Financial Economics, **HEC, Paris**

Problem Set 5

Problem 1:

We consider a financial market composed of **only 2 risky assets** (which are imperfectly correlated) **and 1 risk-free asset**. Risky asset 1 has a Beta equal to: $\beta_1 = 1.2$ and expected returns equal to $E(r_1) = 12.2\%$. Risky asset 2 has a Beta equal to: $\beta_2 = 0.6$ and expected return equal to $E(r_2) = 8.6\%$. You will assume in what follows that **all the assumptions of the CAPM are satisfied**.

a) Find the level of the risk-free rate.
b) Find the expected return of the market portfolio.
c) What are the respective weights of assets 1 and 2 in the **tangent portfolio**?

Problem 2:

Assume that CAPM accurately represents security returns (hence prices). Your portfolio consists of 50% invested in the risk-free asset and the remaining 50% is allocated as follows among four risky securities:

Security (i)	**$E(r_i)$**	**β_i**	**x_i**
1	0.076	0.2	10%
2	0.124	0.8	10%
3	0.156	1.2	10%
4	0.188	1.6	20%

a. What is your current portfolio's expected return and **beta**?
b. If you wished to earn a 12% expected return on your portfolio, one possibility would be to sell some of your holdings in the risk-free asset and invest the proceeds from that sale in the market portfolio (M). If you decide to rebalance your portfolio this way, what would be your new portfolio weights and your portfolio's overall **beta**?
c. Alternatively, you could invest your portfolio only in the market portfolio and the risk-free asset. What would be your portfolio weights to obtain a 12% expected return?
d. Which portfolio rebalancing strategy would you prefer?

Problem 3:

Consider a market with many risky assets and a risk-free security. Asset's returns are not perfectly correlated. All the CAPM assumptions hold and the market is in equilibrium. The risk-free rate is 5%, the expected return on the market is 15%. Mr. T and Mrs. R are two investors with mean-variance utility functions and different risk-aversion coefficients. They both invest into efficient portfolios composed of the market portfolio and the risk-free security.

a. Mr. T's portfolio has an expected return of 11%. What are the weights of his efficient portfolio? What is the beta of his portfolio?
b. Mrs. R's portfolio has a beta of 2.0. What are the weights of her efficient portfolio?
c. Plot the Security Market Line and place portfolios of Mr. T and Mrs. R on the same graph together with the market portfolio.
d. If the market portfolio's standard deviation is 30%, plot the Efficient Frontier in the $E(r) - \sigma$ space. Plot the locations of Mr. T's and Mrs. R's portfolios on the efficient frontier.
e. Find the coefficients of risk aversion for Mr. T. and Mrs. R. Are your results in line with part (c)? Explain.
f. A particular stock X's expected return is 25%. What is its beta? If the stock X returns' standard deviation is 70%, what is the coefficient of correlation between stock X and the market portfolio?

Problem Set "5" – SOLUTIONS KEYS

Problem 1:

a)-b) The following relations (SML) link the beta of the two stocks, their expected returns, the risk-free return and the expected return on the market portfolio:

$$\begin{cases} 0.122 = R_F + \left(E[R_M] - R_F\right) \times 1{,}2 \\ 0.086 = R_F + \left(E[R_M] - R_f\right) \times 0{,}6 \end{cases}$$

The solution of this system yields: R_F =5% and $E[R_M]$ = 11%.

c) The expected return of the market is 11 % = *x* 12.2 % + (1-*x*) 8.6 %. Thus the weights of assets 1 and 2 in the market portfolio are *x* = 2/3 and (*1-x*) = 1/3 respectively. In the CAPM the market portfolio coincides with the tangent portfolio.

Problem 2:

a.

Solve for any two CAPM relationships among securities 1,2,3, and 4 to find $E(r_M)$ and r_F :

$$\begin{cases} E(r_2) = r_F + \beta_2(E(r_M) - r_F) \\ E(r_4) = r_F + \beta_4(E(r_M) - r_F) \end{cases} \Rightarrow \begin{cases} 0.124 = r_F + 0.8(E(r_M) - r_F) \\ 0.188 = r_F + 1.6(E(r_M) - r_F) \end{cases} \Rightarrow \begin{cases} -0.124 = -r_F - 0.8(E(r_M) - r_F) \\ +0.188 = +r_F + 1.6(E(r_M) - r_F) \\ \hline 0.064 = 0.8(E(r_M) - r_F) \end{cases}$$

$$E(r_M) - r_F = 0.08$$

$$r_F = 0.188 - 1.6(E(r_M) - r_F) = 0.188 - 1.6 \times 0.08 = 0.06 = 6\%$$

$$E(r_M) = 0.08 + 0.06 = 0.14 = 14\%$$

$$E(r_P) = x_F r_F + x_1 E(r_1) + x_2 E(r_2) + x_3 E(r_3) + x_4 E(r_4)$$

$$E(r_P) = 0.5 \times 0.06 + 0.1 \times 0.076 + 0.1 \times 0.124 + 0.1 \times 0.156 + 0.2 \times 0.188 = 0.1032 = 10.32\%$$

$$\beta_P = x_F \beta_F + x_1 \beta_1 + x_2 \beta_2 + x_3 \beta_3 + x_4 \beta_4$$

$$\beta_P = 0.5 \times 0 + 0.1 \times 0.2 + 0.1 \times 0.8 + 0.1 \times 1.2 + 0.2 \times 1.6 = 0.54$$

b.

$$E(r_P) = x_M E(r_M) + (0.5 - x_M) r_F + x_1 E(r_1) + x_2 E(r_2) + x_3 E(r_3) + x_4 E(r_4)$$

$$12\% = x_M 14\% + (0.5 - x_M) \times 6\% + .01 \times 7.6\% + 0.1 \times 12.4\% + 0.1 \times 15.6\% + 0.2 \times 18.8\%$$

$$x_M = 0.21$$

sell 39% out of your 50% T - bills and buy the market portfolio : your expected return would be 12%.

$$\beta_P = x_M \beta_M + (0.5 - x_M \beta_F) + x_1 \beta_1 + x_2 \beta_2 + x_3 \beta_3 + x_4 \beta_4$$

$$\beta_P = 0.21 \times 1.0 + 0.29 \times 0 + 0.1 \times 0.2 + 0.1 \times 0.8 + 0.1 \times 1.2 + 0.2 \times 1.6 = 0.75$$

c.

$$E(r_P) = x_M E(r_M) + (1 - x_M) r_F$$
$$x_M [E(r_M) - r_F] = E(r_P) - r_F$$
$$x_M = \frac{E(r_P) - r_F}{E(r_M) - r_F} = \frac{0.12 - 0.06}{0.14 - 0.06} = 0.75$$
$$.x_F = 1 - x_M = 0.25$$
$$\beta_P = x_M \beta_M + (1 - x_M \beta_F)$$
$$\beta_P = 0.75 \times 1.0 + 0.25 = 0.75$$

d. I would prefer strategy (c) as it involves an efficient portfolio. Strategy b) combines the original portfolio with the market portfolio, however nothing guarantees that the initial portfolio is efficient implying that strategy (b) might not lead to an efficient portfolio.

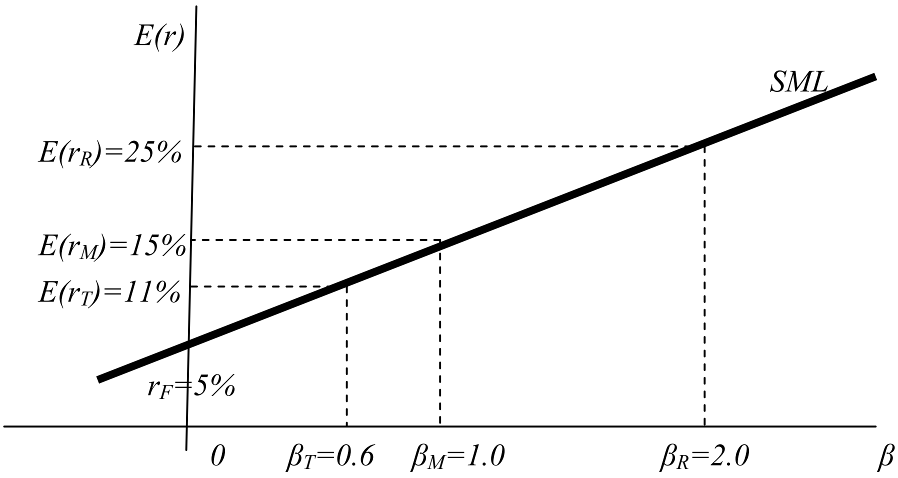
Problem 3:

a.
$$E(r_T) = x_M E(r_M) + x_F r_F = x_M E(r_M) + (1 - x_M) r_F$$
$$E(r_T) - r_F = x_M (E(r_M) - r_F)$$
$$x_M = \frac{E(r_P) - r_F}{E(r_M) - r_F} = \frac{0.11 - 0.05}{0.15 - 0.05} = 0.6 = 60\% \Rightarrow x_F = 1 - x_M = 0.4 = 40\%$$
$$E(r_P) = r_F + \beta_T (E(r_M) - r_F) \Rightarrow \beta_T = \frac{E(r_P) - r_F}{E(r_M) - r_F} = 0.6$$

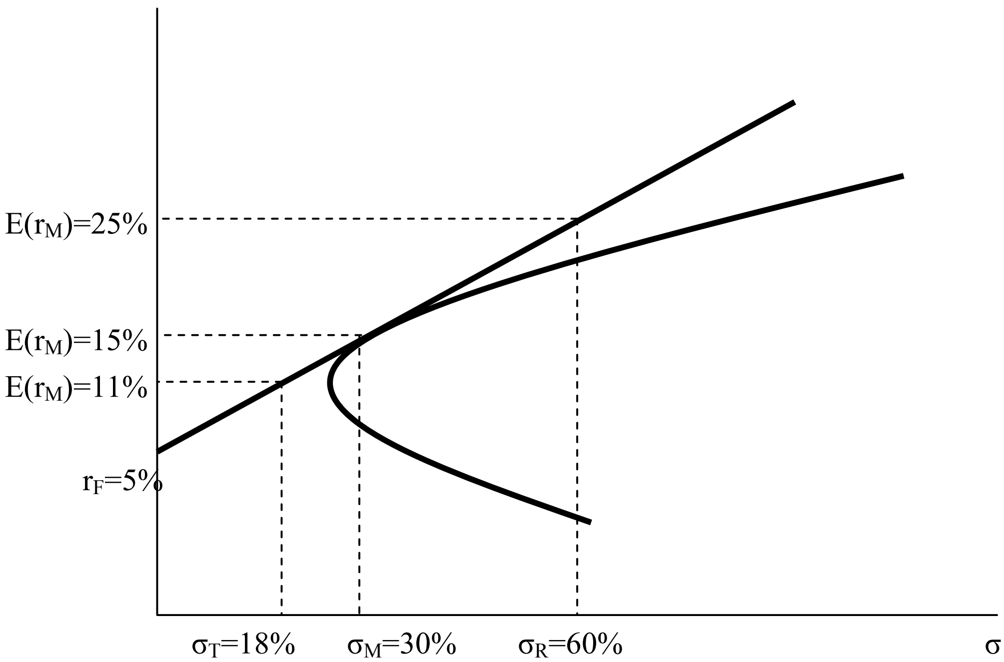
b.
$$E(r_R) = r_F + \beta_R (E(r_M) - r_F) = 0.05 + 2.0(0.15 - 0.05) = 0.25 = 25\%$$
$$x_M = \frac{E(r_R) - r_F}{E(r_M) - r_F} = \frac{0.25 - 0.05}{0.15 - 0.05} = 2.0 = 200\% \Rightarrow x_F = 1 - x_M = -1.0 = -100\%$$

Mrs. R short - sells 100% of the risk - free security (that is, borrows at the risk - free rate) and invests all the proceeds in the market portfolio.

c. Plot the Security Market Line and portfolios of T & R:



d.



$$\sigma_T^2 = x_{T,M}^2\sigma_M^2 + x_{T,F}^2\sigma_F^2 + 2x_{T,M}x_{T,F}\sigma_M\sigma_F\rho_{MF} = x_{T,M}^2\sigma_M^2 = 0.6^2 \times 0.30^2 = 0.0324 \Rightarrow \sigma_T = \sqrt{0.0324} = 0.18$$

$$\sigma_R^2 = x_{R,M}^2\sigma_M^2 + x_{R,F}^2\sigma_F^2 + 2x_{R,M}x_{R,F}\sigma_M\sigma_F\rho_{MF} = x_{R,M}^2\sigma_M^2 = 2.0^2 \times 0.30^2 = 0.3600 \Rightarrow \sigma_R = \sqrt{0.3600} = 0.60$$

e.

$$x_{T,M} = \frac{E(r_M) - r_F}{A_T \sigma_T^2} \Rightarrow A_T = \frac{E(r_M) - r_F}{x_{T,M} \sigma_M^2} = \frac{0.15 - 0.05}{0.6 \times 0.3^2} = 1.85$$

$$x_{R,M} = \frac{E(r_M) - r_F}{A \sigma_R^2} \Rightarrow A_R = \frac{E(r_M) - r_F}{x_{R,M} \sigma_M^2} = \frac{0.15 - 0.05}{2.0 \times 0.3^2} = 0.55$$

$A_R < A_M$ and this makes sense as T is more risk - averse than R.

f.

$$E(r_X) = r_F + \beta_X (E(r_M) - r_F) \Rightarrow \beta_X = \frac{E(r_X) - r_F}{E(r_M) - r_F} = \frac{0.25 - 0.05}{0.15 - 0.05} = 2.0$$

$$\beta_X = \frac{\text{cov}_{X,M}}{\sigma_M^2} = \frac{\rho_{X,M} \sigma_X \sigma_M}{\sigma_M^2} = \frac{\rho_{X,M} \sigma_X}{\sigma_M} \Rightarrow \rho_{X,M} = \frac{\beta_X \sigma_M}{\sigma_X} = \frac{2.0 \times 0.30}{0.70} = +0.86$$

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