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Order flow composition and trading costs in a dynamic limit order market[1]

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Abstract

This article provides a game theoretic model of price formation and order placement decisions in a dynamic limit order market. Investors can choose to either post limit orders or submit market orders. Limit orders result in better execution prices but face a risk of non-execution and a winner's curse problem. Solving for the equilibrium of this dynamic game, closed-form solutions for the order placement strategies are obtained. Thus, testable implications for the cross-sectional behavior of the mix between market and limit orders and trading costs in limit order markets are derived. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Several security markets[2] are organized as limit order markets. In these markets, buyers and sellers carry their trades by submitting either limit orders or

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[2] For instance, the NYSE, the Paris Bourse, the Tokyo Stock Exchange and the Toronto Stock Exchange. Domowitz (1993) reports that 35 financial markets have features of limit order markets.

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market orders.[3] Limit orders are stored in a limit order book, waiting for future execution. This execution is triggered by incoming market orders, which are matched with the best offers in the book. Traders face the following dilemma. With a market order, a trader is executed with certainty, at the posted prices in the market. With a limit order, a trader has the possibility to improve his execution price. But she runs the risk of not being executed. Moreover because their prices are fixed over time, limit orders can become mispriced when new public information arrives. This possibility creates a winner's curse problem for limit order traders since they are more likely to be executed ('picked off') at a loss when their orders become mispriced than when they are not.

What is the behavior of the mix between market and limit orders ('the order flow composition') across securities? Surprisingly, this question has not been addressed yet (to our knowledge), neither empirically,[4] nor theoretically. The objective of this article is to develop a simple model in which the mix between market and limit orders can be characterized, in equilibrium. As explained below, in this way, we obtain testable predictions concerning the cross-sectional behavior of the order flow composition. Furthermore, the model has new testable implications for the cross-sectional behavior of trading costs in limit order markets.

In order to portray, in a natural way, the execution risk and the risk of being picked off, we consider a dynamic model. Traders arrive sequentially. Upon arrival, a trader can choose to post quotes (place a limit order) or to trade at the quotes previously posted by other traders (place a market order). Execution of limit orders is uncertain and the asset value fluctuates, which creates a winner's curse problem for limit order traders. The optimal choice between a market and a limit order and the optimal prices for limit orders depend on the order

[3] A limit order specifies a limit price and a quantity. For a buy limit order, the limit price is the maximum price that a buyer will pay and for a sell limit order, the limit price is the minimum price that a seller will obtain. Market orders are orders to buy or sell a given quantity at any price. Those orders are the main channels through which liquidity is supplied and consumed in limit order markets. Biais et al. (1995) report (Table III, p. 1670) that, for the Paris Bourse, 47.2% of all orders are market orders and 41.3% are limit orders. The other orders are cancellations or applications.

[4] Biais et al. (1995) and Hedvall and Niemeyer (1996) focus on the variations in the order flow due to transient changes in the state of the limit order book. Hamao and Hasbrouck (1995) study the supply of liquidity when there is no market-maker. Hamon et al. (1993), Handa and Schwartz (1996) and Harris and Hasbrouck (1996) focus on optimal order submission strategies. DeJong et al. (1995) compare trading costs in a limit order market and a dealer market. Finally an interesting approach is developed by Angel (1995) and Harris (1995) who analyze optimal order placement strategies in different market conditions (state of the book, rate of arrival of orders...) exogenously specifying traders' beliefs on their environment and proceeding by simulations.

submission choices of the future traders. Solving for the equilibrium of this game, the traders' order placement strategies are characterized, in closed form, as a function of traders' valuations and the best offers in the book. This solution has methodological interest, independent of the issues we address. Actually, to our knowledge, this model is the first to offer a closed form characterization in equilibrium of *both* quotes *and* order placement decisions in a dynamic limit order market.

Our primary finding is that the volatility of the asset is a main determinant of the mix between market and limit orders. When the asset volatility increases, the probability of being picked off and the losses, which ensue, are larger. For this reason, limit order traders ask for a larger compensation for the risk of being picked off in markets with high volatility. But this entails that, in these markets, market order trading is more costly. Thus more traders find optimal to carry their trades using limit orders. As a result, limit orders execution probabilities are lower since market order trading is less frequent. These effects of volatility have two testable implications. First, the proportion of limit orders in the order flow is positively related to asset volatility. Second, the fill rate (the ratio of filled limit orders to the number of submitted limit orders) is negatively related to volatility. It also turns out that posted spreads are positively related to asset volatility. Consequently, another testable hypothesis is that the proportion of limit orders in the order flow is positively related to the size of the spread. Asset volatility decreases with equity capitalization (see, for instance, Hasbrouck, 1991 Table 3, p. 588). According to our results, small firms should have a larger *proportion* of limit orders,[5] lower fill rates and larger spreads than large firms, in limit order markets.

We define the increase in execution risk as an exogenous decrease in the execution probabilities of limit orders at all possible price levels. We find that limit order traders react to an increase in execution risk by posting *larger spreads*. Actually, when execution risk is high, traders are under pressure to trade immediately upon arrival because the probability of being executed with a limit order is small. For this reason, traders are willing to place market orders at more unfavorable prices. This effect allows limit order traders to capture larger rents in equilibrium. It is a well-known stylized fact that spreads enlarge at the end of the trading day in limit order markets (see McInish and Wood (1992) for the NYSE for instance). The model suggests that this observation can be due to the fact that execution risk is larger at the end of the trading period. With regards to this empirical finding, the model yields the additional testable hypothesis that the size of the increase in the spread at the end of the

[5] It is worth stressing that it does not imply that the number of limit orders, per unit of time say, should be larger in markets for small firms. See discussion in Section 6.1.

trading day is negatively related to the level of competition between limit order traders.

We also obtain the result that trading costs for buy and sell market orders are related to the ratio of buy to sell orders (limit and market orders), because of execution risk. To see this point, consider a decrease in the proportion of potential sellers (the traders with low valuations for the asset). It results in lower execution probabilities for buy limit orders, at all price levels. Consequently, execution risk is higher for potential buyers (the traders with high valuations). Limit orders are less attractive for these traders and the maximum ask prices at which they are willing to submit buy market orders increase. But for this reason, limit order trading is more attractive for potential sellers and bid prices must increase to attract sell market orders. Thus the average trading cost for buy (sell) market orders increases (decreases) with the ratio of buy to sell orders. Moreover, the sum of the average trading costs for buy and sell market orders turns out to be concave in the ratio of buy to sell orders, with a maximum when this ratio is equal to 1.

Most of the models in the market microstructure literature do not allow traders to choose between market and limit orders. For this reason, these models cannot derive implications concerning the mix of market and limit orders. This is the case for models which focus explicitly on dealer markets (e.g. Glosten and Milgrom, 1985). This is also the case for the models of limit order trading developed by Biais et al. (1998), Glosten (1994), Rock (1996), Seppi (1996) or Parlour and Seppi (1997). Kumar and Seppi (1993) (in a static setting) and Parlour (1996) (in a dynamic setting) analyze models in which traders can choose between market or limit orders.[6] However, in these models, limit order traders are not exposed to the risk of being picked off. Here, this risk is at the root of the interactions, between volatility and order flow composition, uncovered by the model. The model is in fact most closely linked to the empirical study of Hollifield et al. (1996). They empirically relate the order flow and the quotes to the underlying distribution of traders' valuations as we do theoretically.

The paper is organized as follows. The model is spelled out in Section 2. In Section 3, the equilibrium of the trading game is defined. In Section 4, the benchmark case in which limit order traders behave competitively is analyzed. In Section 5, the equilibrium of the limit order market is derived. Section 6 derives and discusses the empirical predictions of the model. We conclude in Section 7. The Appendix contains all the proofs.

[6] Cohen et al. (1981) also analyze the choice between a market order and a limit order. In contrast with our approach, in their model traders' beliefs on limit order execution probabilities are exogenous.

2. A model of trading with market and limit orders

In this section, a sequential trading process in which traders can choose between market and limit orders is described.

2.1. The process of the asset expected value

Consider the market for a single risky asset. The trading day is divided into discrete time intervals denoted $t = 1, 2, \ldots, \tilde{T}$, where $\tilde{T}$ is *unbounded*. We assume that the payoff date, $\tilde{T}$, is random: At each time t, there is a probability $(1-\rho) > 0$ that the trading process stops and that the payoff of the asset is realized. The random termination assumption allows us to solve for the stationary equilibria of the model and simplifies the presentation of the results. Furthermore, performing comparative statics with respect to ρ offers a convenient way to analyze the impact of execution risk on traders' order placement strategies (more on this below).

Let $\tilde{V}_T = v_0 + \sum_{t=1}^{T} \tilde{\varepsilon}_t$ be the payoff of the asset at the end of the realized number of trading intervals. Furthermore, let v_t be the expected value of $\tilde{V}_T$ conditional on public information at time t. We refer to v_t as the *underlying value of the asset*. This value follows a random walk:

$$\tilde{v}_{t+1} = \tilde{v}_t + \tilde{\varepsilon}_{t+1}, \tag{1}$$

where the innovations, due to the arrival of *public information*, are assumed to be independent and identically distributed. They can take the values $+\sigma$ or $-\sigma$ with equal probabilities.

2.2. The traders and the trading process

Following Glosten and Milgrom (1985) or Easley and O'Hara (1992), the trading process is sequential and all orders are for one unit of the asset. At each time, a new trader arrives in the market. Traders differ by their reservation prices. At time t', the reservation price $\tilde{R}_{t'}$ for the trader who arrives at time $t \leqslant t'$ is:

$$\tilde{R}_{t'} = \tilde{v}_{t'} + \tilde{y}_t \tag{2}$$

The reservation price is the sum of the asset value and a trader specific component (y_t), which is time invariant.[7] The realization of $\tilde{y}_t$ characterizes a trader's *type*. The y's are assumed independent and identically distributed.

[7] Tauchen and Pitts (1983) and Hollifield et al. (1996) use a similar decomposition for reservation prices.

Moreover they are independent from the innovations in the asset value. They can take two values $y_h = +L > 0$ or $y_l = -L$ respectively with probabilities k and $(1-k)$. The dispersion in reservation prices ($L > 0$), for a given asset value, creates gains from trade and is necessary to generate trading.[8]

It is worth stressing that there are no 'noise traders' in the model. All agents are assumed to maximize their expected utility and they form correct expectations on the other traders' trading strategies. They are assumed risk-neutral and the utility of purchasing or selling the asset at price P for an agent of type y, if the final value of the asset turns out to be V_T, is

$$U(y) = (V_T + y - P)q \tag{3}$$

with $q = +1$ ($q = -1$) if the agent has purchased (sold) the asset. The reservation utility of all the agents if they do not trade is normalized to zero (i.e. $U(y)$ is the surplus obtained by agent y if he trades one unit).

2.3. Market structure: orders and information sets

Upon arrival, a trader can choose (a) to submit either a buy or a sell market order or (b) to post a buy and a sell limit order for one unit. In case of indifference between the placement of a market order or limit orders, it is assumed that limit orders are chosen. If there is no offer available (the book is empty), the trader posts a buy and a sell limit order.[9] For tractability, limit orders are assumed to expire after one period. As a consequence, a trader's limit order is not executed if his order is not hit by the next agent. The risk of being picked off exists in real trading situations, because limit order traders do not continuously monitor the market. In order to model this risk in the simplest manner, it is assumed that limit order traders cannot revise (or cancel) their offers once they have been posted.

Let $s_t = (A_t^m, B_t^m)$ denote the best quotes at time t. An empty book is represented by $A_t^m = +\infty$ and $B_t^m = -\infty$. At the time of his trading decision, an agent observes the current state of the book s_t, the current underlying value of the asset v_t and learns his type y_t. Those variables define the *state of the market at time t*. Let us denote this state as $S_t = (v_t, y_t, s_t)$. At a given point in time, all the

[8] The heterogeneity of reservation prices can be justified by differences of opinion as in Harris and Raviv (1993). In a more elaborate framework, the dispersion in reservation prices could stem from disparities in endowments or preferences across agents. For instance, in Glosten and Milgrom (1985) or Parlour (1996), the differences in valuations are due to differences in agents' discount factors.

[9] There is no loss of generality in assuming that a trader places both a buy and a sell limit order if he decides not to submit a market order. Actually he can always post a buy (sell) limit order with zero execution probability if he does not want to buy (sell). We only consider equilibria in pure strategies in this paper. Thus we do not consider possible equilibria in which an indifferent trader picks limit orders with any probability greater than (or equal to) 0 and less than one.

traders observe v_t and no trader has superior information.[10] Fig. 1, in the Appendix, summarizes our assumptions on the probabilistic structure of the model, the trading process and the market structure.

Execution risk: We say that *execution risk increases* if the execution probability of limit orders, at all price levels, decreases. The probability that a limit order will not be executed, *whatever the price chosen for the limit order*, is inversely related to ρ in the model. Thus a lower ρ characterizes a market with a larger execution risk for limit order traders. It follows that we can study the impact of an increase in execution risk on traders' behavior by analyzing the impact of a decrease in ρ.

Winner's curse problem: Suppose that the trader who arrives at time t posts a buy limit order. In addition assume that the asset value decreases between time t and time $t+1$ and that, for this reason, the trader's bid price becomes higher than the asset value. In this case, the trader runs the risk of being picked off by the next trader who arrives in the market. Thus limit order traders face the risk of being picked off in our setting.

This discussion shows that, although the model is very stylized, the basic trade-offs to an investor when choosing between a market order or a limit order are present. We can therefore study the implications of these trade-offs. Our assumptions put some constraints on what can be said with the model, however. First, because all orders are for one unit, we are not able to derive implications for the depth of limit order markets. As usual with sequential trade models, our focus is on the quotes and the trading costs. Second because limit orders last only one period, the book has only two possible states: empty or full, in our model. Thus we cannot analyze the interactions between transient changes in the state of the book and the order flow (as in Biais et al. (1995) for instance). Rather we focus on the cross-sectional behavior of the order flow composition.

2.4. An example

The purpose of this section is to consider a special case, which helps to explain intuitively how the model works and the methodology we use to solve for the equilibrium of the limit order market. Suppose the asset value does not change over time ($\sigma = 0$). Thus there is *no winner's curse problem*. Let v be the constant value of the asset in this case. We also assume (*only in this section*) that traders with type y_h only place buy orders while traders with type y_l only place sell orders. The results of this section hold without this assumption because, in equilibrium, only traders with type y_h (y_l) purchase (sell) the asset, as shown in

[10] Chakravarty and Holden (1995) consider a model with asymetric information in which informed traders can choose to submit limit orders or market orders.

Section 5. Assuming directly that this is the case conveys more rapidly the intuition.

First, consider a trader of type y_h who arrives at time t. Let $B^*(v, +L, t)$ be the bid price chosen by this trader if he posts a buy limit order. This order will be executed only if (i) the game does not stop before the arrival of the next trader (probability ρ) (ii) the next trader has type y_l (probability $(1-k)$) and (iii) the next trader market sells. The probability of the last event is *endogenous*. If the bid price is too low, the trader with type y_l will be better off posting a sell limit order. Denote $C^{s*}(v, -L, t+1)$, the bid price such that the trader with type y_l is indifferent between a market or a limit order. If the trader with type y_h posts a price slightly above this threshold then his execution probability is $\rho(1-k)$ and he obtains an expected gain equal to $\rho(1-k)[v+L-C^{s*}(v, -L, t+1)]$. In fact, this bid price is optimal. Actually, (a) a higher bid price has the same execution probability and (b) a lower bid price has a zero execution probability.

Now, consider the optimal order placement decision of trader y_h. Let $C^{b*}(v, +L, t)$ denote the ask price such that he is indifferent between a buy market order or a buy limit order. This price satisfies:

$$\underbrace{(v+L) - C^{b*}(v, +L, t)}_{\textit{Gain With a Market Order}} = \underbrace{\rho(1-k)[v+L-C^{s*}(v, -L, t+1)]}_{\textit{Expected Gain With a Limit Order}} \quad \forall t < \tilde{T}. \tag{4}$$

If the best ask price in the market, A^m, is greater than $C^{b*}(v, +L, t)$, the trader with type y_h is better off placing a buy limit order with price $B^*(v, +L, t) = C^{s*}(v, -\mathrm{L}, t+1)$; otherwise, he submits a buy market order.

A trader with type y_l faces exactly the same type of problem. Proceeding in a symmetric way, we can write:

$$\underbrace{C^{s*}(v, -L, t) - (v-L)}_{\textit{Gain With a Market Order}} = \underbrace{\rho k(C^{b*}(v, +L, t+1) - (v-L)]}_{\textit{Expected Gain With a Limit Order}} \quad \forall t < \tilde{T}. \tag{5}$$

If the best bid price, B^m, is lower than $C^{s*}(v, -L, t)$, the trader with type y_l posts a sell limit order with price $A^*(v, -L, t) = C^{b*}(v, +L, t+1)$; otherwise, he submits a sell market order.

Note that the solutions of the previous system of recursive equations yield *both* a characterization of the order submission choice and the quotes at each point in time. Thus these solutions characterize the order placement strategies in the limit order market. If the payoff date (T) were deterministic, we could compute the functions $C^{s*}(v, -L, .)$ and $C^{b*}(v, +L, .)$ by backward induction. *However T is not deterministic in our setting.* A trader has always a non-zero probability (ρ) of not being the last trader in the trading day. For this reason, whatever his arrival date, a trader's order placement strategy depends on the order placement strategy of the next trader, conditional on continuation of the trading process. Since the order placement strategy of the latter is itself endogenous, there is no point in time from which we can start solving recursively for the equilibrium order placement strategies. Thus we use a different method. We look for *stationary solutions* of the system of equations defined by Eqs. (4) and (5) i.e. functions $C^{s*}(v, -L, .)$ and $C^{b*}(v, +L, .)$ that do not depend on time. Denote $C^{s*}(., -L)$ and $C^{b*}(., +L)$ these stationary solutions. Eqs. (4) and (5) become

$$(v+L) - C^{b*}(v, +L) = \rho(1-k)[v + L - C^{s*}(v, -L)] \quad \forall t < \tilde{T} \tag{6}$$

and

$$C^{s*}(v, -L) - (v-L) = \rho k[C^{b*}(v, +L) - (v-L)] \quad \forall t < \tilde{T}. \tag{7}$$

Eqs. (6) and (7) reduces to a system of two equations in which the unknowns are the prices $C^{s*}(v, -\mathrm{L})$ and $C^{b*}(v, +L)$. Solving this system, we obtain

$$B^*(v, +L) = C^{s*}(v, -L) = v + L - \frac{(1-\rho k)}{1-\rho^2 k(1-k)}(2L), \tag{8}$$

$$A^*(v, -L) = C^{b*}(v, +L) = v - L + \frac{1-\rho(1-k)}{1-\rho^2 k(1-k)}(2L). \tag{9}$$

Thus if the order placement strategies of the traders who arrive at times $\{t+1, t+2, \ldots\}$ are as described in Eqs. (8) and (9), it is optimal for the trader who arrives at time t to follow the same order placement strategy. It follows that this order placement strategy is an equilibrium.

We proceed in this way[11] below to solve for the equilibrium of the trading game in the more general case in which $\sigma > 0$. This complete and parsimonious characterization of the order placement strategies allows us to compute the mix between market and limit orders and the trading costs, in equilibrium, as a function of the parameters of the model. This is useful for deriving testable implications. Throughout the article, we assume $k = 0.5$. We reconsider the results when $k \neq 0.5$ in Section 6.2.

3. Order placement strategies

This section gives a formal definition of the order placement strategies and the equilibrium concept, which is used to solve the trading game.

[11] If the set of possible prices was discrete, the trader with type y_h, for instance, would choose the first bid price on the grid above the lowest price at which a trader with type y_l submits a sell market order. Thus introducing a positive tick size does not change the analysis but makes the computation of the equilibrium more complex. For simplicity we assume a zero tick size.

3.1. Equilibrium definition

There are two components to a trader's order placement decision: the order type choice and the limit prices of limit orders. The indicator variable Q takes the value $+1$ (-1) if the trader decides to submit a buy (sell) market order and 0 if he decides to place a buy and a sell limit order. In the latter case, A and B denote his ask and bid prices respectively. A trader's order placement strategy is a mapping $O(\cdot)$ from the set of possible values for the state of the market to $\{1, 0, -1\} \times \mathbb{R}^2$. For each possible state of the market, the strategy specifies the order type choice: Market Order ($Q(S_t) \neq 0$) or Limit Orders ($Q(S_t) = 0$) and the quotes ($A(S_t)$, $B(S_t)$) associated to the placement of limit orders. It is worth stressing that we consider stationary order placement strategies (see discussion in Section 2.4).

Let $J(S_t, A, B)$ be the expected utility for an agent who arrives in the state of the market S_t if he chooses to place limit orders:

$$J(S_t, A, B) = \mathrm{E}(I^{\mathrm{s}}(A)(A - (\tilde{v}_{t+1} + y_t))|S_t) + \mathrm{E}(I^{\mathrm{b}}(B)((\tilde{v}_{t+1} + y_t) - B)|S_t) \quad (10)$$

where $I^{\mathrm{s}}(A)$ ($I^{\mathrm{b}}(B)$) is an indicator function which takes the value $+1$ in case of execution of the sell (buy) limit order and 0 otherwise. Let $\Delta\tilde{v}_{t+1}$ denote the change in the asset underlying value between times t and $t+1$. Moreover let $\Gamma(B|S_t)$ and $\Psi(A|S_t)$ be, respectively, the execution probabilities of a buy limit order with price B and a sell limit order with price A, conditional on the state of the market at time t. Using the definitions of reservation prices and the indicator functions, Eq. (10) yields:

$$\begin{aligned} J(S_t, A, B) = {} & \Psi(A|S_t)[A - R_t - \mathrm{E}(\Delta v_{t+1}|S_t, I^{\mathrm{s}}(A) = +1)] \\ & + \Gamma(B|S_t)[R_t + \mathrm{E}(\Delta v_{t+1}|S_t, I^{\mathrm{b}}(B) = +1) - B]. \end{aligned} \quad (11)$$

The objective function of an agent arriving at time t is then

$$\max_{O(S_t) = (Q, A, B)} \mathrm{E}(U(y_t)|S_t) = (v_t + y_t - P(Q))Q + (1 - |Q|)J(S_t, A, B) \quad (12)$$

with $P(+1) = A_t^m$ and $P(-1) = B_t^m$. The first term is the expected surplus if the agent decides to submit a market order. The second term is the expected surplus with the placement of limit orders. The optimal order placement decision at time t depends on the order placement strategy of the trader who arrives at time $t+1$. Actually both the execution probabilities of limit orders and the conditional expectations in Eq. (11) depend on it. The subgame perfect equilibria of this game will be analyzed.

Definition 1: A subgame perfect equilibrium of the limit order market is an order placement strategy $O^*(\cdot)$ such that, for each possible state of the market S_t,

$O^*(S_t)$ maximizes the expected utility of a trader who arrives in the state of the market S_t (i.e. is solution of Eq. (12)) if the other traders follow the order placement strategy $O^*(\,\cdot\,)$.

A useful formulation of the equilibrium order placement strategy is now proposed. Let $A^*(v_t, y_t)$ and $B^*(v_t, y_t)$ be the quotes, which maximize $J(S_t, ., .)$ in state S_t given that the future traders will act according to $O^*(\,\cdot\,)$. Eq. (12) implies that the optimal order type choice $Q^*(S_t)$ must be solution of

$$\max_{Q \in \{-1,\,0,\,1\}} \quad (v_t + y_t - P(Q))Q + (1 - |Q|)J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)). \tag{13}$$

This entails that the optimal order type choice can be described by a simple cutoff rule.

Proposition 1: *The optimal order type choice depends on the best offers in the book. Upon arrival, a trader submits a buy market order if the ask price is lower than or equal to a given price, called the buy cutoff price* (*denoted* $C^{b*}(., .)$) *or a sell market order if the bid price is greater than or equal to a given price, called the sell cutoff price* (*denoted* $C^{s*}(., .)$). *Otherwise he posts limit orders. Buy and sell cutoff prices are functions of the asset value and the trader's type. Moreover* $C^{s*}(v_t, y_t) \geqslant v_t + y_t \geqslant C^{b*}(v_t, y_t)$, $\forall v_t$, $\forall y_t$.

The buy (sell) cutoff price is the highest ask (lowest bid) price at which an agent who arrives in the market is willing to submit a buy (sell) market order instead of placing limit orders. Cutoff prices are given by

$$\underbrace{C^{s*}(v_t, y_t) - (v_t + y_t)}_{\textit{Gain With a Market Order}} = \underbrace{J(S_t, A^*, B^*)}_{\textit{Expected Gains With Limit Orders}} \tag{14}$$

$$\underbrace{(v_t + y_t) - C^{b*}(v_t, y_t)}_{\textit{Gain With a Market Order}} = \underbrace{J(S_t, A^*, B^*)}_{\textit{Expected Gains With Limit Orders}} \tag{15}$$

The cutoff prices are just like reservation prices. But contrary to the R's, they are endogenous. They depend on the expected gain with limit orders.

Let *a quotation strategy* be a pair of functions $\{A(., .), B(., .)\}$ and let *an order choice strategy*, be a pair of functions $\{C^s(., .), C^b(., .)\}$. Proposition 1 yields the following corollary.

Corollary 1: *A subgame perfect Equilibrium* $O^*(\,\cdot\,)$ *of the limit order market is completely characterized by an order choice strategy* $\{C^{s*}(., .), C^{b*}(., .)\}$ *and a quotation strategy* $\{A^*(., .), B^*(., .)\}$ *such that*: (*C*.1) *when the asset value is* v_t, *the offers* $A^*(v_t, y_t)$ *and* $B^*(v_t, y_t)$ *maximize the expected utility of a trader with type*

y_t if he places limit orders given that the other traders' order choice strategy is $\{C^{b*}(.,.), C^{s*}(.,.)\}$ and (C.2):

$$(v_t + y_t) - C^{b*}(v_t, y_t) = J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)) \quad \forall y_t, \quad \forall v_t \tag{16}$$

and

$$C^{s*}(v_t, y_t) - (v_t + y_t) = J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)) \quad \forall y_t, \quad \forall v_t \tag{17}$$

Proof. *Straightforward.* $\square$

As in the example of Section 2.4, Condition (C.1) and Eqs. (16) and (17) will be used to derive a system of equations, whose solutions give the equilibrium cutoff prices (the order choice strategy). These cutoff prices can then be used to compute the closed forms solution for the equilibrium quotation strategy. For brevity, only the equilibrium quotation strategy is reported in the text. The corresponding order choice strategy is derived in the Appendix.

3.2. Two important conditions for bid and ask prices

Limit order traders can always obtain their reservation utilities by posting orders with zero execution probability. Consequently Eq. (11) implies that the quotes posted by a limit order trader satisfy:

$$A(v_t, y_t) \geqslant R_t + \mathrm{E}(\Delta\tilde{v}_{t+1} | I^{s}(A) = +1, S_t) \tag{18}$$

$$B(v_t, y_t) \leqslant R_t + \mathrm{E}(\Delta\tilde{v}_{t+1} | I^{b}(B) = +1, S_t). \tag{19}$$

The right-hand side of Eq. (18) (resp. Eq. (19)) shows that limit order traders post ask (bid) prices at least equal to their initial reservation prices adjusted by the expected change in the asset value conditional on selling (resp. buying) the asset. A sell (buy) limit order is executed when its price is lower (greater) than the buy (sell) cutoff price of the last trader who arrives in the market. Suppose that cutoff prices increase with the asset value (it will be the case in equilibrium). In this case, a sell (buy) limit order trader has a greater probability to be executed when the asset value increases (decreases) than when it decreases (increases). Intuitively this entails that: $\mathrm{E}(\Delta\tilde{v}_{t+1} | I^{b}(A) = +1, S_t) \geqslant 0$ and $\mathrm{E}(\Delta\tilde{v}_{t+1} | I^{s}(B) = +1, S_t) \leqslant 0$ if $\sigma > 0$. As can be seen from Eqs. (18) and (19), rational traders properly account for this adverse selection bias by shading their offers when they place their limit orders. Combining these two equations, it is straightfoward that the spread posted by limit order traders is at least equal to $\mathrm{E}(\Delta\tilde{v}_{t+1} | I^{b}(A) = +1, S_t) - \mathrm{E}(\Delta\tilde{v}_{t+1} | I^{s}(B) = +1, S_t) \geqslant 0$ if $\sigma > 0$. This component of the spread is due to the risk of being picked off. We call it *the reservation spread* since this wedge between ask and bid prices is required for limit orders traders to break-even. Moreover Conditions (18) and (19) yield the following:

Lemma 1. When the asset value is v_t, $A(v_t, +L) \geqslant v_t + L + \sigma$ and $B(v_t, -L) \leqslant v_t - L - \sigma$.

The intuition is as follows. Consider a trader with type y_h who arrives at time t. The ask price, A, chosen by this trader must be higher than his reservation price $v_t + L$. If the asset value decreases, the reservation price of the trader who arrives at the next point in time is lower than $v_t + L$. This implies that the trader can be executed only in case of an increase in the asset value. But in this case: $\mathrm{E}(\Delta v_{t+1} | I^{a}(A) = +1, S_t) = +\sigma$. Thus, according to Eq. (18), the ask price posted by y_h must be at least equal to $v_t + L + \sigma$. A symmetric argument can be used for the bid price of a trader with type y_l.

4. A benchmark: Quotes with competitive behavior

There is no direct price competition among limit order traders in the model. Thus one concern is that the results are dependent on the imperfectly competitive behavior of the limit order traders.[12] In order to better understand the effects, which stem from non-competitive behavior, the model is first solved, in this section, under the postulate that limit order traders behave competitively. In this case, the results are completely driven by the risk of being picked off for limit orders. Comparison of the results obtained in this benchmark case and in equilibrium allows to distinguish which of the determinants of traders' quotes are specifically due to imperfect competition from those which are not.

Let $\{A^{c}(.,.), B^{c}(.,.)\}$ be the quotation strategy when limit order traders are competitive. In this case traders' quotes are such that they break-even: Traders' spreads are equal to their reservation spreads, i.e. Eqs. (18) and (19) are binding. We already know that for traders with type y_h, the ask price in this case is: $A^{c}(v_t, +L) = v_t + L + \sigma$ (from Lemma 1). Thus, for these traders, we just have to characterize the competitive bid price. For a symmetric reason, we just have

[12] However it is worth stressing that limit order traders' market power is limited because traders can choose to trade with market or limit orders. Consider, for instance, a trader who arrives at time t with type y_h. If he posts a bid price which is too low then the trader who arrives at time $t+1$ will not submit a market order and will instead trade with a limit order. This possibility limits the rents that can be captured by limit order traders. In fact Eqs. (14) and (15) show that a market order trader must obtain trading gains which are at least equal to those he can expect with limit order trading.

to characterize the competitive ask price posted by traders with type y_l. To this end, we can use the fact that Eqs. (18) and (19) are binding, i.e.:

$$A^{c}(v_t, y_l) = v_t + y_l + \mathrm{E}(\Delta\tilde{v}_{t+1} | I^{s}(A) = +1, S_t), \tag{20}$$

$$B^{c}(v_t, y_h) = v_t + y_h + \mathrm{E}(\Delta\tilde{v}_{t+1} | I^{b}(B) = +1, S_t). \tag{21}$$

The expected change in the asset value conditional on a sell (buy) limit order being executed depends on the price of the limit order. Thus, finding the competitive quotes requires solving for a fixed point. Define $\bar{\sigma}^{c} = \frac{3}{2}L$. We obtain the following result (details of the computations for the fixed point are in the Appendix).

Proposition 2: (Zero expected profits quotes). *When limit order traders behave competitively, their quotation strategy is*:

1. *If* $0 < \sigma < \bar{\sigma}^{c}$, $A^{c}(v_t, -L) = v_t - L + \frac{1}{3}\sigma$ *and* $B^{c}(v_t, +L) = v_t + L - \frac{1}{3}\sigma$. *The execution probability of these offers is equal to* $\frac{3}{4}\rho$.
2. *If* $\bar{\sigma}^{c} \leqslant \sigma$, $A^{c}(v_t, -L) = v_t - L + \sigma$ *and* $B^{c}(v_t, +L) = v_t + L - \sigma$. *The execution probability of these offers is equal to* $\rho/4$.
3. *In all the cases*, $A^{c}(v_t, +L) = v_t + L + \sigma$ *and* $B^{c}(v_t, -L) = v_t - L - \sigma$. *The execution probability of these offers is zero.*

The competitive quotation strategy has two interesting properties, which will still be obtained with imperfect competition. First traders shade their offers more, relative to their reservation prices, when the asset volatility increases. Actually, the expected change in the asset underlying value conditional on execution of a buy order or a sell order (the amount by which limit order traders shade their offers) increases (in absolute value) with the volatility of the asset. This entails that the spread posted by each type of traders (the reservation spread) enlarges when the volatility of the asset increases.

Second the execution probability of a limit order trader is lower when the volatility is high (larger than $\bar{\sigma}^{c}$) than when it is low (lower than $\bar{\sigma}^{c}$). Traders with type y_h, for instance, quote a lower bid price when the volatility is high than when it is low, other things equal. But this implies that their offer is less attractive. Accordingly the execution probability of their buy limit order is lower.[13]

[13] The execution probability does not decrease continuously with the volatility because the probability distributions for the innovations in the asset value and traders' valuations are discrete.

5. Equilibrium

In this section, we give the closed form characterization of limit order traders' quotation strategy in equilibrium. Then we compare the equilibrium quotation strategy with the competitive quotation strategy. Let $\bar{\sigma}^e$ be equal to $L/(1 + \rho/4)$.

Proposition 3 (*Equilibrium quotes*). *For all values of the parameters* (L, σ, ρ), *there exists a unique stationary equilibrium of the limit order market. In this equilibrium, the quotation strategy is*:

1. *If* $0 \leqslant \sigma < \bar{\sigma}^e$, *then* $A^*(v_t, -L) = v_t - L + (2L - \sigma)(2/(2 + \rho))$ *and* $B^*(v_t, +L) = v_t + L - (2L - \sigma)(2/(2 + \rho))$. *The execution probability of these offers is equal to* $\rho/2$.
2. *If* $\bar{\sigma}^e \leqslant \sigma$, *then* $A^*(v_t, -L) = v_t - L + \sigma + 8L/(4 + \rho)$ *and* $B^*(v_t, +L) = v_t + L - \sigma - 8L/(4 + \rho)$. *The execution probability of these offers is equal to* $\rho/4$.
3. *In all the cases*, $A^*(v_t, +L) = v_t + L + \sigma$ *and* $B^*(v_t, -L) = v_t - L - \sigma$. *The execution probability of these offers is zero.*

The equilibrium quotation strategy is derived using the methodology described in the example of Section 2.4. Details are explained in the Appendix. Using the characterization of traders' cutoff prices, it can be checked that, in equilibrium, only traders with type y_h purchase the asset and only traders with type y_l sell the asset.

Using Proposition 3, we obtain that the spread $(A^*(v_t, y_t) - B^*(v_t, y_t))$ posted by a limit order trader is

$$SPREAD = \underbrace{\sigma}_{\textit{reservation spread}} + \underbrace{(2L - \sigma)(2/(2 + \rho))}_{\textit{rent}} \quad \textit{when } \sigma < \bar{\sigma}^e \tag{22}$$

and

$$SPREAD = \underbrace{2\sigma}_{\textit{reservation spread}} + \underbrace{8L/(4 + \rho)}_{\textit{rent}} \quad \textit{when } \bar{\sigma}^e \leqslant \sigma \tag{23}$$

In equilibrium, limit order traders shade their offers for two different reasons: (i) the winner's curse problem, as in the competitive case and (ii) non-competitive behavior. For the second reason, limit order traders' spreads are larger than in the competitive case and limit order traders' offers have a lower execution probabilities than in the competitive case. The spread posted by a limit order trader can be split in two components. The first component ('reservation spread') is due to the risk of being picked off whereas the second component ('rent') comes from non-competitive behavior and is linked to execution risk.

1. *Reservation spread*: As explained in Section 3.2, the reservation spread accounts for the winner's curse problem. A limit order trader's reservation spread is equal to the difference between the RHS of Eq. (18) (reservation ask price) and Eq. (19) (reservation bid price). Consider a trader with type y_h for instance (computations for y_l are symmetric). His ask price is always equal to $v_t + L + \sigma$. Using Lemma 1, it follows that Eq. (18) is binding for this trader. On the other hand, if $\sigma < \bar{\sigma}^e$, the buy limit order posted by the trader with type y_h is executed if the trading does not stop at the next point in time and if the next trader has type y_l, whether the asset value increases or not (See the proof of Proposition 3). This implies that execution of his buy limit order is not correlated with changes in the asset value, i.e. $E(\Delta v_{t+1}|I^b(B^*(v_t, +L)) = +1, S_t) = 0$. Thus his reservation bid price is $v_t + L$. It follows that the reservation spread is σ, in this case. If $\bar{\sigma}^e \leqslant \sigma$, the buy limit order chosen by the limit order trader with type y_h is executed only if the asset value decreases and thus $E(\Delta v_{t+1}|I^b(B^*(v_t, +L)) = +1, S_t) = -\sigma$. Then the reservation spread is 2σ. In all the cases the reservation spread increases with the volatility. This is the reason why limit order traders' execution probability is weakly decreasing with the asset volatility. These two properties are obtained, for the same reasons, in the competitive case.

2. *Execution risk component*: The difference between the spread posted by a limit order trader and his reservation spread is a measure of his rent in case of execution. This difference is equal to $2(2L - \sigma)/(2 + \rho)$ if $\sigma < \bar{\sigma}^e$ and $8L/(4 + \rho)$ otherwise. Thus the rent captured by a limit order trader decreases as ρ increases. An increase in ρ improves the execution probability of a limit order trader for all possible quotes, i.e. unambiguously decreases the risk of non-execution. As a consequence, the traders who arrive in the market are more 'patient'. Thus the minimum bid price at which a trader is willing to submit a sell market order (his sell cutoff price) increases. In the same way the maximum ask price (the buy cutoff price) at which he is willing to submit a buy market order decreases. Accordingly limit order traders must improve their offers since bid (ask) prices are equal to sell (buy) cutoff prices in equilibrium (as in the example). Through this channel, execution risk determines the rent component of the spread posted in the limit order market.

The existing literature has not pointed to this effect of execution risk on the spread. The effect comes from the fact that traders can choose to trade immediately upon arrival (with a market order) or to delay their trade (with a limit order). Thus it cannot be derived in static models of limit order trading. Furthermore, the closed-form characterization of traders' quotes is necessary in order to sign the effect of execution risk on limit order traders' rent. The empirical implications for trading costs derived in this article come from this effect.

Lack of execution because of termination of the game is not a cost. Thus limit order traders do not need to be compensated for this risk. This is the reason why it does not influence traders' reservation spreads and it does not play a role when traders are competitive (ρ does not influence traders' quotes in the competitive case). As explained above, execution risk determines the fraction of the gains from trade ($y_h - y_l = 2L$) obtained by limit order traders when they behave strategically. Because of these interactions between limit order traders' rent and execution risk, the limit order market can feature a spread even when the risk of being picked off is not an issue. This can be seen by considering the particular case in which $\sigma = 0$. In this case, the spread posted by limit order traders, is (using Eq. (22)): $SPREAD = 4L/(2 + \rho) > 0$.

6. Testable implications

6.1. Implications for the order flow composition

Let $\tilde{M}_{\tilde{T}}$ be the number of market orders in the interval of time $[0, \tilde{T} - 1]$. $T - 1$ is the total number of orders (limit orders and market orders) in this period. The proportion of market orders in the order flow over this period is then $\tilde{m}_{\tilde{T}} = \tilde{M}_{\tilde{T}}/(\tilde{T} - 1)$ and $(1 - \tilde{m}_{\tilde{T}})$ is the proportion of limit orders. Let $\bar{m}_t = E(\tilde{m}_{\tilde{T}} | \tilde{T} = t)$ be the expected proportion of market orders conditional on the total number of orders being $(t - 1)$. Finally let us define $\bar{m} = \lim_{t \to +\infty} \bar{m}_t$. The proportions of market orders and limit orders in the order flow over a long period of time are then: $\bar{m}$ and $(1 - \bar{m})$ respectively.[14] These proportions can be computed using the execution probabilities of limit orders given in Propositions 2 and 3.

Proposition 4: *In equilibrium,* 80% *of all the orders are limit orders if* $\sigma \geqslant \bar{\sigma}^{e}$. *Otherwise* 66.66% *of all the orders are limit orders. In the competitive case,* 80% *of all the orders are limit orders if* $\sigma \geqslant \bar{\sigma}^{c}$. *Otherwise the proportion of limit orders in the order flow is* 57.14%.

All the implications of Proposition 4, which are derived below, are valid *both* in equilibrium *and* in the competitive case. Thus they do *not* rely specifically on the possibility for limit order traders to obtain strictly positive trading profits. Indeed they derive from the fact that when the volatility increases, limit order traders' reservation spreads enlarge and limit order traders' execution probability decreases. These properties are obtained both in equilibrium and in the competitive case. An immediate implication of Proposition 4 is that cross-sectional variations in the asset volatility must generate cross-sectional variations in the mix of market and limit orders. More specifically:

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[14] The limit of $\bar{m}_t$ exists for all values of the parameters.

Corollary 2. *Other things equal, the proportion of limit orders in the order flow increases with the asset volatility.*

When the asset volatility increases, limit order traders are more exposed to the risk of being picked off. For this reason, their reservation spreads enlarge and they post offers which are less attractive. Consequently the cost of market order trading increases and limit order trading turns out to be the optimal trading strategy more frequently. According to Corollary 2 in a regression of the proportion of limit orders on volatility, the sign of the coefficient for the volatility should be positive. The asset volatility (σ) is not directly observable but techniques have been proposed to estimate it. For instance, Hasbrouck (1991) decomposes the mid-quote into a random walk and a residual discrepancy term. He interprets the random walk component as the asset efficient value (v_t) and shows how to estimate its volatility using changes in the mid-quotes and trade innovations. Corollary 4 below offers an alternative way to test the previous corollary.

The fill rate (fr) of limit orders is defined as the total number of limit orders executed divided by the total number of limit orders submitted. Fill rates are a measure of likelihood of execution for limit orders and offer an alternative characterization of the order flow (the mix between filled and unfilled limit orders). Let $E_t(\tilde{fr}|\tilde{l} = l, \tilde{T} \geqslant t + 2)$ be the expected fill rate over the period $[0, t]$, conditional on the total number of limit orders being l and the game not stopping before date $t + 2$. We obtain the following corollary.

Corollary 3. *Other things equal, the expected limit order fill rate decreases with the asset volatility.*

In markets with high volatility, limit order traders shade more their offers because the risk of being picked off is larger. For this reason, their execution probability is low when volatility is high (See Propositions 2 and 3) and the expected fill rate is therefore smaller. Corollary 3 leads to the testable hypothesis that the average fill rate is negatively related to asset volatility.

Hasbrouck (1991) (Table 3) shows that volatility is negatively related to equity capitalization. Thus, according to our results, the proportions of limit orders for stocks with small capitalization must be larger than for stocks with large capitalization. Moreover fill rates must be lower for stocks with small capitalization. Casual empiricism suggests that the number of orders (market and limit), say per unit of time, is lower in small firm markets. This observation does not imply that the prediction of our model is counterfactual, however. Actually the claim is that, *for a given number of orders*, the number of limit orders *relative* to the number of market orders should be higher in small firm markets.

We denote $t(n)$ as the time of the nth transaction and P_n as the associated transaction price. $\tilde{Q}_n$ is an indicator variable which takes the values $+1$ if this transaction is triggered by a buy market order and -1 if it is triggered by a sell market order. In equilibrium, if the nth transaction is triggered by a buy market order, then the trade is consumed at price $A^*(v_{t(n)-1}, -L)$ (remember that the execution probability of an ask price posted by a trader with type y_h is zero). Conversely if it is triggered by a sell market order, the trade is consumed at price $B^*(v_{t(n)-1}, +L)$. Using the closed form solutions given in Proposition 3, $\tilde{P}_n$ can be written:

$$\tilde{P}_n = \tilde{v}_{t(n)-1} + [(2L-\sigma)(2/(2+\rho)) - L]\tilde{Q}_n \quad \text{if} \quad \sigma < \bar{\sigma}^{\mathrm{e}} \tag{24}$$

and

$$\tilde{P}_n = \tilde{v}_{t(n)-1} + [\sigma + 8L/(4+\rho) - L]\tilde{Q}_n \quad \text{if} \quad \sigma \geqslant \bar{\sigma}^{\mathrm{e}} \tag{25}$$

Proceeding in the same way, similar expressions for the transaction prices can be derived in the competitive case. Then the variance of changes in transaction prices $Var(\tilde{P}_{n+1} - \tilde{P}_n)$ can be computed in equilibrium and in the competitive case. Proposition 4 has the following corollary.

Corollary 4. *For a given L, the variance of changes in transaction prices is positively related to the proportion of limit orders and negatively related to the expected limit order fill rate.*

The intuition is as follows. When the volatility of the asset underlying value is large, the proportion of limit orders is large and the expected fill rate is low. At the same time, the variance of changes in transaction prices is large because (i) the volatility of the asset value, per period, is large, (ii) the average interval of time between two transactions is large (because there are less market orders) and (iii) the difference between the prices at which buy and sell market orders are executed is large because traders shade more their offers. The variance of changes in transaction prices is another characterization of the asset volatility. Thus Corollary 4 reinforces the conclusion that more volatile markets should feature a larger proportion of limit orders. Moreover it shows that the variance of changes in transaction prices, in place of an estimation of the unobservable volatility, can be used to test the predictions of Corollaries 2 and 3. The following result is a direct implication of Proposition 4 and Eqs. (22) and (23).

Corollary 5. *For a given L, the spread in the limit order market is positively related to the proportion of limit orders in the order flow.*

When the volatility increases, limit order traders shade more their offers, which entails a decrease in the proportion of market orders. This effect creates a positive correlation between the size of the spread in the limit order market and the proportion of limit orders. Note that this entails a negative relationship between the spread and the proportion of market orders and thereby a negative relationship between the spread and transaction frequency. This result is consistent with empirical observation (e.g. McInish and Wood (1992) for the NYSE). The traditional explanation is that greater trading activity leads to lower spreads because of economies of scale. For limit order markets, the model shows that the winner's curse problem provides another possible interpretation.

6.2. Implications for the trading costs

The trading cost for the nth transaction, denoted $\tilde{TC}_n$, is defined as the premium (discount) between the asset value and the price at which the nth market order is executed (as in Hasbrouck (1993), for instance):[15]

$$\tilde{TC}_n = (\tilde{P}_n - \tilde{v}_{t(n)})\tilde{Q}_n \tag{26}$$

Using Eqs. (24) and (25), it follows that

$$\tilde{TC}_n = -\tilde{\varepsilon}_{t(n)}\tilde{Q}_n - \left(\frac{2}{2+\rho}\right)\sigma + \left(\frac{2-\rho}{2+\rho}\right)L \quad \text{if } \sigma < \bar{\sigma}^{\mathrm{e}} \tag{27}$$

and

$$\tilde{TC}_n = -\tilde{\varepsilon}_{t(n)}\tilde{Q}_n + \sigma + \left(\frac{4-\rho}{4+\rho}\right)L \quad \text{if } \sigma \geqslant \bar{\sigma}^{\mathrm{e}} \tag{28}$$

Using these equations, we obtain the following result.

Proposition 5. *In equilibrium, the expected trading cost in the limit order market is*:

$$E(\tilde{TC}_n) = \left(\frac{4-\rho}{4+\rho}\right)L \textit{ if } \sigma \geqslant \bar{\sigma}^{\mathrm{e}} \textit{ and}$$

$$E(\tilde{TC}_n) = \left(\frac{2-\rho}{2+\rho}\right)L - \left(\frac{2}{2+\rho}\right)\sigma \textit{ otherwise.}$$

It decreases with ρ.

As explained in Section 5, the larger the execution risk for limit order traders, the larger the wedge between their posted spread and their reservation spread.

[15] Here the spread posted by the traders is not a good measure of actual trading costs for market order traders. First, because the asset value fluctuates over time, a limit order price can be stale relative to the fair value of the asset at the time of the transaction. Second, a quote posted by a limit order trader is not necessarily a price at which a transaction will take place. The measure of trading costs, which is defined here, overcomes these two problems. It is worth stressing however that the same results are obtained when we use the quoted spread as a proxy for execution costs.

For this reason, trading costs enlarge when ρ decreases (i.e. when the execution risk of limit order traders increases).

The lower ρ in the model, the larger the probability that a trader will be the last trader of the trading 'day' and will, for this reason, not be executed. In real trading situations, this probability is larger at the end of the trading day. Thus comparing the size of the trading costs, *across equilibria*, when ρ is small and ρ is large in the model, is *like* comparing the size of the trading costs at the end of the trading day and at an earlier point in time during the trading day.[16] Thus the model predicts that trading costs must increase at the end of the trading day. This prediction is consistent with the empirical findings regarding limit order markets (e.g. McInish and Wood, 1992; Kleidon and Werner, 1993; Biais et al., 1995). The interpretation provided here is that execution risk is larger at the end of the trading day. For this reason, traders are willing to trade at more unfavorable prices and limit order traders can extract larger rents from market order traders.

As explained in Section 5, execution risk (ρ) does not influence traders' quotes when limit order traders post zero expected profits quotes. For this reason, the expected trading cost does not depend on ρ in the competitive case. Thus, according to the model, the increase in trading costs at the end of the trading day relies on the possibility for limit order traders to extract rents from market order traders. It follows that the size of the increase in trading costs at the end of the trading day must be negatively related to the level of competition between limit order traders. This implication could be tested in the following way. The posted spread is a proxy for trading costs. The difference, $\Delta SPREAD$, between the spread at the end of the trading day and the spread, say, in the middle of the day measures the extent by which trading costs increase at the end of the day. The testable hypothesis is that, in a cross-sectional analysis, $\Delta SPREAD$ is negatively related to the proxy chosen for the level of competition between limit order traders. Examples of such proxies are the number of markets in which a stock is traded or the number of broker-dealers active in a stock (as in Sandås, 1997).

Consider now the case in which the proportions of traders of type y_h and traders of type y_l are not equal (i.e. $k \neq 0.5$) and there is no winner's curse problem ($\sigma = 0$). The equilibrium quotes in this case have been derived in Section 2.4. Using Eqs. (8) and (9), we obtain the following result.

[16] Since bidding strategies are stationary, we cannot directly analyze the evolution of trading cost over time. Thus we compare equilibria for different values of ρ. Alternatively, we could assume that the closing date is deterministic and that ρ decreases over time (gets closer and closer to 0 as t goes to the closing time). In this case, one can show directly that trading costs increase over time in our model because execution risk increases over time.

Proposition 6. *Suppose* $\sigma = 0$. *In equilibrium, the ask prices posted by traders with type* y_l *and the bid prices posted by traders with type* y_h *increase with* k.

This result is also due to execution risk. Actually, an increase in the proportion of traders with type y_h decreases the execution probability of these traders (because they trade only with agents of type y_l in equilibrium). As a consequence they become more 'impatient' and their buy cutoff prices increases. This effect allows the traders with type y_l to increase their ask prices. For this reason, these traders derive larger trading profits when their sell limit orders are executed. Moreover their execution probability increases as well since the proportion of traders with type y_h is larger. These two effects entail that limit order trading becomes relatively more attractive for traders with type y_l. It follows that the minimum bid price at which they are willing to submit a sell market order increases. Consequently traders with type y_h must improve their bid prices. As a result, the rents of traders with type y_l increase whereas the rents of traders with type y_h decrease.

The previous result is intuitive. In a market in which there are few sellers (traders with type y_l), the execution risk faced by the buyers (traders with type y_h) is higher. Consequently they must leave larger gains from trade to the sellers. This effect is reflected in the expected trading costs for *buy market orders* ($\mathrm{E}(\tilde{TC}_n|Q_n = +1)$) and *sell market orders* ($\mathrm{E}(\tilde{TC}_n|Q_n = -1)$). These expected trading costs are (Using Eqs. (8) and (9)):

$$\mathrm{E}(\tilde{TC}_n|Q_n = +1) = A^*(v, -L) - v = L\left(\frac{1-\rho(1-k)(2-\rho k)}{1-\rho^2 k(1-k)}\right), \tag{29}$$

$$E(\tilde{TC}_n|Q_n = -1) = v - B^*(v, +L) = L\left(\frac{1-\rho k(2-\rho(1-k))}{1-\rho^2 k(1-k)}\right). \tag{30}$$

Finally we denote by STC, the sum of the expected trading costs for buy market order traders and sell market order traders ($\mathrm{E}(\tilde{TC}_n|Q_n = +1) + \mathrm{E}(\tilde{TC}_n|Q_n = -1)$). From the two previous equations, we obtain

$$STC = A^*(v, -L) - B^*(v, +L) = 2L\left(\frac{1-\rho+\rho^2 k(1-k)}{1-\rho^2 k(1-k)}\right). \tag{31}$$

This equation yields the following corollary.

Corollary 6

1. *The expected trading cost for buy market orders increases with* k *whereas the expected trading cost for sell market orders decreases with* k.
2. STC *is a concave function of* k *and is maximum when* $k = 0.5$.

Notice that k is the proportion of buy orders (buy limit orders and buy market orders) in the order flow and that $(1-k)$ is the proportion of sell orders. Thus

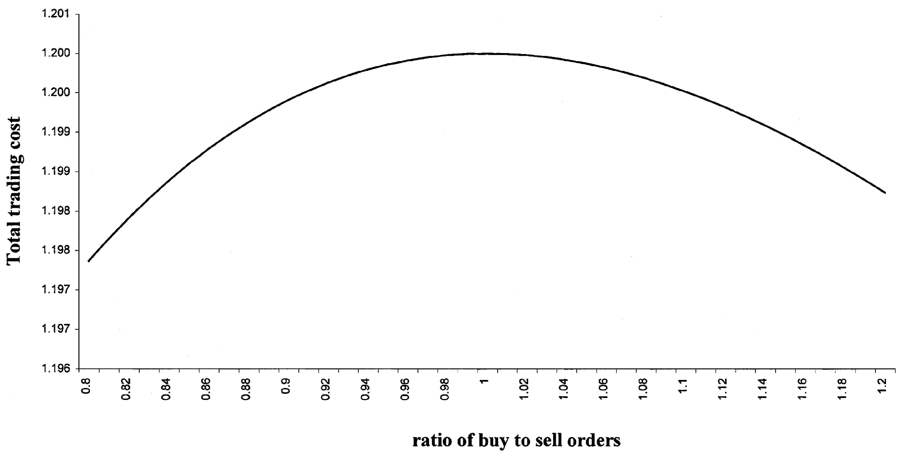

Fig. 2. Total trading cost as a function of the ratio of buy to sell orders.

the ratio of buy to sell orders is given by $i = k/(1-k)$. The previous corollary leads to the following predictions:

1. The average trading cost for buy (sell) market orders is positively (negatively) related to the ratio of buy to sell orders.
2. The sum of the average trading costs for buy market orders and for sell market orders increases with i when $i < 1$ and decreases with i when $i > 1$.

Fig. 2 illustrates the second prediction.

7. Conclusion

This paper computes the subgame perfect equilibrium to a trading game where traders arrive sequentially and choose to submit either a market order or a limit order with a one-period life. A complete characterization, in closed form, of traders' order placement strategy is obtained.

The closed form characterization of the equilibrium is useful for analyzing how the risk of being picked off and the execution risk faced by limit order traders influence (a) the order flow composition and (b) the trading costs, in limit order markets. We summarize below the main testable hypotheses for the cross-sectional analysis of order flow composition and trading costs in these markets:

H1. The proportion of limit orders in the order flow is positively related to asset volatility.

H2. The fill rate (the ratio of filled limit orders to total number of limit orders) is negatively related to asset volatility.

H3. The proportion of limit orders is positively related to the average size of the spread.
H4. The increase in trading costs at the end of the trading day is negatively related to the level of competition[17] between limit order traders.
H5. The size of the sum of trading costs for buy and sell orders is maximum when the ratio of buy to sell orders, is equal to one.

In order to obtain a closed form characterization of the equilibrium, we have assumed that limit orders expire after one period. Our results rely on very basic effects of volatility and execution risk, that (qualitatively) do not depend on the maturity of limit orders. For instance, traders react to an increase in volatility by shading more their offers. Actually when volatility enlarges, the probability of being picked off and the associated loss are larger for limit order traders. Clearly this effect of volatility will still be obtained if limit orders last more than one period. Considering the case in which limit orders last more than one period could be worthwhile, however because it would allow the analysis of the relationships between the order flow and the state of the book. Computing closed form solutions for the optimal order placement strategy seems extremely difficult in this case. But the equilibrium might be solved with numerical methods. This exercise is left for future research.

Appendix A.

Proof of Proposition 1. Consider a given state of the market S_t. In this state, Eq. (13) implies that an agent must submit a buy market order iff

$$v_t + y_t - A_t^m > \mathrm{Max}\{J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)), B_t^m - v_t - y_t\}. \tag{A.1}$$

In the same way, an agent must submit a sell market order iff

$$B_t^m - v_t - y_t > \mathrm{Max}\{J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)), v_t + y_t - A_t^m\}. \tag{A.2}$$

Consider first the two following inequalities:

$$v_t + y_t - A_t^m > J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)), \tag{A.3}$$

$$B_t^m - v_t - y_t > J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)). \tag{A.4}$$

Denote by $C^{\mathrm{b}}(v_t, y_t)(C^{\mathrm{s}}(v_t, y_t))$ the ask (bid) price such that the first (second) inequality holds as an equality (the solutions are given by Eqs. (14) and (15) in Section 3.2). $J \geqslant 0$ because a trader has always the possibility to get his

[17] Measures for the level of competition have been proposed in Section 6.2.

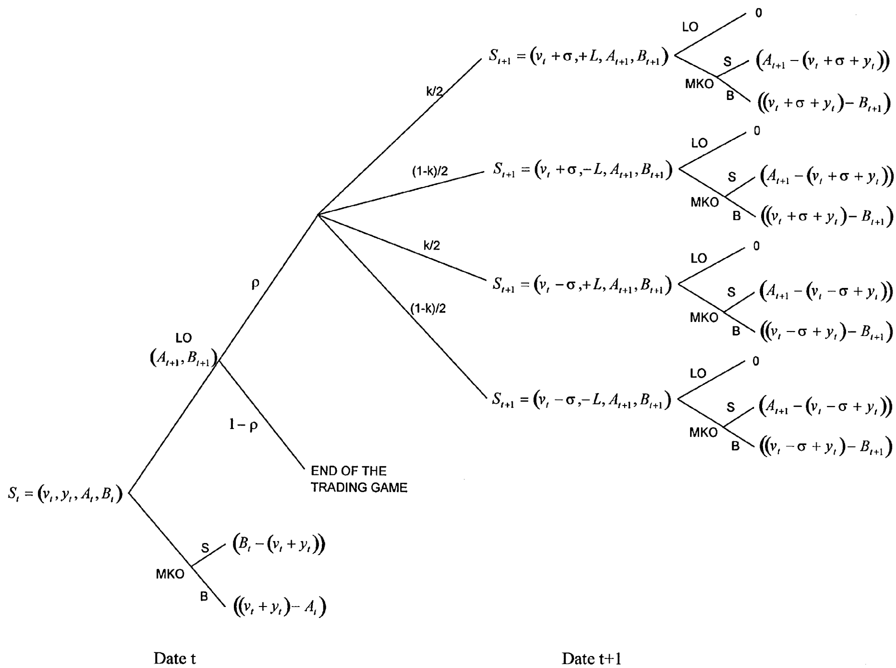

Fig. 1. This tree represents the possible trading decisions, at time t, for a trader of type y_t and the possible payoffs for this trader according to the decision of the trader who arrives at time $t+1$. LO indicates the placement of Limit Orders. MKO indicates the submission of a Market Order. Finally S refers to a Sell order and B to a Buy order.

reservation utility by posting limit orders with a zero execution probability. This implies $C^{s}(v_t, y_t) \geqslant v_t + y_t \geqslant C^{b}(v_t, y_t)$. Since $A_t^m > B_t^m$, if $A_t^m < C^b$ then $v_t + y_t - A_t^m > J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)) \geqslant B_t^m - v_t - y_t$. In the same way, if $B_t^m > C^s$ then $B_t^m - v_t - y_t > J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)) \geqslant v_t + y_t - A_t^m$. Consequently, an agent must submit a buy (sell) market order if the ask (bid) price is lower (greater) than C^b (C^s) and place limit orders otherwise.

Proof of Lemma 1. Consider a trader with type y_l who arrives at time t. The bid price, $B(v_t, -L)$, posted by this trader must be lower than his reservation price $v_t - L$. If the asset value increases, the reservation price $v_t + \sigma + y_t$ of the trader who arrives at the next point in time is necessarily larger than $v_t - L$ since $-L \leqslant y_t$. But then the bid price posted by the trader with type y_l has a zero execution probability when the asset value increases. Thus execution can occur only if the asset value decreases. But then, using Eq. (19), the bid price must be at most $v_t - \sigma - L$ for the trader with type y_l to break-even. The proof is symmetric for the ask price posted by a trader with type y_h. $\square$

Proof of Proposition 2. In the competitive case, limit order traders get zero expected profits, i.e. $J = 0$. Thus their cutoff prices are just equal to their reservation prices. It follows that the execution probability of a given offer depends on its position relative to the possible reservation prices of the future traders. The execution probability of a buy limit order according to its price is given in Table 1.

Step 1: We first look for solutions of $B^c(v_t, +L) = v_t + L + E(\Delta\tilde{v}_{t+1}|I^b(B^c(v_t, +L)) = +1, S_t)$. Let $\pi_t(B)$ be the probability that an increase in the asset value has occurred between time t and $t+1$, conditional on the execution of a buy limit order with price B at time $t+1$. It is the case that:

$$E(\Delta v_{t+1}|I^b(B) = +1, S_t) = \pi_t(B)\sigma - (1 - \pi_t(B))\sigma = \sigma(2\pi_t(B) - 1) \qquad \text{(A.5)}$$

and

$$\pi_t(B) = \frac{Prob(I^b(B) = +1|\varepsilon_{t+1} = +\sigma)\, Prob(\varepsilon_{t+1} = +\sigma)}{Prob(I^b(B) = +1)}. \qquad \text{(A.6)}$$

Since $\pi_t(B) \geqslant 0$, the competitive bid price is at least equal to $B^c(v, +L) = v + L - \sigma$. Suppose first that $0 < \sigma < \frac{3}{2}L$. Then $v_t + L - \frac{1}{3}\sigma \geqslant \text{Max}\{v_t + \sigma - L, v_t - \sigma + L\}$. For this reason a bid price B^c equal to $v_t + L - \frac{1}{3}\sigma$ has a probability ρ of being executed conditional on a decrease in the asset value and $\rho/2$ conditional on an increase in the asset value (see Table 1, Cases 1 and 2). Thus

$$\pi_t(B^c) = \frac{\frac{\rho}{2}\frac{1}{2}}{\frac{\rho}{2}\frac{1}{2} + \frac{\rho}{2}} = \frac{1}{3}. \qquad \text{(A.7)}$$

Thus $E(\Delta\tilde{v}_{t+1}|I^b(B^c(v_t, +L)) = +1, S_t) = -\frac{1}{3}\sigma$, which means that $B^c(v_t, +L) = v_t + L - \frac{1}{3}\sigma$ is solution if $0 < \sigma < \frac{3}{2}L$. Using the first column of Table 1, we obtain $Prob\,(I^b(B^c(v_t, +L) = +1) = \frac{3}{4}\rho$ in this case. If $\sigma \geqslant \frac{3}{2}L$ then $v_t + L - \frac{1}{3}\sigma \leqslant v_t - L + \sigma$. This implies that a bid price B lower than or equal to $v_t + L - \frac{1}{3}\sigma$ has a zero execution probability conditional on an increase in the asset value (see Case 1 in Table 1). It follows that $\pi_t(B) = 0$ if $B \leqslant v_t + L - \frac{1}{3}\sigma$ and if $\sigma \geqslant \frac{3}{2}L$. This means that the possible solution is either equal to $v_t + L - \sigma$ or strictly greater than $v_t + L - \frac{1}{3}\sigma$. Using Table 1 (in Case 1), computations show that a bid strictly larger than $B = v_t + L - \frac{1}{3}\sigma$ cannot be solution. Consequently $B^c(v_t, +L) = v_t + L - \sigma$ is the only solution if $\frac{3}{2}L \leqslant \sigma$. In this case (using the first column of Table 1, case 1), $Prob(I^b(B^c(v_t, +L) = +1) = \frac{1}{4}\rho$. The same argument yield the ask prices solutions of $A^c(v_t, -L) = v_t - L + E(\Delta\tilde{v}_t|I^b(A^c(v_t, -L)) = +1, S_t)$ and their execution probabilities.

Table 1
Buy limit orders execution probabilities in the competitive case

Bid price	Execution probability	Execution probability conditional on a decrease in the asset value	Execution probability conditional on an increase in the asset value
Case 1: $L < \sigma$			
$\leqslant v_t - \sigma - L$	0	0	0
$\in]v_t - \sigma - L, v_t - \sigma + L]$	$\rho/4$	$\rho/2$	0
$\in]v_t - \sigma + L, v_t + \sigma - L]$	$\rho/2$	ρ	0
$\in]v_t + \sigma - L, v_t + \sigma + L]$	$3\rho/4$	ρ	$\rho/2$
$> v_t + \sigma + L$	ρ	ρ	ρ
Case 2: $L \geqslant \sigma$			
$\leqslant v_t - \sigma - L$	0	0	0
$\in]v_t - \sigma - L, v_t + \sigma - L]$	$\rho/4$	$\rho/2$	0
$\in]v_t + \sigma - L, v_t - \sigma + L]$	$\rho/2$	$\rho/2$	$\rho/2$
$\in]v_t - \sigma + L, v_t + \sigma + L]$	$3\rho/4$	ρ	$\rho/2$
$> v_t + \sigma + L$	ρ	ρ	ρ

Note: This table gives the conditional and unconditional execution probability of a buy limit order according to the position of its price relative to traders cutoff prices, in the competitive case. It is used in the proof of Proposition 2. Consider Case 1 for instance and suppose that the price of the buy limit order is in the interval $(v_t + \sigma - L, v_t + \sigma + L]$. In case of a decrease in the asset value, all the traders, whatever their type, find optimal to place a sell market order at this price. Thus the conditional execution probability of the buy limit order is ρ. In case of an increase in the asset value, only the traders with type $y_l = -L$ find optimal to place a sell market order. It follows that the conditional execution probability is $\rho/2$. Consequently the unconditional execution probability of a buy limit order with a price in the interval $(v_t + \sigma - L, v_t + \sigma + L]$ is $0.5\rho + 0.5\rho/2 = 3\rho/4$. Other entries are derived following the same reasoning.

Step 2: Lemma 1 implies that the largest possible bid price posted by a trader with type y_l who arrives at time t is $v_t - \sigma - L$. It has a zero execution probability (see Table 1) since it is lower than the possible reservation prices for the trader who arrives at time $t + 1$. In the same way, the lowest possible ask price posted by a trader with type y_h is $v_t + L + \sigma$ which has a zero execution probability. This gives the last part of the proposition. □

Proof of Proposition 3. In what follows, traders quotation strategy is derived under the conjecture that cutoff prices are increasing in the asset value in equilibrium. Then, it is checked, using the closed-form solution, that this conjecture is indeed correct. Consider a possible candidate $\{C^{s*}(.,.), C^{b*}(.,.)\}$ for the equilibrium order choice strategy. Table 2 gives the execution probability of

Table 2
Buy limit orders execution probabilities in equilibrium

Bid price	Execution probability	Execution probability conditional on a decrease in the asset value	Execution probability conditional on an increase in the asset value
Case 1: $L/(1+\rho/4) \leqslant \sigma$			
$\in C^{s*}(v_t-\sigma,-L)$	0	0	0
$\in]C^{s*}(v_t-\sigma,-L), C^{s*}(v_t-\sigma,+L)]$	$\rho/4$	$\rho/2$	0
$\in]C^{s*}(v_t-\sigma,+L), C^{s*}(v_t+\sigma,-L)]$	$\rho/2$	ρ	0
$\in]C^{s*}(v_t+\sigma,-L), C^{s*}(v_t+\sigma,+L)]$	$3\rho/4$	ρ	$\rho/2$
$> C^{s*}(v_t+\sigma,+L)$	ρ	ρ	ρ
Case 2: $L/(1+\rho/4) > \sigma$		0	0
$\leqslant C^{s*}(v_t-\sigma-L)$	0		
$\in]C^{s*}(v_t-\sigma,-L), C^{s*}(v_t+\sigma,+L)]$	$\rho/4$	$\rho/2$	0
$\in]C^{s*}(v_t+\sigma,-L), C^{s*}(v_t-\sigma,+L)]$	$\rho/2$	$\rho/2$	$\rho/2$
$\in]C^{s*}(v_t-\sigma,+L), C^{s*}(v_t+\sigma,+L)]$	$3\rho/4$	ρ	$\rho/2$
$> C^{s*}(v_t+\sigma,+L)$	ρ	ρ	ρ

Note: This table gives the conditional and unconditional execution probability of a buy limit order according to the position of its price relative to traders' cutoff prices, in equilibrium. It is used in the proof of Proposition 3. Consider Case 1 for instance and suppose that the price of the buy limit order is in the interval $(C^{s*}(v_t+\sigma,-L), C^{s*}(v_t+\sigma,+L)]$. In case of a decrease in the asset value, all the traders, whatever their type, find optimal to place a sell market order at this price. Thus the conditional execution probability of the buy limit order is ρ. In case of an increase in the asset value, only the traders with type $y_l = -L$ find optimal to place a sell market order. It follows that the conditional execution probability is $\rho/2$. Consequently, the unconditional execution probability of a buy limit order with a price in the interval $(C^{s*}(v_t+\sigma,-L), C^{s*}(v_t+\sigma,+L)]$ is $0.5\rho + 0.5\rho/2 = 3\rho/4$. Other entries are derived following the same reasoning.

a bid price posted at time t according to its position relative to the sell cutoff prices of the trader who arrives at time $t+1$.

Step 1: The proof of the last part of the proposition is as Step 2 in the proof of Proposition 2.

Step 2: Consider a trader with type y_h who arrives at time t, who chooses to place a buy limit order with a strictly positive execution probability. It follows from Table 2 that he must optimally choose a bid price slightly higher than the *lower bound* of one of the 4 possible intervals for the bid price. Actually a higher price could be decreased without changing the execution probability of the limit order. A bid price higher than $C^{s*}(v_t + \sigma, +L)$ cannot be optimal since it is larger than the maximum valuation of the trader with type y_h. In Case 1 (of Table 2), a limit order with a bid price B in the interval $(C^{s*}(v_t - \sigma, +L), C^{s*}(v_t + \sigma, -L)]$ is executed only if the asset value decreases. It follows that for a bid price in this interval, $\mathrm{E}(\Delta v_{t+1} | I^{b}(B) = +1, S_t) = -\sigma$. But then Condition (19) cannot be satisfied since $B > C^{s*}(v_t - \sigma, +L) \geqslant v_t + L - \sigma$. In Case 2, a buy limit order with a price in the interval $(C^{s*}(v_t - \sigma, +L), C^{s*}(v_t + \sigma, +L)]$ can be executed whether the asset value increases or decreases. Execution in case of a decrease in the asset value cannot be profitable, however. Actually the bid price in this case is larger than $C^{s*}(v_t - \sigma, +L)$, which is itself larger than the reservation price of a trader with type y_h, conditional on a decrease in the asset value. Thus, in this case, an offer in the interval $(C^{s*}(v_t + \sigma, -L), C^{s*}(v_t - \sigma, +L)]$ dominates an offer in the interval $(C^{s*}(v_t - \sigma, +L), C^{s*}(v_t + \sigma, +L)]$. It follows from these remarks that the possible bid price posted by a trader with type y_h is either slightly higher than $C^{s*}(v_t - \sigma, -L)$ or slightly higher than $C^{s*}(v_t + \sigma, -L)$. A symmetric argument implies that a trader with type y_l in state S_t must choose an ask price, which is either slightly lower than $C^{b*}(v_t - \sigma, +L)$ or slightly lower than $C^{b*}(v_t + \sigma, +L)$.

Step 3: We consider two cases now.

Case 1: Assume $L/(1 + \rho/4) \leqslant \sigma$. Suppose that $C^{s*}(v_t + \sigma, -L) \geqslant C^{s*}(v_t - \sigma, +L)$ and that $C^{b*}(v_t + \sigma, -L) \geqslant C^{b*}(v_t - \sigma, +L)$ for all possible values for the asset. Moreover suppose that:

$$\tfrac{1}{4}[v_t + L - \sigma - C^{s*}(v_t - \sigma, -L)] \geqslant \tfrac{1}{2}[v_t + L - C^{s*}(v_t + \sigma, -L)] \quad \forall v_t. \tag{A.8}$$

Using the fact that $C^{s*}(v_t + \sigma, -L) \geqslant C^{s*}(v_t - \sigma, +L) \geqslant v_t + L - \sigma$, we get

$$\tfrac{1}{2}[v_t + L - C^{s*}(v_t + \sigma, -L)] \geqslant \tfrac{1}{2}[v_t + L - C^{s*}(v_t + \sigma, -L)] + \tfrac{1}{4}[v_t + L - \sigma - C^{s*}(v_t + \sigma, -L)]. \tag{A.9}$$

Using Eq. (A.8), it follows that

$$\frac{\rho}{4}[v_t + L - \sigma - C^{s*}(v_t - \sigma, -L)] \geqslant \frac{\rho}{2}[v_t + L - C^{s*}(v_t + \sigma, -L)] + \frac{\rho}{4}[v_t + L - \sigma - C^{s*}(v_t + \sigma, -L)]. \tag{A.10}$$

The LHS is the expected profit of a trader with type y_h if he places a buy limit order with a price slightly higher[18] than $C^{s*}(v_t - \sigma, -L)$. The expected profit of the trader if he places a buy limit order with a price slightly higher than $C^{s*}(v_t + \sigma, -L)$ is (using the conditional execution probabilities in Table 2):

$$\frac{\rho}{2}(v_t + L - \sigma - C^{s*}(v_t + \sigma, -L)) + \frac{\rho}{4}(v_t + L + \sigma - C^{s*}(v_t + \sigma, -L)). \tag{A.11}$$

Some algebra shows that this equation is equal to the RHS of Eq. (A.10). Consequently, under our hypotheses, it is optimal for a trader with type y_h to place a buy limit order with a bid price slightly higher than $C^{s*}(v_t - \sigma, -L)$. Proceeding in the same way, it is optimal for agents of type y_l to quote an ask price slightly below $C^{b*}(v_t + \sigma, +L)$ if

$$\tfrac{1}{4}[C^{b*}(v_t + \sigma, +L) - (v_t + \sigma - L)] \geqslant \tfrac{1}{2}[C^{b*}(v_t - \sigma, +L) - (v_t - L)] \quad \forall v_t. \tag{A.12}$$

Given these choices by the different types of traders, the 2 following equalities must be satisfied in equilibrium (see Eqs. (16) and (17) of Corollary 1):

$$v_t + L - C^{b*}(v_t, +L) = \frac{\rho}{4}[v_t + L - \sigma - C^{s*}(v_t - \sigma, -L)] \quad \forall v_t, \tag{A.13}$$

$$C^{s*}(v_t, -L) - (v_t - L) = \frac{\rho}{4}[C^{b*}(v_t + \sigma, +L) - (v_t + \sigma - L)] \quad \forall v_t. \tag{A.14}$$

The last equation implies:

$$C^{s*}(v_t - \sigma, -L) - (v_t - \sigma - L) = \frac{\rho}{4}[C^{b*}(v_t, +L) - (v_t - L)]. \tag{A.15}$$

Using Eqs. (A.13) and (A.15), an equation with unknown $C^{b*}(v_t, +L)$ is obtained. Solving for $C^{b*}(v_t, +L)$, yields

$$C^{b*}(v_t, +L) = v_t + L - \frac{\rho}{4+\rho}(2L). \tag{A.16}$$

Proceeding in the same way

$$C^{s*}(v_t, -L) = v_t - L + \frac{\rho}{4+\rho}(2L). \tag{A.17}$$

[18] In this proof, we proceed as if the bid prices were just equal to $C^{s*}(v_t - \sigma, -L)$ or $C^{s*}(v_t + \sigma, -L)$ since they can be chosen as close as desired to these cutoff prices.

Using the expressions for $C^{s*}(., -L)$ and $C^{b*}(., +L)$ given by Eqs. (A.16) and (A.17), one obtains that Eqs. (A.8) and (A.12) are satisfied *iff* $L/(1+\rho/4) \leqslant \sigma$. It is also straightforward to check that $C^{s*}(v_t+\sigma, -L) \geqslant C^{s*}(v_t-\sigma, +L)$ and $C^{b*}(v_t+\sigma, -L)C^{b*}(v_t-\sigma, +L)$ as supposed. Consequently, the quotation strategy $A^*(v_t, -L) = C^{b*}(v_t+\sigma, +L)$, $B^*(v_t, +L) = C^{s*}(v_t-\sigma, -L)$, $A^*(v_t, +L) = v_t + L + \sigma$ and $B^*(v_t, -L) = v_t - L - \sigma$ and the associated order choice strategy is an equilibrium iff $L/(1+\rho/4) \leqslant \sigma$. Moreover, Eqs. (A.16) and (A.17) can be used to obtain closed forms for the equilibrium ask price of traders with type y_l and the equilibrium bid price of traders with type y_h. They can be written as in Proposition 3 after straightforward manipulations. A trader with type y_h chooses a bid price slightly above $C^{s*}(v_t-\sigma, -L)$. Thus the execution probability of his bid price is $\rho/4$ (See Table 2). By symmetry, the execution probability of the ask price posted by a trader with type y_l is also $\rho/4$.

Case 2: $L/(1+\rho/4) > \sigma$. Suppose that $C^{s*}(v_t+\sigma, -L) < C^{s*}(v_t-\sigma, +L)$ and $C^{b*}(v_t+\sigma, -L) < C^{b*}(v_t-\sigma, +L)$ for all possible values for the asset. Under this conjecture, necessary and sufficient conditions for $B^*(v_t, +L) = C^{s*}(v_t+\sigma, -L)$ and $A^*(v_t, -L) = C^{b*}(v_t-\sigma, +L)$ to be optimal in equilibrium are (using Table 2, case 2):

$$\tfrac{1}{4}[v_t + L - \sigma - C^{s*}(v_t-\sigma, -L)] < \tfrac{1}{2}[v_t + L - C^{s*}(v_t+\sigma, -L)] \quad \text{(A.18)}$$

and

$$\tfrac{1}{4}[C^{b*}(v_t+\sigma, +L) - (v_t+\sigma-L)] < \tfrac{1}{2}[C^{b*}(v_t-\sigma, +L) - (v_t - L)]. \quad \text{(A.19)}$$

If these conditions are satisfied, then according to Eqs. (16) and (17), cutoff prices are given by

$$v_t + L - C^{b*}(v_t, +L) = \frac{\rho}{2}[v_t + L - C^{s*}(v_t+\sigma, -L)], \quad \text{(A.20)}$$

$$C^{s*}(v_t, -L) - (v_t - L) = \frac{\rho}{2}[C^{b*}(v_t-\sigma, +L) - (v_t - L)]. \quad \text{(A.21)}$$

Using the same procedure as in Case 1, this system can be solved for the cutoff price functions. We obtain $C^{s*}(v_t, -L) = v_t - L + (2L-\sigma)\rho/(2+\rho)$ and $C^{b*}(v_t, +L) = v_t + L - (2L-\sigma)\rho/(2+\rho)$. Since $L > \sigma$, in Case 2, it can be checked that our initial conjecture on cutoff prices in this case is satisfied. Moreover, using closed form solutions for cutoff prices, it turns out that Conditions (A.18) and (A.19) are satisfied if $L/(1+\rho/4) > \sigma$ as supposed. Consequently the quotation strategy: $A^*(v_t, -L) = C^{b*}(v_t-\sigma, +L)$, $B^*(v_t, +L) = C^{s*}(v_t+\sigma, -L)$, $A^*(v_t, +L) = v_t + L + \sigma$ and $B^*(v_t, -L) = v_t - L - \sigma$ and the associated order type choice strategy is an equilibrium if $L/(1+\rho/4) > \sigma$. As in

Case 1, the closed form solution for the cutoff prices can be used to derive directly the closed form solution for the quotation strategy. Using Table 2 (Case 2), the execution probability of the bid price (ask price) of a trader with type y_h (y_l) is $\rho/2$.

Existence and uniqueness. Note that for each case above, the conjecture that cutoff price functions are increasing in v_t is satisfied. On the other hand, there is no set of parameters for which there exists no equilibrium in pure strategy. This proves existence. Moreover, for a given set of parameters, it is possible to show that no other equilibrium than those derived above, can be obtained. Uniqueness follows. The computations which are necessary to show uniqueness are quite long. As they do not provide any additional intuition, they are omitted for brevity. □

Proof of Proposition 4. Let $Abs(x)$ be the absolute value of x. Using the definition of $\tilde{Q}_i$ (given in Section 3), note that: $Abs(\tilde{Q}_i) = 1$ if a market order is submitted at time i (Event 1) and $Abs(\tilde{Q}_i) = 0$ if a limit order is submitted at time i (Event 2). It follows that $\tilde{M}_t = \sum_{i=0}^{i=t-1} Abs(\tilde{Q}_i)$. Then $\bar{m}_t = (\sum_{i=0}^{i=t-1} \mathrm{E}(Abs(\tilde{Q}_i)))/(t-1)$, for $t \geqslant 2$. Call π_{ij} the probability of each of the two previous events ($j \in \{1, 2\}$) at time $i \leqslant t-1$, conditional on the game stopping at time t (note that $\pi_{02} = 1$). It follows that

$$\bar{m}_t = \left(\sum_{i=0}^{i=t-1} \pi_{i1} \right) \Big/ (t-1). \tag{A.22}$$

Furthermore

$$\begin{aligned} \pi_{i1} &= Prob(Abs(Q_i) = 1 | \tilde{T} > i, Q_{i-1} = 0) Prob(Q_{i-1} = 0 | \tilde{T} > i) \\ &\quad + Prob(Abs(Q_i) = 1 | \tilde{T} > i, Abs(Q_{i-1}) = 1) \\ &\quad \times Prob(Abs(Q_{i-1}) = 1 | \tilde{T} > i) \ \forall_i \geqslant 1. \end{aligned} \tag{A.23}$$

If a market order is placed at time $i-1$, then the book is empty at time i. In this case, no market order can be placed at time i. It follows that $Prob(Abs(Q_i) = 1 | \tilde{T} > i, Abs(Q_{i-1}) = 1) = 0$. On the other hand, $Prob(Q_{i-1} = 0 | \tilde{T} > i) = \pi_{(i-1)2}$. Thus the previous equation is rewritten:

$$\pi_{i1} = Prob(Abs(Q_i) = 1 | \tilde{T} > i, Q_{i-1} = 0) \pi_{(i-1)2} \quad \forall i \geqslant 1 \tag{A.24}$$

$Prob(Abs(Q_i) = 1 | -\tilde{T} > i, Q_{i-1} = 0)$ is the execution probability of a limit order trader who places limit orders at time $i-1$, conditional on the trading process not being stopped at time i. From Proposition 3, we know that this conditional execution probability is $\frac{1}{2}$ if $\sigma < \bar{\sigma}^{\mathrm{e}}$. Since $\pi_{(i-1)2} = 1 - \pi_{(i-1)1}$, we obtain:

$\pi_{i1} = (1/2) - (1/2)\pi_{(i-1)1}, \quad \forall i \geqslant 1$. Then Eq. (A.22) can be written

$$\bar{m}_t = \frac{1}{2} - \frac{1}{2}\frac{t-2}{t-1}\bar{m}_{t-1} \quad \forall t \geqslant 2. \tag{A.25}$$

Then taking the limit when t goes to infinity on both side yields $\bar{m} = 1/3$. The same types of computation can be used in the case $\sigma \geqslant \bar{\sigma}^{\mathrm{e}}$. The only difference is that $Prob(Abs(Q_i) = 1 | \tilde{T} > i, Q_{i-1} = 0) = \frac{1}{4}$. One obtains $\bar{m} = \frac{1}{5}$ in this case. The reasoning is the same in the competitive case. The proportion of market orders in this case is $\bar{m} = \frac{3}{7}$ when $0 < \sigma < \sigma^{\mathrm{c}}$ and $\bar{m} = \frac{1}{5}$ if $\sigma \geqslant \sigma^{\mathrm{c}}$. □

Proof of Corollary 2. Consider two levels of volatility σ_h and σ_l with $\sigma_h > \sigma_l$. If σ_h and σ_l are such that $\sigma_h \geqslant \bar{\sigma}^{\mathrm{e}} > \sigma_l$ then, from Proposition 4, we know that the proportion of limit orders is higher when the volatility is σ_h than when it is σ_l. In all the other cases, the proportion of limit orders is the same when the volatility is σ_h and when it is σ_l. This shows that in equilibrium the proportion of limit orders increases with volatility. The reasoning is exactly the same in the competitive case. □

Proof of Corollary 3. Remember that each limit order trader posts 2 limit orders (a buy limit order and a sell limit order). Thus the arrival of l limit orders entails that $l/2$ traders have decided to place limit orders. Using this remark, conditional on the arrival of l limit orders, the fill rate $\tilde{fr}$ can be written:

$$\tilde{fr} = \sum_{i=1}^{i=l/2} \tilde{F}_i/l \tag{A.26}$$

with $\tilde{F}_i = 0$ if none of the limit orders placed by the ith limit order trader is executed and $\tilde{F}_i = 1$ if one of the order is executed (the model is such that at most one can be executed). Consider the case in which $\sigma < \bar{\sigma}^{\mathrm{e}}$. From Proposition 3, we know that each limit order trader has a probability $\rho/2$ of being executed in this case. *Conditional on the game not stopping before the arrival of the next trader*, the execution probability of each trader is then 1/2. Thus $\mathrm{E}_t(\tilde{F}_i | \tilde{l} = l, T \geqslant t+2) = \frac{1}{2}$ and $\mathrm{E}_t(\tilde{fr} | \tilde{l} = l, \tilde{T} \geqslant t+2) = \frac{1}{4}$. If $\bar{\sigma}^{\mathrm{e}} \leqslant \sigma$, the execution probability of a limit order is $\rho/4$. Following the same reasoning, we obtain that $\mathrm{E}_t(\tilde{fr} | \tilde{l} = l, \tilde{T} \geqslant t+2) = \frac{1}{8}$ in this case. The rest of the proof is similar to the proof of the previous corollary. The methodology is the same in the competitive case. □

Proof of Corollary 4. In all cases, the nth transaction price can be written $\tilde{P}_n = \tilde{v}_{t(n)-1} + (D - L)\tilde{Q}_n$. The constant D varies with the parameters and has different values in equilibrium and in the competitive case. For instance, in equilibrium, $D = (2L - \sigma)(2/(2+\rho))$ if $\sigma < \bar{\sigma}^e$ and $D = \sigma + 8L/(4+\rho)$

otherwise (See Eqs. (24) and (25)). Thus

$$\Delta \tilde{P}_n = \tilde{P}_{n+1} - \tilde{P}_n = \tilde{v}_{t(n+1)-1} - \tilde{v}_{t(n)-1} + (D-L)(\tilde{Q}_n - \tilde{Q}_{n+1}). \tag{A.27}$$

The symmetry of the model when $k = 0.5$ implies that in all cases: $Prob(\tilde{Q}_n = +1) = Prob(\tilde{Q}_n = -1) = 0.5$. Some algebra gives:

$$Var(\Delta \tilde{P}_n) = \mathrm{E}(\tilde{t}(n+1) - \tilde{t}(n))\sigma^2 + 2(D-L)^2 - 2\mathrm{E}(\tilde{\varepsilon}_{t(n)}\tilde{Q}_n). \tag{A.28}$$

$\mathrm{E}(\tilde{t}(n+1) - \tilde{t}(n))$ is the average time between two transactions. It depends on the parameters values since the transaction frequency depends on them. Using the characterization of order placement strategies in equilibrium, the average time between two transactions in equilibrium is shown to be 3 periods if $\bar{\sigma}^{\mathrm{e}} \geqslant \sigma$ and 5 periods otherwise. Moreover $\mathrm{E}(\tilde{\varepsilon}_{t(n)}\tilde{Q}_n) = 0.5\mathrm{E}(\tilde{\varepsilon}_{t(n)}|\tilde{Q}_n = +1) - 0.5\mathrm{E}(\tilde{\varepsilon}_{t(n)}|\tilde{Q}_n = -1)$. Now consider the case in which $\sigma \geqslant \bar{\sigma}^{\mathrm{e}}$. The quotes chosen by the limit order traders in equilibrium are such that a buy (sell) market order is observed only if the asset innovation is positive (negative) (See proof of Proposition 3). This entails: $\mathrm{E}(\tilde{\varepsilon}_{t(n)}\tilde{Q}_n) = 0.5\sigma - 0.5(-\sigma) = \sigma$ in this case. When $\sigma < \bar{\sigma}^e$, buy (sell) market orders are placed only by traders of type $y_h(y_l)$ whatever the innovation in the asset value. This entails: $\mathrm{E}(\tilde{\varepsilon}_{t(n)}\tilde{Q}_n) = 0$ in this case.

Consider two levels of volatility σ_h and σ_l with $\sigma_h > \sigma_l$. If $\sigma_h \geqslant \bar{\sigma}^{\mathrm{e}} > \sigma_l$, using the expressions for D, $\mathrm{E}(\varepsilon_{t(n)}Q_n)$ and $\mathrm{E}(\tilde{t}(n+1) - \tilde{t}(n))$, the variance of transaction prices is greater when the volatility is σ_h than when it is σ_l. It is also the case that the proportion of limit orders is higher when the volatility is σ_h. In the other cases, the variance of transaction prices and the proportion of limit orders when the volatility is high are the same as when the volatility is low. Thus, overall there is a positive relationship between the variance of transaction prices and the proportion of limit orders. We can proceed in the same way to show that there is a negative relationship between the variance of transaction prices and the fill rate. In the competitive case, the average time between two transactions is 7/3 periods if $\sigma < \bar{\sigma}^{\mathrm{c}}$ and 5 periods otherwise. Moreover $\mathrm{E}(\tilde{\varepsilon}_{t(n)}\tilde{Q}_n)$ takes the values $\sigma/3$ and σ, respectively, according to the position of σ with respect to $\bar{\sigma}^{\mathrm{c}}$. Finally $D = (\sigma/3)$ if $\sigma < \bar{\sigma}^{\mathrm{e}}$ and $D = \sigma$ otherwise. The result is then proved as in the equilibrium case. □

Proof of Corollary 5. Eqs. (22) and (23) give the possible sizes for the spread posted by limit order traders. Then we can consider two levels of volatility σ_h and σ_l with $\sigma_h > \sigma_l$ and proceed as in the proof of Corollary 2 in order to prove the result. □

Proof of Proposition 5. Take the expectations in Eqs. (27) and (28). They depend on $E(\tilde{\varepsilon}_{t(n)}\tilde{Q}_n)$. We have shown in the proof of Corollary 4 that $\mathrm{E}(\tilde{\varepsilon}_{t(n)}\tilde{Q}_n) = \sigma$ if $\sigma \geqslant \bar{\sigma}^{\mathrm{e}}$ and 0 otherwise. Then it is then straightfoward to obtain the expressions for the expected trading costs. □

Proof of Proposition 6. Using Eqs. (8) and (9), it is obtained that

$$\frac{\partial A^*(v, -L)}{\partial k} = \frac{2\rho L}{(1-\rho^2 k(1-k))^2}[1 - k\rho + \rho(1-k)(1-(1-k)\rho)] \geqslant 0 \tag{A.29}$$

and

$$\frac{\partial B^*(v, +L)}{\partial k} = \frac{2\rho L}{(1-\rho^2 k(1-k))^2}[1 - \rho + k\rho(2 - \rho k)] \geqslant 0. \tag{A.30}$$

□

Proof of Corollary 6. The first part of the corollary is straightforward. On the other hand

$$\frac{\partial STC}{\partial k} = \frac{2L(2-\rho)\rho^2(1-2k)}{(1-\rho^2 k(1-k))^2}. \tag{A.31}$$

This is positive for $k < 0.5$, equal to zero for $k = 0.5$ and negative for $k > 0.5$. Moreover

$$\frac{\partial^2 STC}{\partial^2 k} = \frac{-4L(2-\rho)\rho^2}{(1-\rho^2 k(1-k))^3}[1 - \rho^2 + 3k\rho^2(1-k)] < 0 \tag{A.32}$$

which proves the second part of the proposition. □

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