Financial Markets
2: Bonds

Stefano Lovo

HEC, Paris

What is a bond (or fixed income security)?

- Tradable loan
  - Issuer borrows money
  - Initial purchaser on primary market lends money
  - Security can be traded on secondary market until maturity ⇒ identity of creditors changes over time

- Name “fixed income” initially came from nature of cash flows (fixed in advance)
  - No longer true, financial innovation

- Large spectrum of issuers — Values in trillion USD

	EU/UK	USA
Governments (“sovereign bonds”)	12	18
Corporations (financial and non-financial)	15	22
Cities		
...		

Overview

1. Bond basics
2. Interest rates & yield-to-maturity
3. Arbitrage pricing
4. Yield curve & forward rates
5. Interest rate risk
6. Default risk

Bonds

Definition

A bond is a financial asset that promises a stream of known cash flows in the future.

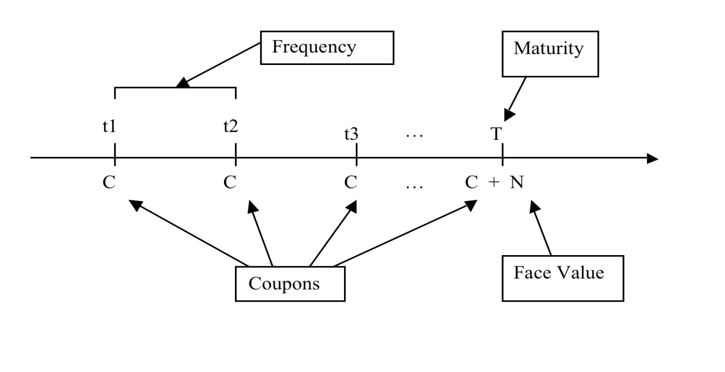
Example

Table: Coupon Bond

Labels	year 1	year 2	year 3	year 4
Cash-flows	€5	€5	€5	€105

The features of characterizing a bond

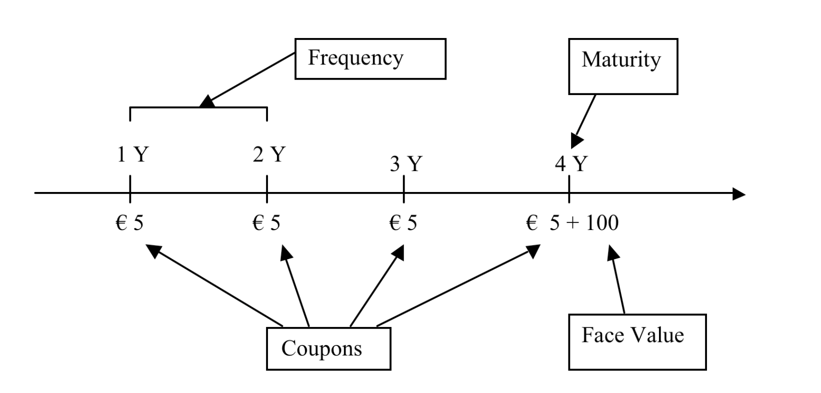
- The **Maturity** T: The date on which the last payment to the bondholder is due
- The **Coupon** C: the interest payment that is made to each bond holder at periodic dates.
- The **Face Value** N: The final payment that is made at maturity with the last coupon. (In general $N = 100$)
- The **Frequency** z with which coupons are paid (examples: once every year once every semester)



Bonds: Example 1

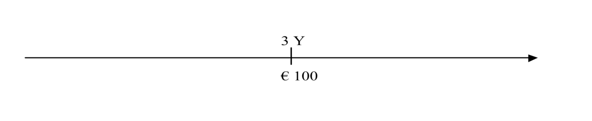
Example

Bond A has maturity 4 years, face value €100, coupon €5 frequency 1 year:

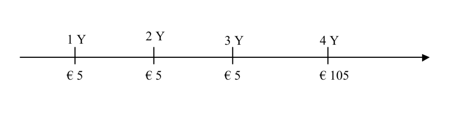


Main types of bonds

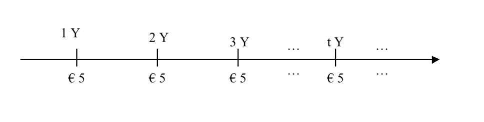
- **Zero Coupon Bond**: A bond without coupon ($C = 0$)
  Example: 3-year zero coupon bond with face value €100:



- **Coupon Bond**:A bond with a finite maturity and strictly positive coupon.
  Example: 4-year coupon bond with face value €100 and annual coupon of €5:



- **Perpetuity**: A bond with an infinite maturity and strictly positive coupon
  Example: A perpetuity with annual coupon of €5:



Quick check questions

Consider the following bonds:

Bond Name	Maturity in years	face value	Coupon	frequency
Bond A	4	€100	€8	6 months
Bond B	2.5	€1000	€50	1year
Bond Zcb	3	€200	€0	1year
Bond Pe	∞	€1000	€20	2 year

Q1 What are the cash flows of Bond A?

Q2 What are the cash flows of Bond B?

Q3 What are the cash flows of Bond Zcb?

Q4 What are the cash flows of Bond Pe?

Stripping or how to create a zero-coupon bond out of a coupon bond

- Remove the coupons from the coupon bond and sell the separate parts as Separate Trading of Registered Interest and Principal of Securities (STRIPS)

Pension fund demand drives rise in Treasury ‘strips’ activity

US tax code changes trigger rush for retirement plans to hedge long-term liabilities

Joe Rennison in New York SEPTEMBER 6, 2018

0

Pension fund demand for longer-dated US government bonds has sparked a sharp rise in Treasury “strips” activity ahead of a tax break ending this month.

With a new 21 per cent rate set to replace the old 35 per cent level later this month, pension plans have been rushing to buy strips, whereby existing 30-year Treasury bonds are separated into two types of security: one that only pays a fixed rate of interest, or coupon, over a set period, and another that is sold at a steep discount as it pays out only its principal value upon maturity.

Source

Types of bonds

- **Floating-rate bonds**: adjustable coupon depending on the level of a benchmark interest rate (example: LIBOR+1%)

- **Inflation-indexed bonds**: coupon indexed on inflation

- **Asset-Backed Securities (ABS)**: coupon payments are based on revenues of underlying assets (mortgage loans, car loans, credit card receivables, etc.); issued through securitization, make up roughly 25% of bond market

Hybrid bonds

- Callable bonds: The bond issuer has the right to buy back ("called") the bond at (typically) par value plus one coupon payment after a certain number of years.
- Convertible bonds: the bondholder has the right to convert the bond for a predetermined number of shares of the company's stock after an initial waiting period. The bondholder will convert only if it is profitable to do so.

Q1 A convertible bond is trading at €900. It can be converted into 100 shares of the company's stock. The stock is trading at €10.
Would you convert?

Q2 A firm has issued a convertible bond and a callable bond, both with same maturity, same face value, and same coupon.
Which one has the higher price on the market?

Q3 A firm has issued a convertible bond and a non-convertible bond, both with same maturity, same face value, and same price.
Which one has the higher coupon?

Hybrid bonds

Tesla Stock's Surge Puts Convertible Bonds in the Money

By Randall Forsyth • Aug. 7, 2018 2:45 p.m. ET

The surge in Tesla's (TSLA) shares after CEO Elon Musk's tweet about going private has had huge, salutary effect on its key convertible bonds and its financing position. By pushing the common stock's price above the $360 level, his tweet has put a $900 million convert issue above its conversion price, which would effectively let the electric-auto maker pay off that obligation in stock instead of cash.

Tesla has a $920 million 0.25% convertible issue due Feb. 27, 2019, which would be convertible into common at a price of $359.8676 per share. If the stock trades below that price at the conversion price, the holder would be better off taking cash. That would put pressure on Tesla while it continues to burn cash, as of the second quarter. But by being able to redeem the debt in stock would obviate the obligation to come up with over $900 million in cash.

Source

Bond trading

- Bonds typically trade in over-the-counter (OTC), and nowadays electronic, platforms

Robin Wigglesworth and **Joe Rennison** in New York AUGUST 16, 2017 3

Goldman Sachs has expanded its algorithmic corporate bond trading programme, more than trebling the number of securities it quotes since last summer to more than 7,000 — and is now eyeing an expansion into areas such as junk bonds later this year.

It comes as both banks and investors, such as hedge funds and asset managers, are focusing on automating smaller-size trades, in a bid to cut costs and free up dealers for larger transactions.

The bank's algorithm scrapes publicly-available pricing data for thousands of bonds to automatically generate firm, tradable prices for investors. Earlier this year it broke into the ranks of the top-three dealers on MarketAxess in US investment grade odd-lots — defined as smaller slivers of debt below $1m, according to Goldman Sachs.

Source

Bond yileds

Consider a bond with coupon C, frequency z, face value N, maturity T, and is currently trading for price P

- The bond's **coupon rate** is equal to the annualized coupon divided by its face value

$$i = \frac{zC}{N}$$

- The **current yield** is equal to the annualized coupon divided by its price

$$y = \frac{zC}{P}$$

- The **yield to maturity** is the discount rate y that makes the present value of the bond's cash-flows equal to its current price.

$$P = \frac{C}{(1+y)^{t_1}} + \frac{C}{(1+y)^{t_2}} + \cdots + \frac{C+N}{(1+y)^{T}}$$

where t_i is the years to wait for receiving the i-th coupon.
If $z = 1$, and T is integer, then from the annuity formula:

$$P = \frac{C}{y}\left(1 - \left(\frac{1}{1+y}\right)^{T}\right) + \frac{N}{(1+y)^{T}}$$

Quick check questions

Consider the following bonds:

Bond Name	Maturity in years	face value	Coupon	frequency	current price
Bond A	4	€100	€8	6 months	€120.5
Bond B	2.5	€1000	€50	1year	€1010
Bond Zcb	3	€200	€0	1year	€195
Bond P	∞	€1000	€20	2 year	€2000

Fill-in the following table:

Bond Name	coupon rate	current yield	yield to maturity
Bond A	?	?	?
Bond B	?	?	?
Bond Zcb	?	?	?
Bond P	?	?	?

Quick check questions

Consider the following bonds:

Bond Name	Maturity in years	face value	Coupon	frequency	current price
Bond A	4	€100	€8	6 months	€120.5
Bond B	2.5	€1000	€50	1year	€1010
Bond Zcb	3	€200	€0	1year	€195
Bond P	∞	€1000	€20	2 year	€2000

These bonds yields are:

Bond Name	coupon rate	current yield	yield to maturity
Bond A	16%	13.28%	9.93%
Bond B	5%	4.95%	5.64%
Bond Zcb	0%	0%	0.85%
Bond P	1%	0.5%	0.499%

More examples

Example 4. A coupon bond has a face value of €100, an annual coupon, a maturity of 10 years, a yield of 5%, and a price of €115.443.What is the bond's coupon?

$$P_0 = \frac{C}{y}\left[1 - \frac{1}{(1+y)^T}\right] + \frac{N}{(1+y)^T} \Leftrightarrow C = y\frac{P_0 - \frac{N}{(1+y)^T}}{1 - \frac{1}{(1+y)^T}} = €7$$

Example 5. A coupon bond has a face value of €100, an annual coupon of €4, a yield of 3%, and a price of €103.717.What is the bond's maturity?

$$P_0 = \frac{C}{y} + \frac{N - \frac{C}{y}}{(1+y)^T} \Leftrightarrow (1+y)^T = \frac{N - \frac{C}{y}}{P_0 - \frac{C}{y}} \Leftrightarrow T = \log\left(\frac{N - \frac{C}{y}}{P_0 - \frac{C}{y}}\right) / \log(1+y) = 4 \text{ years}$$

Example 6. A coupon bond has a face value of €100,000, coupons of €2,000 that are received *semi-annually*, a maturity of 5 years, and a yield to maturity of 5%.What is the bond's price?

Use the formula $P_0 = \frac{C}{y}\left[1 - \frac{1}{(1+y)^T}\right] + \frac{N}{(1+y)^T}$ at the semi-annual frequency, i.e.,

where C = €2,000 are semi-annual coupons, T = 10 semesters, and y is the semi-

annually compounded bond yield. Since (1 + semi-annually compounded yield)^2

$y = 1.05^{0.5} - 1 = 2.47\%$. Finally, $P_0 = \frac{2,000}{0.0247}\left[1 - \frac{1}{1.0247^{10}}\right] + \frac{100,000}{1.0247^{10}}$ = €95,880

Yield to maturity of zero coupon bonds

Theorem

The yield to maturity of a **zero coupon bond** *with face value N, maturity in T years and current price of P is*

$$r(T) = \left(\frac{N}{P}\right)^{1/T} - 1$$

Example: Consider a 4-year maturity ZCB with face value N =€ 100 and current price P=€ 88.85. Then

$$r(4) = \left(\frac{100}{88.85}\right)^{\frac{1}{4}} - 1 = 3\%$$

Yield to maturity of ZCB: more examples

Bond name	Maturity	Face value	Coupon	Price
ZCB1	1 year	€100	0	€98.04
ZCB2	2 year	€100	0	€96.11
ZCB3	3 year	€100	0	€88.90
ZCB4	4 year	€100	0	€88.85

Then

$$r(1) = \left(\frac{100}{98.04}\right) - 1 = 2\%$$

$$r(2) = \left(\frac{100}{96.11}\right)^{\frac{1}{2}} - 1 = 2\%$$

$$r(3) = \left(\frac{100}{88.90}\right)^{\frac{1}{3}} - 1 = 4\%$$

$$r(4) = \left(\frac{100}{88.85}\right)^{\frac{1}{4}} - 1 = 3\%$$

Yield to maturity of zero coupon bonds: few remarks

Remarks:

- $r(T)$ is also called the T-year interest rate or T-year spot rate or T-year zero-coupon rate.
- $r(T)$ is an annual effective rate even if T is not one year.
- **Interpretation:** If I buy for price P a ZCB of maturity T year and keep it until maturity, it is as if I was investing for T years an amount P in a bank account at an effective annual rate $r(T)$.
- The price of a T-year maturity ZCB with face value N is equal to

$$P = \frac{N}{(1 + r(T))^T}$$

Applying the low of one price to determine the price of a bond

Bond name	Maturity	Face value	Coupon	frequency	Price
ZCB1	1 year	€100	0	-	€98.04
ZCB2	2 year	€100	0	-	€96.11
ZCB3	3 year	€100	0	-	€88.90
ZCB4	4 year	€100	0	-	€88.85
Bond A	4 year	€1000	100	1 year	**??**

Q1 What are bond A's cashflows?

Q2 What is the composition of a portfolio that replicates Bond A's cashflows?

Q3 What is the no arbitrage price of Bond A?

Applying the low of one price to determine the price of a bond

Bond name	Maturity	Face value	Coupon	frequency	Price
ZCB1	1 year	€100	0	-	€98.04
ZCB2	2 year	€100	0	-	€96.11
ZCB3	3 year	€100	0	-	€88.90
ZCB4	4 year	€100	0	-	€88.85
Bond A	4 year	€1000	100	1 year	**??**

Q1 What are bond A's cashflows?

Time	Bond A
Year 1	€100
Year 2	€100
Year 3	€100
Year 4	€1100

Applying the low of one price to determine the price of a bond

Bond name	Maturity	Face value	Coupon	frequency	Price
ZCB1	1 year	€100	0	-	€98.04
ZCB2	2 year	€100	0	-	€96.11
ZCB3	3 year	€100	0	-	€88.90
ZCB4	4 year	€100	0	-	€88.85
Bond A	4 year	€1000	100	1 year	**??**

Q2 What is the composition of portfolio that replicates Bond A's cashflows?

Applying the low of one price to determine the price of a bond

Bond name	Maturity	Face value	Coupon	frequency	Price
ZCB1	1 year	€100	0	-	€98.04
ZCB2	2 year	€100	0	-	€96.11
ZCB3	3 year	€100	0	-	€88.90
ZCB4	4 year	€100	0	-	€88.85
Bond A	4 year	€1000	100	1 year	**??**

Q2 What is the composition of portfolio that replicates Bond A's cashflows?

Time	Bond A	ZCB1	ZCB2	ZCB3	ZCB4
Year 1	€100	€100	0	0	0
Year 2	€100	0	€100	0	0
Year 3	€100	0	0	€100	0
Year 4	€1100	0	0	0	€100

Applying the low of one price to determine the price of a bond

Bond name	Maturity	Face value	Coupon	frequency	Price
ZCB1	1 year	€100	0	-	€98.04
ZCB2	2 year	€100	0	-	€96.11
ZCB3	3 year	€100	0	-	€88.90
ZCB4	4 year	€100	0	-	€88.85
Bond A	4 year	€1000	100	1 year	**??**

Q2 What is the composition of portfolio that replicates Bond A's cashflows?

Time	Bond A	ZCB1	ZCB2	ZCB3	ZCB4
Year 1	€100	€100	0	0	0
Year 2	€100	0	€100	0	0
Year 3	€100	0	0	€100	0
Year 4	€1100	0	0	0	€100

Bond A replicating portfolio :

	ZCB1	ZCB2	ZCB3	ZCB4
Bond A =	1	1	1	11

Applying the low of one price to determine the price of a bond

Bond name	Maturity	Face value	Coupon	frequency	Price
ZCB1	1 year	€100	0	-	€98.04
ZCB2	2 year	€100	0	-	€96.11
ZCB3	3 year	€100	0	-	€88.90
ZCB4	4 year	€100	0	-	€88.85
Bond A	4 year	€1000	100	1 year	**??**

Q2 What is the composition of portfolio that replicates Bond A's cashflows?

Time	Bond A	ZCB1	ZCB2	ZCB3	ZCB4
Year 1	€100 =	€100 $\times y_{ZCB1}$			
Year 2	€100 =		€100 $\times y_{ZCB2}$		
Year 3	€100 =			€100 $\times y_{ZCB3}$	
Year 4	€1100=				€100 $\times y_{ZCB4}$

Bond A replicating portfolio :

	ZCB1	ZCB2	ZCB3	ZCB4
Bond A =	1	1	1	11

Applying the low of one price to determine the price of a bond

Bond name	Maturity	Face value	Coupon	frequency	Price	yiled
ZCB1	1 year	€100	0	-	€98.04	2%
ZCB2	2 year	€100	0	-	€96.11	2%
ZCB3	3 year	€100	0	-	€88.90	4%
ZCB4	4 year	€100	0	-	€88.85	3%
Bond A	4 year	€1000	100	1 year	**??**	

Q3 What is the no arbitrage price of Bond A?

Applying the low of one price to determine the price of a bond

Bond name	Maturity	Face value	Coupon	frequency	Price	yiled
ZCB1	1 year	€100	0	-	€98.04	2%
ZCB2	2 year	€100	0	-	€96.11	2%
ZCB3	3 year	€100	0	-	€88.90	4%
ZCB4	4 year	€100	0	-	€88.85	3%
Bond A	4 year	€1000	100	1 year	??	

Q3 What is the no arbitrage price of Bond A?

Bond A replicating portfolio =

ZCB 1	ZCB 2	ZCB 3	ZCB 4
1	1	1	11

$$P_A = 1 \times 98.04 + 1 \times 96.11 + 1 \times 88.90 + 11 \times 88.85 = 1260.40$$

Low of one price and zero rates

Bond name	Maturity	Face value	Coupon	frequency	Price	Yield
ZCB1	1 year	€100	0	-	€98.04	2%
ZCB2	2 year	€100	0	-	€96.11	2%
ZCB3	3 year	€100	0	-	€88.90	4%
ZCB4	4 year	€100	0	-	€88.85	3%
Bond A	4 year	€1000	100	1 year	€1260.40	

Bond A's cashflows=

year 1	year 2	year 3	year 4
€100	€100	€100	€1100

$$P_A = 1 \times 98.04 + 1 \times 96.11 + 1 \times 88.90 + 11 \times 88.85 = 1260.40$$

Recalling that for a ZCB $P = \frac{N}{(1+r(T))^T}$, then

$$P_A = 1 \times \frac{100}{(1.02)^1} + 1 \times \frac{100}{(1.02)^2} + 1 \times \frac{100}{(1.04)^3} + 11 \times \frac{100}{(1.03)^4} = 1260.40$$

Low of one price and zero rates

Bond name	Maturity	Face value	Coupon	frequency	Price	Yield
ZCB1	1 year	€100	0	-	€98.04	2%
ZCB2	2 year	€100	0	-	€96.11	2%
ZCB3	3 year	€100	0	-	€88.90	4%
ZCB4	4 year	€100	0	-	€88.85	3%
Bond A	4 year	€1000	100	1 year	€1260.40	

Bond A's cashflows=

year 1	year 2	year 3	year 4
€100	€100	€100	€1100

$$P_A = 1 \times \frac{100}{(1.02)^1} + 1 \times \frac{100}{(1.02)^2} + 1 \times \frac{100}{(1.04)^3} + 11 \times \frac{100}{(1.03)^4} =$$

Low of one price and zero rates

Bond name	Maturity	Face value	Coupon	frequency	Price	Yield
ZCB1	1 year	€100	0	-	€98.04	2%
ZCB2	2 year	€100	0	-	€96.11	2%
ZCB3	3 year	€100	0	-	€88.90	4%
ZCB4	4 year	€100	0	-	€88.85	3%
Bond A	4 year	€1000	100	1 year	€1260.40	

Bond A's cashflows=

year 1	year 2	year 3	year 4
€100	€100	€100	€1100

$$P_A = 1 \times \frac{100}{(1.02)^1} + 1 \times \frac{100}{(1.02)^2} + 1 \times \frac{100}{(1.04)^3} + 11 \times \frac{100}{(1.03)^4} =$$

$$= \frac{100}{(1.02)^1} + \frac{100}{(1.02)^2} + \frac{100}{(1.04)^3} + \frac{1100}{(1.03)^4} = 1260.40$$

Low of one price and zero rates

Bond name	Maturity	Face value	Coupon	frequency	Price	Yield
ZCB1	1 year	€100	0	-	€98.04	2%
ZCB2	2 year	€100	0	-	€96.11	2%
ZCB3	3 year	€100	0	-	€88.90	4%
ZCB4	4 year	€100	0	-	€88.85	3%
Bond A	4 year	€1000	100	1 year	€1260.40	

Bond A's cashflows=

year 1	year 2	year 3	year 4
€100	€100	€100	€1100

$$P_A = 1 \times \frac{100}{(1.02)^1} + 1 \times \frac{100}{(1.02)^2} + 1 \times \frac{100}{(1.04)^3} + 11 \times \frac{100}{(1.03)^4} =$$

$$= \frac{100}{(1.02)^1} + \frac{100}{(1.02)^2} + \frac{100}{(1.04)^3} + \frac{1100}{(1.03)^4} = 1260.40$$

Low of one price and zero-rates

Theorem

- *Consider a bond paying cashflows at dates* $t_1, t_2, \ldots t_T$.
- *Let* $C(t_i)$ *be the cashflow that the bond pays at date* t_i, $i = 1, 2, \ldots T$.
- *Let* $r(t_i)$ *be the* t_i *year zero-coupon rate.*

Then the no-arbitrage price of the bond is

$$P = \frac{C(t_1)}{(1 + r(t_1))^{t_1}} + \frac{C(t_2)}{(1 + r(t_2))^{t_2}} + \cdots + \frac{C(T)}{(1 + r(T))^{T}}$$

Term structure of interest rate (or yield curve)

Definition

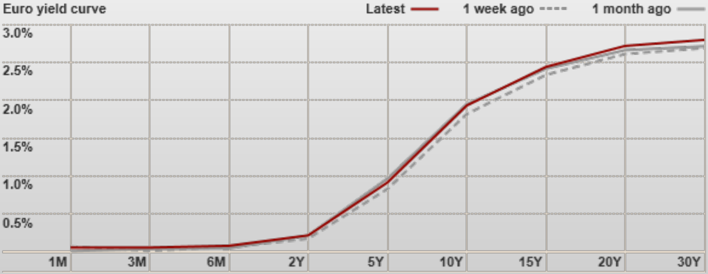
The term structure of interest rate (or yield curve) is the relation between the maturity of zero coupon bonds and their yield to maturity:

$$\{\ldots r(0.5), \ldots r(1), r(2), \ldots r(t), \ldots\}$$

Definition

The term structure of interest rate (or yield curve) is said to be flat if $r(t)$ does not vary with t.

Term structure of interest rate: real examples



October 2013

FT

Using the yield curve to price bonds

ZCB	Maturity	Face value	Price	$r(t)$
ZCB1	1 year	€100	€98.04	2%
ZCB2	2 year	€100	€96.11	2%
ZCB3	3 year	€100	€88.90	4%
ZCB4	4 year	€100	€88.85	3%

What are the prices of the following bonds?

Bond name	Maturity	Face value	Coupon	frequency	Price
Bond 1	1 year	€200	0	-	??
Bond 2	2 year	€1000	50	1 year	??
Bond 3	3 year	€50	1	1 year	??
Bond 4	4 year	€100	10	2 year	??

Using the yield curve to price bonds

ZCB	Maturity	Face value	Price	$r(t)$
ZCB1	1 year	€100	€98.04	2%
ZCB2	2 year	€100	€96.11	2%
ZCB3	3 year	€100	€88.90	4%
ZCB4	4 year	€100	€88.85	3%

What are the prices of the following bonds?

Bond name	Maturity	Face value	Coupon	frequency	Price
Bond 1	1 year	€200	0	-	€196.08
Bond 2	2 year	€1000	50	1 year	€1058.25
Bond 3	3 year	€50	1	1 year	€47.28
Bond 4	4 year	€100	10	2 year	€107.35

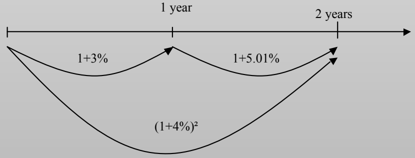
Forward rates

Definition

A forward interest rate is the rate implied by the current term structure over a given period in the future.

Example: The one-year interest rate is $r(1Y) = 3\%$ and the 2-year interest rate is $r(2Y) = 4\%$. What is the annual interest rate between year 1 and 2?

$$(1 + 3\%)(1 + r(1Y, 2Y)) = (1 + 4\%)^2 \Rightarrow r(1Y, 2Y) = 5.01\%$$



Forward rates

Theorem

Given any two dates $t_1 < t_2$ and a term structure, the forward rate from t_1 to t_2 is equal to

$$r(t_1, t_2) = \left(\frac{(1 + r(t_2))^{t_2}}{(1 + r(t_1))^{t_1}}\right)^{\frac{1}{t_2 - t_1}} - 1$$

Proof: Note that the forward rate $r(t_1, t_2)$ must satisfy

$$(1 + r(t_2))^{t_2} = (1 + r(t_1))^{t_1}(1 + r(t_1, t_2))^{t_2 - t_1}$$

Forward rates: example

r(1Y) =	2%
r(2Y) =	2%
r(3Y) =	4%
r(4Y) =	3%

Then

$$r(1Y,2Y) = \frac{1.02^2}{1.02} - 1 = 2\%$$
$$r(2Y,3Y) = \frac{1.04^3}{1.02^2} - 1 = 8\%$$
$$r(3Y,4Y) = \frac{1.03^4}{1.04^3} - 1 = 0.06\%$$

What is $r(2Y, 4Y)$? 4.01%

Any investor can lock in the forward rate implied by a term structure.

Example

The one-year interest rate is $r(1Y) = 3\%$ and the 2-year interest rate is $r(2Y) = 4\%$. Investor can both borrow and lend at these rates. The forward rate from year 1 to year 2 is
$r(1Y, 2Y) = 1.04^2/1.03 - 1 = 5.01\%$.

Trade	today	Year 1	year 2
Borrow Eu 100 for 1 year	+100	-103	0
Invest Eu 100 for 2 years=	-100	0	$100 * 1.04^2$
	0	**-103**	**108.16**

Note that $108.16 = 103 * (1 + r(1y, 2y)) = 103 * 1.0501$

Forward rates: interpretations

- If agents are risk neutral the forward rate from years t to $t+n$ corresponds to what investors expect to be in t years the n-year interest rate.
- The t-year interest rate can be interpreted as the composition of the first $t-1$ years forward rates:

$$(1+r(t))^t = \sqcap_{i=0}^{t-1}(1+r(iY,(i+1)Y))$$

- If agents are risk neutral then a decreasing (increasing),term structure indicates agents expects short term interest rate to decrease (resp. increase).

The yield curve and the business cycle

What does the shape of the term structure imply?

- Upward sloping: normal times

  This is how commercial banks make money: borrow short term, lend long term

- Steeply upward sloping: usually forecast economic expansion

  Why?

- Flat or inverted: rare but a sign of trouble

  2006 yield curves

Risks of a bond portfolio

1. Interest rate risk
   - Fluctuations of market price of bond due to changes in yield curve
   - Can be hedged with portfolio immunization (see later) or interest rate derivatives (see forward rates)
   - I care about it if I want to resell the bond before maturity
2. Default risk
   - Risk that whatever is owed by the issuer is not paid
   - Gives rise to default premium
   - Can be hedged with credit derivatives (not covered in the course but see problem 3 in problem set on bonds)
   - I care about it if a plan to hold the bond until matuirty

Interest rate risk

Definition

Interest rate risk is the risk that the bond price changes due to fluctuations in market interest rates (even if there is no risk of default)

Exmple: You hold a bond that matures in 2 years, makes annual coupon payments, whose yield curve is currently flat at 5% and is trading at par $P_0 = N = \$1000$

- Q0 What are the cash-flows that this bond will pay?
- Q1 What is the YTM and coupon of the bond?
- Q2 What is your holding period return over the course of next year if interest rates do not change?
- Q3 What if interest rates increase by 1 percentage point (and the yield curve remains flat) between today and next year?
- Q4 What if they decrease by 1 percentage point?

Duration

- Definition: The duration (D) of a bond is the sensitivity (or elasticity) of its price to interest rates changes:

$$\boxed{\frac{\triangle P_0}{P_0} \simeq -D\frac{\triangle y}{1+y}}$$

  - If the yield changes by $\triangle y$, the % change in the bond price is approximately $-D\frac{\triangle y}{1+y}$
  - The definition has a $\simeq$ because it is an approximation valid for small $\triangle y$
  - The larger a bond's duration, the "less" I know about the resale price of the bond if there is a change in the interest rate.

Duration

- How to calculate the duration?
  - Formula: D = weighted-average maturity of cash flows

$$D = \frac{1 \times \frac{C}{1+y} + 2 \times \frac{C}{(1+y)^2} + \ldots + T \times \frac{C+N}{(1+y)^T}}{P_0}$$

    (detailed calculation on the next slide)

  - Intuition: longer duration when cash flows further away in the future
  - D is in years
  - The duration of a portfolio of bonds is the weighted average of the durations of the bonds in the portfolio

Proof of the duration formula

The definition of duration is

$$D = -\frac{\frac{dP_0}{P_0}}{\frac{d(1+y)}{1+y}} = -\frac{dP_0}{dy} \times \frac{1+y}{P_0} \qquad (1)$$

The bond price is equal

$$P_0 = \frac{C}{1+y} + \frac{C}{(1+y)^2} + \ldots + \frac{(C+N)}{(1+y)^T}$$

which we differentiate with respect to y

$$\frac{dP_0}{dy} = -\frac{C}{(1+y)^2} - \frac{2C}{(1+y)^3} - \ldots - \frac{T(C+N)}{(1+y)^{T+1}} \qquad (2)$$

Replacing (2) into (1), we obtain

$$D = \frac{1 \times \frac{C}{1+y} + 2 \times \frac{C}{(1+y)^2} + \ldots + T \times \frac{C+N}{(1+y)^T}}{P_0}$$

Duration: Example

	Coupon	Face value	Maturity	Frequency	P
Bond 1	Eu 2	Eu 100	2 Y	annual	Eu 98.09
Bond 2	Eu 2	Eu 100	3 Y	annual	Eu 97.17

$r(1Y) = r(2Y) = r(3Y) = 3\%$

For Bond 1:

$$P_1 = \frac{2}{1.03} + \frac{102}{1.03^2} = 1.94 + 96.15 = 98.09$$

$$\omega_1 = \frac{1.94}{98.09} = 0.0198 \quad \text{and} \quad \omega_2 = \frac{96.15}{98.09} = 0.9802$$

$$D_1 = 1 * 0.0198 + 2 * 0.9802 = 1.98$$

For Bond 2:

$$D_2 = 1\frac{2}{1.03}\frac{1}{97.17} + 2\frac{2}{1.03^2}\frac{1}{97.17} + 3\frac{102}{1.03^3}\frac{1}{97.17} = 2.94$$

Duration: Example

Now suppose that becomes $r(1Y) = r(2Y) = r(3Y) = 4\%$.
The new prices of Bond 1 and Bond 2 are:

$$
\begin{aligned}
P'_1 &= \frac{2}{1.04} + \frac{102}{1.04^2} = 96.23 \\
P'_2 &= \frac{2}{1.04} + \frac{2}{1.04^2} + \frac{102}{1.04^3} = 94.45
\end{aligned}
$$

The percentage changes in the prices are:
For Bond 1:

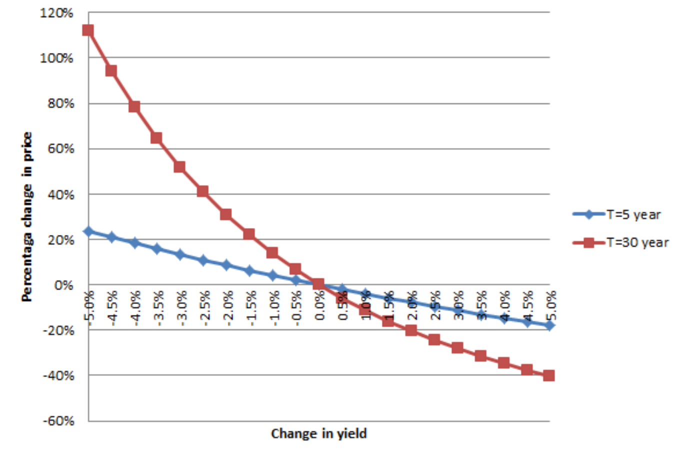
$$\frac{96.23 - 98.09}{98.09} = -1.89\%$$

For Bond 2:

$$\frac{94.45 - 97.17}{97.17} = -2.8\%$$

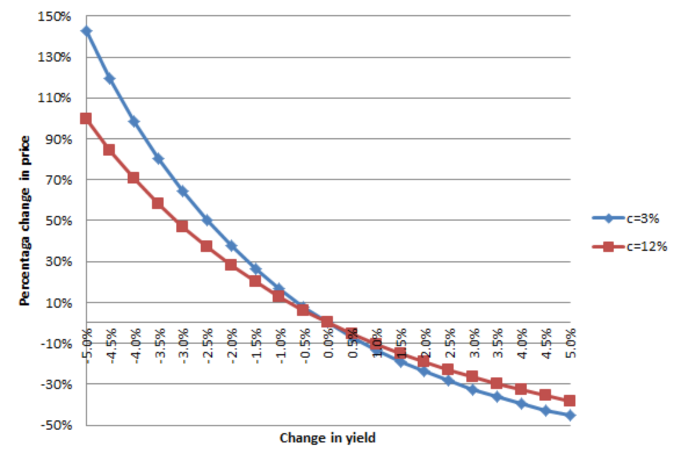
Factors influencing interest rate risk

- The longer the time to maturity (T) of a bond, the *more* its value will be affected by a change in interest rates



Factors influencing interest rate risk

- The larger the coupon (C) of a bond, the *less* its value will be affected by a change in interest rates



Exercise 1

Q 1 What is the duration of a 3-year zero-coupon bond?

Exercise 1

Q 1 What is the duration of a 3-year zero-coupon bond? D= 3 years

Q 2 Calculate the price and the duration of an annual coupon-paying bond with 3 years to maturity, par of €1,000, yield to maturity of 5% and a coupon rate of 10%

Exercise 1

Q 1 What is the duration of a 3-year zero-coupon bond? D= 3 years

Q 2 Calculate the price and the duration of an annual coupon-paying bond with 3 years to maturity, par of €1,000, yield to maturity of 5% and a coupon rate of 10%
$P_0 = 1136.16$, $D = 2.75$ years

Q 3 If the yield curve shifts upward by 1 percentage point, what will be the % change in the bond's price?

Exercise 1

Q 1 What is the duration of a 3-year zero-coupon bond? D= 3 years

Q 2 Calculate the price and the duration of an annual coupon-paying bond with 3 years to maturity, par of €1,000, yield to maturity of 5% and a coupon rate of 10%
P_0 = 1136.16, D = 2.75 years

Q 3 If the yield curve shifts upward by 1 percentage point, what will be the % change in the bond's price?
-2.57%, Approx. -2.62%

Q 4 If interest rates decrease by 50 basis points, what will be the €-change in the bond's price?

Exercise 1

Q 1 What is the duration of a 3-year zero-coupon bond? D= 3 years

Q 2 Calculate the price and the duration of an annual coupon-paying bond with 3 years to maturity, par of €1,000, yield to maturity of 5% and a coupon rate of 10%
$P_0 = 1136.16$, $D = 2.75$ years

Q 3 If the yield curve shifts upward by 1 percentage point, what will be the % change in the bond's price?
-2.57%, Approx. -2.62%

Q 4 If interest rates decrease by 50 basis points, what will be the €-change in the bond's price? €15.03, Approx. €15.88

Exercise 2

Q 1 Rank the following bonds in order of descending duration

Bond	Coupon rate (%)	Time to maturity (years)	Yield to maturity (%)
A	15	20	10
B	15	15	10
C	8	20	10
D	0	20	10

Exercise 2

Q 1 Rank the following bonds in order of descending duration

Bond	Coupon rate (%)	Time to maturity (years)	Yield to maturity (%)
A	15	20	10
B	15	15	10
C	8	20	10
D	0	20	10

$$D(D) > D(C) > D(A) > D(B)$$

Q 2 Consider a long-short portfolio composed of a €1 million long position in Bond A and €1 million short position in Bond C (thus, the initial value of the portfolio is €0). How is the value of the portfolio affected by a rise in interest rates?

- Consider a portfolio of n different bonds.
- Let x_i be the **weight** of bond i in the portfolio, i.e., $\sum_i^n x_i = 1$

Then, the duration of the portfolio is

$$D_p = x_1 D_1 + x_2 D_2 + \cdots + D_n x_n$$

where D_i is the duration of bond i.

Bond A is a ZCB with maturity 1 year and Bond B is a ZCB with maturity 30 years.
If you invest $x_A > 0$ of your wealth in Bond A, what should be the weight of Bond B so that your portfolio has zero duration? (assume your portfolio has no other bonds)

Example of portfolio immunization

Bond A is a ZCB with maturity 1 year and Bond B is a ZCB with maturity 30 years.
If you invest $x_A > 0$ of your wealth in Bond A, what should be the weight of Bond B so that your portfolio has zero duration? (assume your portfolio has no other bonds)

$$D_p = x_A \times 1 + x_B \times 30 + (1 - X_A - X_B) \times 0 = 0$$

$$\Rightarrow$$

$$x_B = -X_A \frac{1}{30}$$

Default risk

- Default risk (also called credit risk) is the risk that the issuer defaults on its obligation
  - Corporates and sovereigns are subject to default risk (General Motors 2009, Greece 2011, Banca Monte dei Paschi 2017, etc.)
- Credit ratings by rating agencies evaluate the likelihood of default

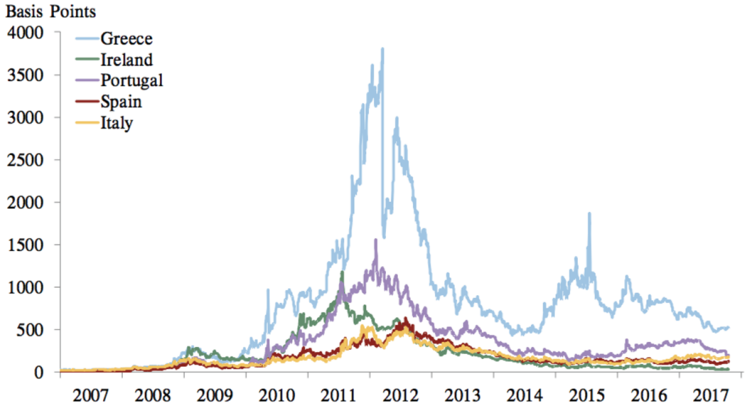
	S&P	Moody's	Fitch
Investment grade	AAA	Aaa	AAA
	AA	Aa	AA
	A	A	A
	BBB	Baa	BBB
High-yield	BB	Ba	BB
	B	B	B
	CCC	Caa	CCC
	CC	Ca	
	C		
In default	D	C	D

Default risk: spread

- Default risk is reflected in a higher bond yield through the default premium
- These days the German Treasury Bill is considered the safest bond.
- A bond spread is the difference between the bond yield to maturity and the yield to maturity of the German Treasury Bill

Default risk: sovereign bonds

- Government bonds are not necessarily default-risk-free.
  - 10-year sovereign bond spreads above risk-free rate (= German rate)



NOTE: The chart shows the spread, or difference, in interest rates between 10-year government bonds for various countries and German 10-year government bonds.

Default risk: high-yield bonds

- High-yield bonds are also called “junk bonds”

can be “original junk issues”

Altice returns to bond market with $3bn sale

Robert Smith JULY 17, 2018 8

Altice’s French unit completed a nearly $3bn high-yield debt sale on Tuesday, raising junk bonds for the first time since concerns around the cable group’s debt pile spooked investors at the end of last year.

The group raised $1.75bn of dollar bonds at 8.125 per cent yield and €1bn of euro bonds at 5.875 per cent yield, the highest yields its French unit has been charged in both markets since it was created out of the merger of Numericable and SFR in 2014.

Source

or “fallen angels”

Bond investors wary of threat from potential ‘fallen angels’

Analysts expect more companies to complete slide from investment grade to junk

Eric Platt in New York JULY 25, 2016 1

Investors in the most highly rated US corporate bonds have enjoyed a buoyant 2016, with the Barclays index returning nearly 9 per cent. However, the market now faces an unsettling threat from a potential new wave of fallen angels — companies that first sold debt with investment grade status but have since been downgraded to junk.

The list of companies now on the brink of junk includes watchmaker Fossil Group, which suffered a 9 per cent drop in sales in its first quarter, and internet security company Symantec after it agreed to purchase Blue Coat for $4.65bn with $2.8bn of new debt. They join multinationals such as Rémy Cointreau, LG Electronics and miner Goldcorp sitting on the edge of speculative rating territory.

Source

Default risk: examples

Compare the 10-year interest rates of:

GE-IT: What can you conclude?

GE-IT: The *realized* return on the Italian bond will be higher than that on the German bond: true or false?

The *expected* return on the Italian bond is higher than that on the German bond: true or false?

FR-US: What can you conclude?

Consider the following Zero-coupon bonds

	Rating	N	T	yield to maturity	Price
Bond 1	AAA	Eu 100	1 Y	5.2%	Eu 95
Bond 2	BB	Eu 100	1 Y	11.11%	Eu 90
Bond 3	BB	Eu 100	2 Y	12%	Eu 79.71
Bond 4	AAA	Eu 100	2 Y	5.5%	Eu 89.86

Remark: Keeping fixed the features of a bond, the higher its default risk the lower its market price, the higher its yield to maturity.

What are the term structure for AAA bonds and for BB bonds?

Maturity	term structure for AAA	term structure for BB
1 Year	5.2%	11.11%
2 Year	5.5%	12%

Maturity	term structure for AAA	term structure for BB
1 Year	5.2%	11.11%
2 Year	5.5%	12%

What is the no-arbitrage price of a bond with the following properties?
$C = Eu\ 2$
$N = Eu\ 100$
$T = 2\ years$
Frequency = annual
Rating: BB

$$P = \frac{2}{1.1111} + \frac{102}{1.12^2} = Eu\ 83.11$$

Default risk: arbitrage and pricing

As a general rule, the bonds I can use to build a replicating portfolio must be as similar as possible to the bond I want to price.

- Bonds denominated in the same currency
- Bonds with the same rating
- Issued by firms with the same characteristics (ex. sector of activity, size, country etc.)

	T	C	yield to maturity	Rating
Bond 4	4 year	Eu 5	5%	AAA
Bond 3	3 year	Eu 3	4%	AAA
Bond 2	2 year	Eu 0	3%	AAA
Bond 1	1 year	Eu 2	2%	AAA
Bond 1bis	1 year	Eu 2	10%	B

Suppose that coupons are paid every year and that the face value is Eu 100 for each bond.

1. What are the prices of these bonds? $P1bis = 92.72$, $P1 = 100$, $P2 = 94.26$, $P3 = 97.22$, $P4 = 100$.
2. What is the term structure of interest rate for AAA Bonds? $r(1Y) = 2\%$, $r(2Y) = 3\%$, $r(3Y) = 4.04\%$, $r(4Y) = 5.13\%$.
3. Suppose that after six months, the term structure is flat at 2%. What will be the price of Bond 3? 103.91

Conclusion

By now you should know:

- What a bond is and what cash-flows it generates.
- What the yield curve is.
- How to price a bond using the no-arbitrage method and/or the yield curve.
- Why bonds are risky and how to measure default and interest rate risks.