39}{39} = -2.6\%$
- Yield component : $\frac{4}{39} = 10.3\%$

When an investors buys a stock:

- He knows the price he pays.
- He does not know the dividend he will receive.
- He does not know the resale price.

Example

The actual rate of return of a stock depends on performance of the company and of the whole economy
You spent *Eu*100 to buy 1 share of XYZ Corp.

Event	P_1	D	r
Recession	*Eu* 80	*Eu* 0	−20%
Normal	*Eu* 103	*Eu* 7	10%
Boom	*Eu* 110	*Eu* 10	20%

Contingent return rates

Example

		Annual return rates				
Event	Probability	Alcatel	BNP	BMW	...	Treasury bill
Recession	0.7	−10%	−5%	−15%	...	1.5%
Normal	0.25	12%	10%	20%	...	1.5%
Boom	0.05	18%	30%	22%	...	1.5%

Probability

Let Ω be the set of all possible events or scenarios.
Let $\omega \in \Omega$ be one event.

Definition

The probability of an event ω, is the likelihood that the event will happen, it is denoted $Pr(\omega)$ and satisfies:

1. $0 \leq Pr(\omega) \leq 1$.
2. If ω is certain, then $Pr(\omega) = 1$.
3. If ω is impossible, then $Pr(\omega) = 0$.
4. If two events ω and ω' are mutually exclusive, then
$$Pr(\omega \text{ or } \omega') = Pr(\omega) + Pr(\omega')$$
5. $Pr(\Omega) = 1$

Probability QCQ

		Annual return rates				
Event	Probability	Alcatel	BNP	BMW	...	Treasury bill
Recession ω_1	0.7	-10%	-5%	-15%	...	1.5%
Normal ω_2	0.25	12%	10%	20%	...	1.5%
Boom ω_3	0.05	18%	30%	22%	...	1.5%

- What is the probability that the economy is not in recession?
- What is the probability that the rate of return of Alcatel is less than 5%?
- What is the probability that the rate of return of BMW is positive and the rate of return of BNP is negative?

Random variable

Definition

A random variable is a function that maps the set of possible events into a real number: $\widetilde{r} : \Omega \to \mathbb{R}$

Example

$r_{BMW}(\omega_1) = -15\%$; $r_{BMW}(\omega_2) = 20\%$; $r_{BMW}(\omega_3) = 22\%$

Definition

The distribution function of a random variable represents the probability that the realization of a random variable reaches a given value. $\pi_{\widetilde{r}} : \mathbb{R} \to [0, 1]$

Example

$\pi_{BMW}(-15\%) = 0.7$; $\pi_{BMW}(20\%) = 0.25$; $\pi_{BMW}(22\%) = 0.05$

Assumptions on the event space and notation

Assumption: There is a finite number *n* of possible events (or state of the economy)

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$$

Notation:

$$\pi_i := Pr(\omega_i)$$

$$r_{A,i} := \widetilde{r}_A(\omega_i)$$

with

$$\widetilde{r}_A \in \{r_{A,1}, r_{A,2}, \ldots, r_{A,i}, \ldots, r_{A,n}\}$$

Event	Probability	Alcatel
Recession ω_1	0.7	-10%
Normal ω_2	0.25	12%
Boom ω_3	0.05	18%

- $r_{Alcatel,2} = ?$
- $\pi_1 = ?$
- $Pr(\tilde{r}_{Alcatel} > 0) = ?$

Expectation of a random variable

How can we make prediction about the rate of return of an asset?

Definition

The expected return of an asset with return rate $\widetilde{r}$ is:

$$E\left[\widetilde{r}\right] = \sum_{i=1}^{n} \pi_i r_i = \pi_1 r_1 + \pi_2 r_2 + \ldots + \pi_n r_n$$

Example

Event	Probability	Alcatel
Recession ω_1	0.7	-10%
Normal ω_2	0.25	12%
Boom ω_3	0.05	18%

$$E\left[\widetilde{r}_{Alcatel}\right] = 0.7(-10\%) + 0.25 * 12\% + 0.05 * 18\% = -3.1\%$$

The expectation is a linear operator

Let $\widetilde{r}_A$ and $\widetilde{r}_B$ be two random variables and let α and β be two real numbers. Then

$$E\left[\alpha\widetilde{r}_A + \beta\widetilde{r}_B\right] = \alpha E[\widetilde{r}_A] + \beta E[\widetilde{r}_B]$$

Proof:

$$\begin{aligned} E\left[\alpha\widetilde{r}_A + \beta\widetilde{r}_B\right] &= \sum_{i=1}^{n} \pi_i(\alpha r_{A,i} + \beta r_{B,i}) = \\ = \sum_{i=1}^{n} \pi_i(\alpha r_{A,i}) + \sum_{i=1}^{n} \pi_i(\beta r_{B,i}) &= \alpha \sum_{i=1}^{n} \pi_i r_{A,i} + \beta \sum_{i=1}^{n} \pi_i r_{B,i} = \\ &= \alpha E[\widetilde{r}_A] + \beta E[\widetilde{r}_B] \end{aligned}$$

Example

Let $E[\widetilde{r}_A] = 12\%$ and $E[\widetilde{r}_B] = 33\%$. Then

$$E\left[\frac{1}{3}\widetilde{r}_A + \frac{2}{3}\widetilde{r}_B\right] = \frac{1}{3}12\% + \frac{2}{3}33\% = 26\%$$

Dispersion of realized return

Is the expected return a good predictor of the actual return?

Example

Event	Prob	$\widetilde{r}_A$	$\widetilde{r}_B$	$\widetilde{r}_C$
ω_1	0.5	5%	5.5%	18%
ω_2	0.5	5%	4.5%	−8%

$$E\left[\widetilde{r}_A\right] = E\left[\widetilde{r}_B\right] = E\left[\widetilde{r}_C\right] = 5\%$$

Expectation is a perfect forecast for $\widetilde{r}_A$, a good forecast for $\widetilde{r}_B$ and a vague forecast for $\widetilde{r}_C$.

Question: How can we measure the quality of the expected return as forecast for the realized return?

Variance and Standard Deviation

If I predict that the actual return on C will be $E\left[\tilde{r}_C\right]$, how far should I expect to be my prediction from the true r_C?

Definition

The variance of a random variable $\tilde{r}$ is equal to

$$Var\left[\tilde{r}\right] := E\left[\left(\tilde{r} - E\left[\tilde{r}\right]\right)^2\right] = \sum_{i=1}^{n} \pi_i \left(r_i - E\left[\tilde{r}\right]\right)^2$$

Definition

The standard deviation of a random variable $\tilde{r}$ is equal to

$$\sigma_{\tilde{r}} = \sqrt{Var\left[\tilde{r}\right]}$$

Interpretation: Variance and standard deviation measure the dispersion of realized return around the expected return.

Example

Event	Prob	$\tilde{r}_A$	$\tilde{r}_B$	$\tilde{r}_C$
ω_1	0.5	5%	5.5%	18%
ω_2	0.5	5%	4.5%	−8%

$$Var\left[\tilde{r}\right] = 0.5(r_1 - E\left[\tilde{r}\right])^2 + 0.5(r_2 - E\left[\tilde{r}\right])^2$$

$$\begin{aligned}
Var\left[\tilde{r}_A\right] &= 0.5(0.05 - 0.05)^2 + 0.5(0.05 - 0.05)^2 = 0 \\
\sigma_{\tilde{r}_A} &= \sqrt{0} = 0 \\
Var\left[\tilde{r}_B\right] &= 0.5(0.055 - 0.05)^2 + 0.5(0.045 - 0.05)^2 = 0.000025 \\
\sigma_{\tilde{r}_B} &= \sqrt{0.000025} = 0.005 = 0.5\% \\
Var\left[\tilde{r}_C\right] &= 0.5(0.18 - 0.05)^2 + 0.5(-0.08 - 0.05)^2 = 0.0169 \\
\sigma_{\tilde{r}_C} &= \sqrt{0.0169} = 13\%
\end{aligned}$$

Properties of the variance

1. $$Var\left[\tilde{r}\right] = E\left[\tilde{r}^2\right] - E\left[\tilde{r}\right]^2$$
2. If k is a constant, then
$$Var\left[k\tilde{r}\right] = k^2 Var\left[\tilde{r}\right]$$
3. Let $\tilde{r}_A$ and $\tilde{r}_B$ be two random variables and α and β two real numbers, then
$$Var\left[\alpha\tilde{r}_A + \beta\tilde{r}_B\right] = \alpha^2 Var\left[\tilde{r}_A\right] + \beta^2 Var\left[\tilde{r}_B\right] + 2\alpha\beta Cov\left[\tilde{r}_A, \tilde{r}_B\right]$$
where $Cov\left[\tilde{r}_A, \tilde{r}_B\right]$ is the covariance between $\tilde{r}_A$ and $\tilde{r}_B$.

Covariance

Definition

The covariance between two random variables $\tilde{r}_A$ and $\tilde{r}_B$ is equal to

$$Cov\left[\tilde{r}_A, \tilde{r}_B\right] \quad : \quad = E\left[\left(\tilde{r}_A - E\left[\tilde{r}_A\right]\right)\left(\tilde{r}_B - E\left[\tilde{r}_B\right]\right)\right] =$$

$$= \sum_{i=1}^{n} \pi_i (r_{A,i} - E\left[\tilde{r}_A\right])(r_{B,i} - E\left[\tilde{r}_B\right])$$

Interpretation: The covariance measures the degree to which two random variables move together.

Positive covariance: example

- If $Cov\left[\widetilde{r}_A, \widetilde{r}_B\right] > 0$, then it is likely that when $\widetilde{r}_A > E\left[\widetilde{r}_A\right]$, also $\widetilde{r}_B > E\left[\widetilde{r}_B\right]$

Example

Evolution of the stock prices of Renault and Michelin

Negative covariance: example

- If $Cov\left[\tilde{r}_A, \tilde{r}_B\right] < 0$, then it is likely that while $\tilde{r}_A > E\left[\tilde{r}_A\right]$, $\tilde{r}_B < E\left[\tilde{r}_B\right]$

Example

Rate of return of Gold and CAC40

Properties of the Covariance

1. $$Cov\left[\widetilde{r}_A, \widetilde{r}_B\right] = E\left[\widetilde{r}_A \widetilde{r}_B\right] - E\left[\widetilde{r}_A\right] E\left[\widetilde{r}_B\right]$$
2. $$Cov\left[\widetilde{r}_A, \widetilde{r}_A\right] = Var\left[\widetilde{r}_A\right]$$
3. $$Cov\left[\widetilde{r}_A, \widetilde{r}_B\right] = Cov\left[\widetilde{r}_B, \widetilde{r}_A\right]$$
4. If α and β are two real numbers, then

$$Cov\left[\widetilde{r}_C, \alpha\widetilde{r}_A + \beta\widetilde{r}_B\right] = \alpha Cov\left[\widetilde{r}_C, \widetilde{r}_A\right] + \beta Cov\left[\widetilde{r}_C, \widetilde{r}_B\right]$$

Correlation coefficient

Definition

The correlation coefficient between two random variables $\tilde{r}_A$ and $\tilde{r}_B$ is equal to

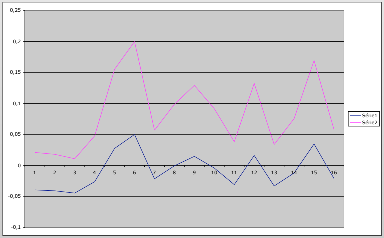
$$\rho_{\tilde{r}_A,\tilde{r}_B} = \frac{Cov\left[\tilde{r}_A, \tilde{r}_B\right]}{\sigma_{\tilde{r}_A}\sigma_{\tilde{r}_B}}$$

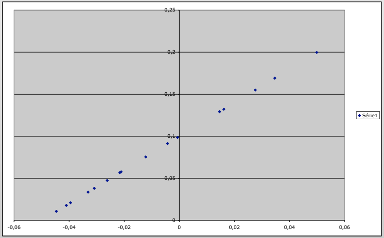
properties:

1. $-1 \leq \rho_{\tilde{r}_A,\tilde{r}_B} \leq 1$
2. if $\rho_{\tilde{r}_A,\tilde{r}_B} = -1$, then there exists $\alpha < 0$ and β such that $\tilde{r}_A = \alpha\tilde{r}_B + \beta$
3. if $\rho_{\tilde{r}_A,\tilde{r}_B} = 1$, then there exists $\alpha > 0$ and β such that $\tilde{r}_A = \alpha\tilde{r}_B + \beta$

Correlation coefficient: example

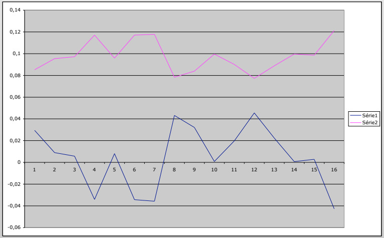
$$\rho = 1$$

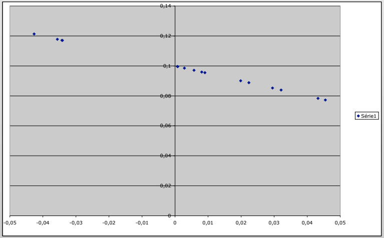




Correlation coefficient: example

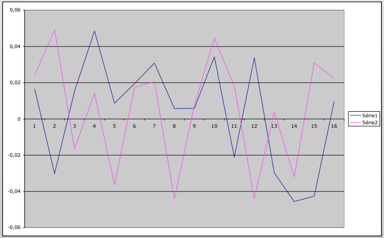
$$\rho = -1$$

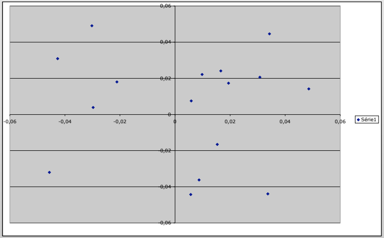




Correlation coefficient: example

$$\rho = 0$$





Estimation of expected return, variance and covariance

If we do not know the distribution probabilities of return rates how can we estimate an asset expected return, variance and its covariance with another asset?

- We are in year $T + 1$.
- Let $R_{i,t}$ be the realized annual return of asset i in a past year t.
- Suppose you observe the realized return of assets A and B during the last T years:

$$\{R_{A,1}, R_{A,2}, \dots R_{A,T}\} \text{ and } \{R_{B,1}, R_{B,2}, \dots R_{B,T}\}$$

Estimating $E\left[\tilde{r}_A\right]$, $Var\left[\tilde{r}_A\right]$ *and* $Cov\left[\tilde{r}_A, \tilde{r}_B\right]$

- Sample mean of historical returns (an estimation of $E\left[\tilde{r}_A\right]$):

$$\hat{r}_A := \frac{1}{T}\sum_{t=1}^{T} R_{A,t}$$

- Sample variance of historical return (an estimation of $Var\left[\tilde{r}_A\right]$):

$$\hat{\sigma}_A^2 := \frac{1}{T-1}\sum_{t=1}^{T} \left(R_{A,t} - \hat{r}_A\right)^2$$

- Sample covariance of historical return (an estimation of $Cov\left[\tilde{r}_A, \tilde{r}_B\right]$):

$$\hat{\sigma}_{A,B} := \frac{1}{T-1}\sum_{t=1}^{T} \left(R_{A,t} - \hat{r}_A\right)\left(R_{B,t} - \hat{r}_B\right)$$

Caveat: These estimations are not correct if the true probabilities underlying the past return rates are not stationary.