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Informational cascades with endogenous prices: The role of risk aversion

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Abstract

In this paper, we show that long run market informational inefficiency and informational cascades can easily happen when trades occur at market clearing prices. We consider a sequential trade model where: (i) the investors' set of actions is discrete; (ii) dealers and investors differ in risk aversion; (iii) investors' information is bounded. We show that informational cascade occurs as soon as traders' beliefs do not differ too sharply. Thus, prices cannot fully incorporate the private information dispersed in the economy.

JEL classification: G1; G14; D82; D83

Keywords: Informational cascades; Endogenous prices; Risk aversion

1. Introduction

This last decade economists have made important progress in developing rational models of herding and cascades that aim at explaining outcomes that at a first glance appear anomalous. The seminal model of Bikhchandani et al. (1992) (BHW henceforth) assumes that an investment opportunity is available to a series of investors at a fixed price. They show that rational investors can engage in imitative behavior leading to informational inefficiency, that is the failure in the aggregation of investors' private information regarding the quality of the

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investment.[1] BHW pioneered an impressive wealth of papers that helps us understand the basis for uniformity of behavior, informational cascades and inefficient outcomes. However, this literature[2] shares a particular feature with BHW: The models do not allow for a price system.

This assumption is crucial, as shown by classic results in sequential trade literature.[3] If prices are endogenous, then in the long run they will incorporate all available private information leading to full social learning. In these models, a price system leads to informational efficiency even when tradable quantities are restricted to being discrete, and private information is bounded, that are two typical assumption required to obtain informational cascade in the herding literature. Thus, at a first glance, results of herd behavior cannot be directly extended to markets where price is endogenously fixed. This is what Chari and Kehoe (2004) call the "price critique" to herding models. In this paper, we explore the relation between endogenous price formation and social learning further. To this purpose, we generalize a standard sequential trade model in the Glosten and Milgrom (1985) style. Our main finding is that endogenous prices cannot fully incorporate private information if the following three conditions coexist. First, the investors' set of actions is discrete, second, dealers and investors differ in risk aversion, third, investors' information is bounded. In this instance full social learning is impossible even when transactions occur at market clearing prices.

Our paper belongs to the recent literature that attempts to adapt the studies on "herd behavior" to markets where the prices are endogenously determined. This literature focuses on the existence of informational cascades, being situations where the actions of informed agents cease to be informative for an observer. Cascades with endogenous prices have been studied by Chari and Kehoe (2004), Avery and Zemsky (1998), Lee (1998), and Cipriani and Guarino (2003). Chari and Kehoe (2004) study a situation where each investor is endowed with a risky project that requires them to exert a fixed amount of effort to become viable. Moreover, investors have the opportunity to buy one additional project or to sell the project they hold. These trades occur at market clearing prices. Thus, investors' decisions have two dimensions. On the one hand, they choose whether or not to "invest" effort in their project, and on the other hand, they have to make a trading decision on the project's market. Chari and Kehoe show that past market trading can trigger a cascade on the "investment" choice. For example, as soon as the information provided by the trading activity in the market is positive enough, it becomes optimal for all investors to invest the fixed effort cost in order to make their projects viable, thus an investment cascade. However, in their model, market cascades are impossible, i.e., informational cascades in the trading action never occur. In fact, once all projects have been undertaken, market activity would continue to provide information on the value of the project. Thus, nothing prevents endogenous prices from eventually incorporating all private information dispersed in the economy. Although Chari and Kehoe's (2004) result is useful to explain how herding in investment at fixed cost starts, in our opinion, it does not completely overturn the "price critique" regarding trade herding and long run social learning. Namely, they state that an informational cascade would not occur in a model without the investment decision, i.e., when agents only have to take the trade decision. In our model, contrary to Chari and Kehoe (2004), there is no investment dimension. Investors's decisions only regard trades that occur at endogenous prices, nevertheless full social learning is impossible and informational cascades can

[1] An analogous result is also obtained independently by Banerjee (1992).

[2] See for instance Chamley (2004) for an extensive study on rational herding.

[3] See O'Hara (1995) for a review of financial microstructure models.

occur. Avery and Zemsky (1998) (hereafter simply AZ) introduce multidimensional uncertainty in a Glosten and Milgrom (1985) style model and show that in the short run imitative behaviors can occur. Nevertheless, in the long run these phenomena vanish and all private information is incorporated into prices. Thus, in their model informational cascades cannot occur and prices eventually incorporate all information available in the economy. Informational cascades in the presence of endogenous prices are obtained by Lee (1998). He introduces an exogenous transaction cost and shows that when profits from trading are smaller than the transaction costs, traders stop trading and the flow of information stops. However, in his model cascades are impossible in the absence of trading cost. Differently from Lee (1998), we do not need to assume any exogenous trading costs to derive our result. Closer to our analysis, Cipriani and Guarino (2003) obtains informational cascades in a Glosten and Milgrom (1985) model. They assume that the traders' valuation for the risky asset is equal to the expected fundamental value of the asset multiplied by an exogenous term stemming for gain (or loss) to trade. However, the nature of this linear scaling valuation is unclear. In our model agents are fully rational but differ in their risk aversion and in their initial endowments. Furthermore, contrary to Cipriani and Guarino (2003) agents are not restricted to trade one unit of the asset but are allowed to exchange any integer amount of it.

More precisely, we consider a sequential trade model similar to Glosten and Milgrom (1985) and Glosten (1989): in each period risk averse investors choose the amount of their trade on a quantity grid. Then, risk neutral dealers quote a price at which they clear investors' demand. Note that a risk averse investor that is privately informed trades for two reasons. On the one hand, he wants to exploit his private information on the intrinsic quality of the traded item. On the other hand, he trades for risk hedging, or in other words to smooth consumption across states of the nature. The latter follows from investor's risk aversion and is not directly related to the quality of the item. Intuitively, our result can be explained as follows. Suppose that past history of trades generates enough consensus about fundamentals, that is to say that agent's beliefs do not differ too sharply; then an investor's belief will only be slightly affected by a bounded private signal. Thus, as an investor can demand only discrete quantities of the asset, a small change in his beliefs will not affect his demand.[4] As a consequence, all investors will eventually only trade to hedge. From this point on, the flow of trades will no longer be informative on fundamentals and the social learning process stops. Moreover, as trading prices correspond to the expected value of the item's fundamentals given the history of trades, trading price will never converge to the true fundamentals. If the learning process stops when public expectation is wrong, then there will be large, steady discrepancy between the price and the actual fundamentals, (the informational cascade is in the wrong direction), and consequently the long term pricing error will be significant.[5]

It is interesting to compare our result to those in market microstructure literature, where many papers have separately considered the discrepancies in risk aversion and discrete trading. None of these papers found cascades or long run informational inefficiency, and this independently from the boundedness of private information. For instance, in Glosten and Milgrom (1985), in Easley and O'Hara (1992) and in Avery and Zemsky (1998) the set of actions is discrete, but market makers and informed investors are both risk neutral. As a consequence, prices eventually

[4] A similar intuition drives the result of Lee (1993) who shows the existence of wrong informational cascades in a generalized BHW's setting. The crucial difference with our study is that we allow for a price system.

[5] The long term pricing error can be measured with the distance between the trading prices in the long run and the expected value of the asset for someone who has the combined knowledge of all agents in the economy.

converge to fundamentals. This shows that if market participants are risk neutral, then market imperfection, due to discreteness of trade, would not be sufficient to generate long run market inefficiency. On the other hand, risk aversion alone is not enough to generate market inefficiency as shown by Glosten (1989), Vives (1995) and Biais et al. (2000). In these models, risk neutral agents make a market for risk averse informed traders, but agents can trade a continuum of quantities, and for this reason the order flow is always informative. Thus, our contribution to this literature is to show that the coexistence of a grid for tradable quantities along with a discrepancy in risk aversion leads to informational inefficiency. In other words, the efficiency result of standard microstructure literature is not robust.

In Section 2 we present the model. Section 3 shows the main result. Section 4 gives an example. Section 5 concludes. The proofs are in the Appendix.

2. The model

In the following we will refer to the market for a generic asset, however the model is general enough to fit markets for investment projects, commodities and financial assets. We consider an infinite horizon economy with periods denoted $t = 0, 1, \ldots$ where investors and dealers mutually exchange an asset. We denote by $\mathbf{v} = \mathbf{V} + \boldsymbol{\epsilon}$ the fundamental value of the asset where $\mathbf{V}$ is a random variable that takes value in the compact set $\Omega \subset \mathbb{R}^+$ with generalized probability density function[6] (abbreviated g.p.d.f) $\mu_{\mathbf{V}}(.)$. The random variable $\boldsymbol{\epsilon}$ admits a g.p.d.f that satisfies $E[\boldsymbol{\epsilon}|V] = 0$ and $\mathrm{Var}[\boldsymbol{\epsilon}|V] > 0$ for all $V \in \Omega$. Thus, $E[\mathbf{V}]$ is an unbiased estimator of $\mathbf{v}$. We assume that aggregating all private information dispersed in the economy discloses the realization of $\mathbf{V}$ but not that of $\boldsymbol{\epsilon}$. One can interpret $\mathbf{V}$ as a realized shock for which agents are asymmetrically informed, whereas $\boldsymbol{\epsilon}$ can be seen as shocks on fundamentals for which realization is unknown to everybody. This reflects many situations in the real world, think for example of the asset being a capital market instrument where private information regards cash-flows paid in the short run, but not the cash-flows that will be paid in the long run. Similarly, if the asset is interpreted as a productive project, $\mathbf{V}$ could represent the project's intrinsic quality that only partially affects the project's actual cash flows. In Section 4 we show that the presence of the noise component $\boldsymbol{\epsilon}$ is not crucial to obtain our result. We restrict the tradable quantities of the asset to the set $\mathbb{Z}$ of *integer amounts*. Markets are discrete in nature, and we think that in many cases it is reasonable to assume that only integer quantities can be exchanged.

2.1. Agents

There is an infinity of risk averse investors and risk neutral dealers. An investor's expected utility obtained from an amount X of the asset and M of cash is $E[u(M + X\mathbf{v})]$ where $u : \mathbb{R} \to \mathbb{R}$ is increasing and strictly concave. Apart from this assumption, investors can differ in their utility functions u and in their initial endowment of asset and cash, which we will denote x and m, respectively. Thus, the triple $\theta = (u, x, m)$ identifies the investor's type and it is privately known by the investor. We denote with Θ the set of all possible investor' types and we assume that the number of possible types is finite, that (x, m) is bounded for all types and that $x \in \mathbb{Z}$. We will refer to x as the investor's inventory.

[6] Throughout the paper we consider generalized probability density functions (g.p.d.f). A g.p.d.f designates a distribution that can be either discrete either absolutely continuous. We use here the terminology of DeGroot (1970) page 19.

2.2. Trading mechanism

Trading occurs sequentially. At the beginning of each period an investor receives a private signal $\mathbf{s}$ and comes to the market. He announces the quantity $Q \in \mathbb{Z}$ of the asset he wants to trade and dealers compete in price to satisfy the investor's demand. We assume that investors leave the market after they have had the opportunity to trade. The probability that the investor arriving at time t is of type θ is exogenous, orthogonal to $\mathbf{v}$ and constant across time.

2.3. Information structure

Each investor receives a partially informative private signal $\mathbf{s}$ that takes value in a compact set Σ and is independently distributed from his type θ. Conditional on the realization of $\mathbf{V}$, private signals are i.i.d. according to the g.p.d.f $\mu_{\mathbf{s}|\mathbf{V}}(.|.)$ that satisfies $0 < \underline{\pi} \leq \mu_{\mathbf{s}|\mathbf{V}}(s|V) \leq \overline{\pi}$ for all $V \in \Omega$ and $s \in \Sigma$, where $\underline{\pi}$ and $\overline{\pi}$ are finite constants. That means that private signals are not perfectly informative as each realization of the signal is compatible with all realizations of $\mathbf{V}$. We assume that the distribution of $\mathbf{s}$ only depends on the realization of $\mathbf{V}$, and that by aggregating all private signals in the economy it is possible to know the actual realization of $\mathbf{V}$. We denote H_t the history of trades (quantities and prices) up to time $t-1$. All the agents observe H_t but they do not know the types and signals of past investors. Last, $\mu_{\boldsymbol{\epsilon}|\mathbf{V}}(.|.)$, the conditional g.p.d.f of the shock $\boldsymbol{\epsilon}$ given $\mathbf{V}$ is assumed to be independent of the signal $\mathbf{s}$ and of history H_t.

2.4. Agents' behavior and equilibrium concept

We denote with $\tilde{P}_t(Q)$ the unit price at which the investor arriving at time t expects to trade a quantity $Q \in \mathbb{Z}$, with the convention that a positive Q corresponds to a buy order. Thus, an investor of type $\theta = (u, x, m)$ who received the signal $s \in \Sigma$ and expects a price function $\tilde{P}_t(.) : \mathbb{Z} \to \mathbb{R}$ will demand the quantity

$$Q^*(\theta, \tilde{P}_t, H_t, s) = \arg\max_{Q \in \mathbb{Z}} E[u(m + (x + Q)\mathbf{v} - \tilde{P}_t(Q)Q)|H_t, s].$$

Apart from the discreteness in the tradable quantities, competition among an infinity of risk neutral dealers is modelled as in Glosten (1989) and Kyle (1985). At any given period t the price competition among risk neutral dealers will lead their expected profits to zero. Thus the price $P_t(Q)$ at which a trade of size Q is executed must satisfy

$$-Q(E[\mathbf{v}|H_t, Q^*(\theta, \tilde{P}_t, H_t, \mathbf{s}) = Q] - P_t(Q)) = 0.$$

An equilibrium in trading period t is a price function $P_t^* : \mathbb{Z} \to \mathbb{R}$ such that: (i) period-t-investor correctly anticipates the prices at which quantities in $\mathbb{Z}$ will be traded; (ii) dealers expected profit from trading any given quantity is zero. Formally, $\forall Q \in \mathbb{Z}$,

$$P_t^*(Q) = E[\mathbf{v}|H_t, Q^*(\theta, P_t^*, H_t, \mathbf{s}) = Q]. \tag{1}$$

That is, in period t the market clearing price is equal to the expectation of $\mathbf{v}$ conditional on the information provided by past and current trades.

3. Informational cascades and inefficiency

In the long run social learning is complete if the observation of the investors' actions eventually gather all the information that is dispersed in the economy. As in our model private information only regards $\mathbf{V}$, learning can only regard the realization of $\mathbf{V}$.

Definition 1. Learning is said to be *complete* only if the random variable $(E[\mathbf{V}|H_t])_{t\geq 0}$ converges to $\mathbf{V}$ almost surely when t tends to infinity.

Note that complete learning is also characterized by the equality[7] $\lim_{t\to\infty} \mathrm{Var}[\mathbf{V}|H_t] = \lim_{t\to\infty} E[(\mathbf{V} - E[\mathbf{V}|H_t])^2|H_t] = 0$. Note also that from $E[\boldsymbol{\epsilon}|\mathbf{V}] = 0$ and Eq. (1), it follows that complete learning occurs only if the trading prices eventually converge to the realization of $\mathbf{V}$. Thus complete learning coincides with market strong-form informational efficiency.

An opposite case is when an informational cascade occurs, i.e., nothing can be deduced from the investor's actions because all investors take actions that do not change with their private signal. It is useful to provide a definition of non-informative trade:

Definition 2. An investor of type θ is said to make a *non-informative trade* at time t, if the quantity he demands is not affected by his private signal, i.e., for all signal s and s' in Σ, we have

$$Q^*(\theta, P_t^*, H_t, s) = Q^*(\theta, P_t^*, H_t, s').$$

According to this definition, an investor's trade is non-informative when the observation of his trade and the knowledge of his type θ provides no information regarding the realization of his private signal.

If for all types $\theta \in \Theta$, type-θ investor makes non-informative trade, then all investor's action will be independent of their private information and the observation of trades will not provide any additional information regarding $\mathbf{V}$. Consequently, an informational cascade occurs and complete learning is impossible. In the following, we show that an informational cascade necessarily happens as soon as there is enough agreement on the asset's fundamentals, i.e., $\mathrm{Var}[\mathbf{V}|H_t]$ is sufficiently small. In order to understand why investors' orders eventually cease to be informative, note first that as signals are not perfectly informative about $\mathbf{V}$, the impact of private information on investor's choice decreases as $\mathrm{Var}[\mathbf{V}|H_t]$ approaches 0. In other words, if investors are quite sure about the realization of $\mathbf{V}$, a partially informative private signal will have little impact on their beliefs. Now, as investors are risk averse and in addition can demand only discrete quantities of the asset, a small change in their belief will in general not be sufficient to affect their demand and so, for any given $\theta \in \Theta$, we will have $Q^*(\theta, P, H_t, s) = Q^*(\theta, P, H_t, s')$ for all $s,s' \in \Sigma$. In this instance the flow of trade is not informative anymore and an informational cascade happens. The following Theorem states this result formally:

Theorem 1. *There exist $\eta > 0$ such that if $Var(\mathbf{V}|H_t) < \eta$, then in all trading periods $\tau \geq t$ the equilibrium is unique and such that:*

(i) *The price schedule satisfies $P_\tau^*(Q) = E[\mathbf{V}|H_t]$ for all $Q \in \mathbb{Z}$.*
(ii) *An investor with inventory x will trade exactly $-x$ no matter the signal $\mathbf{s}$ he received.*

[7] This follows from a standard result of convergence for martingales (see for instance Durrett (1996) pages 252–253).

Theorem 1 characterizes the unique equilibrium for $\mathrm{Var}[\mathbf{V}|H_t]$ sufficiently small. Considering that signals are not perfectly informative, no single order can fully reveal $\mathbf{V}$ and so full social learning cannot occur in a finite number of steps. Therefore, only two outcomes are possible. First, $\mathrm{Var}[\mathbf{V}|H_t]$ remains larger than the threshold η and heterogeneity of types leads to what Smith and Sørensen (2000) call confounded learning, i.e. a situation where history offers no decisive lesson for anyone and full social learning is never reached. Second, $\mathrm{Var}[\mathbf{V}|H_t]$ is eventually smaller than η but strictly positive, and so Theorem 1 implies that the flow of trade ceases to provide information and an informational cascade starts. The latter situation does not rely on heterogeneity of investors' type, but rather on their risk aversion. In both outcomes the conditional variance $\mathrm{Var}[\mathbf{V}|H_t]$ remains bounded away from zero and therefore we can conclude that $E[\mathbf{V}|H_t]$ will remain bounded away from $\mathbf{V}$ with probability 1.

Note that in our set-up an informational cascade implies uniformity of actions, or herd-like behavior, among investors of the same type but not necessarily among investors of different types. In fact, in the presence of an informational cascade, two traders will choose different actions provided they differ in their inventory. In addition, Theorem 1 states that as soon as $\mathrm{Var}[\mathbf{V}|H_t]$ is small, the equilibrium price schedule must be $P_\tau(Q) = E[\mathbf{V}|H_t]$ for all tradable quantities and all following periods $\tau \geq t$. Thus, endogenous trading prices cannot incorporate all private relevant information even in the long run. Interestingly, if the cascade occurs when market beliefs are substantially wrong, the difference between the true realization of $\mathbf{V}$ and the long term equilibrium price $E[\mathbf{V}|H_t]$ will be sizable.

Note finally that there are simple empirical tests to detect informational cascade in markets. The model predicts that in the presence of an informational cascade, the price reaction to the volume of trade should be significantly smaller compared to a situation where traders' orders are informative. In other words, the deepness of the market, measured as the quantity of assets that one can trade without affecting the trading price, should be significantly larger when an informational cascade occurs.

4. Example

In order to understand to what extent the different assumptions in the model are necessary to obtain our result, we study the case where traders have negative exponential utility function with the same risk aversion coefficient γ. We assume that $\boldsymbol{\epsilon}$ is normally distributed, that $\Omega = \{V_1, V_2\}$ with $V_1 < V_2$ denoting $\Pr(\mathbf{V} = V_2|H_t) = \mu_t$, and that $\Sigma = \{l, h\}$ with $\Pr(\mathbf{s} = l|V_1) = \Pr(\mathbf{s} = h|V_2) = \pi \in (1/2, 1)$. In this framework $\mathrm{Var}(\mathbf{V}|H_t) = \mu_t(1-\mu_t)(V_2 - V_1)^2$ is small for μ_t close to 1 or close to 0. The following Proposition characterizes the belief thresholds $\underline{\mu}$ and $\overline{\mu}$ beyond which an informational cascade will occur.

Proposition 1. *Let $\overline{\mu}$ (resp. $\underline{\mu}$) be the minimum $\mu > 1/2$ (resp. maximum $\mu < 1/2$) such that the following two expressions are satisfied*

$$\mathrm{e}^{-\gamma(\mu V_2 + (1-\mu)V_1 + \gamma\sigma_\epsilon^2/2)} \leq \mu^h \mathrm{e}^{-\gamma V_2} + (1-\mu^h)\mathrm{e}^{-\gamma V_1}, \tag{2}$$

$$\mathrm{e}^{\gamma(\mu V_2 + (1-\mu)V_1 - \gamma\sigma_\epsilon^2/2)} \leq \mu^l \mathrm{e}^{\gamma V_2} + (1-\mu^l)\mathrm{e}^{\gamma V_1}, \tag{3}$$

where $\mu^h = \mu\pi/(\mu\pi + (1-\mu)(1-\pi))$ and $\mu^l = \mu(1-\pi)/(\mu(1-\pi) + (1-\mu)\pi)$. An informational cascade occurs as soon as $\mu_t < \underline{\mu}$ or *$\mu_t > \overline{\mu}$.*

Note first that if π is close to 0.5, the information content of signals is low and thus μ^h is close to μ^l. In this instance, because of the convexity of the exponential function, inequalities

(2) and (3) will be satisfied even if σ_ϵ is arbitrarily small. This suggests that the presence of the additional noise ϵ is not a necessary condition to obtain informational inefficiency. Thus, even if the aggregation of all private information could virtually resolve uncertainty completely, when investors have imprecise signals they will neglect their information and trade only for hedging. This will impede the convergence of prices to fundamental.

Note also that if γ is sufficiently large, then inequalities (2) and (3) will be met at all levels of the beliefs μ and thus independently on the precision of private signals.[8] This happens because when investors are sufficiently risk averse, they only trade to reduce the risk of their portfolio even if they have perfect information on $\mathbf{V}$. Consequently, the informational content of their order vanishes.

Finally, notice that for σ_ϵ sufficiently large the two inequalities are satisfied no matter what the level of public belief or the information content of the private signal. This means that if the uncertainty coming from the noise ϵ is sufficiently large with respect to the information provided by the component $\mathbf{V}$, then even signals that are perfectly informative about $\mathbf{V}$ will not be reflected in traders' orders. Indeed, the asset will be too risky to be held even by investors that are perfectly informed about one component of the asset's fundamental value.

To sum up, when (i) the investors' risk aversion is high; or (ii) the precision of private signals is low; or (iii) the volatility in the asset fundamentals is mostly due to shocks for which there is no information, then even an infinite sequence of trades will not allow the market to aggregate the relevant private information dispersed among investors.

5. Conclusion

We studied the possibility of informational inefficiency and cascades in markets where trades occur at market clearing prices. We show that as soon as agents' beliefs do not differ too sharply, an informational cascade occurs and the price mechanism fails to aggregate all remaining relevant private information dispersed in the economy. This result is obtained assuming that agents can trade integer amounts and that risk-neutral dealers make a market for risk averse investors that have bounded private information. In a simplified model, Décamps and Lovo (2004) show that informational cascades and long term mispricing can also occur when investors are risk neutral and dealers are risk averse. This suggests that what leads to inefficiency is not the absence of risk neutral investors but the absence of investors whose utility functions are identical to those of dealers. The comparison of our result with those of other papers on sequential discrete trading with endogenous price, suggests that market clearing prices lead to full social learning only under very specific conditions. Namely, prices should be fixed in a way that some investors are always willing to follow their private signal, no matter how small the information provided by this signal. This is not the case when market makers and investors differ in their risk aversion.

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[8] Indeed, an increase in γ increases the convexity of the exponential. Moreover, a sufficiently large increase in γ reduces the left hand sides of expressions (2) and (3) more than the right hand sides.

Appendix A

Proof of Theorem 1. Take an investor of type $\theta = (u, x, m)$ with private signal s who expects a price function $\tilde{P}_t$. After observing history H_t his demand is

$$Q^*(\theta, \tilde{P}_t, H_t, s) = \arg\max_{Q \in \mathbb{Z}} E[U(Q, \tilde{P}_t(Q), \mathbf{v})|H_t, s],$$

where, to simplify notation, we defined $U(Q, P, \mathbf{v}) := u(m + (x + Q)\mathbf{v} - QP)$. Theorem 1 will be deduced from the following Proposition.

Proposition 2. *There exists $\eta_\theta > 0$ such that if $Var(\mathbf{V}|H_t) < \eta_\theta$ then, in any equilibrium the type-θ investor will demand to trade exactly $-x$ no matter the realization of his private signal.*

Throughout the proof, given two random variables $\mathbf{y}$ and $\mathbf{z}$, we will use the notation $\mu_{\mathbf{y}|\mathbf{z}}(.)$ to denote the g.p.d.f of random variable $\mathbf{y}$ given the realization of random variable $\mathbf{z}$. The proof of our proposition goes through a series of steps. The first is to remark that in the extreme case where the private signal s provides no information and the investor expects to trade all quantities at unit price exactly equal to $E[\mathbf{V}|H_t]$ then, his demand will indeed be exactly $-x$. This follows directly from risk aversion:

$$\arg\max_{Q \in \mathbb{Z}} E[U(Q, E[\mathbf{V}|H_t], \mathbf{v})|H_t] = -x.$$

In other words, if a risk averse agent can trade a risky asset at a unit price equal to the expected value of the risky asset, then he will optimally choose to have a neutral position and he will sell his inventory x. Moreover, strict concavity of u and the fact that only discrete quantities are tradable implies that there exists $\Delta > 0$ such that

$$E[U(-x, E[\mathbf{V}|H_t], \mathbf{v})|H_t] - E[U(Q, E[\mathbf{V}|H_t], \mathbf{v})|H_t] > \Delta \quad (4)$$

for all $Q \in \mathbb{Z} - \{-x\}$.

In a second step, consider the same investor and suppose he receives an informative signal s. We denote thereafter λ the maximum distance between his belief and those of an agent that has only observed the history H_t:

$$\lambda \equiv \sup_{V \in \Omega, s \in \Sigma} |\mu_{\mathbf{V}|H_t}(V) - \mu_{\mathbf{V}|\mathbf{s}, H_t}(V|s)|.$$

We show that if λ is sufficiently small then an agent who expects to trade any quantity at price $E[\mathbf{V}|H_t]$ will demand exactly $-x$. More precisely we have:

Lemma 1. *If $\lambda > 0$ sufficiently small, then*

$$\arg\max_{Q \in \mathbb{Z}} E[U(Q, E[\mathbf{V}|H_t], \mathbf{v})|H_t, s] = -x$$

for all $s \in \Sigma$, and

$$E[U(-x, E[\mathbf{V}|H_t], \mathbf{v})|H_t, s] - E[U(Q, E[\mathbf{V}|H_t], \mathbf{v})|H_t, s] > \Delta_s \quad (5)$$

for all $Q \in \mathbb{Z} - \{-x\}$ and some $\Delta_s > 0$.

Proof. Fix any finite quantity and prices (Q, P). Let

$$\Phi_{Q,P} \equiv \max_{V \in \Omega} \int_{\epsilon \in \mathbb{R}} |U(Q, P(Q), V + \epsilon)| \mu_{\boldsymbol{\epsilon}|\mathbf{V}}(\epsilon|V) \mathrm{d}\zeta(\epsilon).$$

Accordingly to DeGroot (1970), the differential $d\zeta(\epsilon)$ indicates that the integral may be either the integral of a probability density function or the sum of the values of a discrete probability function. Note that $\Phi_{Q,P}$ is finite as u is continuous and Ω is compact. We have

$$\begin{aligned}
&|E[U(Q, P, \mathbf{v})|H_t] - E[U(Q, P, \mathbf{v})|H_t, s]|\\
&\quad = \left| \int_{V\in\Omega} \int_{\epsilon\in\mathbb{R}} U(Q, P, V+\epsilon)(\mu_{\mathbf{V},\boldsymbol{\epsilon}|H_t}(V,\epsilon) - \mu_{\mathbf{V},\boldsymbol{\epsilon}|\mathbf{s},H_t}(V,\epsilon|s))d\zeta(\epsilon)d\nu(V)\right|\\
&\quad = \left| \int_{V\in\Omega} \left(\int_{\epsilon\in\mathbb{R}} U(Q, P, V+\epsilon)\mu_{\boldsymbol{\epsilon}|\mathbf{V}}(\epsilon|V)d\zeta(\epsilon)\right)(\mu_{\mathbf{V}|H_t}(V) - \mu_{\mathbf{V}|\mathbf{s},H_t}(V|s))d\nu(V)\right|\\
&\quad \le \int_{V\in\Omega} \left(\int_{\epsilon\in\mathbb{R}} |U(Q, P, V+\epsilon)|\mu_{\boldsymbol{\epsilon}|\mathbf{V}}(\epsilon|V)d\zeta(\epsilon)\right)|(\mu_{\mathbf{V}|H_t}(V) - \mu_{\mathbf{V}|\mathbf{s},H_t}(V|s))|d\nu(V)\\
&\quad < \lambda\Phi_{Q,P},
\end{aligned} \tag{6}$$

where the second equality comes from the fact that the distribution of $\boldsymbol{\epsilon}$ conditional to $\mathbf{V}$ is orthogonal to $\mathbf{s}$ and to H_t, which implies $\mu_{\mathbf{V},\boldsymbol{\epsilon}|H_t}(V,\epsilon) = \mu_{\boldsymbol{\epsilon}|\mathbf{V}}(\epsilon|V)\mu_{\mathbf{V}|H_t}(V)$ and $\mu_{\mathbf{V},\boldsymbol{\epsilon}|\mathbf{s},H_t}(V,\epsilon|s) = \mu_{\boldsymbol{\epsilon}|\mathbf{V}}(\epsilon|V)\mu_{\mathbf{V}|\mathbf{s},H_t}(V|s)$. Again the differentials $d\zeta(\epsilon)$ and $d\nu(V)$ indicate that each of the integrals may actually be either the integral of a probability density function, or the sum of the values of a discrete probability function.

Now, take $\lambda < \Delta/\Phi_{(1-x),E[\mathbf{V}|H_t]}$ where Δ satisfies (4) then, for all $s \in \Sigma$, we have:

$$\begin{aligned}
&E[U(-x, E[\mathbf{V}|H_t], \mathbf{v})|H_t, s]\\
&\quad = E[U(-x, E[\mathbf{V}|H_t], \mathbf{v})|H_t] > E[U(1-x, E[\mathbf{V}|H_t], \mathbf{v})|H_t] + \Delta\\
&\quad > E[U(1-x, E[\mathbf{V}|H_t], \mathbf{v})|H_t, s] - \lambda\Phi_{(1-x),E[\mathbf{V}|H_t]} + \Delta\\
&\quad > E[U(1-x, E[\mathbf{V}|H_t], \mathbf{v})|H_t, s]
\end{aligned} \tag{7}$$

where the first and second inequalities follow from (4) and (6), respectively. In the same vein, when $\lambda < \Delta/U_{-1-x,E[\mathbf{V}|H_t]}$, we have

$$E[U(-x, E[\mathbf{V}|H_t], \mathbf{v})|H_t, s] > E[U(-1-x, E[\mathbf{V}|H_t], \mathbf{v})|H_t, s]$$

that together with (7) shows that $-x$ is a local maximum. Considering that $E[U(Q, E[\mathbf{V}|H_t], \mathbf{v})|H_t, s]$ is concave in the traded quantity Q, a local maximum will also be a global maximum. Finally, inequality (5) follows from the strict concavity of u and the fact that only discrete quantities are tradable. □

The last step of the proof of Proposition 2 relies on two lemma. Lemma 2 states that, given a prior distribution with most of the probability concentrated on a state, a private signal will only slightly affect posterior beliefs. Lemma 3 studies the maximum range of price function that will be adopted by the dealers.

Lemma 2. *Let denote $\delta_{E[\mathbf{V}|H_t]}$ the point mass distribution concentrated on $E[\mathbf{V}|H_t]$, that is $\delta_{E[\mathbf{V}|H_t]}(V) = 1$ if $V = E[V|H_t]$ and 0 otherwise. The following holds: For all $\xi > 0$ there exists $\alpha > 0$ such that, if for all $V \in \Omega$, $|\mu_{\mathbf{V}|H_t}(V) - \delta_{E[\mathbf{V}|H_t]}(V)| < \alpha$ then, $|\mu_{\mathbf{V}|\mathbf{s},H_t}(V) - \delta_{E[\mathbf{V}|H_t]}(V)| < \xi$.*

Proof. First note that, since signals' precision is bounded ($0 < \underline{\pi} < \mu_{\mathbf{s}|\mathbf{V}}(s|V) < \overline{\pi}$ for all $(V, s) \in \Omega \times \Sigma$), $\mu_{\mathbf{V}|\mathbf{s},H_t}(V|s) = \frac{\mu_{\mathbf{s}|\mathbf{V}}(s|V)\mu_{\mathbf{V}|H_t}(V)}{\int_{W\in\Omega}\mu_{\mathbf{s}|\mathbf{V}}(s|W)\mu_{\mathbf{V}|H_t}(W)\,d\nu(W)}$ is well defined for all $(V, s) \in \Omega \times \Sigma$. Algebraic manipulations yield then to

$$
\begin{aligned}
|\mu_{\mathbf{V}|\mathbf{s},H_t}(V) - \delta_{E[\mathbf{V}|H_t]}(V)| &\le \frac{\mu_{\mathbf{s}|\mathbf{V}}(s|V)}{\int_{W\in\omega}\mu_{\mathbf{s}|\mathbf{V}}(s|W)\mu_{\mathbf{V}|H_t}(W)\,\mathrm{d}\nu(W)}\,\alpha \\
&\quad + \left|\frac{\mu_{\mathbf{s}|\mathbf{V}}(s|V)}{\int_{W\in\omega}\mu_{\mathbf{s}|\mathbf{V}}(s|W)\mu_{\mathbf{V}|H_t}(W)\,\mathrm{d}\nu(W)} - 1\right|\delta_{E[V|H_t]}(V) \\
&\le \frac{\overline{\pi}}{\underline{\pi}}\alpha + \left|\frac{\mu_{\mathbf{s}|\mathbf{V}}(s|V)}{\int_{W\in\Omega}\mu_{\mathbf{s}|\mathbf{V}}(s|W)\mu_{\mathbf{V}|H_t}(W)\,\mathrm{d}\nu(W)} - 1\right|\delta_{E[\mathbf{V}|H_t]}(V) \\
&\le \frac{\overline{\pi}}{\underline{\pi}}\alpha + \left|\frac{\mu_{\mathbf{s}|\mathbf{V}}(s|E[\mathbf{V}|H_t])}{\int_{W\in\Omega}\mu_{\mathbf{s}|\mathbf{V}}(s|W)\mu_{\mathbf{V}|H_t}(W)\,\mathrm{d}\nu(W)} - 1\right|. \qquad (8)
\end{aligned}
$$

Remark now that

$$
\begin{aligned}
\int_{\Omega}\mu_{\mathbf{s}|\mathbf{V}}(s|V)\mu_{\mathbf{V}|H_t}(V)\,\mathrm{d}\nu(V) &= \int_{\Omega}\mu_{\mathbf{s}|\mathbf{V}}(s|V)(\mu_{\mathbf{V}|H_t}(V) - \delta_{E[V|H_t]}(V))\,\mathrm{d}\nu(V) \\
&\quad + \mu_{\mathbf{s}|\mathbf{V}}(s|E[\mathbf{V}|H_t]) \le \alpha\overline{\pi}K + \mu_{\mathbf{s}|\mathbf{V}}(s|E[\mathbf{V}|H_t]),
\end{aligned}
$$

where $K \equiv \int_{\Omega}\mathrm{d}\nu(V)$ is finite, since Ω is compact. It follows that:

$$
\frac{\mu_{\mathbf{s}|\mathbf{V}}(s|E[\mathbf{V}|H_t])}{\mu_{\mathbf{s}|\mathbf{V}}(s|E[\mathbf{V}|H_t]) + \alpha\overline{\pi}K} \le \frac{\mu_{\mathbf{s}|\mathbf{V}}(s|E[\mathbf{V}|H_t])}{\int_{\Omega}\mu_{\mathbf{s}|\mathbf{V}}(s|V)\mu_{\mathbf{V}|H_t}(V)\,\mathrm{d}\nu(V)} \le \frac{\mu_{\mathbf{s}|\mathbf{V}}(s|E[\mathbf{V}|H_t])}{\mu_{\mathbf{s}|\mathbf{V}}(s|E[\mathbf{V}|H_t]) - \alpha\overline{\pi}K}
$$

which together with (8) implies Lemma 2. □

Lemma 3. *For any finite history H_t there exists signals $\overline{s}_t$ and $\underline{s}_t$ in the set Σ such that in equilibrium the price function satisfies*

$$E[\mathbf{V}|H_t, \mathbf{s} = \underline{s}_t] \le P(Q) \le E[\mathbf{V}|H_t, \mathbf{s} = \overline{s}_t] \qquad \forall Q \in \mathbb{Z}.$$

Proof. Simply remark that for any finite history H_t it is always possible to find two signals $\underline{s}_t$ and $\overline{s}_t$ in the compact set Σ, such that

$$E[\mathbf{V}|H_t, \mathbf{s} = \underline{s}_t] \le E[\mathbf{V}|H_t, \mathbf{s} = s] \le E[\mathbf{V}|H_t, \mathbf{s} = \overline{s}_t], \qquad \forall s \in \Sigma. \qquad (9)$$

Now, note that dealers cannot infer from an investor's order more than what the investor knows himself, and that his inventory provides no information about $\mathbf{V}$. Then, Lemma 3 follows from Eq. (1). □

We now conclude the proof of Proposition 2. Notice that the lower the conditional variance $Var(\mathbf{V}|H_t)$, the more the distribution of $\mathbf{V}|H_t$ is concentrated around its mean $E[\mathbf{V}|H_t]$. Then Lemma 2 implies that for any choice of $\lambda > 0$ and $\beta > 0$, it is possible to fix $\eta_\theta > 0$ sufficiently small in order to have $\sup_{V\in\Omega, s\in\Sigma}|\mu(V|H_t) - \mu(V|H_t, s)| \le \lambda$ and in addition $\sup_{s\in\Sigma}|E[\mathbf{V}|H_t] - E[\mathbf{V}|H_t, s]| < \beta$. Namely one can choose λ so that inequality (5) is met. Lemma 3 implies that dealers prices will be in the interval $[E[\mathbf{V}|H_t] - \beta, E[\mathbf{V}|H_t] + \beta]$. Thus by fixing a sufficiently small $\beta > 0$, expression (5) and the continuity of u lead to

$$E[U(-x, \tilde{P}(-x), \mathbf{v})|H_t, s] > E[U(Q, \tilde{P}(Q), \mathbf{v})|H_t, s]$$

for all $Q \in \mathbb{Z} - \{-x\}$. In other words the investor optimally chooses to trade $-x$. □

Theorem 1 is easily deduced from Proposition 1. Notice that by choosing $\eta = \min_{\theta\in\Theta}\eta_\theta$, we obtain that when $\mathrm{Var}(V|H_t) < \eta$, all investors submit non-informative orders. Consequently, when $\mathrm{Var}(V|H_t)$ is small, there exists no equilibria where investors' orders provide information on the asset's fundamentals. Now we prove that the non-informative equilibrium exists for these levels of $Var(V|H_t)$. It is sufficient to observe that if trades are not informative, then from Eq. (1) the equilibrium price function can only be $P^*(Q) = E[\mathbf{v}|H_t]$. Thus, as soon as $\mathrm{Var}(V|H_t) < \eta$, the equilibrium is unique and satisfies $P^*(Q) = E[\mathbf{v}|H_t]$ for all $Q \in \mathbb{Z}$, and $Q^*(\theta, E[\mathbf{v}|H_t], H_t, s) = -x$ for all $s \in \Sigma$ and all $\theta \in \Theta$. ■

Proof of Proposition 1. As the expression $E[u(m + (x + q)\mathbf{v} - E[\mathbf{V}|H_t]q)|H_t, \mathbf{s}]$ is a strictly concave function in the traded quantity $q \in \mathbb{R}$, then it will have a unique maximum. Thus, in order to find $\bar{\mu}$ (resp. $\underline{\mu}$), it is sufficient to find the minimum $\mu_t \geq 1/2$ (resp. maximum $\mu_t \leq 1/2$) such that the investor prefers to trade $-x$ rather than $-x - 1$ or $-x + 1$ for both $s = h$ and $s = l$. That is to say

$$u(m + E[\mathbf{V}|H_t]x) > \max\{E[u(m + \mathbf{v} + (x - 1)E[\mathbf{V}|H_t])|H_t, s], \\ E[u(m - \mathbf{v} + (x + 1)E[\mathbf{V}|H_t])|H_t, s]\} \quad (10)$$

for $s \in \{h, l\}$. Considering that $u(w) = -\mathrm{e}^{-\gamma w}$ and that $\boldsymbol{\varepsilon} \hookrightarrow N(0, \sigma_\varepsilon)$, we have that expression (10) is satisfied only if both inequalities in Proposition 1 are met. □

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