Financial Economics
4: Portfolio Theory

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What is a portfolio?

Definition

A portfolio is an amount of money invested in a number of financial assets.

Example

Portfolio A is worth *Eu* $100,000$ and consists of

- *Eu* $25,000$ invested in Alcatel shares.
- *Eu* $20,000$ invested in Michelin shares.
- *Eu* $18,000$ invested in JP Morgan shares.
- *Eu* $22,000$ Invested in gold.
- *Eu* $15,000$ invested in German Treasury bills.

Vector representation:

$$X_A = \{x_{Al}, x_M, x_{Jp}, x_g, x_{Tb}\} = \{0.25, 0.2, 0.18, 0.22, 0.15\}$$

Notation

- Let $S = \{s_1, s_2, \dots, s_N\}$ be the set of financial assets available in the economy.
- Let $\{v_1, v_2, \dots, v_N\}$ be the amount of money invested in each one of the N assets to obtain portfolio A.

Today the value of portfolio A is

$$V_A = \sum_{i=1}^{N} v_i = v_1 + v_2 + \cdots + v_n.$$

The composition of the portfolio A is $X_A = \{x_1, x_2, \dots, x_N\}$ where

$$x_i = \frac{v_i}{V_A}.$$

and

$$\sum_{i=1}^{N} x_i = x_1 + x_2 + \cdots + x_n = 1$$

Some portfolios

$$S = \{s_1, s_2, s_3\} = \{Alcatel, BMW, Treasury\ bill\}$$

1. Only long positions

$$\begin{aligned} V_A &= 5,000 + 2,000 + 10,000 = 17,000 \\ X_A &= \{\tfrac{5,000}{17,000}, \tfrac{2,000}{17,000}, \tfrac{10,000}{17,000}\} \simeq \{0.29, 0.12, 0.59\} \end{aligned}$$

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2. Short position

$$\begin{aligned} V_B &= 15,000 + 6,000 - 4,000 = 17,000 \\ X_B &= \{\tfrac{15,000}{17,000}, \tfrac{6,000}{17,000}, \tfrac{-4,000}{17,000}\} \simeq \{0.882, 0.353, -0.235\} \end{aligned}$$

Short selling an asset

Example

Let $X_B = \{0.882, 0.353, -0.235\}$, then portfolio B has a short position in asset s_3.

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How to short sell an asset i

- At time 0:
  1. You borrow the asset i from your broker.
  2. You sell the asset in the stock market at price $P_i(0)$.

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How to short sell an asset *i*

- At time 0:
  1. You borrow the asset i from your broker.
  2. You sell the asset in the stock market at price $P_i(0)$.
- At time 1:
  1. You pay your broker whatever dividend the asset has paid at time 1.
  2. You buy the asset in the stock market at price $P_i(1)$.
  3. You return the asset to your broker.
- The broker charges you a fee for lending you the asset and asks for a collateral.

Portfolio return rate example

Asset	$p_i(0)$	$p_i(1)$	D_i	r_i
Alcatel	50	49	4	6%
BMW	20	22	0	10%
T.B.	100	100	2	2%

You invest *Eu* 100 into portfolio

$$X_C = \{x_{Al}, x_{BM}, x_{TB}\} = \{0.8, 0.2, 0\}.$$

What is the rate of return of your portfolio?

At $t = 1$ portfolio C is worth

$$100 \times 0.8(1 + r_{Al}) + 100 \times 0.2(1 + r_{BM}) = 80(1.06) + 20(1.1) = 106.8$$

hence

$$\begin{aligned} r_C &= \frac{106.8 - 100}{100} = 6.8\% = \frac{80(1.06) + 20(1.1) - (80 + 20)}{100} = \\ &= \frac{80 * 0.06 + 20 * 0.1}{100} = 0.8 \times 6\% + 0.2 \times 10\% = x_{Al} r_{Al} + x_{BM} r_{BM} \end{aligned}$$

Return rate of a portfolio

- Return rate on a long position (buy) on portfolio A:

    1. At time 0 you buy the portfolio at $X_a = \{x_1, x_2, \ldots, x_n\}$.
    2. At time 1 you receive dividends from each stock
    3. At time 1 you resell the portfolio.

The return rate r_A on your investment in Portfolio A is

$$r_A = \sum_{i=1}^{n} r_i x_i = r_1 x_1 + r_2 x_2 + \cdots + r_n x_n.$$

Uncertainty

Theorem

- *Let* $S = \{s_1, s_2, \ldots, s_N\}$ *be the set of available assets.*
- *Let* $\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_N$ *be N random variables representing the return rates on assets* $s_1, s_2, \ldots, s_N$*, respectively.*

Then, the rate of return of a portfolio A with composition

$$X_A = \{x_1, x_2, \ldots, x_N\}$$

is a random variable

$$\widetilde{r}_A := \sum_{i=1}^{N} x_i \widetilde{r}_i$$

Uncertainty

Example

The rate of return of a portfolio composed of risky asset is uncertain:

$$S = \{Alcatel, BMW\}$$

Portfolio A:

$$X_A = \{0.8, 0.2\}$$

Event	$\widetilde{r}_{Al}$	$\widetilde{r}_{BM}$	$\widetilde{r}_A$
The economy is booming	12%	20%	0.8$*$ 12%+0.2$*$ 20%=13.6%
The economy is in recession	-3%	-15%	-0.8$*$ 3%-0.2$*$ 15% =-5.4%

Expected return on a portfolio

Theorem

The expected return rate on a portfolio with composition $X = \{x_1, x_2, \ldots, x_n\}$ *is:*

$$E\left[\tilde{r}_X\right] = x_1 E\left[\tilde{r}_1\right] + x_2 E\left[\tilde{r}_2\right] + \cdots + x_n E\left[\tilde{r}_n\right] = \sum_{i=1}^{n} x_i E\left[\tilde{r}_i\right].$$

Example

S={AI,BM} X_A={0.8,0.2} E$\left[\tilde{r}_{AI}\right]$=1.5% E$\left[\tilde{r}_{BM}\right]$=-4.5%

Event	Prob	$\tilde{r}_{AI}$	$\tilde{r}_{BM}$	$\tilde{r}_A$
Boom	0.3	12%	20%	13.6%
Recession	0.7	-3%	-15%	-5.4%

$$\begin{aligned} E\left[\tilde{r}_A\right] &= 0.3 \times 13.6\% - 0.7 \times 5.4\% = 0.3\% \\ &= 0.8 \times 1.5\% - 0.2 \times 4.5\% = 0.3\% \end{aligned}$$

Expected return on a portfolio

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Expected return on a portfolio: QCQ

Assets	s_1	s_2	s_3	s_4
$E\left[\tilde{r}_i\right]$	30%	25%	8%	2%

What are the expected returns of the following portfolios?

	x_1	x_2	x_3	x_4
X_A	0	1	0	0
X_B	0.2	0.1	0.3	0.4
X_C	0.3	0.8	0.6	-0.7
X_D	0.5	-0.2	0.8	-0.1

Answers: $E\left[r_A\right] = 25\%$, $E\left[r_B\right] = 11.7\%$, $E\left[r_C\right] = 32.4\%$, $E\left[r_D\right] = 16.2\%$.

Variance of the return rate of a portfolio

Theorem

The variance of the return rate on a portfolio with composition $X = \{x_1, x_2, \ldots, x_n\}$ is:

$$\sigma_X^2 := Var\left[\widetilde{r}_X\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij}$$

where $\sigma_{ii} = Var\left[\widetilde{r}_i\right]$ and $\sigma_{ij} := Cov\left[\widetilde{r}_i, \widetilde{r}_j\right]$.

for $n = 2$:

$$\sigma_X^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2 Cov\left[\widetilde{r}_1, \widetilde{r}_2\right]$$

Since $x_1 + x_2 = 1$ and $Cov\left[\widetilde{r}_i, \widetilde{r}_j\right] = \rho_{1,2}\sigma_1\sigma_2$,

$$\sigma_X^2 = x_1^2\sigma_1^2 + (1 - x_1)^2\sigma_2^2 + 2x_1(1 - x_1)\rho_{1,2}\sigma_1\sigma_2$$

Variance of the return rate of a portfolio: example

Example

$$S = \{s_1, s_2\}$$

	s_1	s_2	
σ_i	30%	12%	
ρ_{12}			-0.56

If the composition of portfolio A is $X_A = \{0.8, 0.2\}$, then

$$\begin{aligned} \sigma_A^2 &= x_1^2\sigma_1^2 + (1-x_1)^2\sigma_2^2 + 2x_1(1-x_1)\rho_{12}\sigma_1\sigma_2 = \\ &= 0.8^2 \times 0.3^2 + 0.2^2 \times 0.12^2 + \\ &- 2 \times 0.8 \times 0.2 \times 0.56 \times 0.3 \times 0.12 \\ &= 0.052 \end{aligned}$$

$$\sigma_A = \sqrt{\sigma_A^2} = \sqrt{0.052} = 22.7\%$$

Summing-up about portfolios

1. **Ingredients**: Set of available assets

$$S = \{s_1, s_2, \dots, s_n\}$$

2. **Recipe**: Portfolio composition $X = \{x_1, x_2, \dots, x_n\}$ satisfying

$$\sum_{i=1}^{n} x_i = 1.$$

3. **Gain**: Expected return of a portfolio

$$E\left[\widetilde{r}_X\right] = \sum_{i=1}^{n} x_i E\left[\widetilde{r}_i\right].$$

4. **Risk**: variance of the portfolio's return

$$Var\left[\widetilde{r}_X\right] = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}$$

Definition

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1. The investor only cares about $E\left[\tilde{r}_X\right]$ and σ_X^2.

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2. For a given level of risk σ_X^2 , he/she prefers the portfolio with the largest expected return $E\left[\tilde{r}_X\right]$.

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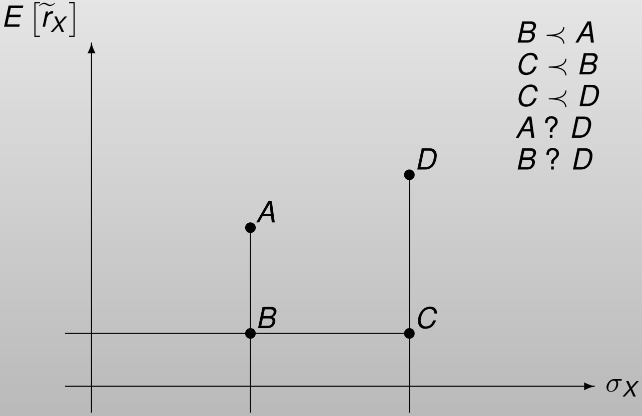
3. For a given level of expected return $E\left[\tilde{r}_X\right]$, he/she prefers the portfolio with the lowest risk σ_X^2 .
4. A Mean-Variance investor chooses the portfolio that maximizes the following utility function

$$U(X) := E\left[\tilde{r}_X\right] - \frac{a}{2}\sigma_X^2$$

where a is the investor's personal degree of **risk aversion**.

Risk aversion: Graphically

Portfolio X is at least as good as portfolio X' iff $U(X) \geq U(X')$.



$B \prec A$
$C \prec B$
$C \prec D$
$A \; ? \; D$
$B \; ? \; D$

Diversification

Rule 0: Diversification of your investment allows to reduce the risk of your portfolio.

Example

Event	Prob.	$\widetilde{r}_1$	$\widetilde{r}_2$
ω_1	0.25	0%	5%
ω_3	0.25	0%	15%
ω_2	0.25	20%	5%
ω_4	0.25	20%	15%

$$
\begin{aligned}
E\left[\widetilde{r}_1\right] &= E\left[\widetilde{r}_2\right] = 10\%, \;\; \sigma_1 = 10\%, \sigma_2 = 5\% \\
Cov\left[\widetilde{r}_1, \widetilde{r}_2\right] &= \rho_{1,2} = 0
\end{aligned}
$$

If $X = \{0.2, 0.8\}$, then

$$
\begin{aligned}
E\left[\widetilde{r}_X\right] &= 10\% \\
\sigma_X &= \sqrt{0.2^2 * 0.1^2 + 0.8^2 * 0.05^2} = 4.47\% < 5\%
\end{aligned}
$$

Example

Event	Prob.	$\widetilde{r}_1$	$\widetilde{r}_2$
ω_1	0.25	0%	15%
ω_3	0.25	0%	15%
ω_2	0.25	20%	5%
ω_4	0.25	20%	5%

$$\begin{aligned} E\left[\widetilde{r}_1\right] &= E\left[\widetilde{r}_2\right] = 10\%, \;\; \sigma_1 = 10\%, \sigma_2 = 5\% \\ \rho_{1,2} &= -1 \end{aligned}$$

If $X = \{\frac{1}{3}, \frac{2}{3}\}$, then

$$\begin{aligned} E\left[\widetilde{r}_X\right] &= 10\% \\ \sigma_X &= \sqrt{\left(\frac{1}{3}0.1\right)^2 + \left(\frac{2}{3}0.05\right)^2 - \frac{4}{9}0.05 * 0.1} = 0 \end{aligned}$$

Minimum Variance Portfolio

$S = \{s_1, s_2\}$. What is the composition of the portfolio (x_1, x_2) that has the minimum variance?

$$\begin{aligned} \min_{x_1,x_2} x_1^2\sigma_1^2 \quad &+ \quad x_2^2\sigma_2^2 + 2x_1x_2\rho_{12}\sigma_1\sigma_2 \\ \text{s.t. } x_1 \quad &+ \quad x_2 = 1 \end{aligned}$$

$$x_1^{\min} = \frac{\sigma_2(\sigma_2-\rho_{12}\sigma_1)}{\sigma_1^2+\sigma_2^2-2\rho_{12}\sigma_1\sigma_2}, \quad x_2^{\min} = 1 - x_1^{\min} \quad \sigma_{\min}^2 = \frac{(1-\rho_{12}^2)\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2-2\rho_{12}\sigma_1\sigma_2}$$

Remarks:

- If $\rho_{12} \ll 1$, then $x_1^{\min} > 0$ and $x_2^{\min} > 0$
- If $\sigma_1 > \sigma_2$ and ρ_{12} is close enough to 1 then short sell s_1.
- If $\rho_{12} = \pm 1$, then $\sigma_{\min} = 0$.

Limit of risk reduction through diversification

Definition

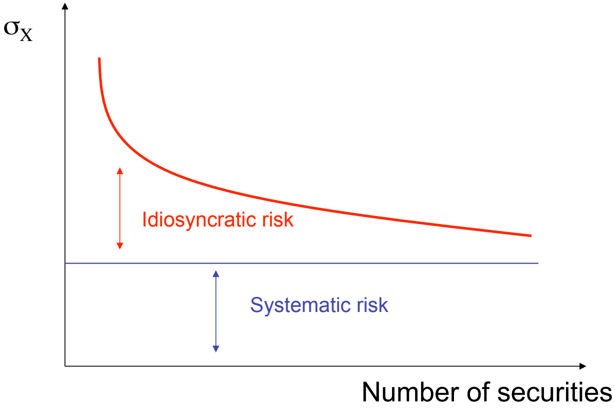
- The idiosyncratic risk on stock i is the uncertainty on $\tilde{r}_i$ linked to risk factors that are specific to firm i.
  Examples: changes in management, production process innovation, local strikes, etc.

- The systematic risk on stock i is the uncertainty on $\tilde{r}_i$ linked to risk factors that affect the whole economy and are common to other stocks.
  Examples: Wars, political change, general economic downturn, pandemics, etc.

Theorem

Diversification intended as buying a large number of different assets can eliminate the idiosyncratic risk but not the systematic risk.

Limit of diversification



Portfolio choice

Assumptions

1. $S = \{s_1, s_2, \ldots, s_n\}$
2. Any assets $i < n$ is a risky asset with return $\widetilde{r}_i$.
3. Asset n is a risk-free asset with $\widetilde{r}_n = r_f$. (Ex. Treasury bill)
4. Investors are mean-variance investors.

Implication:

Choose the$X = \{x_1, \ldots, x_f\}$ that maximizes

$$U(X) := E\left[\widetilde{r}_X\right] - \frac{a}{2}\sigma_X^2$$

$$\begin{aligned}
\text{subject to } 1 &= x_1 + x_2 + \cdots + x_f, \\
E\left[\widetilde{r}_X\right] &= \sum_{i=1}^{n} x_i E\left[\widetilde{r}_i\right], \\
Var\left[\widetilde{r}_X\right] &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij}.
\end{aligned}$$

Definition

A portfolio A is said to be efficient if there exists no other portfolio B satisfying:

$$\sigma_B^2 \leq \sigma_A^2 \text{ and } E\left[\tilde{r}_B\right] \geq E\left[\tilde{r}_A\right]$$

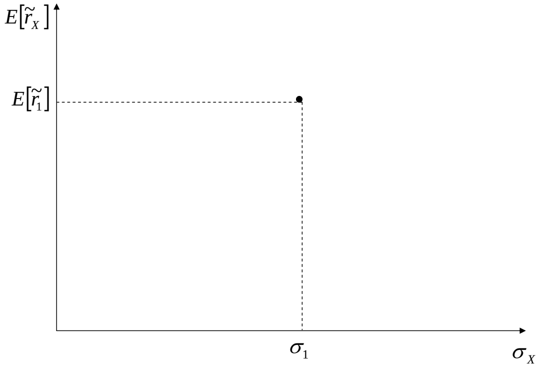
with at least one strict inequality.

Portfolio choice methodology

1. Determine the set of couple risk-return $(E\left[\tilde{r}_X\right], \sigma_X^2)$ reachable by combining the available assets S.
2. Identify the set of portfolios that are efficient.
3. Among the set of efficient portfolios choose the one that best fits the investor's risk aversion.

Return-risk region. Case 0

Just one risky asset:
$S = \{s_1\}$ with $E\left[\widetilde{r}_1\right] > 0$ and $\sigma_1^2 > 0$.



Return-risk region. Case 1

One risky asset and one risk-free asset:
$S = \{s_1, s_f\}$ with $E\left[\widetilde{r}_1\right] > 0$, $\sigma_1^2 > 0$, $\widetilde{r}_f = r_f < E\left[\widetilde{r}_1\right]$ and $\sigma_f^2 = 0$.

$$\begin{aligned} E\left[\widetilde{r}_X\right] &= x_1 E\left[\widetilde{r}_1\right] + (1 - x_1) r_f \\ \sigma_X^2 &= x_1^2 \sigma_1^2 \\ E\left[\widetilde{r}_X\right] &= r_f \pm \frac{E\left[\widetilde{r}_1\right] - r_f}{\sigma_1} \sigma_X \end{aligned}$$

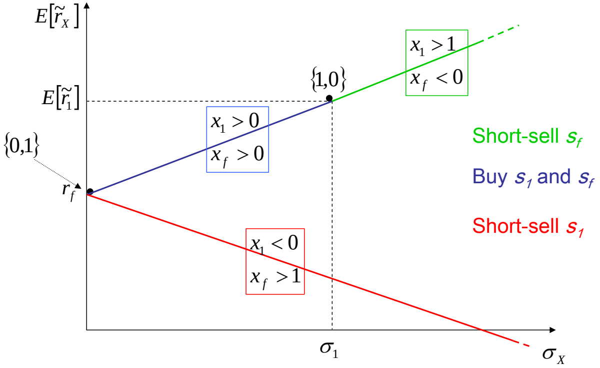
Example

$E\left[\widetilde{r}_1\right] = 20\%$, $\sigma_1^2 = 0.01$, $\widetilde{r}_f = 2\%$ and $\sigma_f^2 = 0$

- What is the composition of a portfolio with $E\left[\widetilde{r}_X\right] = 11\%$?
- What is the composition of a portfolio with risk $\sigma_X = 2\%$?
- The expected return on portfolio A is $E\left[\widetilde{r}_A\right] = 40\%$. What is its risk σ_A?

Return-risk region. Case 1

One risky asset and one risk-free asset: $S = \{s_1, s_f\}$



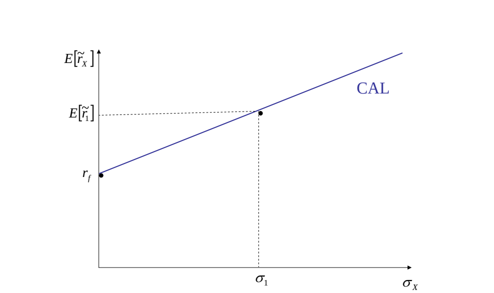
Capital Allocation Line

Theorem

In Case 1 a risk-return combination corresponds to an efficient portfolio if and only if

$$E\left[\tilde{r}_X\right] = r_f + \lambda\sigma_X$$

where $\lambda := \frac{E\left[\tilde{r}_1\right]-r_f}{\sigma_1}$

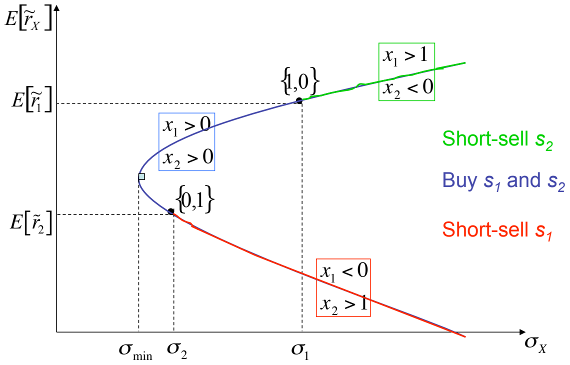


Two risky assets :
$S = \{s_1, s_2\}$ with $E\left[\widetilde{r}_1\right] > 0$, $\sigma_1^2 > 0$, $E\left[\widetilde{r}_2\right] > 0$, $\sigma_2^2 > 0$.

$$
\begin{aligned}
E\left[\widetilde{r}_X\right] &= x_1 E\left[\widetilde{r}_1\right] + (1 - x_1) E\left[\widetilde{r}_2\right] \\
\sigma_X^2 &= x_1^2 \sigma_1^2 + (1 - x_1)^2 \sigma_2^2 + 2 x_1 (1 - x_1) \rho_{12} \sigma_1 \sigma_2. \\
\sigma_X^2 &= \left(\frac{E[\widetilde{r}_X] - E[\widetilde{r}_2]}{E[\widetilde{r}_1] - E[\widetilde{r}_2]} \right)^2 \sigma_1^2 + \left(\frac{E[\widetilde{r}_1] - E[\widetilde{r}_X]}{E[\widetilde{r}_1] - E[\widetilde{r}_2]} \right)^2 \sigma_2^2 + \\
&+ 2 \frac{(E[\widetilde{r}_X] - E[\widetilde{r}_2])(E[\widetilde{r}_1] - E[\widetilde{r}_X])}{(E[\widetilde{r}_1] - E[\widetilde{r}_2])^2} \rho_{12} \sigma_1 \sigma_2.
\end{aligned}
$$

Return-risk region. Case 2

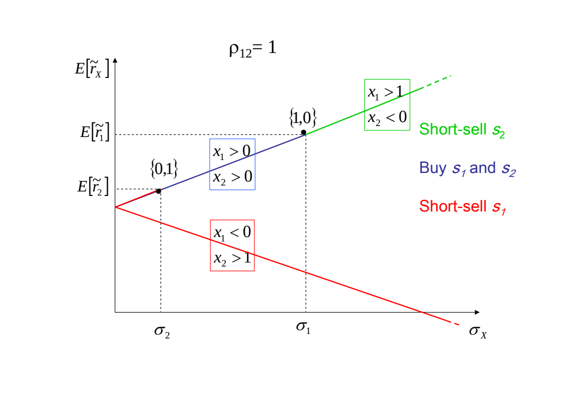
Two risky assets : $S = \{s_1, s_2\}$



Case 2 with perfect positive correlation

Show that if $S = \{s_1, s_2\}$ and $\rho = 1$, then

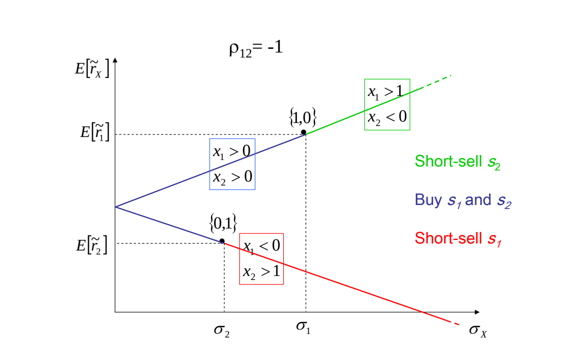
1. the return risk region is as depicted below.
2. find the composition, the expected return and the risk of the minimum Var portfolio
3. find the equation of the two lines.



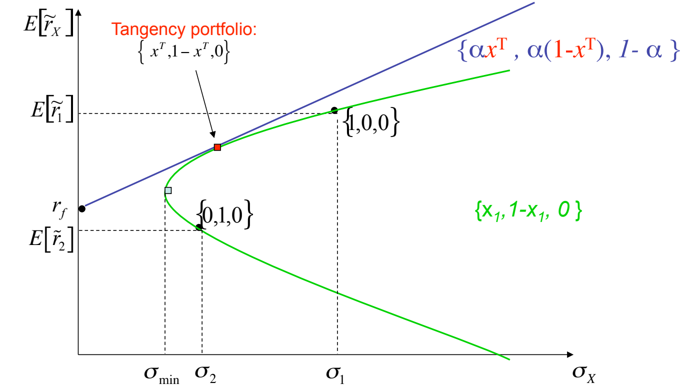
Return-risk region. Case 2 with perfect negative correlation

Show that if $S = \{s_1, s_2\}$ and $\rho = -1$, then

1. the return risk region is as depicted below.
2. find the composition, the expected return and the risk of the minimum Var portfolio.
3. find the equation of the two lines.



Two risky assets and one risk-free asset: $S = \{s_1, s_2, s_f\}$
with $E\left[\widetilde{r}_1\right] > 0$, $\sigma_1^2 > 0$, $E\left[\widetilde{r}_2\right] > 0$, $\sigma_2^2 > 0$, $\widetilde{r}_f = r_f$, $\sigma_f^2 = 0$.



Case 3: efficient portfolios

Theorem

In Case 3 a risk-return combination corresponds to an efficient portfolio if and only if:

1. $$E\left[\tilde{r}_X\right] = r_f + \lambda\sigma_X$$

   where

   $$\lambda := \frac{E\left[\tilde{r}_T\right] - r_f}{\sigma_T}$$

   and $E\left[\tilde{r}_T\right]$ and σ_T are the expected return and risk of the tangency portfolio.

2. *It is obtained combining the tangency portfolio with the risk-free asset.*

Tangency Portfolio composition for Case 3: $S = \{s_1, s_2, s_f\}$

What is the composition of the tangency portfolio?

$$\max_{x_1^T} \lambda = \max_{x_1^T} \frac{E[\widetilde{r}_T] - r_f}{\sigma_T}$$

subject to

$$\begin{aligned} E\left[\widetilde{r}_T\right] &= x_1^T E\left[\widetilde{r}_1\right] + (1 - x_1^T) E\left[\widetilde{r}_2\right] \\ \sigma_T &= \left((x_1^T \sigma_1)^2 + ((1 - x_1^T)\sigma_2)^2 + 2x_1^T(1 - x_1^T)\rho_{1,2}\sigma_1\sigma 2\right)^{\frac{1}{2}} \end{aligned}$$

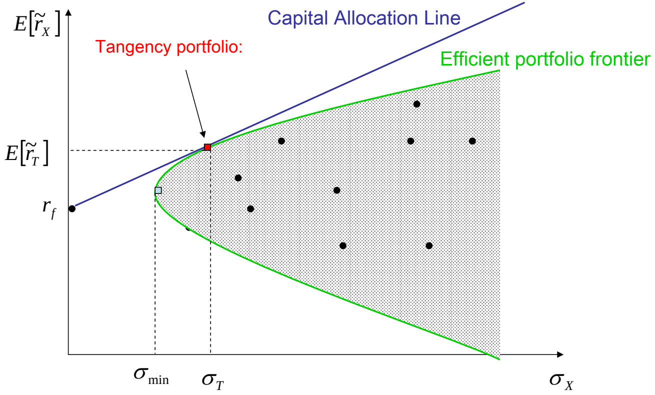
This gives

$$x_1^T =$$

$$\frac{(E\left[\widetilde{r}_2\right] - r_f)\rho_{1,2}\sigma_1\sigma_2 - (E\left[\widetilde{r}_1\right] - r_f)\sigma_2^2}{(E\left[\widetilde{r}_1\right] + E\left[\widetilde{r}_2\right] - 2r_f)\rho_{1,2}\sigma_1\sigma_2 - (E\left[\widetilde{r}_1\right] - r_f)\sigma_2^2 - (E\left[\widetilde{r}_2\right] - r_f)\sigma_1^2}$$

General case

$n-1$ risky assets and one risk-free asset:
$S = \{s_1, \ldots, s_{n-1}, s_f\}$



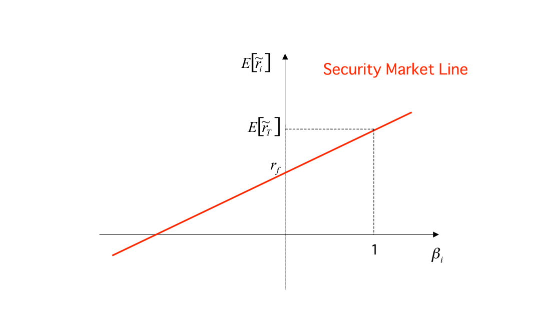
Theorem

For any asset or portfolio s_i,

$$E\left[\widetilde{r}_i\right] - r_f = \beta_i \left(E\left[\widetilde{r}_T\right] - r_f\right)$$

where

$$\beta_i := \frac{Cov\left[\widetilde{r}_i, \widetilde{r}_T\right]}{\sigma_T^2}$$

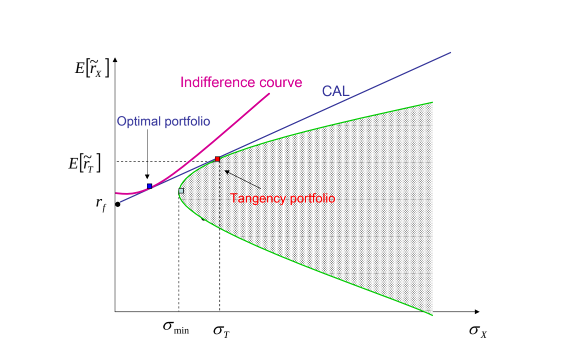


Optimal Portfolio

Theorem

An investor with a mean-variance utility function and a risk aversion A will choose an efficient portfolio such that:

$$x_T^* = \frac{E[\tilde{r}_T] - r_f}{A\sigma_T^2}, \; x_f^* = 1 - x_T^*$$



Exercise

Suppose that

- $S = \{s_1, s_2, s_3, s_f\}$
- $\{E[\widetilde{r}_1], E[\widetilde{r}_2], E[\widetilde{r}_3], r_f\} = \{9\%, 23\%, -1\%, 1\%\}$
- The tangency portfolio is $X_T = \{1.48, 0.3, -0.78, 0\}$ and $\sigma_T = 18\%$.

Questions:

1. What is $E[\widetilde{r}_T]$? (Ans.: 21%)
2. What are $\beta_1, \beta_2, \beta_3$? (Ans.: $0.4, 1.1, -0.1$)
3. What is the composition of the optimal portfolio for an investor with risk aversion $A = 4$? (Ans.: 1.543 in the tangency portfolio and -0.543 in the risk free asset, that is $X_P = \{2.284, 0.463, -1.205, -0.543\}$)

Summary

- Definition of efficient portfolio.
- Mean-Variance investors choose efficient portfolios.
- An efficient portfolio is composed of the tangency portfolio and the risk free asset.
- if X is efficient, then

$$E\left[\widetilde{r}_X\right] = r_f + \frac{E\left[\widetilde{r}_T\right] - r_f}{\sigma_T}\sigma_X$$

- The tangency portfolio satisfies

$$E\left[\widetilde{r}_i\right] - r_f = \beta_i\left(E\left[\widetilde{r}_T\right] - r_f\right)$$

where

$$\beta_i = \frac{Cov\left[\widetilde{r}_T, \widetilde{r}_i\right]}{\sigma_T^2}$$