Financial Economics, **HEC, Paris**

Problem Set 3

Exercise 1

A fund manager can invest in 3 securities. Securities A and B are risky with the following characteristics : $E(r_A)=0.2$; $E(r_B)=0.12$; $\sigma(r_A)=0.3$ and $\sigma(r_B)=0.15$. The correlation between the returns of A and B is 0.1. The third security is risk-free with a rate of return equal to 0,08.

a) What are the investment proportions of the minimum-variance portfolio of the two risky securities and what is the expected value and standard deviation of its rate of return?
b) Tabulate and draw the investment opportunity set of the two risky securities. Use investment proportions, i.e. weights, for security A of zero to 100 % in increments of 20 %.
c) Draw a tangent from the risk-free rate to the opportunity set. What does your graph show for the expected return and standard deviation of return of the tangent portfolio?

Exercise 2

The probability distribution for security X is :

Probability	0.1	0.2	0.4	0.2	0.1
Rate of return	0.6	0.4	0.3	0.2	-1

Furthermore, there exists another risky security, Y, whose rate of return is linked to that of X by the following relation: $r_Y = 0.06 + 0.2\ r_X$. If there also existed a risk-free security with a rate of return equal to 0.06 which profitable operation could you realize and under what conditions? Be sure to describe in details the operation you have in mind.

Exercise 3

Consider an economy with two risky assets s_1 and s_2. Suppose that the expected rate of return of asset 1 is 10% and that σ_1=10%. The expected rate of return of asset 2 is 5% and σ_2=20%. The correlation coefficient between s_1 and s_2 is $\rho_{1,2} = 0.875$.

a) What is the composition, the expected return and the standard deviation of the minimum variance portfolio?
b) Represent in a graph all the combination risk/return that can be obtained with portfolios containing s_1 and s_2. Clearly identify s_1, s_2 and the minimum variance portfolio.
c) Harry is a mean-variance investor with risk aversion A=3. Harry is not allowed to short-sell asset s_2 while he can short-sell asset s_1.

- c.1) In a new graph, represent risk/return combinations that are available to Harry.
- c.2) What is the composition of Harry's optimal portfolio?

d) Let us introduce a risk free asset s_f with return r_f =10%. Now all short-sales are allowed:

- d.1) In a third graph represent, s_1, s_2 and s_f, the set of risk/return combination that can be obtained with portfolios composed only of s_1 and s_2, the tangency portfolio T, and the capital allocation line.
- d.2) Let portfolio D have the composition $X_D = \{x_1=1/3,\ x_2=1/3,\ x_f=1/3,\}$. Is portfolio D efficient? Explain carefully but concisely why it is or why it is not.
- d.3) What is the composition of Harry's optimal portfolio in this case (recall that Harry cannot short-sell asset s_2)

Elements of answer PS3

Exercise 1

a)

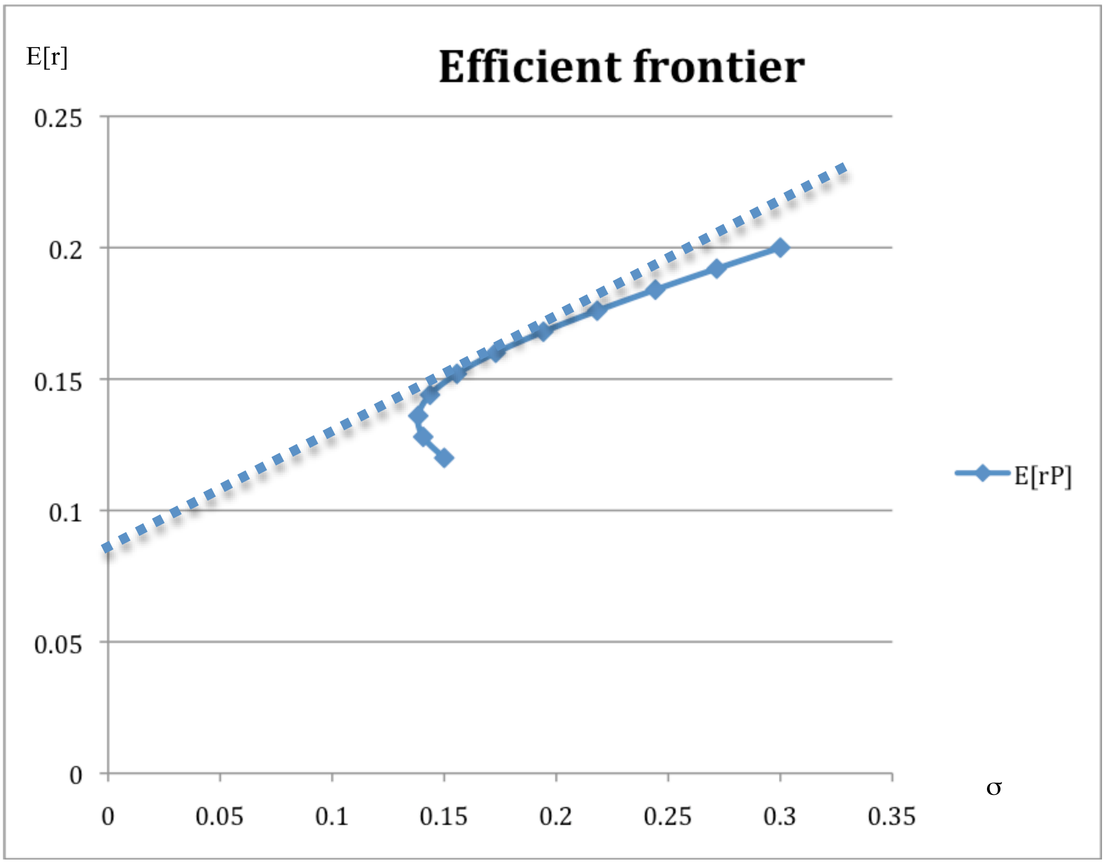
Weight of sec. A in min variance portfolio : $x = \dfrac{Var(R_B) - \rho(R_A, R_B)\sigma(R_A)\sigma(R_B)}{Var(R_A) + Var(R_B) - 2\rho(R_A, R_B)\sigma(R_A)\sigma(R_B)}$

$$x = \frac{0.15^2 - (0.1)(0.3)(0.15)}{0.3^2 + 0.15^2 - 2(0.1)(0.3)(0.15)} = 0.1739$$ and (1-x) = 0,8261.

$E(R_P) = 0.1739 \times 0.2 + 0.8261 \times 0.12 = 0.1339$
$Var(R_P) = 0.1739^2 \times 0.3^2 + 0.8261^2 \times 0.15^2 + 2 \times 0.1739 \times 0.8261 \times 0.1 \times 0.3 \times 0.15 = 0.01037$
$\sigma(R_P) = \sqrt{0.01037} = 0.1392$

b)

Weight A	Wight B	Sigma[rP]	E[rP]	Var[rP]
0	1	0.15	0.12	0.0225
0.1	0.9	0.1406165	0.128	0.019773
0.2	0.8	0.138390751	0.136	0.019152
0.3	0.7	0.143655839	0.144	0.020637
0.4	0.6	0.155653461	0.152	0.024228
0.5	0.5	0.172988439	0.16	0.029925
0.6	0.4	0.194236969	0.168	0.037728
0.7	0.3	0.21825902	0.176	0.047637
0.8	0.2	0.244237589	0.184	0.059652
0.9	0.1	0.271611855	0.192	0.073773
1	0	0.3	0.2	0.09



Exercise 2

Remark : Securities X and Y are perfectly positively correlated.
$E(R_X) = 0.1\times0.6 + 0.2\times0.4 + 0.4\times0.3 + 0.2\times0.2 + 0.1\times(-1) = 0.2$
$E(R_Y) = 0.06 + 0.2\times0.2 = 0.1$

Let us look up if we can find a riskless portfolio. The return rate of a portfolio P composed of x of asset X and $(1\text{-}x)$ of asset Y is

$R_P = x\, r_X + (1\text{-}x)\, r_Y = r_X\,(x + (1\text{-}x)0.2) + (1\text{-}x)0.06.$

P is a riskless portfolio if $(x + (1\text{-}x)0.2)=0$, i.e. $x = -0.25$.
As an alternative observe that :
$Var(R_X) = 0.172$
$\sigma(R_X) = 0.4147$
$\sigma(R_Y) = 0.2 \times \sigma(R_X) = 0.0829$
The minimum variance portfolio is:

For $$x = \frac{\sigma(R_Y)}{\sigma(R_Y) - \sigma(R_X)} = \frac{0.0829}{0.0829 - 0.4147} = -0.25$$

and $(1\text{-}x) = 1.25$ we get a risk-free portfolio. Hence, if short-sales are allowed one can construct a risk-free portfolio.

Expected return on this risk-free portfolio = $-0.25\times0.2 + 1.25\times0.1 = 0.075$
This is larger than the rate of return of the risk-free security which is 0.06 $\Rightarrow$ there exists an arbitrage opportunity: borrow at the risk-free rate and invest in the synthetic riskless portfolio (-0,25 ; 1,25).

Exercise 3

Consider an economy with two risky assets s_1 and s_2. Suppose that the expected rate of return of asset 1 is 10% and that σ_1=10%. The expected rate of return of asset 2 is 5% and σ_2=20%. The correlation coefficient between s_1 and s_2 is $\rho_{1,2} = 0.875$.

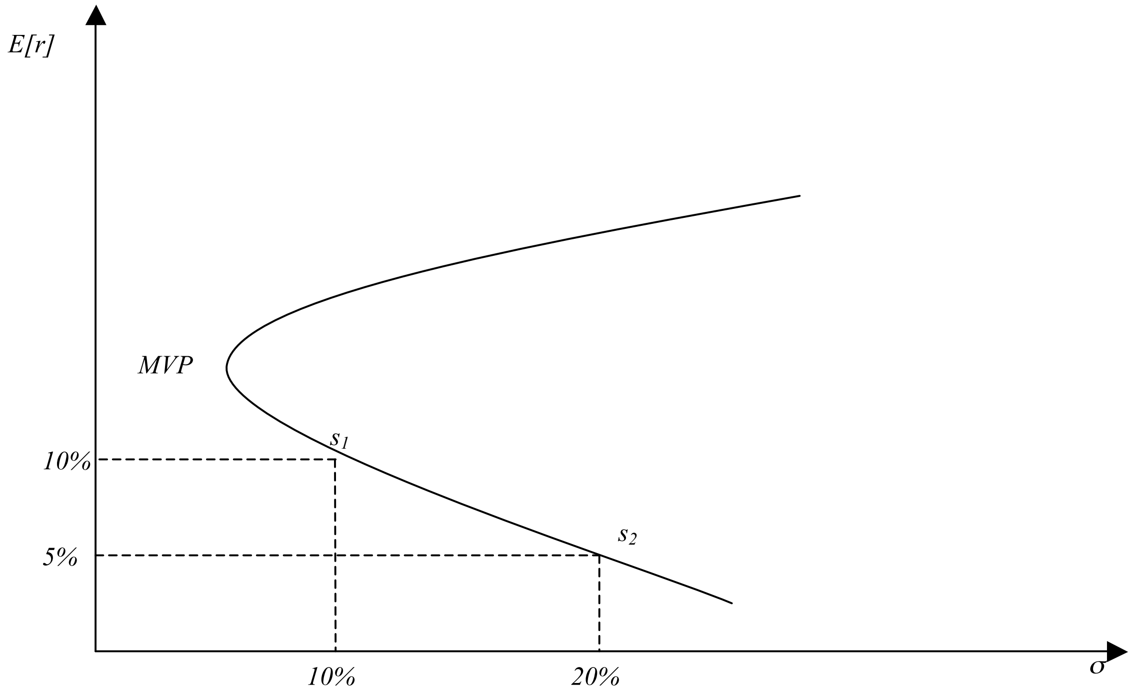
a) What is the composition, the expected return and the standard deviation of the minimum variance portfolio?

$x_{1,MVP} = 0.2\times (0.2 - 0.875 \times 0.1)/(0.2^2+0.1^2- 2\times0.875\times0.1\times0.2) = 1.5$
$x_{2,MVP} = -0.5$

$E[r_{MVP}] = 1.5\times10\% - 0.5\times5\% = 12.5\%$

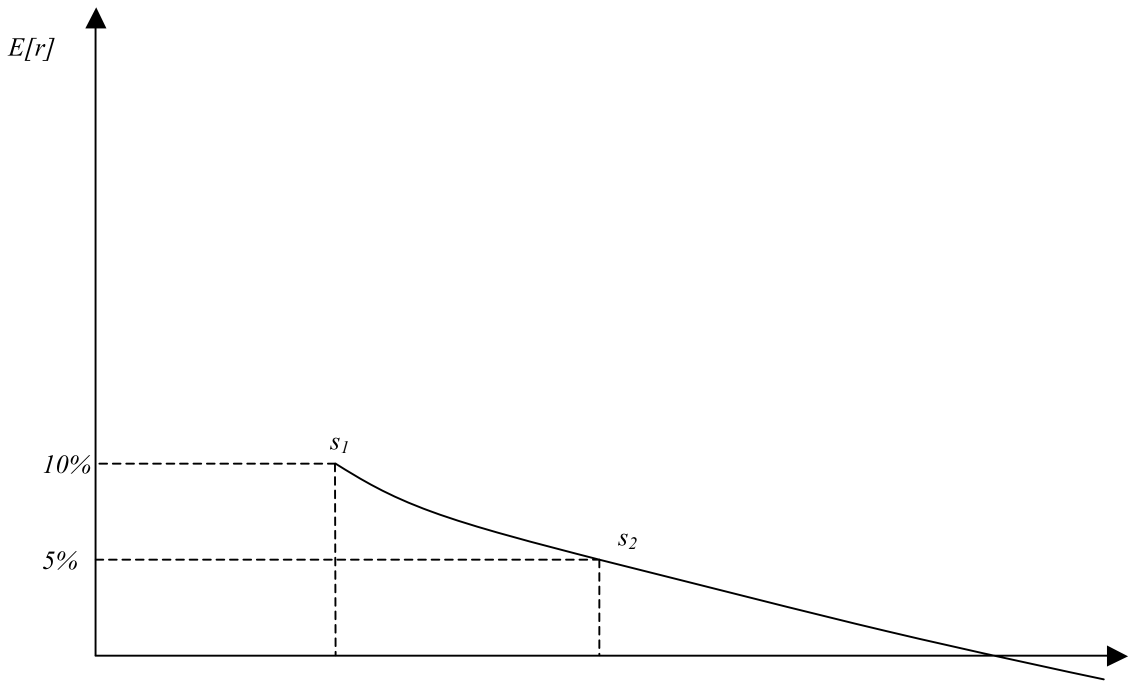
$\sigma_{MVP} =((1.5\times0.1)^2+(-0.5\times0.2)^2-2\times1.5\times0.5\times0.875\times0.2\times0.1)^{1/2} =7.9\%$

b) Represent in a graphic all the combination risk/return that can be obtained by combining s_1 and s_2. Clearly identify s_1, s_2 and the minimum variance portfolio.



c) Harry is a mean-variance investor with risk aversion A=3. Harry is not allowed to short sell asset s_2 while he can short sell asset s_1.

c.1) In a new graph, represent the combinations risk/return that are available to Harry.

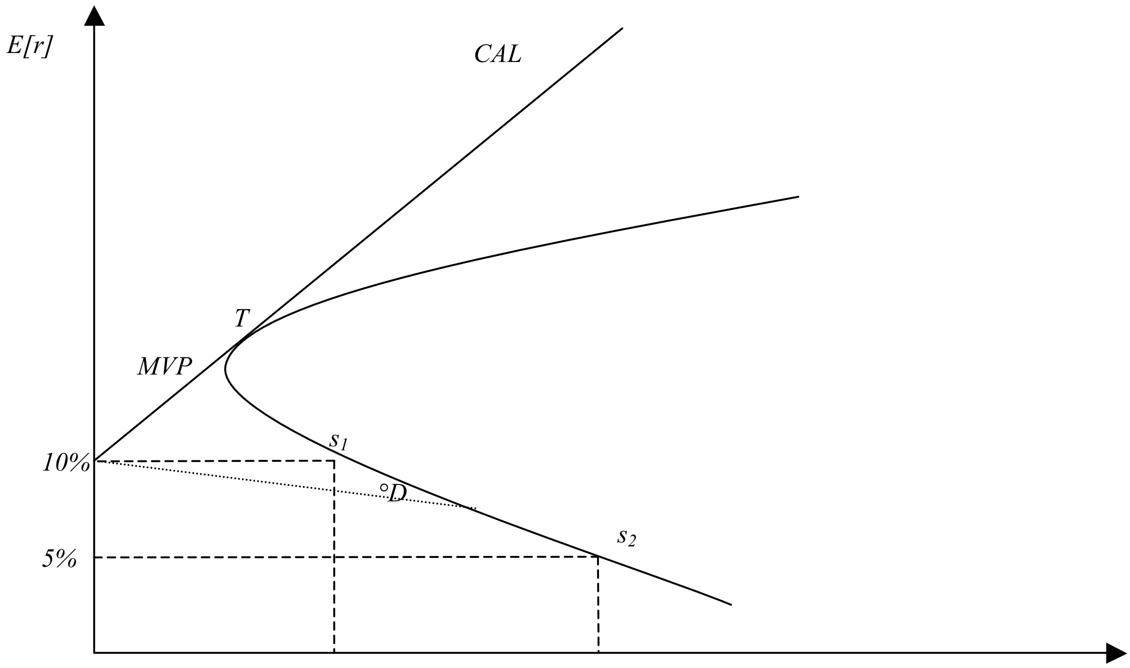


c.2) What is the composition of Harry optimal portfolio?

Harry will invest all his wealth in asset 1 as this is the only efficient portfolio given his short-selling constraint.

d) Let introduce a risk free asset s_f with return is r_f =10%. All short sales are possible.

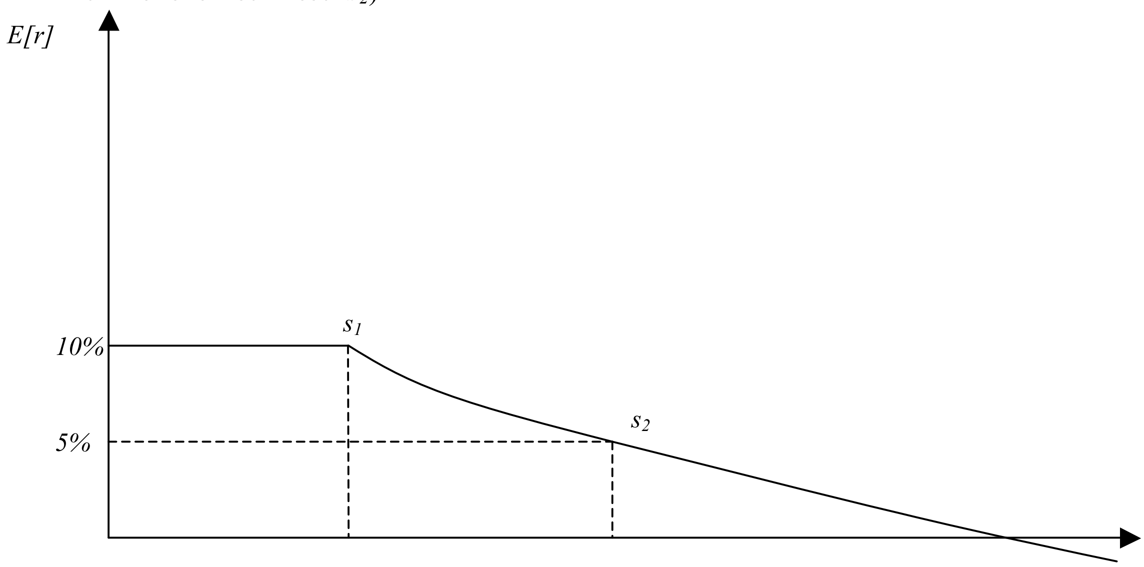
d.1) In a third graph represents, s_1, s_2 and s_f , the set of risk/return combination that can be obtained with portfolios composed only of s_1 and s_2 , the tangency portfolio, and the capital allocation line.



d.2) Let portfolio D has the composition $X_D = \{x_1=1/3, x_2=1/3, x_f=1/3,\}$. Is portfolio D efficient? Explain carefully but concisely why it is or why it is not.

The tangency portfolio is obtained by short selling 2 and to buy 1. An efficient portfolio is a combination of the risk free asset and the tangency portfolio. Hence it must have a negative weight of asset 2 if it has a positive weight of asset 1. Portfolio D has positive weight of asset 1 and 2 hence it is not a combination of T and s_f, hence it is not efficient. (See picture)

d.3) What is the composition of Harry optimal portfolio in this case? (Recall that Harry cannot short sell asset s_2)



Now Harry can reach all points on the line and the only efficient portfolio is obtained by investing all his wealth in s_f.