Financial Economics
5: Capital Asset Pricing Model

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Capital Asset Pricing Model

What explains the difference in prices and returns of different financial asset?

Assumptions

1. There are two dates: $t = 0$ and $t = 1year$.
2. Set of available assets: $S = \{s_1, \ldots, s_{n-1}, s_f\}$.
3. Population is composed of Mean-Variance investors.
4. Investors behave competitively (price taker).
5. Investors have homogeneous beliefs reflecting the distribution of financial assets' returns.
6. There is no transaction cost.
7. The risk-free asset is in zero net supply.
8. Short sales are allowed.

CAPM: timitng

-Investors form their portfolios.

-Asset prices are such that:

Supply = Demand

Assets return are realized.

Notation:

- Q_i: Number of outstanding shares of stock (or security) i.
- p_i: Price of one share of stock i at time $t = 0$.
- $K_i := Q_i p_i$ *Market capitalization* of security i.
- $K := K_1 + K_2 + \cdots + K_{n-1}$ *Total market capitalization*.

Definition

The market portfolio is $M = \{\omega_1, \omega_2, \ldots, \omega_{n-1}, 0\}$ where

$$\omega_i := \frac{K_i}{K}$$

The market portfolio is:

- Composed only of risky assets ($\omega_f = 0$).
- Its composition reflects the relative size of all listed companies.

Total market cap of 100 biggest companies \$20,500,000 million in 2018.

(Source: www.statista.com)

Ranking of the companies rank 1 to 100	Market value in billion U.S. dollars
Apple	926.9
Amazon.com	777.8
Alphabet	766.4
Microsoft	750.6
Facebook	541.5
Alibaba	499.4
Berkshire Hathaway	491.9
Tencent Holdings	491.3
JPMorgan Chase	387.7
ExxonMobil	344.1
Johnson & Johnson	341.3
Samsung Electronics	325.9
Bank of America	313.5

Notation

- L: Number of investors in the economy.
- W^h: Investor h's wealth.

Remark: Mean-variance investors demand efficient portfolios.
$\Rightarrow$ Investors' portfolio are combinations of the risk-less asset and the tangency portfolio:

$$X^h = \{x_T^h, x_f^h\}$$

with

- x_T^h: fraction of W^h invested in the tangency portfolio.
- $x_f^h = 1 - x_T^h$: fraction of W^h invested in the risk-free asset.

Remarks:

1. If two investors h and h' differ in risk aversion, then
$$x_T^h \neq x_T^{h'}.$$
2. All investors share the same beliefs on the probability distribution of assets return $\Rightarrow$ all investors agree on the composition of the tangency portfolio

$$\begin{aligned} X_T &= \left\{x_{T,1}, x_{T,2}, \ldots, x_{T,n-1}, 0\right\} \\ &\Downarrow \\ D_i^h &= W^h x_T^h\, x_{T,i} \frac{1}{p_i} \end{aligned}$$

Investor h's demand for asset i

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$$\Downarrow$$
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Varies across investors ($W^h x_T^h$)

The same for all investors ($x_{T,i} \frac{1}{p_i}$)

Demand: QCQ

$$
\begin{aligned}
S &= \{s_1, s_2, s_3, s_f\} \\
X_T &= \{0.6, 0.3, 0.1, 0\} \\
p_1 &= Eu\,5 \\
W^h = EU\,20,000 \quad &, \quad X_T^h = 0.4 \\
W^{h'} = EU\,30,000 \quad &, \quad X_T^{h'} = 0.5
\end{aligned}
$$

1. What are investors h and h''s demand for asset 1.

$$
\begin{aligned}
D_i^h &= 20,000 \times 0.4 \times 0.6 \times \frac{1}{5} = 960 \text{ shares of asset 1} \\
D_i^{h'} &= 30,000 \times 0.5 \times 0.6 \times \frac{1}{5} = 1,800 \text{ shares of asset 1}
\end{aligned}
$$

2. Is investor h more or less averse to risk than investor h'?

Equilibrium

Definition

An equilibrium of the economy is a situation where

1. Each investor chooses the portfolio that maximizes his utility function, given the current stock prices.
2. The stocks' prices are such that the aggregate demand for each asset equals its aggregate supply:

$$\sum_{h=1}^{L} D_i^h = Q_i, \forall i = 1, 2, \ldots, n-1, f$$

Theorem

In equilibrium the composition of the Tangency portfolio is equal to the composition of the Market portfolio:

$$X_T = \{x_{T,1}, \ldots, x_{T,n-1}, 0\} = \{\omega_1, \ldots, \omega_{n-1}, 0\} = M$$

Proof:

$$\sum_{h=1}^{L} D_i^h = \sum_{h=1}^{L} W^h x_T^h \frac{x_{T,i}}{p_i} = \frac{x_{T,i}}{p_i} \sum_{h=1}^{L} W^h x_T^h.$$

$$\sum_{h=1}^{L} D_i^h = Q_i \Rightarrow x_{T,i} \sum_{h=1}^{L} W^h x_T^h = p_i Q_i = K_i \Rightarrow$$

$$x_{T,i} = \frac{K_i}{\sum_{h=1}^{L} W^h x_T^h}$$

What is the total wealth invested in risky assets?

$$\sum_{h=1}^{L} W^h x_T^h = K_1 + K_2 + \cdots + K_{n-1} = K \Rightarrow$$

$$x_{T,i} = \frac{K_i}{K} = \omega_i$$

Security Market Line

Since in equilibrium, the Market portfolio = Tangency portfolio, the Market portfolio inherits the properties of the tangency portfolio:

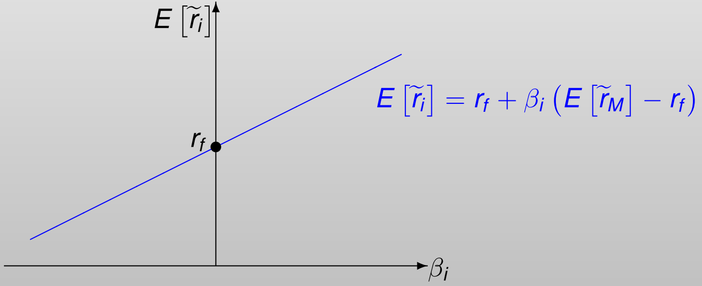
Theorem

In equilibrium the expected rate of return of an asset is a linear function of its beta:

$$\begin{aligned} E\left[\widetilde{r}_i\right] &= r_f + \beta_i\left(E\left[\widetilde{r}_M\right] - r_f\right) \\ \text{with } \beta_i &= \frac{Cov\left[\widetilde{r}_i, \widetilde{r}_M\right]}{\sigma_M^2} \end{aligned}$$

Security Market Line: properties

1. There is a linear relation between the expected rate of return of any asset and its beta.
2. The slope of the SML is $E\left[\tilde{r}_M\right] - r_f$.
3. The intercept of the SML is r_f.



Example

Suppose

$$r_f = 5\%,\ E\left[\widetilde{r}_M\right] = 16\%$$

Consider the following assets:

1. Asset 1: $\beta = 2$ and $E\left[\widetilde{r}_1\right] = 27\%$.
2. Asset 2: $\beta = 0$ and $E\left[\widetilde{r}_2\right] = 5\%$.
3. Asset 3: $\beta = -0.2$ and $E\left[\widetilde{r}_3\right] = 2.8\%$

Security Market Line: Examples

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Indeed, $E\left[\widetilde{r}_i\right] = r_f + \beta_i\left(E\left[\widetilde{r}_M\right] - r_f\right)$:

1. Asset 1: $E\left[\widetilde{r}_1\right] = 5\% + 2(16\% - 5\%) = 27\%$.
2. Asset 2: $E\left[\widetilde{r}_2\right] = 5\% + 0(16\% - 5\%) = 5\%$.
3. Asset 3: $E\left[\widetilde{r}_3\right] = 5\% - 0.2(16\% - 5\%) = 2.8\%$

Security Market Line: Interpretation

Definition

The expected risk premium on asset i is the extra expected return required by investors to hold risky asset i rather than the risk-free asset: $E\left[\tilde{r}_i\right] - r_f$

SML states that an asset's risk premium is proportional to the asset's beta.

WHY?

Where the risk of a portfolio comes from?

- Investor *h*'s portfolio: $X^h = \{x_M^h, 1 - x_M^h\} \Rightarrow \sigma_{X^h} = x_M^h \sigma_M$.

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  $\omega_1 Cov\left[\widetilde{r}_1, \widetilde{r}_M\right] + \omega_2 Cov\left[\widetilde{r}_2, \widetilde{r}_M\right] + \cdots \omega_{n-1} Cov\left[\widetilde{r}_{n-1}, \widetilde{r}_M\right]$
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- Investor h's portfolio: $X^h = \{x_M^h, 1 - x_M^h\} \Rightarrow \sigma_{X^h} = x_M^h \sigma_M$.
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  $\Rightarrow$
- The larger an asset's contribution to the risk, the larger is the risk premium that risk averse investors require to hold such asset in their portfolios.

Beta and Risk

- A risky asset with a negative Beta reduces the risk of the market portfolio
  $\Rightarrow$ Investors will hold it at a negative risk premium:
  $E\left[\tilde{r}_i\right] < r_f$
- σ_i, the standard deviation of $\tilde{r}_i$, is not a good measure of asset *i*'s risk.
- The correct measure for an asset's risk is its β_i, i.e., its contribution to the market portfolio's risk.
- There is no relation between σ_i and $E\left[\tilde{r}_i\right]$.

Example

Let $r_f = 2\%$ and $E\left[\tilde{r}_M\right] = 16\%$

	β_i	$E\left[\tilde{r}_i\right]$	σ_i
Asset 1	2	30%	15%
Asset 2	0	2%	20%
Asset 3	-0.1	0.6%	16%

SML: QCQ

- None of the following situations is consistent with the CAPM predictions. Why?
  1. $r_f = 3\%$, $E\left[\widetilde{r}_1\right] = 4\%$, $\beta_1 = 0$.
  2. $E\left[\widetilde{r}_M\right] = 15\%$, $E\left[\widetilde{r}_2\right] = 30\%$, $\beta_2 = 1$.
  3. $E\left[\widetilde{r}_3\right] = 15\%$, $\beta_3 = 1.5$, $E\left[\widetilde{r}_4\right] = 20\%$, $\beta_4 = 0.8$.
- Suppose $E\left[\widetilde{r}_1\right] = 27\%$, $\beta_1 = 2$, $r_f = 2\%$.
  1. What is $E\left[\widetilde{r}_M\right]$? (Ans. 14.5%)
  2. What is $E\left[\widetilde{r}_6\right]$ if $\beta_6 = 3$? (Ans. 39.5%)

Beta and Correlation

General Electric vs. Dow

Gold vs. Dow

Source www.FT.com

1. Is the beta of GE positive or negative?
2. How does GE's expected return rate compares to $E\left[\tilde{r}_M\right]$ and r_f?
3. Is the beta of Gold positive or negative?
4. How does Gold's expected return rate compares to $E\left[\tilde{r}_M\right]$ and r_f?

How to compute the beta of a portfolio

Theorem

Consider portfolio $X_A = \{x_1, x_2, \ldots, x_n\}$. Then

$$\beta_A := \frac{Cov\left[\tilde{r}_A, \tilde{r}_M\right]}{\sigma_M^2} = x_1\beta_1 + x_2\beta_2 + \cdots + x_n\beta_n$$

Proof:

$$\begin{aligned}
r_f + \beta_A \left(E\left[\tilde{r}_M\right] - r_f\right) &= E\left[\tilde{r}_A\right] = \sum_{i=1}^{n} x_i E\left[\tilde{r}_i\right] = \\
= \sum_{i=1}^{n} x_i \left(r_f + \beta_i \left(E\left[\tilde{r}_M\right] - r_f\right)\right) &= r_f \sum_{i=1}^{n} x_i + \left(E\left[\tilde{r}_M\right] - r_f\right) \sum_{i=1}^{n} x_i\beta_i \\
&= r_f + \left(E\left[\tilde{r}_M\right] - r_f\right) \sum_{i=1}^{n} x_i\beta_i
\end{aligned}$$

Beta of a portfolio: example

Example

	β_i	$E\left[\widetilde{r}_i\right]$	σ_i
Asset 1	2	30%	15%
Asset 2	0	2%	20%
Asset 3	-0.1	0.6%	16%

The beta of a portfolio $X_P = \{0.2, 0.5, 0.3\}$ is

$$\beta_P = 0.2 * 2 + 0.5 * 0 + 0.3 * (-0.1) = 0.37$$

Beta and efficient portfolios

- What is the beta of the market portfolio?

$$\beta_M = \frac{Cov\left[\widetilde{r}_M, \widetilde{r}_M\right]}{\sigma_M^2} = 1$$

- What is the beta of the risk free asset?

$$\beta_f = \frac{Cov\left[r_f, \widetilde{r}_M\right]}{\sigma_M^2} = 0$$

- What is the composition of an efficient portfolio with $\beta_P = 0.7$

$$\begin{aligned} X_P^* &= \{x_M, 1 - x_M\} \\ 0.7 &= \beta_P = x_M\beta_M + (1 - x_M)\beta_f = x_M 1 + (1 - x_M)0 = \\ &= x_M \Longrightarrow X_P^* = \{0.7, 0.3\} \end{aligned}$$

How to build an efficient portfolio?

- A portfolio is efficient if it is a combination of the Tangency portfolio and the risk free asset.
- Under the CAPM, the Tangency portfolio is equal to the Market portfolio.

$$\Downarrow$$

- Choose either the desired level of risk or the desired level of expected return for your portfolio.
- Combine the risk-free asset with the market portfolio in such a way that you reach your desired level of risk or return.

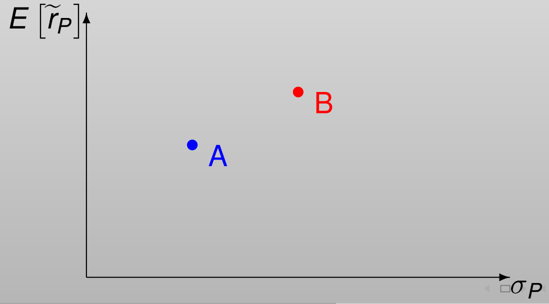
Can you beat the market?

Example

You have to evaluate two mutual funds managers:

- Mutual fund A: $E\left[\tilde{r}_A\right] = 10\%$, $\sigma_A = 16\%$.
- Mutual fund B: $E\left[\tilde{r}_B\right] = 20\%$, $\sigma_B = 22\%$.

Suppose $E\left[\tilde{r}_M\right] = 15\%$, $\sigma_M = 20\%$ and $r_f = 3\%$.



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- Mutual fund A: $E\left[\tilde{r}_A\right] = 10\%$, $\sigma_A = 16\%$.
- Mutual fund B: $E\left[\tilde{r}_B\right] = 20\%$, $\sigma_B = 22\%$.

Suppose $E\left[\tilde{r}_M\right] = 15\%$, $\sigma_M = 20\%$ and $r_f = 3\%$.

$$\text{CML: } E\left[\tilde{r}_P\right] = r_f + \frac{E\left[\tilde{r}_M\right] - r_f}{\sigma_M}\sigma_p$$

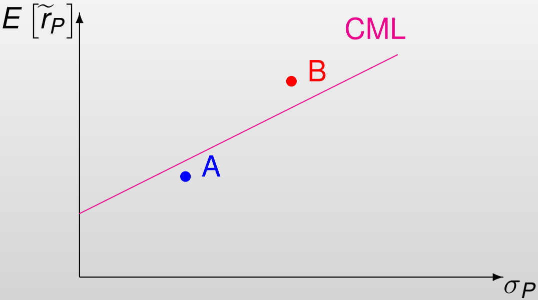
What is the expected return of an efficient portfolio with risk $\sigma = 16\%$.

$$E\left[\tilde{r}_P\right] = 3\% + \frac{15\% - 3\%}{20\%}16\% = 12.6\% > E\left[\tilde{r}_A\right] = 10\%$$

What is the expected return of an efficient portfolio with risk $\sigma = 22\%$.

$$E\left[\tilde{r}_P\right] = 3\% + \frac{15\% - 3\%}{20\%}22\% = 16.2\% < E\left[\tilde{r}_B\right] = 20\%$$

Can you beat the market?



- Mutual fund A is beaten by the market.
- Mutual fund B beats the market.

Applications of CAPM: The choice of OCC for risky investments

Rule: The OCC used to compute the NPV of a project is the interest rate one can gain from an alternative investment with the same risk.

Result: According to the CAPM, the expected return on a risky investment only depends on its beta.

Rule: The OCC used to compute the NPV should be the expected return on an asset that has the same beta of the project.

Example

An investment project requires *Eu* $2,000,000$. In one year the project will generate one random cash flow, $\widetilde{F}_1$, with $E\left[\widetilde{F}_1\right] = 2,161,654$. Also, $Cov\left[\widetilde{F}_1, \widetilde{r}_M\right] = 200$, $E\left[\widetilde{r}_M\right] = 10\%$, $r_f = 6\%$, $\sigma_M^2 = 0.001$.
Is it worthwhile to undertake the project?

1. OCC
2. NPV

The choice of OCC for risky investments

Example

Solution:

1. OCC= $r_f + \beta_{proj}(E[\tilde{r}_M] - r_f)$ where

$$\beta_{proj} = \frac{Cov[\tilde{r}_{proj},\tilde{r}_M]}{\sigma_M^2} \quad , \quad \tilde{r}_{proj} = \frac{\tilde{F}_1}{2{,}000{,}000} - 1$$

$$Cov[\tilde{r}_{proj},\tilde{r}_M] = E[(\tilde{r}_{proj} - E[\tilde{r}_{proj}])(\tilde{r}_M - E[\tilde{r}_M])]$$

$$= \frac{Cov[\tilde{F}_1,\tilde{r}_M]}{2{,}000{,}000} = 0.0001 \Longrightarrow$$

$$\beta_{proj} = \frac{Cov[\tilde{r}_{proj},\tilde{r}_M]}{\sigma_M^2} = 0.1 \Longrightarrow OCC = 6\% + 0.1 * 4\% = 6.4\%$$

2. $NPV = -2,000,000 + \frac{2{,}161{,}654}{1.064} = 31,629.7 > 0$

CONCLUSION

- Under the CAPM assumptions the tangency portfolio is equal to the market portfolio
- An efficient portfolio is a combination of the risk free asset and the market portfolio.
- The Beta of an asset represents the risk contribution of this asset to the risk of the market.
- The risk premium required by investors to hold a risky asset in their portfolio is proportional to the asset's beta Beta.