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FINANCIAL MARKETS

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Contents

Chapter 1

Stocks

1 Definition

A share of common stock (also referred to as stock or equity) is a financial contract that represents ownership of a specific portion of the company that issued it. Common stocks have a limited liability clause: in the case of bankruptcy, the shareholders can lose their initial investment in the company's shares, but not more. Stocks are also residual claims in the sense that if the company were to be liquidated, stockholders would have access to the cash flows arising from the sale of the company's assets only after all other claimholders (tax authorities, employees, suppliers, bondholders) are fully paid off. When the company operates as an on-going concern, shareholders have a right to the operating income of the company after operating expenses, interest on bonds and taxes have been fully paid off. In turn, stockholders are given the right to vote on decisions concerning the corporate governance of the firm during the annual shareholders' meetings (for example, they vote on the appointment of the directors of the company, who oversee the company's management). A share of common stock also entitles its owner to cash flows that are associated with the ownership. Although most stocks pay an annual dividend, many young firms do not pay dividends. For example, Google has not yet started to pay out dividends in 2020.

On a typical day, $60 billion worth of stocks change hands on the New York Stock Exchange. In this chapter we examine how investors evaluate the price of a common stock given the fundamentals of the company. The fundamentals refer to the future cash flows associated with a share of company. To do so, we will put to use a tool that we have acquired so far: the discounting of future cash flows. There will be slight twist, however, as the expected cash flows (dividend payments) will be growing for most of the companies that we will consider.

2 The Valuation Problem

As of October 23, 2020 the price of one share of Apple common stock was $115.04. Given that Apple had about 17.1 billion shares outstanding at the time, the market capitalization of the company was $1,967 billion. In contrast, the book value of the company was only $72 billion. Why do market participants believe that the Microsoft stock should be priced more than ten times its book value? To understand the reasons for a market-to-book ratio of 27 we have to remember that the book value is based on accounting rules that are used to allocate the historical cost of acquiring the assets of the firm over time. Such rules are inevitably arbitrary from the point of view of finance professionals. The market value, on the other hand, represents the value of the firm as an on-going economic entity. As a result, when valuing the company's share, market participants try to measure the present value of all future cash flows associated with a share of common stock.

For example, consider the stock of a particular company that is selling for a price of €50 today. The company is expected to make a dividend payment of €4 in exactly one year from today, and at that point the stock is forecasted to have a price of €55 (for now take the forecasted stock price as given). Imagine a scenario in which you buy the stock today, hold it for a year, and sell it at that point (after receiving the dividend): what would be your expected Holding Period Return (HPR) on the stock?

$$\text{Expected HPR} = E(r) = \frac{E(D_1) + (E(P_1) - P_0)}{P_0} = \frac{4 + (55 - 50)}{50} = 18\%.$$

Note that the expected return (HPR) on the stock is a combination of $E(D_1)/P_0$, the dividend yield and $(E(P_1) - P_0)/P_0$, the capital appreciation (or depreciation if $E(P_1) < P_0$). Based on this expected return, should you buy the stock today? That decision will depend on how the "expected rate of return" $E(r)$ compares with the "required rate of return" k:

$$k = r_f + \beta(E(r_m) - r_f).$$

This is a formula coming from the CAPM. Suppose that the risk free rate is $r_f = 3\%$, the systematic risk of the company in question (i.e., the beta) is 1.2, and that the market premium is $E(r_m) - r_f = 10\%$, then

$$k = 0.03 + 1.2 \times 0.10 = 15\%.$$

This suggests that, given the expected dividends and the expected price one year from now, and given the expected return on the stock, we should buy this stock: the expected rate of return is higher than the required rate of return.

We could attack the same problem from an alternative point of view: we can calculate the "fundamental value" (or "intrinsic value") of the stock given the required rate of return (k) and compare it with the actual price of the stock in the market:

$$V_0 = \frac{E(D_1) + E(P_1)}{1 + k} = \frac{4 + 55}{1 + 0.15} = 51.30.$$

Unsurprisingly, we come to the same conclusion: we should buy this stock since its fundamental value exceeds its current price in the market. This means that, given our estimates of the expected cash flows associated with the stock, our valuation (intrinsic value) differs from the consensus in the market place that is, the current stock price. In other words, the current market price is a reflection of the expectations of market participants on average. When the intrinsic value is higher (lower) than the market price, the stock is said to be undervalued (overvalued) and in this case buying (shorting) the stock is a positive NPV investment.

2.1 Dividend Discount Models

Let's go back to our intrinsic value formula for today ($t = 0$):

$$V_0 = \frac{E(D_1) + E(P_1)}{1+k}$$

and think about what would happen if we were to expand the holding period from one to two years. Since we will not be selling the stock until the end of year 2, we would have to come up with the intrinsic value of the stock at the end of year 1 and replace it in the formula for the forecasted stock price $E(P_1)$:

$$\begin{aligned} V_1 &= \frac{E(D_2) + E(P_2)}{1+k} \\ V_0 &= \frac{E(D_1) + V_1}{1+k} = \frac{E(D_1)}{1+k} + \frac{E(D_2) + E(P_2)}{(1+k)^2}. \end{aligned}$$

Thus, the intrinsic value today becomes the present value of the dividend payments at the end of years 1 and 2, plus the present value of the forecasted-price at which the stock can be sold at the end of year 2, all discounted at the required rate of return on the stock (k).

In fact, nothing stops us from obtaining the intrinsic value of the stock in question for a holding period of T years:

$$\begin{aligned} V_T &= \frac{E(D_{T+1}) + E(P_{T+1})}{1+k} \\ V_0 &= \frac{E(D_1)}{1+k} + \frac{E(D_2)}{(1+k)^2} + \cdots + \frac{E(D_{T+1}) + E(P_{T+1})}{(1+k)^{T+1}} \end{aligned}$$

You can see the similarity between this stock valuation formula and the NPV formula from the Financial Economics class. The difference is that here we are dealing with expected dividend payments and the expected sales price for the stock at the end of the holding period. Given the fact firms would be expected to be in operation forever, we would obtain the following infinite series:

$$V_0 = \frac{E(D_1)}{1+k} + \frac{E(D_2)}{(1+k)^2} + \cdots + \frac{E(D_t)}{(1+k)^t} + \cdots . \tag{1}$$

The formula states that the intrinsic value of the firm is equal to the present value of a perpetual series of expected dividends. This formula is called the **Dividend Discount Model (DDM)**. It should be noted that even though the formula only focuses on dividends, the capital gains are also implicitly in the formula, since any future price at which the stock can be sold is the present value of future dividends from that point onward.

The difficulty with the above formulation of DDM is that we have to come up with the forecasts of an infinite series of expected dividends. That exercise becomes significantly easier if we assume that dividends are expected to grow at a constant rate g over time:

$$\begin{aligned} E[D_1] &= D_0(1+g) \\ E[D_2] &= D_0(1+g)^2 \\ &\cdots \\ E[D_t] &= D_0(1+g)^t \end{aligned}$$

Under this assumption, the DDM formula becomes:

$$V_0 = \frac{D_0(1+g)}{1+k} + \frac{D_0(1+g)^2}{(1+k)^2} + \cdots + \frac{D_0(1+g)^t}{(1+k)^t} + \cdots = \sum_{t=1}^{\infty} \frac{D_0(1+g)^t}{(1+k)^t}.$$

The formula remains an infinite series, but this perpetual series has a limit if $k > g$:

$$V_0 = \frac{D_0(1+g)}{k-g} = \frac{D_1}{k-g}. \tag{2}$$

This model of stock valuation is called the **Constant Growth Dividend Discount Model** or the Gordon Model. This formula is one of the most commonly used stock valuation tools. It should be noted that: (i) the formula is only valid if $k > g$; (ii) if $k = g$ or if $k < g$ the formula explodes as the series has no longer a finite limit; (iii) if $g = 0$, that is if dividends remain the same over the course of time, the formula becomes one for valuing a "perpetuity".

Example 1. *What is the price of a preferred stock that promises to pay a dividend of €2 each year forever? Assume that the required rate of return for this company's preferred stock is 10%.*

Example 2. *Show that the intrinsic value of a stock, V_0, as given by the constant growth DDM formula, is (i) increasing in the expected dividend per share; (ii) increasing in the growth rate of dividends; and (iii) decreasing in the expected rate of return.*

Example 3. *Show that under the constant growth DDM, the expected growth rate of dividends g is also the expected growth rate of the stock price (that is, show that $E[P_1] = P_0(1+g)$).*

If the constant growth DDM model implies that the growth rate of dividends g is the rate at which the stock price will grow, then g becomes the capital gains rate:

$$E(r) = \text{dividend yield} + \text{capital gains yield} = \frac{D_1}{P_0} + \frac{E[P_1] - P_0}{P_0} = \frac{D_1}{P_0} + g.$$

If the stock is selling at its intrinsic value (that is, if the stock is neither over- nor under-valued), then we have another way of obtaining the required rate of return on the stock:

$$k = E(r) = \frac{D_1}{P_0} + g.$$

In other words, if the stock is fairly priced (i.e., $P_0 = V_0$), we can obtain the required rate of return on the stock by observing the dividend yield and estimating the growth rate of dividends. In fact, this formula can be more useful than one could give it credit for. Remember that, when our valuation problem involved only one period (buying the stock at $t = 0$, receiving its dividend and selling it at $t = 1$) we used the CAPM to obtain the required rate of return. This is in the spirit of the CAPM since the model is a one period one (remember the CAPM assumptions from your Financial Economics class). When we have a multi-period valuation problem, from a theoretical perspective, the CAPM needs not give us the best required rate of return unless we assume that the same rate of return will prevail also in the future. This is a very strong assumption. Yet, many in the industry are willing to make such a strong leap of faith. One alternative is to use the formula above that equates the required rate of return to the dividend yield plus the capital gains yield. As you can see, however, in a multi-period valuation problem, this approach also presumes that current dividend and capital gains observed this year will remain constant over time.

The previous paragraph highlights the difficulty of evaluating intrinsic values for stocks: using the CAPM or another method for obtaining required rates of return involves strong assumptions that may not be met in practice. This is the reason why many stock analysts conduct "sensitivity analyses".

These would involve examining how sensitive the intrinsic value obtained from the Gordon model is to small changes in the parameters. If the price estimates remain "robust" then we can have confidence in the results. If, on the other hand, small changes, for example in the required rate of return or the growth rate of dividends, lead to very large changes in the intrinsic value that we calculate, then we should be skeptical of the results. This is one reason why stock analysts in the industry use a number of valuation methods, including for example the "multiples" approach which are beyond the scope of this course and will be covered in upper level classes.

Example 4. *DigiPro, a firm producing digital projection equipment is expected to pay a dividend of €1 in exactly one year from now, has a dividend growth rate of 3% and a required rate of return of 10%. Suppose that DigiPro lands a major deal in selling digital projection equipment to European movie theaters. After such a large deal, the management thinks that it can increase the growth rate of dividends from 3% to 5% in the future without reducing next year's dividend payment and without changing the riskiness of the stock. How will the company's stock price be affected? How will the expected rate of return on the company's stock be affected?*

2.2 Intrinsic Value, Stock Price and Investment Opportunities

Let's take the case of two companies, CashUsIn (C) and Growthorama (G), that are identical in terms of risk ($k^C = k^G = 10\%$) and earnings per share over the course of next year ($E[D_1^C] = E[D_1^G] = €4$). If these two companies were to distribute all of their earnings as perpetual dividends to their shareholders, then the value of these companies would be:

$$V_0^C = \frac{D_1^C}{k^C} = V_0^G = \frac{D_1^G}{k^G} = \frac{4}{0.1} = €40.$$

Note that, if these two companies distribute all of their earnings, they will have no opportunity to grow, since they are not retaining any of their earnings to reinvest in existing or new projects. Hence, their dividends would not grow and remain constant over time. Note also that the earnings that we refer to are earnings after setting aside funds necessary to cover the economic depreciation of assets in place. In other words, the earnings per share that we deal with are earnings that are over and above the amount that needs to be set aside to maintain the assets in place in working order perpetually. As such, we are dealing with "economic" earnings per share (E) rather than "accounting" earnings per share reported in the company's Income Statement (remember from Financial Economics that financial statements represent accountants' view of the world given a set of inevitably arbitrary rules and not the financial economists', who tend to think in terms of market values).

Let's suppose that Growthorama started its operations with a capital of 200 million euros all equity-financed and had issued 5 million shares of common stock. Company's total earnings E will be total equity time the Return on Equity (ROE):

$$€200 \text{ million} \times \text{ROE} = 200 \text{ million} \times 0.1 = €20 \text{ million per year.}$$

Since there are 5 million shares outstanding, the Earnings Per Share (EPS) is €20/5 = €4 per share, the figure we had above.

Suppose now that in fact Growthorama can earn 14% on new projects it plans to undertake. If Growthorama's management decides to retain (or "plow back") some of the 4 euros of earnings per share in order to invest in the new project the company's, then its shareholders will be better off. In order to understand this point, let's assume that Growthorama reduces its dividends from 4 euros per share to 2 euro per share, plowing back 2 euros in the new investment. Now, the firm invests 50% of its total earnings in the new project: €20 million × 0.50 = €10 million, in other words total earnings times the plowback ratio. Thus the total earning will be 0.1 × 200 million from the old project plus 0.14 × 10 million from the new project, that is to say €21.4 million. If only 50% of the earnings are distributed to shareholders, then the new EPS will be €10.7/5 = 2.14. As a result, the growth rate of firm's dividends will be 7%:

$$g^G = \text{ROE}^{NewProject} \times b = 14\% \times 50\% = 7\%,$$

where $\text{ROE}^{NewProject}$ is the rate of return on the new project.

If this growth rate is sustainable in the long run, the new intrinsic value under the 50% plowback policy:

$$V_0^G = \frac{D_1^G}{k^G - g^G} = \frac{2}{0.1 - 0.07} = €66.67.$$

This means that, if the stock price in the market place is equal to the intrinsic value of the firm, Growthorama's stock price should increase by 67% when the firm invests 50% of its earnings in the new project that earns 14% return on capital while paying 2 euros per share as dividends to its shareholders.

In fact, one could think of the share price of any company as having two components:

$$P_0 = \text{No-Growth Value Per Share} + \text{Present Value of Growth Opportunities}.$$

Or equivalently:

$$P_0 = \frac{E_1}{k} + PVGO. \tag{3}$$

In our example: 66.67 = 40 + 26.67.

Of course, companies would benefit from plowing back part of their earnings as long as they have new projects with higher returns than the existing projects in place, that is as long as new project's ROE $> k$. To see this, let's assume that CashUsIn also plows back 50% of its earnings in its existing projects (which earn the same old 10% return on equity). Following the same reasoning as above, growth rate of dividends in this case will be: ROE $\times\, b$ = 10% × 0.50 = 5% and the intrinsic value under this plow-back policy will be the same as before:

$$V_0^C = \frac{D_1^C}{k^C - g^C} = \frac{2}{0.1 - 0.05} = €40.$$

In other words, the new intrinsic value will be the same as before since the firm made additional investment in assets in place when it had no growth opportunities, that is when its PVGO = 0.

This means that, given its market required rate of return of $k^G = 10\%$, when Growthorama does not retain (or plow back) any of its earnings, its shareholders must find alternative ways to invest the dividends that they receive at 10%, the required rate of return associated with the stock that generates the dividends.

2.3 Company Life Cycles and Multistage Growth Models

Many young firms do not pay any dividends during their initial rapid growth phase. When trying to value companies that grow rapidly the constant dividend growth model would not be the appropriate model. In such cases, one needs to adopt a more flexible valuation model that takes into account the "life-cycles" that the firm goes through. Consider a biomedical firm that finds the cure for a certain disease. In the early years it has sizeable growth opportunities and low payout ratios and plows back most of its earnings to make the most of its growth opportunities. In a second stage, the maturing firm increases its production capacity, but its patent expires, which means competitors enter the market. New versions of the old drug may be found, but these new growth opportunities will not be long-lived as the competitors will already have had access to the old patent and can also develop new versions. In a more mature stage the growth opportunities will decline further and the firm will increase (decrease) its payout (plow back) ratio. One way to attack such a valuation problem is to use different growth rates during different stages of the company's life cycle. Such models are called, multi-stage dividend growth models.

Example 5. *The technology stock FatTrix has just announced that it will pay its first dividend of 1 euro today. The company is expected to grow at a rate of 20% over the course of the next five years. Then the company is expected to settle-down to a growth rate of 10% starting with the sixth year as its patent will expire and competition will enter its market. The required rate of return the company's stock, which is 15%, is not expected to change over the life of the company. What must be the company's stock price today?*

3 Price-Earnings Ratios

You will sooner than later hear about the "price-earnings" ratio in the financial press or news. How does the P/E ratio fit into our analysis thus far? We can modify the valuation formula (3) above to get some of the insight:

$$\frac{P_0}{E_1} = \frac{1}{k}\left(1 + \frac{\text{PVGO}}{E_1/k}\right).$$

When the present value of growth opportunities is zero (PVGO $= 0$) the P/E ratio is equal to $1/k$, the inverse of the required rate of return. The latter part of the formula is the ratio of the PVGO to the earnings associated with the assets in place (PVGO$/(E_1/k)$). Thus the intuition behind the P/E ratio is that higher price-earnings multiples are indicative of higher growth opportunities.

Another way to get further insight is to go back to our constant-growth dividend discount model. After a few modifications of the formula (1) we obtain:

$$P_0 = E_1 \times \frac{1-b}{k - b \times \text{ROE}^{NewProjects}}.$$

Thus, another formulation for the price-earnings ratio:

$$\frac{P_0}{E_1} = \frac{1-b}{k - b \times \text{ROE}^{NewProjects}}.$$

Example 6. *Show that the P_0/E_1 ratio increases with $ROE^{NewProjects}$ (holding all else constant).*

Example 7. *GoodCo is expected to earn 4 euros over the course of next year, and is expected to pay dividends of 2 euros at the end of next year. The expected return on equity in new projects is 15% per year. The required rate of return on the company stock is 12.5%. What are GoodCo's expected growth rate, stock price, and P/E ratio?*

Our previous intuition regarding firms with no growth opportunities is still present. If ROE $= k$, then the price-earnings multiple is equal to the inverse of the required rate of return: the investors are indifferent between the firm investing the earnings in the assets in place, or distributing it to investors who then reinvest them in the market at the same required rate of return. Note also that the higher plow-back ratio (b) only leads to higher P/E ratio if the expected rate of return (ROE) is higher than the required rate of return k.

While very intuitive in theory, the price-earnings ratio reported in the financial press and news incorporates some serious flaws. As the left-hand-side of these formulas indicates, our price-earnings multiple is the ratio of the current stock price to the expected earnings per share next year. Now, that is very different from what the investors can observe at time $t = 0$: price today and last year's accounting earnings. One can easily see the flaw in such a price-earnings ratio which presumes (a) that accounting earnings are reflective of economic earnings (which they are not, remember our earlier discussion), and (b) that next year's earnings will be the same as last year's.

Another major flaw is that the Generally Accepted Accounting Principles (GAAP) gives a lot of discretion to the management which can be (and typically is) used to manage earnings. And the empirical evidence and accounting scandals suggest that firms do typically manage their earnings. Thus, observed accounting earnings may not even represent the "true" accounting earnings, much less the economic earnings that the theoretical P/E ratio is based upon. Finally, the P/E ratio varies a lot across the business cycle and with the inflation.

4 Market Efficiency

Now, suppose for a moment that somehow you could combine all publicly available information on a particular company's management, investment plans, its competitors, or its industry to forecast the company's stream of dividends and thus its fundamental value. Could you use this information to your benefit and buy under-valued stocks and short-sell over-valued stocks? The Efficient Market Hypothesis (EMH) suggests that you cannot.

This was one of the puzzling observations that economists first made when they started to use the computers in the 1950s to analyze stock market data. Instead of finding predictable patterns in stock prices, they observed that stock prices did not seem to follow any discernible pattern, making it very difficult (or impossible?) to predict. Today, we know that instead of being indicative of "animal spirits" the randomness of stock prices is indicative of the informational efficiency of the markets.

The idea is that, as you are trying to identify over- or under-valued stocks, thousands of investment companies and millions of investors are trying to do exactly what you are trying to do: use publicly available information to pick stocks. As a result, all of today's publicly available information would be reflected in today's stock prices. This is the essence of market **informational efficiency**: as new information that may become available in the future cannot be predicted today, it makes sense that the stock prices follow a “random walk”.

If financial markets are informationally efficient, it implies that it is impossible even for the smartest investor to consistently predict future stock returns, and thus impossible to consistently generate superior risk-adjusted returns. The behavior of stock prices around news announcements is often cited as evidence in favor of semi-strong form efficiency. The track record of actively managed mutual funds also brings support to the hypothesis, since those funds do not beat passive investment strategies in stock indexes on average.

On the other hand, the existence of bubbles and other market anomalies suggest that financial markets are not always informationally efficient. For example, there are periods during which stock prices seem too high to be justified by economic fundamentals (the internet bubble of the late 1990s is a famous case in point) and yet investors do not seem to take advantage of such mis-valuation. Recent research in finance has pointed out “limits of arbitrage” to explain such anomalies. For instance, even when sophisticated investors like hedge funds know that there is a bubble in the stock market, betting against the bubble is a risky strategy because it might be losing in the short run, even if it is winning in the long run. Because fund managers are always at risk of losing their investors, they can be induced to favor short-term strategies and ride the bubble, neglecting economic fundamentals, and thus sustaining the bubble.

Let us finally note that financial economists often refer to the notion of informational efficiency as simply “market efficiency”. Do not be abuse by this shortcut, as there exist other notions of efficiency. Informational efficiency has a precise definition: it means that security prices accurately reflect economic fundamentals. It is different from the notion of “allocative” efficiency, which refers to the efficiency with which capital markets allocate resources across economic agents. Clearly, the issue of whether stock market bubbles exist (informational efficiency) is different from the issue of whether stock market bubbles have real economic consequences. The last issue is an active area of academic research.

Solutions of examples

Solution of Example 1. A constant dividend is a special case of the constant growth Dividend Discount Model with $g = 0$. The stock price in this case is

$$P_0 = \frac{2}{0.10 - 0} = 20\text{ €}$$

Solution of Example 2. The constant growth DDM says that the intrinsic value of a stock is given by

$$V_0 = \frac{(1+g)D_0}{k-g}.$$

Therefore, the intrinsic value is larger when the dividend per share (D_0) is larger; it is lower when the discount rate (k) is lower; and it is higher when the growth rate of dividends (g) is higher.

Solution of Example 3. The constant growth DDM implies that the stock price at t=0 is

$$P_0 = \frac{(1+g)D_0}{k-g}$$

and the stock price at t=1 is

$$P_1 = \frac{(1+g)D_1}{k-g}$$

Since the dividend is expected to grow at rate g, then expected dividend at t=1 is equal to dividend at t=0 times $(1+g)$, or $E[D_1] = (1+g)D_0$. Plugging this into the expression for P_1, we obtain

$$E[P_1] = \frac{(1+g)(1+g)D_0}{k-g} = (1+g)P_0$$

that is, the stock price is expected to grow at rate g.

Solution of Example 4. The stock price before the deal is

$$P_0^{\text{before}} = \frac{1}{0.10 - 0.03} = 14.29\text{ €}$$

and after the deal

$$P_0^{\text{after}} = \frac{1}{0.10 - 0.05} = 20\text{ €}$$

Thus, the company's stock price increases by

$$\frac{20 - 14.29}{14.29} = 40\%$$

The expected rate of return on the stock remains equal to $k = 10\%$.

Solution of Example 5. We cannot directly apply the constant growth DDM because the growth rate of dividends is not constant. We apply the principle that the stock price is equal to the discounted value of cash flows for an investor buying the stock today and holding it until year 5:

$$P_0 = D_0 + \frac{D_1}{1.15} + \frac{D_2}{1.15^2} + \frac{D_3}{1.15^3} + \frac{D_4}{1.15^4} + \frac{D_5 + P_5}{1.15^5}$$

where dividends are equal to $D_1 = 1.2 \times D_0 = 1.2$ €, $D_2 = 1.2^2 \times D_0 = 1.44$ €, $D_3 = 1.2^3 \times D_0 = 1.72$ €, $D_4 = 1.2^4 \times D_0 = 2.07$ €, $D_5 = 1.2^5 \times D_0 = 2.49$ €, and P_5 is the re-sell value of the stock on year 5.

P_5 can be calculated using the constant growth DDM because dividends will grow at a constant rate of 10% from year 5 on:

$$P_5 = \frac{D_5 \times 1.10}{0.15 - 0.10} = 54.56 \text{ €}$$

We can now calculate $P_0 = 33.81$ €

Solution of Example 6. The price-earnings ratio is equal to

$$\frac{P_0}{E_1} = \frac{1 - b}{k - b \times \text{ROE}^{NewProjects}}$$

so it is higher when $\text{ROE}^{NewProjects}$ is higher.

Solution of Example 7. The dividend payout ratio is equal to $1 - b = 2/4 = 0.5$, so the plow back ratio is $b = 0.5$. The company's expected growth rate is thus equal to

$$g = b \times \text{ROE}^{NewProjects} = 0.5 \times 0.15 = 7.5\%$$

The company's stock price is

$$P_0 = \frac{D_1}{k - g} = \frac{2}{0.125 - 0.075} = 40 \text{ €}$$

The company's price-earnings ratio is

$$\frac{P_0}{E_1} = \frac{40}{4} = 10$$

Chapter 2

Bonds

1 Definition

A bond is a debt contract that promises **a stream of known cash-flows** in the future. These cash-flows are typically a percentage of the borrowed amount and are paid periodically by the issuer (a firm or a government) until the maturity of the bond. The cash-flows, which are due to be paid by the issuer, say at date t, are paid to the investors who hold the bond at this time. Bonds, like stocks have a primary market where they are issued to the public, and a secondary market where already issued bonds are traded.

A bond is defined by the following features:

- **The Maturity** (T): this is the date at which the last payment to the bondholders is due. The bond is redeemed at this date.

- **The Coupon** (C): the coupon is the interest payment that is made by the issuer to each bondholder, at periodic dates.

- **The Face Value** (N) (or par value): this is the final payment which is made at maturity with the last coupon. This corresponds to the reimbursement of the principal to the bondholders.

- **The Frequency** (k) with which coupons are paid (e.g. once every year or once every semester).

- **The Coupon Rate** (i): by definition, the coupon rate is $i = k\frac{C}{N}$.

It is worth stressing that all these features are chosen by the issuer, at the time the bond is issued. These features and other factors that we study below, determine the market price of a bond at any instant.

Example: The holder of a bond with maturity 2 years, face value €1000, frequency 6 months and coupon rate 4% will receive the following cash flows (be careful that a frequency of 6 months means $k = 2$):

6 months	1 year	18 months	2 years
€20	€20	€20	€1020

1.1 Different Types of Bonds

1. **Pure discount bonds (or zero-coupon bonds)**: The coupon of a pure discount bond is zero. This means that a pure discount bond offers a single payment, equal to the face value, at its maturity date. In this case, the interest received by the bondholder is equal to the difference between the face value and the purchase price for the bond.

2. **Perpetual bonds**: For a perpetual bond, the maturity is infinite ($T = +\infty$). U.K. Government consols are a good example of perpetual bonds.

3. **Coupon bonds**: is a bond with a finite maturity and strictly positive coupon payments.

4. **Floating rate bonds**: These are bonds for which the coupon rate is indexed on another interest rate (e.g. $i =$ LIBOR+2

5. **Convertible bonds**: They can be converted into shares of common stocks of the issuing firms.

6. **Callable bonds**: They can be bought back ("called") by the issuer at typically par value plus one coupon payment after a certain number of years.

Note that a coupon bond can be seen as a portfolio of zero-coupon bonds with maturities 1, 2, $\ldots$, $T-1$ and face values C and one zero-coupon bond with face value $N+C$ and maturity T. This fact is at the origin of the process known as **coupon stripping**. Roughly speaking, coupon stripping consists in buying coupon bonds and then selling separately the coupons and the principal amount of these bonds, in the form of zero-coupon bonds. For instance, consider a 10%, 3 years coupon bond issue with \$10 million in face value and annual coupon payments. Stripping this issue would consist in selling three zero-coupon bonds with maturities 1 year, 2 years and 3 years and respective face value totaling \$1 million, \$1 million and \$11 million. The reason for coupon stripping is that in many countries Treasuries issue coupon bonds and their investors prefer default-risk-free zero-coupon bonds. Actually, Treasury departments may even strip the coupons of their coupon paying bonds. STRIPS are U.S. Government bonds that have been stripped by the U.S. Treasury.

The **default risk** is the risk that the issuer will not honor the promised payments (e.g. in case of bankruptcy). Bonds differ according to their default risk. Some investment agencies (Moody's, Standard & Poors) are specialized in evaluating this risk and bonds are ranked according to letter grades ("credit ratings") given by these agencies. These grades vary according to the probability of default (as measured by the agency). Table 1 provides a list of ratings by Moody's and Standard and Poor's.

Bonds issued by governments with strong fiscal positions are in general considered as bonds with the lowest default risk. High yields bonds are bonds of low quality (high default risk) and are called "junk bonds". Junk bonds can be original issues of high risk bonds, or "fallen angels", that is, downgraded bonds.

2 Current Yield, Yield-To-Maturity and Par Yields

Let P_0 be the current price of a bond and assume that there are still T years to maturity. The coupons are paid annually.

Table 1: Credit Ratings by Rating Agencies

Moody's	
Aaa	Best Quality
Aa	High quality
A	Upper medium grade
Baa	Medium grade
Ba	Possesses speculative elements
Caa	Poor standing; may be in defaults
Ca	Speculative in a high degree; often in default
C	Lowest grade; extremely poor prospects
Standard & Poor's	
AAA	Highest rating; extremely strong capacity to pay interest and principal
AA	Very strong capacity to pay
A	Strong capacity to pay
BBB	Adequate capacity to pay
BB	Uncertainties which could lead to inadequate capacity to pay
B	Greater vulnerability to default, but currently has the capacity to pay
CCC	Vulnerable to default
CC	For debt subordinated to that with CC-rating
C	For debt subordinated to CCC-rating; or bankruptcy petition has been filled
D	In payment default

Definition 1. *The current yield is equal to the coupon of a bond multiplied by the frequency divided by its price:* $k\dfrac{C}{P_0}$.

Example: Consider a bond with price $P_0 = 90$ face value $N = 100$ and coupon $C = 3$ paid every semester. The coupon rate of this bond is i=6% and its current yield is $2 \times 3/90 \approx 6.67\%$.

Definition 2. *The yield-to-maturity is the interest rate y such that the following equation is satisfied:*

$$P_0 = \frac{C}{1+y} + \frac{C}{(1+y)^2} + \frac{C}{(1+y)^3} + \cdots + \frac{C+N}{(1+y)^T}.$$

Note that the yield-to-maturity is exactly the Internal Rate of Return of the bond if there is no default.

Example: Consider a bond with maturity in 10 years, with price $P_0 = 82$, face value $N = 100$, and coupon $C = 3$ paid every year. The yield-to-maturity solves:

$$82 = \frac{3}{y}\left(1 - \frac{1}{(1+y)^{10}}\right) + \frac{100}{(1+y)^{10}},$$

that is to say $y \approx 5.37\%$.

Note that, in general, the yield-to-maturity and the current yield are different. There are some properties of the yield-to-maturity that are useful to know:

- Property 1: $P_0 = N \Leftrightarrow y = i$.
- Property 2: $P_0 > N_T \Leftrightarrow y < i$.
- Property 3: $P_0 < N_T \Leftrightarrow y > i$.
- Property 4: There is an inverse relationship between the price of a bond and the yield-to-maturity.

When the price of the bond is equal to (greater than, less than) its face value, then the bond is selling at par (at a premium, at a discount). At issuance bond coupon rates are set equal to (or as close as possible as) the yield to maturity so that the bond sells at (near) its par value in the primary market.

Definition 3. *The par yield for a certain bond is the coupon rate that causes the bond price to equal its face value.*

3 The Term Structure of Interest Rates

3.1 Definitions

Consider a zero-coupon bond that matures in t years. The price of the bond is P_t^0 and its face value is N_t^0. Then the yield-to-maturity of this zero-coupon bond satisfies:

$$P_t^0 = \frac{N_t^0}{(1 + y_t)^t},$$

which gives:

$$y_t = \left(\frac{N_t^0}{P_t^0}\right)^{1/t} - 1.$$

Note that in this case the yield-to-maturity is the annual rate of return that is offered by the bond. Assume that the zero-coupon bond is **without default risk**. This rate of return must be equal to the riskless interest rate, per year, for a placement of t periods (why?).

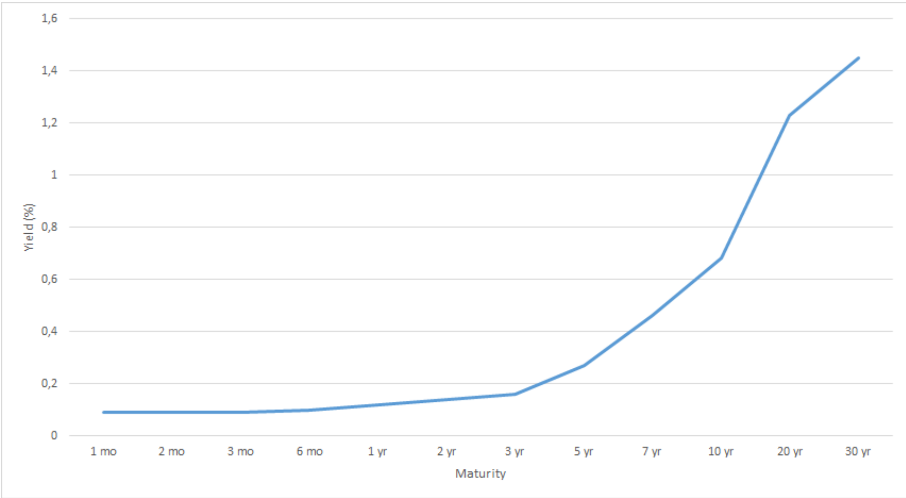
Example: Consider the following example. You observe the prices of 3 zero-coupon bonds with maturities one year, two years and three years. Their prices are: $P_1^0 = \$950$, $P_2^0 = \$880$, $P_3^0 = \$800$. The face value of each bond is \$1,000. We deduce that the interest rates for placements of one year, two years and three years are respectively: $r_1 = 5.26\%$, $r_2 = 6.6\%$, $r_3 = 7.72\%$.

Let r_t be the interest rate for a placement of t years, it is also known as the t-year zero rate. Note that the interest rate depends on the number of years.

Definition 4. *The relationship between the yield-to-maturity of zero-coupon bonds and their time to maturity is the term structure of interest rates (or the yield curve). The term structure of interest rates is depicted by the curve which represents the function r_t.*

Figure 1 is an example of the term structure of interest rates built using U.S zero-coupon treasury securities. We should not expect any specific shape for the term structure: it can be increasing (the larger is the maturity, the larger is the interest rate) or decreasing (the larger is the maturity, the lower is the interest rate). The term structure is said to be flat if the interest rate does not depend on the maturity.

Figure 1: U.S. Government Bonds Term Structure of Interest Rates (October 2020)



Source: U.S. Treasury website.

Definition 5. *The discount factor* d_t *is:*

$$d_t = \frac{1}{(1+r_t)^t}.$$

It follows that the price of a zero-coupon bond with maturity t is $P_t = d_t N_t^0$.

We have built the term structure using zero-coupon bonds with no default risk. We could also use zero-coupon bonds with default risk (or coupon bonds, see section 4.3). However, for the term structure to be meaningful, it is important for the bonds that are used to build the term structure to have the same default risk. In this way, the time to maturity is the only factor that influences the interest rate. For instance, one could consider building the term structure using zero-coupon bonds issued by corporations having a grade of B or lower. The default risk premium is the difference between the yield-to-maturity of a risky zero-coupon bond of a given default risk and the yield-to-maturity of a zero-coupon-bond without default risk, for a given maturity. This premium is the compensation required by bondholders for the given default risk. Note that this default risk premium can change over time.

3.2 Forward Rates

A forward interest rate is the rate implied by the current term structure over a given period in the future. The year t forward rate, denoted $f_{t\to t+1}$, is the implicit interest rate for the time period between the end of year t and the end of year $t+1$. In what follows we will see how the forward rates can be deduced from the term structure of interest rate.

For example suppose that the term structure is such that the 1-year interest rate is $r_1 = 3\%$ and the 2-year interest rate is $r_2 = 4\%$. Then the year 1 forward rate $f_{1\to 2}$, is the interest rate implied by the term structure for the time period between the end of the first year and the end of the second year. To illustrate how $f_{1\to 2}$ is calculated, note that investing €1 in a zero-coupon bond with maturity two years provides a final cash flow equal to $(1+r_2)^2 = 1.04^2$ at the end of the second year. This cash flow should be equal to an investment at the 1-year interest rate $r_1 = 3\%$ during the first year and an investment at the year-1 forward rate $f_{1\to 2}$, between the end of the first year and the end of the second year:

$$1.04^2 = 1.03 \times (1 + f_{1\to 2}),$$

thus $f_{1\to 2} = 1.04^2/1.03 - 1 = 5.0097\%$. We said "should be equal to" because otherwise arbitrageurs will intervene and their transactions in the market will make sure that the above formula holds.

For the general case we have the following formula:

$$f_{t\to t+1} = \frac{(1 + r_{t+1})^{t+1}}{(1 + r_t)^t} - 1.$$

Moreover, the following relation holds:

$$(1 + r_t)^t = (1 + r_1) \times (1 + f_{1\to 2}) \times \cdots \times (1 + f_{t-1\to t}).$$

This means that the t-year interest rate r_t can be seen as the composition (or "average") of the first t forward rates.

Example: Suppose that an investor can invest or borrow at r_1 and r_2 by buying or shorting zero-coupon bonds of maturity 1 and 2 years respectively. Then the investor can lock in the forward rate. If he borrows €100 at $r_1 = 3\%$ and invests the money for two years at $r_2 = 4\%$, then it results a cash outflow of €103 at the end of the first year and an inflow of €100 $\times$ 1.04^2 at the end of the second year. This is equivalent to investing €103 at $f_{1\to 2} = 5.0097\%$ between the end of the first year and the end of the second year.

	t=0	t=1 year	t=2 years
Borrow €100 for 1 year	100	-103	
Invest €100 for two years	-100		100×1.04^2
Total	0	-103	108.16

Remark: The forward rate associated to an investment between year t and year $t + T$, $f_{t\to t+T}$, is also sometimes denoted by $r_t(T)$.

4 Bond Pricing

In this section, we explain how bonds can be priced. We want to find the price of a coupon bond with maturity T and coupon C. We assume that we are able to observe the market prices $P_1^0, \ldots, P_T^0$ of T zero-coupon bonds with maturities 1, 2, ..., T, respectively. The face value of the bond that matures at date T is denoted N.

4.1 The Arbitrage Method

Arbitrage is a method used to value financial assets. It consists in using the prices of, say, N assets to derive the price of another security. In this section, we describe the arbitrage method and in the next subsection we apply it to bond pricing. We will use again this method to value futures and option contracts.

Consider asset A that offers cash-flows CF_1, CF_2, ..., CF_T at dates 1, 2, ..., T. These cash flows might be uncertain. Now consider another asset B with exactly the same cash-flows. Exactly means that in all possible future scenarii, the two assets have identical cash flows.

Example:

Assets	Cash flow if economy is booming	Cash flow if Economy is in recession
A	100	20
B	100	20

Law of One price: Asset A and asset B must have the same price: $P_A = P_B$.

Why? Suppose that $P_A > P_B$. Then, we can short sell A and we buy B. At date 0, we book a sure profit equal to $P_A - P_B > 0$. In the future, we use the cash flows that we receive from asset B to pay the cash flows that are due on asset A. The net cash flow is zero, whatever occurs in the future. This type of sure profit opportunity is called an **arbitrage opportunity**. Investors should quickly buy asset B (this pushes its price up) and sell asset A (this pushes its price down) so that the prices of these two assets become equal.

A well functioning market is characterized by an **Absence of Arbitrage Opportunities** since by exploiting these opportunities, investors bring back prices in line immediately. This fact puts some constraints on securities prices and can sometimes be used to compute the price of a security given the prices of other securities, as shown by the next example.

Let us consider the example again and assume that assets C and D have the following cash flows at date 1.

Assets	Cash flow if economy is booming	Cash flow if Economy is in recession
C	100	0
D	0	100

The market prices of C and D are respectively: $P_C = 50$ and $P_D = 70$. Can we deduce the price of A? Imagine that we could build a portfolio R with C and D such that the cash flows of the portfolio are the cash flows of asset A, in every possible future states of the economy. Then, the law of one price implies that:

$$P_A = n_C P_C + n_D P_D,$$

where n_C (n_D) is the number of units of assets C (D) in portfolio R. How to choose n_C and n_D? For portfolio R to yield the same cash flows as asset A, we need:

$$\begin{cases} n_C \times 100 + n_D \times 0 = 100 \\ n_C \times 0 + n_D \times 100 = 20 \end{cases}$$

It follows that $n_C = 1$ and $n_D = 0.2$. Therefore $P_A = 50 + 0.2 \times 70 = 64$.

Question: What should we do if instead $P_A = 70$?

A portfolio which has the same cash flows as another asset is called a **replicating portfolio**. As shown by the previous example, in order to price a security, we can try to construct a replicating portfolio that replicates the cash flows of the security we want to price. This is the method that we will use below to derive the price of a coupon bond (and later prices of futures and option contracts).

4.2 The Price of a Bond

Let $\{d_t\}_{t=1}^{t=T}$ be the discount factors associated with the T zero-coupon bonds.

Result 1. *The price of a coupon bond with a coupon C and a face value N is:*

$$P_0 = d_1 \times C + d_2 \times C + \cdots + d_T \times (C + N) = \sum_{t=1}^{t=T} d_t C + d_T N$$

This result is proved by arbitrage. The proof consists in building a replicating portfolio with the T zero-coupon bonds that yield the same stream of cash flows as the coupon bond (i.e., a coupon C during T periods and a face value N in the T^{th} period). Note that the price of the bond is just the discounted value of its stream of cash flows. This is another example of application of the Discounted Cash Flow Formula.

Example: Suppose that the term structure is as given in Example 1. Consider a bond with coupon $C = \$100$, a face value equal to \$1,000 and a maturity of three years. The price of this bond is:

$$P = \frac{100}{1 + 5.2\%} + \frac{100}{(1 + 6.6\%)^2} + \frac{1100}{(1 + 7.72\%)^3} = \$1063.1$$

4.3 Term Structure without Discount Bonds

It is clear that discount factors are crucial to price bonds. How can we obtain these discount factors when zero-coupon bonds prices are not available (because they have not been issued for all possible maturities)? Suppose that we observe the prices $\{P_k\}_{k=1}^{k=T}$ of T coupon bonds with maturities $\{T_k\}_{k=1}^{k=T}$. We will assume that there is one bond such that $T_k = T$ and that the bonds have the same default risk.

$$\begin{aligned}
P_1 &= d_1 \times C_1 + d_2 \times C_1 + \cdots + d_{T_1} \times (C_1 + N_1) \\
P_k &= d_1 \times C_k + d_2 \times C_k + \cdots + d_{T_k} \times (C_k + N_k) \\
P_T &= d_1 \times C_T + d_2 \times C_T + \cdots + d_{T_T} \times (C_T + N_T)
\end{aligned}$$

Result 1 means that the price of each bond can be written as the discounted value of its stream of cash flows, where the discount factors are *identical* for each bond. This means: the discount factors are the unknowns of this system of T equations. Since there are T equations with T unknowns, we can solve this system for the discount factors $\{d_t\}_{t=1}^{t=T}$ (if the bonds have been adequately chosen). This shows how coupon bonds, instead of discount bonds, can be used to build the term structure of interest rates.

5 Interest Rate Risk

5.1 Why are Bonds Risky?

We have already mentioned that default risk was one source of risk for bonds. In this section, we describe another source of risk, called interest rate risk which exists even for bonds without default risk. We start by explaining what interest rate risk is.

Let us assume for simplicity that the term structure is flat.[1] We denote r the annual interest rate. In this case, the price of a bond is:

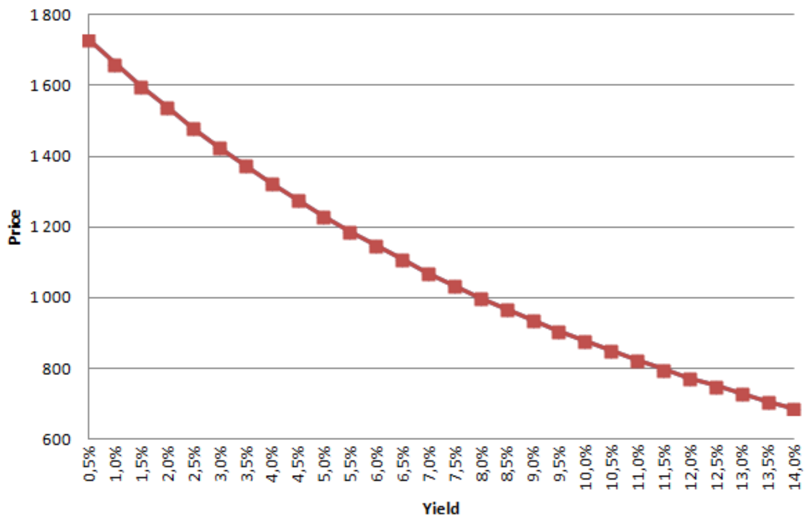
$$P(r) = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C+N}{(1+r)^T}.$$

It is straightforward that:

$$\frac{dP}{dr} = -\left[\frac{C}{(1+r)^2} + \frac{2C}{(1+r)^2} + \cdots + \frac{t(C+N)}{(1+r)^{T+1}}\right] < 0. \qquad (1)$$

Thus the price of a bond decreases (increases) when the interest rate increases (decreases). Figure 2 depicts the bond price as a function of r.

Figure 2: Bond Price and Interest Rate



The inverse relationship between interest rates and bond prices means that bond prices will fluctuate as interest rates fluctuate over time. As a result, the price at which a bond can be resold before its maturity is uncertain. This is the reason why bonds are risky investments. This risk is called the interest rate risk.

Example: Let us consider a bond with a face value of \$1,000 and a coupon rate $i = 10\%$. The maturity of the bond is two years. The interest rate at time $t = 0$ is $r = 10\%$. It follows that the initial price of the bond is $P_0(10\%) = \$1,000$. Now assume that at the beginning of the second year, the interest rate is either $r = 11\%$ or $r = 9\%$. In the first case, the price of the bond is:

$$P_1(11\%) = \frac{1100}{1.11} = \$990.99$$

[1]The arguments in this section can be generalized to the case in which the term structure is not flat.

In this case, the realized rate of return on investing in the bond after one year will be:

$$\frac{100}{1000} + \frac{990.99 - 1000}{1000} = 9.09\%.$$

In the second case, the price of the bond is:

$$P_1(9\%) = \frac{1100}{1.09} = \$1009.17$$

and the realized rate of return on the bond after one year will be:

$$\frac{100}{1000} + \frac{1009.17 - 1000}{1000} = 10.91\%.$$

Thus the rate of return is uncertain and depends on the evolution of the interest rate during the first year. In no case, this rate of return is equal to the 10% interest rate, the expected rate of return at $t = 0$.

5.2 Factors that Influence the Interest Rate Risk

Roughly speaking, the larger the sensitivity of a bond price to a change in the level of interest rates, the larger is the exposure of this bond to interest rate risk. There are two factors that determine this sensitivity: (i) the coupon rate and (ii) the time to maturity.

- **Property 1 (Maturity Effect):** The larger is the time to maturity, the larger is the exposure to interest rate risk.
- **Property 2 (Coupon Effect):** The larger is the coupon rate, the lower is the exposure to interest rate risk.

These two properties are illustrated in Figure 3. Note that Property 2 implies that we cannot simply use the maturity of a bond to characterize its exposure to interest rate risk. We now propose a measure of this exposure that takes into account both the coupon effect and the maturity effect.

5.3 Duration: A Measure of Interest Rate Risk

We still assume that the term structure is flat. A natural measure of the exposure to interest rate risk is the percentage variation in the bond price (in absolute value) for a small percentage variation of the interest rate. The larger is this percentage variation, the larger is the change in the bond price following a change in the level of interest rate. Thus this measure is a measure of the bond exposure to interest rate risk and this measure is called **duration**. It is worth nothing that duration measures the sensitivity of the bond price to a small parallel shift to the yield curve that increases all interest rates by the same small amount. Formally, let D be the duration, that is:

$$D = -\frac{dP/P}{dr/(1+r)} = -\frac{dP}{dr}\frac{1+r}{P},$$

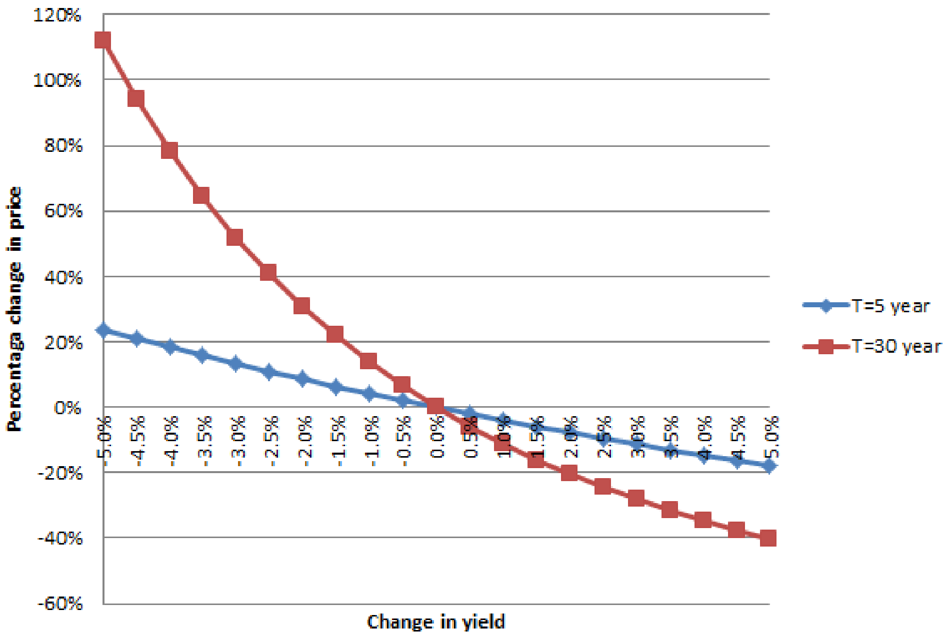
where P is the price of the bond. Using Equation (1) we obtain the following expression for the duration:

$$D = -\sum_{t=1}^{T} w_t t \quad \text{with} \quad w_t = -\frac{C}{(1+r)^t}\frac{1}{P} \text{ for } t < T$$

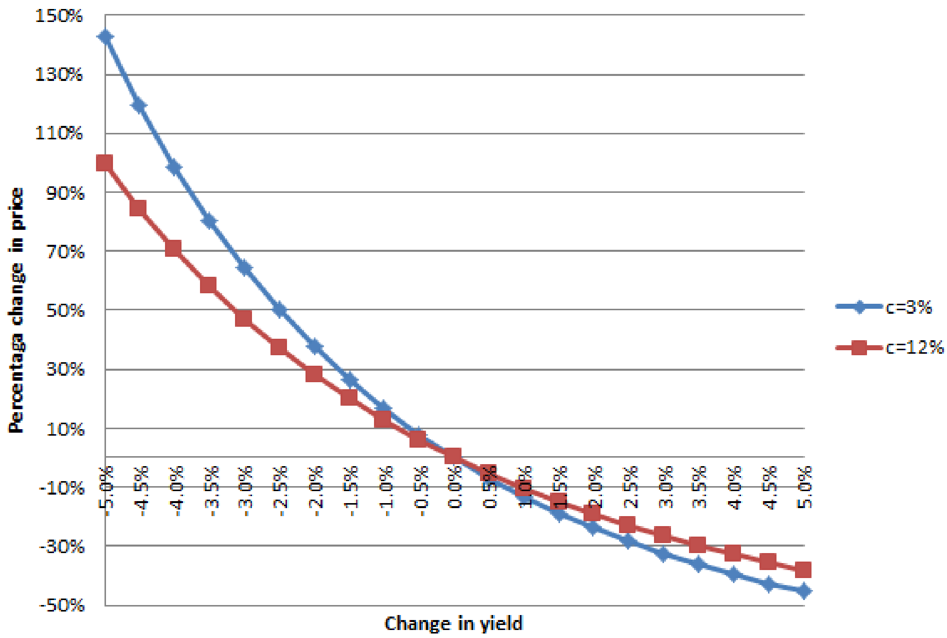
$$w_T = -\frac{C+N}{(1+r)^T}\frac{1}{P}.$$

Figure 3: Interest Rate Sensitivity

Property 1: Influence of Maturity



Property 2: Influence of Coupon



Note that $0 \leq w_t \leq 1$ and $\sum_{t=1}^{T} w_t = 1$. This means that the duration is a weighted average of the dates at which payments of the bond are due to the bondholders. This explains why bonds with high coupon rates are less exposed to interest rate risk. For these bonds, the weights on the intermediate payments are large relative to the weight on the last payment, which means that the bond duration is lower. The formula also shows that in general the maturity is insufficient to characterize the exposure to interest rate risk. Thus, the duration can be interpreted as a measure of how long, on average, the holder of the bond has to wait before receiving cash payments. In fact, a coupon bond with maturity

T years has a duration of less than T years.

Question: For which types of bonds, is the duration equal to the maturity? Why?

The duration is one tool that portfolio managers commonly rely on. It is however important to point out that a higher sensitivity of bond prices to interest rates does not necessarily translate into more interest rate risk for the investor. This is because investors with long horizons also care about the interest rate at which to reinvest intermediate cash-flows. In particular, a portfolio manager can fine-tune the exposure of her portfolio to a change in the level of interest rates by adequately choosing the duration of the bonds in her portfolio, a technique called portfolio immunization.

Chapter 3

Forward and Futures Contracts

1 Forward Contracts

1.1 Definition

A forward contract is an agreement to buy/sell a financial asset or a commodity (the underlying asset) at a specified date, $t = T$ (the maturity), at a price (delivery price) fixed at the time of the agreement (say $t = 0$).

The seller of the forward contract is the agent who has the obligation to sell the asset at maturity. He/she is said to have a short position in the forward contract. The buyer of the forward contract has the obligation to buy the underlying asset at maturity. He/she is said to have a long position in the forward contract.

1.2 The Value and the Price of a Forward Contract

It is worth stressing that there is no exchange of money between the buyer and the seller of the contract at the time the two agents enter into the contract (date $t = 0$). This means that the value of the contract to both parties is zero at date $t = 0$. The forward price is the delivery price, that we denote F_0. This price is chosen in such a way that the value of the contract is zero at the time it is entered into. As shown below, by arbitrage, the forward price will depend on the current value of the underlying asset (the spot price) at date $t = 0$. We will denote this price S_0. Note that the spot price will fluctuate over time whereas the forward price is fixed at time $t = 0$. For this reason, at any point after date $t = 0$, the forward contract will in general have some value (positive or negative). For instance, if the spot price rises (decreases), the value of the forward contract will become positive (negative).

Forward contracts are traded outside exchanges, in Over The Counter (OTC) markets. They often involve a financial institution and its corporate clients.

Example 1. *A farmer and a cereal company enter into the following forward contract on wheat on April, 1st:*

- *Delivery date: September 1st.*
- *Size of the contract: 100,000 bushels of wheat.*

- *Forward price:* $F_0 = \$2$ *per bushel.*

The farmer is the seller of the contract and the cereal company is the buyer.

Example 2. *Forward contracts on currencies are very frequently used by corporations. In this way corporations transfer the risk associated with fluctuations in the exchange rate of a given currency to the bank with which they sign the contract.*

1.3 Payoffs

Let S_T be the spot price of the underlying asset at the maturity date. The payoff from taking a long position in a forward contract on one unit of the underlying asset is:

$$S_T - F_0.$$

This is the difference between the price at which the buyer can resell the asset he obtains from the forward contract and the price at which he must acquire the asset from the seller of the contract. Note that S_T is not known at date 0. This means that the payoff of a long position in a forward contract is uncertain. Note that it can be positive ($S_T > F_0$) or negative ($S_T < F_0$).

For the seller of the contract, the payoff is:

$$F_0 - S_T.$$

This is the difference between the price at which the seller delivers the asset to the buyer of the contract and the price at which the seller can purchase the underlying asset at maturity. Note that the payoff of the seller is just opposite to the payoff of the buyer.

For some forward contracts, there is no physical delivery of the underlying asset. Rather, the contract is settled in cash ("Cash Settlement"). In this case, the buyer of the contract receives $S_T - F_0$ from the seller if this amount is positive or pays this amount to the seller if the amount is negative.

2 Futures Contracts

2.1 Definition

Futures contracts are defined in the same way as forward contracts. However, in contrast to forward contracts, futures contracts are traded on an exchange. For this reason, their features are standardized. That is, futures contracts are formalized along:

- The quality of the underlying asset: This is necessary for commodities such as orange juice for which there are variations in quality.

- Contract size: The amount of the asset that must be delivered.

- The place of delivery: It is particularly important to specify the place of delivery for commodities for which transportation costs can be quite high.

- The maturities: In general, for a given underlying asset, futures contracts for only few maturities are available.

Example 3. *The Wheat Futures Contract of the Chicago Board of Trade (CBOT). The size of this contract is 5,000 bushels. The CBOT specifies the grades that can be delivered and the possible places for delivery. Finally contracts for 5 delivery months (March, May, July, September and December) are available (for up to 18 months in the future).*

There is a wide variety of underlying assets to futures contracts. These assets include: sugar, wool, cotton, pork bellies, gold, stock indices, treasury bonds, interest rate, currencies, etc.

Example 4. *Table 1 provides the contract specifications for the 15,000 pounds of FCOJ (Frozen Concentrate Orange Juice) and 100 oz. gold future contracts on ICE (Intercontinental Exchange).*

Table 1: Contract Specifications

FCOJ Futures	
Contract Size	15,000 pounds of orange juice solids (3% or less)
Price Quotation	Cents and hundredths of a cent to two decimal places
Contract Listing	January, March, May, July, September, November
Minimum Price Movement	5/100 of a cent per pound ($7.50/contract)
Settlement	Physical delivery
Grade/Standards/Quality	US Grade A with a Brix value of not less than 62.5 degrees
Delivery Points	Exchange licensed warehouses in Florida, New Jersey, and Delaware
Deliverable Origins	U.S., Brazil, Costa Rica and Mexico
100 oz. Gold Futures	
Contract Size	100 fine troy ounces
Contract Months	Every calendar month
Price Quotation	U.S. dollars and cents per oz
Tick Size	$0.10 per oz, or $10 per contract
Last Trading Day	Third to last business day of the contract month at 1:30pm New York time
Final Settlement	Physical delivery

Source: ICE website.

A future price at date t, F_t is the price at which the agent who purchases (sells) the futures contract at date t commits to buy (sell) the underlying asset at maturity, if he does not close his position before (see below). The parties involved in a futures contract do not meet directly. The risk of default by one party in the contract could be a serious impediment to the development of futures markets. For this reason, the exchange imposes a variety of mechanisms that guarantee that the contracts will be honored. We now describe these mechanisms.

2.2 Organization of Futures Markets

2.2.1 Marking to Market

Futures contracts are **marked-to-market**. Suppose that one buyer B purchases one future contract to one seller S at date $t = 0$. The future price is F_0. At the end of the first day, the future price is F_0.

At the end of this day, if $F_1 > F_0$, the buyer receives $F_1 - F_0$, which is paid by the seller. If $F_1 < F_0$, then this is the opposite. Now suppose that at the end of the second day, the future price is F_2. The buyer receives (or pays depending on the sign) $F_2 - F_1$ and the seller pays (or receives) $F_1 - F_2$. At date t, the amount that has been accumulated (or lost) by the buyer is:

$$(F_1 - F_0) + (F_2 - F_1) + \cdots + (F_t - F_{t-1}) = F_t - F_0,$$

whereas the seller has accumulated (or lost):

$$(F_0 - F_1) + (F_1 - F_2) + \cdots + (F_{t-1} - F_t) = F_0 - F_t.$$

In fact everything is *as if* futures contracts had a lifetime of one day, with a cash settlement at the end of everyday and *as if* B and S were entering into a new contract every day. Note that the price at which delivery will take place is F_T, the future price quoted at maturity. At maturity, buyer B has received (or paid) an amount equal to $F_T - F_0$ and he pays F_T to the seller. Therefore the price that he actually pays is $F_T - (F_T - F_0) = F_0$, the future price at which he entered into the contract. Similarly, the seller eventually receives: $F_T + (F_0 - F_T) = F_0$.

2.2.2 Convergence of Futures Prices to Spot Prices

At maturity, an arbitrage argument shows that the future price must be equal to the spot price: $F_T = S_T$. (Why?)

2.2.3 Margin Accounts

We consider the case of the buyer B. The same rules apply to the seller. In practice, the buyer's broker, say, will require the buyer to deposit an amount of money in a margin account at time $t = 0$. The daily payments associated with the buyer's position will then be made to or withdrawn from this account. The initial amount which is requested by the broker is called the initial margin. Any amount in excess of the initial margin can be withdrawn by the broker's customer. Furthermore, the broker makes sure that the amount in the margin account does not fall below a specific amount called the maintenance margin. If this occurs, the broker's customer receives a margin call and must restore the account to its initial level. This limits the risk that the holder of a futures contract will default on his obligations and causes losses to the broker or possibly to the futures exchange.

Example 5. *Table 2 provides the Initial margin and the maintenance margin for future contracts on metals in CME (Chicago Mercantile Exchange).*

Table 2: Maintenance Margins

Metals group	Maintenance margin
100 Oz. Gold	$4,250
5000 Oz. Silver	$5,400

Source: CME website.

2.2.4 Closing out a position

Closing out a position in a futures contract consists of entering into a trade which is opposite to the initial trade. For instance suppose that at date t, Buyer B decides to sell his futures contract. The contract is purchased by one new buyer, C, at price F_t. In this case, the payoff to buyer B is $F_t - F_0$. Note that if buyer C and initial seller S do not close their positions before maturity then, S will have to deliver the asset to (say) buyer C.

Question: What are the prices actually received by seller S and paid by buyer C?

2.2.5 Clearinghouse

In practice, a clearinghouse (which is owned by the exchange) acts as an intermediary between all the buyers and all the sellers. For instance, at maturity, this is the clearinghouse that matches buyers and sellers for delivery. The clearinghouse also computes the amount that is due at the end of each day by the buyers or the sellers. It makes sure that the marking-to-market related transfers are made. If, following a margin call, a trader does not restore its margin account, the clearinghouse immediately closes the position of the trader. In case of default, the clearinghouse substitutes to this agent for the payments (however, thanks to the maintenance margin mechanism described above, default happens very rarely). This is an additional guarantee for the traders who take position in futures markets.

2.3 Differences Between Forward and Futures Contracts

To sum up, the main differences between futures and forward contracts are as follows:

1. The features (maturities, size, etc.) of forward contracts can be tailored to the specific needs of the two parties whereas futures contracts are completely standardized.

2. Forward contracts are negotiated Over the Counter (absence of anonymity) whereas futures contracts are traded on exchanges (anonymity).

3. Positions in forward contracts are difficult to close out whereas positions in futures contracts can be very easily closed.

4. Unlike forward contracts, futures contracts are marked-to-market.

5. In futures markets, a clearinghouse guarantees payments and delivery in case of default of one party. For this reason, counterparty risk is less of a concern with futures contracts.

2.4 What Are Futures Contracts Useful For?

A main interest of futures contracts is that they can be used to **hedge** positions in the asset underlying the contract. In fact, derivatives contracts can be used to manage the exposure to the underlying asset. We show how this is achieved with futures contracts with the following example.

Consider a farmer which anticipates a crop of Q bushels of wheat at date T. The price, S_T, at which he will be able to sell this wheat is uncertain. The farmer is not willing to bear this risk and he decides to take a position in N futures contract on wheat with maturity T. The current price of this future contract is F_0. The sign of N indicates the direction of the position ($N < 0$ means that the farmer sells contracts). At maturity T, the farmer receives:

$$V_T = Q \times S_T + N \times (F_T - F_0) = Q \times S_T + N \times (S_T - F_0).$$

Now note that if $N = -Q$, we obtain $V_T = Q \times F_0$, that is, the value of the farmer's position is not uncertain anymore and the farmer receives with certainty the futures price. By taking a position in the futures contract, the farmer has been able to entirely get rid of the risk of his initial position. The hedge is said to be perfect.

Note that hedging with futures suppresses the risk of a price decline in wheat for the farmer. But it also suppresses the possibility of taking advantage of a price increase. Therefore futures hedging is not without cost: the risk of a loss has been reduced at the cost of giving up profits in case of a price increase.

In practice, it is difficult to build a perfect hedge for (at least) two reasons:

1. The asset whose price must be hedged can be different from the underlying asset of the futures contract that is used for the hedge.

2. The date at which the price must be hedged is not necessarily the maturity of the future contract that is used for the hedge.

Consider the previous example again but assume that the wheat must be sold at some date t before the maturity T of the futures contract on wheat. The value of the position at date t is:

$$V_t(N) = Q \times S_t + N \times (F_t - F_0).$$

It is not the case that $F_t = S_t$ because the date at which the position in the futures contract on wheat must be closed out is not the maturity date $t = T$ of the contract. In this case, selling Q futures contract will not remove the risk. In this case, we obtain:

$$V_t(-Q) = Q \times (S_t - F_t) + Q \times F_0.$$

As $S_t - F_t$ is uncertain, $V_t(-Q)$ is uncertain. In fact, the difference between the spot price and the future price, at the date at which the position is closed, is a source of risk that cannot be suppressed. This risk is called the **basis risk**. Note that, at any date, the difference between the spot price and the future price is called the **basis**: $b_t = S_t - F_t$. In this case, one way to determine the number of contracts to sell is to determine N so that the variance of V_t is minimized.

Of course, hedgers are not the only traders in futures markets. Futures contracts can also be sold or purchased for purely **speculative** reasons. For instance, a trader who expects the future price of wheat to increase over time will find it profitable to take a long position in a futures contract on wheat. This is much less costly than taking a position in the cash market in order to speculate, as we shall see shortly.

3 The Valuation of Forward and Futures Contracts

The purpose of this section is to explain how the price of a forward contract can be determined **by arbitrage**. In this way, the price of forward contract is related to the spot price of the underlying asset. For futures contracts, pricing is more complicated because of daily settlements. However, the method is similar. Furthermore, if interest rates are constant, forward prices and futures prices on the same underlying asset for the same maturity must be equal. For these reasons, we mainly focus on forward contracts.

3.1 Forward on Securities

We start by considering the case of a forward contract on a security that provides no income. Stocks that pay no dividends until the maturity of the contract or discount bonds are examples of such securities. We use the following notations:

- F_0 is the forward price a date 0 (the date at which the parties enter the contract).
- S_t is the price of the underlying security at date t.
- r_f is the riskless rate of interest per period.
- T is the number of periods until maturity.

Then we obtain the following result.

Result 1 (**Spot-forward parity**). *The price F_0 of a forward contract that matures at date $t = T$ and the spot price must satisfy the following relationship:*

$$F_0 = (1 + r_f)^T S_0.$$

Proof: Cash and Carry Arbitrage

Case 1: Suppose that $F_0 > (1 + r_f)^T S_0$. In this case, consider the following portfolio at date 0, and the cash flow of this portfolio at date 0:

1. A long position in one share of underlying security: $-S_0$.
2. A short position which is worth S_0 in the riskless asset (that is, we borrow S_0): $+S_0$.
3. A short position in one forward contract: 0.

The cost of this portfolio at date 0 is zero. Now at date T, the cash flows associated with the portfolio are:

1. Forward contract: $F_0 - S_T$
2. Underlying asset: $+S_T$
3. Short position in the riskless asset: $-(1 + r_f)^T S_0$

The net cash flow is $F_0 - (1+r_f)^T S_0 > 0$. Thus if $F_0 > (1+r_f)^T S_0$, there is an arbitrage opportunity.

Case 2: Suppose that $F_0 < (1+r_f)^T S_0$. Then the following table shows that there is an arbitrage opportunity by considering a portfolio in which the positions are opposite to those considered in Case 1.

	$t=0$	$t=T$
Long position in the forward	0	$S_T - F_0$
Short-sell the underlying asset	S_0	$-S_T$
Invest S_0 for T years	$-S_0$	$(1+r_f)^T S_0$
Total	0	$(1+r_f)^T S_0 - F_T > 0$

Thus the only case which is consistent with the Absence of Arbitrage Opportunities is $F_0 = (1+r_f)^T S_0$. QED

Note that we have used the fact that at date 0, the value of the forward contract is zero. By slightly changing the previous proof, it is possible to show that, at any date t, the value V of the forward contract with price F_0 must be: $V = F_0 - (1+r_f)^{T-t} S_t$. At future dates, the price F_t must adjust in such a way that its value is always zero, which means $F_t = (1+r_f)^{T-t} S_t$.

Now we assume that the security provides a known dividend (or coupon) D_t at some date t before maturity.

Result 2. *The forward price and the spot price must satisfy the following relationship:*

$$F_0 = (1+r_f)^T \left(S_0 - \frac{D_t}{(1+r_f)^t} \right).$$

Question: How do you prove this result?

3.2 Forward on Commodities

For forward contracts on commodities, we can follow the same type of reasoning. In this case, however, it is important to take into account the storage cost. For instance, consider a forward contract on gold with maturity T. The storage cost of 1 ounce of gold during T periods is C_0 (we suppose the storage cost is paid entirely at date 0). We obtain the following result.

Result 3. *The forward price and the spot price of gold must satisfy the following relationship:*

$$F_0 = (1+r_f)^T (S_0 + C_0).$$

Note that if $F_0 < (1+r_f)^T (S_0 + C_0)$, there is an arbitrage opportunity only if it is possible to short-sell the underlying commodity and if the short-seller can receive the storage cost, which is in general not possible. However, when the underlying commodity is owned mainly for investment purposes (as it is the case for gold), investors who own the commodity, will find profitable to (i) sell the commodity, (ii) save the storage costs, (iii) invest the proceeds in the riskless asset and (iv) buy the forward contract if $F_0 < (1+r_f)^T (S_0 + C_0)$. Thus for commodities that are held only for investment purposes, we can expect the previous relationship to hold. If the underlying commodity is owned for other reasons ("consumption commodities"), then we can just assert that: $F_0 \leq (1+r_f)^T (S_0 + C_0)$.

3.3 Forward Foreign Exchange Contract *[not at the exam]*

A forward foreign exchange contract is an agreement, at date $t = 0$, to exchange an amount of foreign currency in the future. The exchange rate at which the currency will be traded is fixed at the signature of the contract. A foreign currency has the property that the holder of the currency can earn interest at the risk-free rate prevailing in the foreign country. This can be done by investing in default risk-free bonds denominated in the foreign currency. We will denote r_f^* this foreign risk-free rate and r_f the domestic risk-free rate. We also denote by S_0 the amount of domestic currency needed today to buy one unit of the foreign currency.

Result 4. *The forward exchange rate F_0 and the spot exchange rate S_0 must satisfy the following relationship, called the Covered Interest Rate Parity:*

$$F_0 = S_0 \left(\frac{1 + r_f}{1 + r_f^\star} \right)^T .$$

To understand this relationship, suppose that $F_0 > S_0 \left(\frac{1+r_f}{1+r_f^\star}\right)^T$, then an investor could:

1. borrow $S_0/(1 + r_f^\star)^T$ domestic currency on the domestic market for T years at rate r_f;
2. buy $1/(1 + r_f^\star)^T$ foreign currency on the spot market;
3. invest $1/(1 + r_f^\star)^T$ of foreign currency for T year at rate $r_f^\star$;
4. take a short position in a forward contract for one unit of the foreign currency.

The flow of domestic currency generated by this strategy is:

	$t = 0$	$t = T$
Borrow in the domestic market at rate r_f	$S_0/(1 + r_f^\star)^T$	$-S_0 \left(\frac{1+r_f}{1+r_f^\star}\right)^T$
Buy the foreign currency on the spot market	$-S_0/(1 + r_f^\star)^T$	0
Short position in the forward contract	0	F_0
Total	0	$F_0 - S_0 \left(\frac{1+r_f}{1+r_f^\star}\right)^T > 0$

whereas the flow of foreign currency generated by this trading strategy is:

	$t = 0$	$t = T$
Buy the foreign currency on the spot market	$1/(1 + r_f^\star)^T$	0
Invest the foreign currency at rate $r_f^\star$	$-1/(1 + r_f^\star)^T$	1
Short position in the forward contract	0	-1
Total	0	0

This strategy provides a positive net cash flow of domestic currency at date T and zero net cash flows of the foreign currency.

Similarly if $F_0 < S_0 \left(\frac{1+r_f}{1+r_f^\star}\right)^T$, then an investor could borrow $1/(1+r_f^\star)^T$ of the foreign currency at rate $r_f^\star$ for T years; sell the foreign currency for $S_0/(1+r_f^\star)^T$ of the domestic currency; invest this amount at the domestic rate r_f for T years; and take a long position in the forward contract for one unit of the foreign currency. This strategy provides a net cash flow in domestic currency of $S_0 \left(\frac{1+r_f}{1+r_f^\star}\right)^T - F_0 > 0$ at year T.

Chapter 4

Options

1 What Is An Option?

Definition 1. *An option is a security which gives its owner the right to:*

1. *buy (or sell) another asset (the underlying asset),*

2. *at a pre-specified price (the strike price or exercise price),*

3. *at or until a specific date (maturity or expiration date).*

The seller of the option has the obligation to sell (or to buy) the asset if the buyer of the option exercises his/her right.

It is important to note that the buyer of the option has a right and not an obligation to carry a trade at some future point in time. In contrast, the seller of the option has a commitment to carry a trade if the option buyer exercises her option. The seller of the option must receive compensation in order to endorse such a commitment. For this reason, acquiring an option is costly. This is a main difference between options and futures contracts. The price of an option is called the **option premium**.

The buyer of an option is said to have a long position in the option whereas the seller is said to have a short position. There are two basic types of options:

1. A **call option** is an option that gives the right to purchase the underlying asset at a predetermined price called the strike or exercise price (K).

2. A **put option** is an option that gives the right to sell the underlying asset at a predetermined price called the strike or exercise price (K).

Moreover, a **European option** is an option that can only be exercised on the expiration date $t = T$, whereas an **American option** is an option that can be exercised at any time $t \leq T$ up to the expiration date.

Option contracts can be traded on exchanges or in Over the Counter markets (OTC). Option contracts are negotiated on a wide variety of underlying assets, including stocks, foreign currencies, various commodities, futures and stock indices. The "size" of an option contract is the number of units of the underlying asset that must be delivered if the option is exercised. For instance, the size of stock options is in general 100 shares. The total amount that must be paid to acquire an option is equal to its size times the option price (that is, option prices are posted for one unit of the underlying asset).

Example Consider a European call option on stock ABC with maturity December 2017 and a strike price $K = 70$. The premium of the option is $C = 2.5$. The current price of one share of stock ABC is $S_0 = 62$. Let S_T be the price of ABC on the expiration date. December, we can distinguish two cases:

- Case 1: $S_T > K$. For instance $S_T = 72$. In this case, it is optimal for the option's buyer to exercise her option. In this way, she obtains a payoff equal to $S_T - K = 2$ per share. Her net profit (ignoring discounting) is equal to $2 - 2.5 = -0.5$.
- Case 2: $S_T > K$. For instance $S_T = 65$. In this case, the option's buyer is better off not exercising her option (there is no point in paying 70 for a stock that can be purchased 65 on the spot market). The net loss is equal to the option premium: the option buyer loses all of his/her initial investment in the option.

Options that trade on exchanges have standardized features. The exchanges choose the sizes of the option, the expiration dates and the strike prices at which options can be traded.

In the rest of this chapter, we will consider stock options. The concepts and the results apply to other types of options, however. Let S_t be the spot price at date t of the underlying stock of an option with strike price K.

- The call (put) option is said to be *in the money* if $S_t > K$ ($S_t < K$).
- The call (put) option is said to be *out of the money* if $S_t < K$ ($S_t > K$).
- The call or put option is said to be *at the money* if $S_t > K$.

An option will be exercised on the expiration date if it is in the money. It might not be optimal to exercise an American option before the expiration date, even if it is in the money.[1]

2 Options Payoffs

The payoff of a long position in a European option is defined as the gain which is obtained by the holder of the option at maturity.

For a **call** option, the payoff of a long position is:

$$\max(S_T - K, 0).$$

For a **put** option, the payoff of a long position is:

$$\max(K - S_T, 0).$$

[1]For instance, it can be shown that early exercise is never optimal for an American call on a stock which does not pay any dividend up to maturity. Early exercise of an American put can sometimes be optimal. We will come back to this point later.

Note that for a trader with a short position in the option, the payoffs are the opposite of the payoffs of a trader with a long position. Figures 1 and 2 represent the payoffs of a call option and a put option with a strike equal to K, for a buyer and a seller respectively. These graphics are called payoff diagrams and they are very frequently used to determine the payoffs of investment strategies using options (see the next section).

Figure 1: Payoff of a long or short position in a Call option at maturity

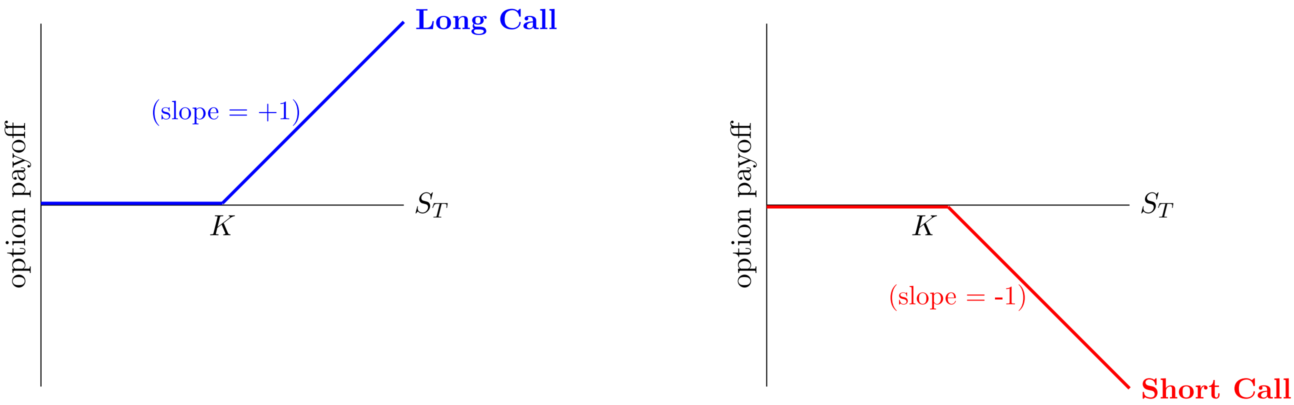
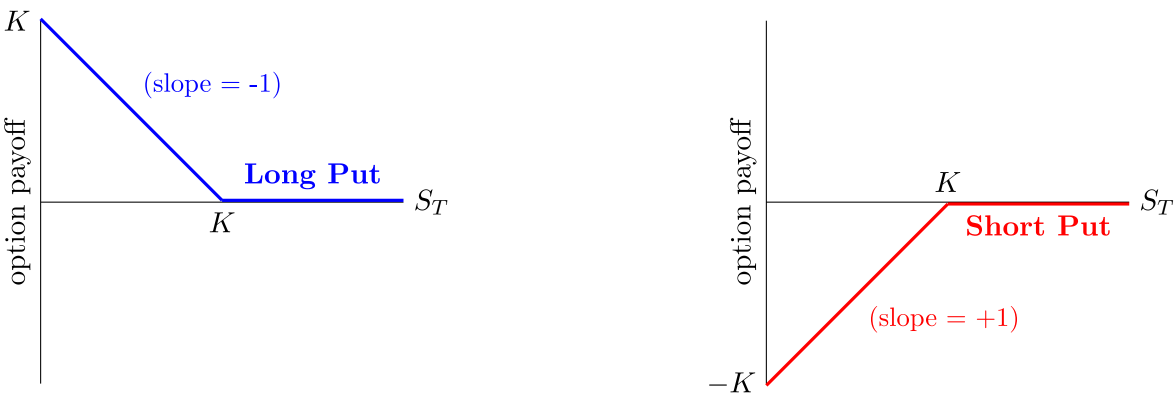


Figure 2: Payoff of a long or short position in a Put option at maturity



Definition 2. *The intrinsic value of an option at date* $t < T$ *is the payoff of the option if it were exercised immediately.*

- *The intrinsic value of a call option is* $\max(S_t - K, 0)$.
- *The intrinsic value of a put option is* $\max(K - S_t, 0)$.

Note that the price of an American option must at least be equal to its intrinsic value. (Why?) For European options (in particular European puts) that might not necessarily be the case. The price of an option is never negative, however. The difference between the price of an option and its intrinsic value is called the **time value**.[2] For options that are out of the money, the price is positive because of the time value (that is the value of the possibility for the option of expiring in the money)

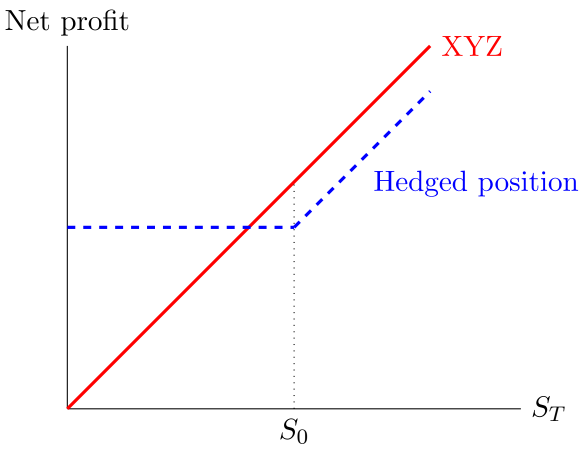
[2]Though this has nothing to do with the time value of money.

3 Insuring With Options

Options can be used to insure a position in a stock or in a portfolio of stocks against a decline in the price. We illustrate this point with an example.

Consider investor A who owns one share of stock XYZ.[3] The current price of XYZ is S_0. Investor A considers closing his position in this stock at date T. He is concerned by the fact that the price of XYZ could decline between date 0 and date T. In fact, without taking positions in other securities, the net profit on his portfolio (ignoring discounting) is as described by the plain line in Figure 3. Now investor A decides to buy one European put with maturity T on XYZ and a strike price $K = S_0$. The price of the put is P_0. Thus investor A now owns one portfolio that contains a long position in stock XYZ and a long position in a put on stock XYZ. The net profit on this portfolio depends on S_T and is depicted by the dashed line in Figure 3. (Why?)

Figure 3: Insuring with options



Note that the portfolio with the put option is such that investor A can still benefit from an increase in the stock price but is protected against a decline in this price. This is the main advantage of using options compared to futures. Of course, this insurance is obtained at a cost which is equal to the option price.

This example is very simple. More complex insurance strategies can be built with options. Consider the following one.

Question: What is the net profit on a portfolio that contains a long position in a European call option with maturity T on stock XYZ and a long position in discount bond with maturity T and face value K, where K is the strike price of the call?

3.1 Portfolios of Options

One of the attractions of options is that they can be used to create a very wide range of payoff patterns. In particular a speculator can build a portfolio of options on the basis of his anticipations regarding the

[3]The argument does not depend on the size of the position that we try to insure.

future evolution of the underlying asset price. In the following we discuss the most common trading strategies involving options.

3.1.1 Spreads

A spread trading strategy consists in taking positions in two or more options of the same type (for example two or more puts with the same maturity and the same underlying asset) but with different strike prices.

Examples:

- A **bull spread** involves a long position in a call option on a stock with a certain strike price and a short position in a call of the same type and a higher strike price. The cash flow generated by a bull spread at maturity of the option is represented in Figure 4. An investor that buys a bull spread profits from an increase in the price of the underlying asset but gives up some of the upside potential by selling a call option with a higher strike price. The short position in the call reduces the cost of this trading strategy.

- A **bear spread** can be created by buying a put option on a stock with a certain strike price and selling a put of the same type but with a lower strike price. The cash flow generated by a bull spread at maturity of the option is represented in Figure 4. An investor that buys a bull spread profits from a decrease in the price of the underlying asset but gives up some of the downside potential by selling a put option with a lower strike price. The short position in the call reduces the cost of this trading strategy.

- A **butterfly spread** involves position in options with three different strike prices. Its cash flow at maturity is illustrated in the bottom diagram of Figure 4. It can be obtained with a long position in a call option with a low strike price K_1, a long position in a call option with high strike price $K_3 > K_1$ and selling two call options with a strike price $K_2 = (K_1 + K_3)/2$ halfway between K_1 and K_2. A butterfly spread provides the highest profit when the price of the underlying asset is equal to K_2 and therefore it represents an appropriate strategy for traders who feel that the price of the underlying asset will remain close to K_2.

Question: Is it possible to build a bull spread, a bear spread and a butterfly spread with portfolios of put options and call options respectively? How?

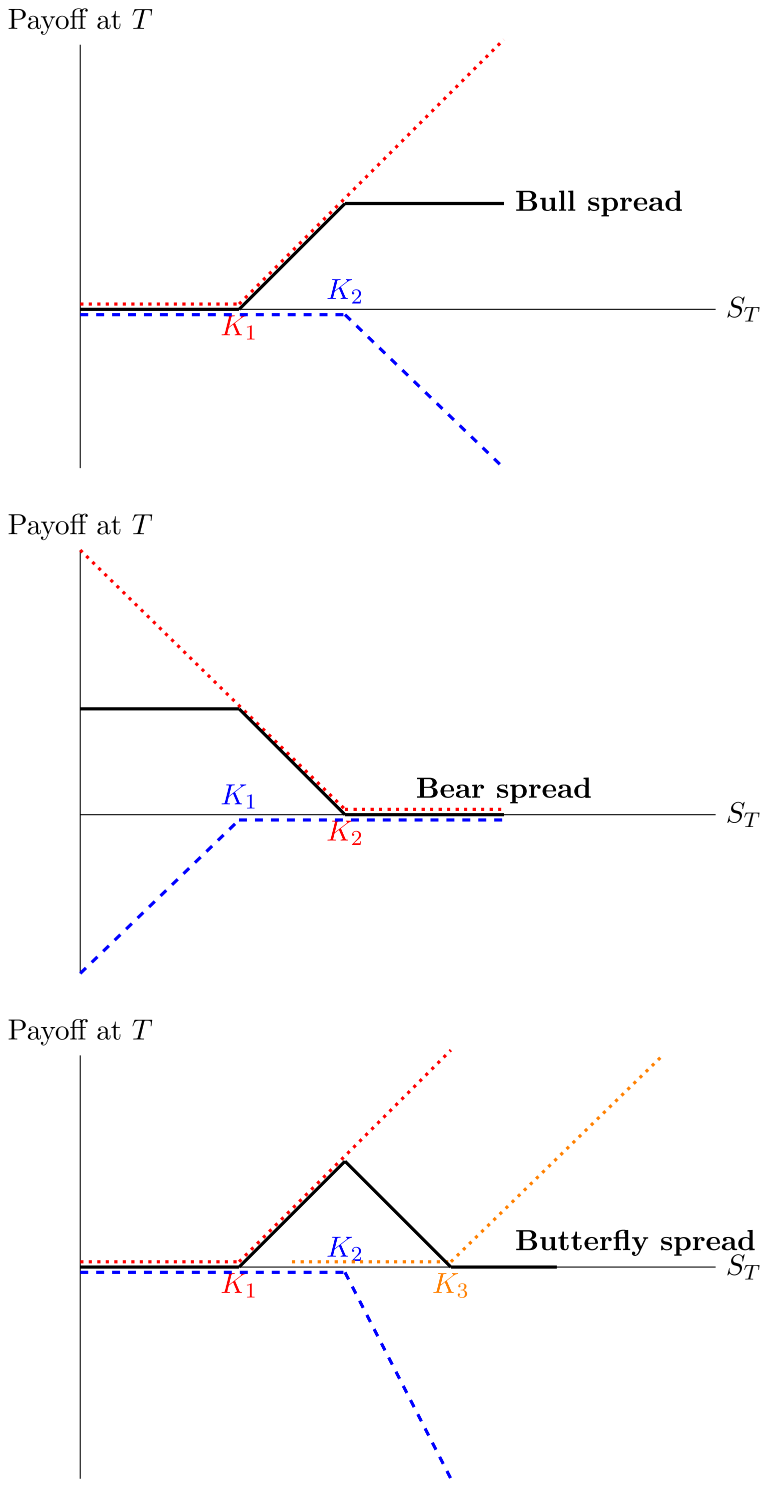
3.1.2 Combinations

A combination is an option trading strategy that involves taking positions in both calls and puts option on the same underlying asset.

Examples:

- A **straddle** involves buying a call and a put option with the same strike price. This strategy provides large payoffs when there is a sufficiently large price movement in either direction. When the strike price of the call and put options are different we obtain a **strangle**. The trader that implements one of these two trading strategies expects that the price of the underlying asset will

Figure 4: Spreads Payoffs

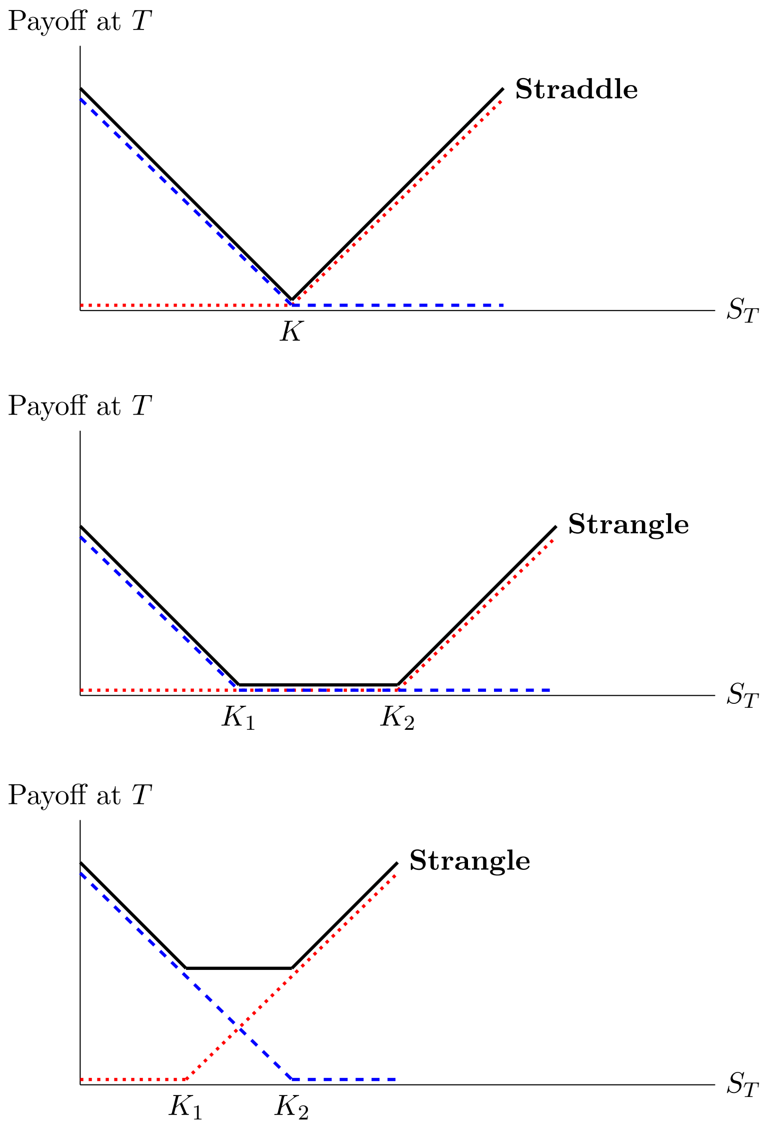


have a large movement but he is uncertain as to its direction. Note that for a given price of the put option in the portfolio a strangle is cheaper than a straddle.

- A **strip** consists of a long position in one call option and two put options with the same strike price and expiration date. A **strap** consists of a long position in two calls and one put with the same strike price and expiration date.

Question: What are the payoff profiles generated by a strip and a strap? What are the expectation of an investor that buys a strip?

Figure 5: Combinations Payoffs



4 Option Pricing

Option pricing consists of finding the relationship between the price of an option and the price of the underlying asset. In general, the option price will be influenced by other factors such as the time to maturity, the volatility of the price of the underlying asset, the strike price and the riskless interest rate. The purpose of option pricing is also to uncover the impact of a change in each of these factors on the price of an option. Note that the impact of these factors can be different for put and call options.

As for futures contracts, we can use the arbitrage approach to obtain the price of the option. There are two ways to use this approach for option pricing:

1. In the first case, no specific assumptions are made on the behavior of the underlying stock price

over time. In this case, it is possible to determine upper and lower bounds for the price of an option but it is not possible to determine exactly the option price. We will give one example below.

2. In the second case, a specific model for the behavior of the underlying stock price is postulated. With such a model, in some cases (European options), it is possible to obtain by arbitrage the price of an option. We will give an example of this approach by considering the binomial model in the case of a European call option. For American options, the analysis can be much more complex because of the possibility of early exercise.

4.1 Arbitrage Bounds: An Example

Let us consider a European call with strike K. The expiration date is in T periods and the riskless interest rate is r_f. Let S_0 be the price of the underlying stock at date $t = 0$. Let C_0 be the price of the European call at the same date. This price must satisfy the following relationships.

Result 1. *The price of a European call:*

- *must be larger than* $\max\left(S_0 - \frac{K}{(1+r_f)^T}, 0\right)$,
- *must be lower than the price of the stock* S_0.

If the price of the European call does not satisfy one of these conditions, then there is an arbitrage opportunity.

4.2 The Put-Call Parity

Consider two European options written on the same underlying stock. One option is a put option and the other is a call option. They have the same strike price K and they expire at the same date. Let C_0 and P_0 be the prices of these two options at date 0. Finally let S_0 be the price of the underlying asset at date 0 and let r_f be the riskless discount rate.

Result 2. *The prices of the call, the put and the underlying stock must satisfy the following relationship:*

$$C_0 = S_0 - \frac{K}{(1+r_f)^T} + P_0.$$

The previous equation is called the **put-call parity**. This relationship is proved by arbitrage and holds only for put and call options that are written on the same underlying asset and that expire on the same date. There are two interesting implications:

1. Once we know the price of a European call, we can deduce the price of a European put using the put-call parity relationship, and vice-versa. This is the reason why below we only focus on the pricing of a European call.

2. The previous relationship (and the proof of Result 2) shows that a call can be viewed as a portfolio that contains (i) a long position in the underlying stock, (ii) a short position in a discount bond with face value K and (iii) a long position in a European put. This provides a method to build a *synthetic call*.

Figure 6: Binomial Trees

A: One Period

$$S_t \nearrow S_{t+\Delta} = u\,S_t$$
$$S_t \searrow S_{t+\Delta} = d\,S_t$$

B: Two Periods

$$S_0 \nearrow S_1 = u\,S_0 \nearrow S_2 = u^2\,S_0$$
$$S_1 = u\,S_0 \searrow S_2 = u\,d\,S_0$$
$$S_0 \searrow S_1 = d\,S_0 \nearrow S_2 = u\,d\,S_0$$
$$S_1 = d\,S_0 \searrow S_2 = d^2\,S_0$$

4.3 Using Binomial Trees to Price Options *[only the one-period binomial model is at the exam]*

4.3.1 Modeling the Behavior of Stock Prices

Let S_t be the price of a stock at date t. Our purpose is to propose a model that describes the evolution of the stock price over time. The binomial approach consists in assuming that, in a given period of time, the return on a stock can take only two values that we will denote u and d (short-hands for up and down). Let Δ be the length of a period (1 week, 1 day, etc.). The binomial approach assumes that:

$$S_{t+\Delta} = \begin{cases} u\,S_t & \text{with probability } p \\ d\,S_t & \text{with probability } 1-p \end{cases}$$

It is usual to represent graphically this process as in panel A of Figure 6. Eventually, all the possible paths for the stock price over a given interval of time, say $[0,T]$, can be represented graphically in the form of a tree, called the binomial tree. For instance, if the interval $[0,T]$ is cut in two subperiods of length: $\Delta = T/2$, the possible paths for the stock prices are given in the binomial tree which is represented in panel B of Figure 6.

Note that the number of final (and intermediate) nodes becomes larger as the number of subperiods increases (that is, as Δ decreases). Thus by shortening the period of time between price changes, we can obtain a more and more accurate description of the distribution of stock prices at date T. Of course, u and d must get closer to one when Δ decreases (the size of the absolute variation in the price of a stock is not the same in one second or in one week).

The binomial approach to option pricing has been developed by Cox, Ross and Rubinstein (1979).[4] It is widely used in practice.

[4]Cox, J., Ross, S. and Rubinstein, M. (1979): "Option Pricing: A Simplified Approach", Journal of Financial Economics, 229-264.

4.3.2 Option Pricing

Now, we will show how to value a European call option on a stock, assuming that the stock price follows a binomial process. Let r_f be the riskless rate of interest. Absence of arbitrage requires: $u > 1 + r_f > d$, an assumption that we will make in what follows.

We present the method using an example in which $u = 110\%$, $d = 95\%$, $r_f = 7\%$. The strike price of the call option is $K = 100$. Let assume first that we are at date $t = 1$, the option expires at date $t = 2$. The price of the stock is $S_0 = 110$. The possible prices for the stock on the expiration date are given by Figure 6. Let C_t be the price of the call at date t. On the expiration date, the value of the call is equal to its intrinsic value, that is $C_2 = \max(S_2 - K, 0)$. Since the stock price can take two values at date 2, there are two possible values for the call on the expiration date. If the stock price increases, then $C_2 = 121 - 100 = 21$. If the stock price decreases, then $C_2 = 104.5 - 100 = 4.5$.

In order to determine the price of the call at date 1, we construct a replicating portfolio R that replicates the payoff of the call at date 2. The replicating portfolio is a combination of the riskless asset and the stock. We denote n_{S_1} the number of shares of the stock in this portfolio and n_{B_1} the amount that is invested in the riskless asset. They must satisfy the following constraints:

$$\begin{cases} 21 = n_{S_1} 121 + n_{B_1}(1 + 7\%) \\ 4.5 = n_{S_1} 104.5 + n_{B_1}(1 + 7\%) \end{cases}$$

The solution of this system of equations is $n_{S_1} = 1$ and $n_{B_1} = -93.45$. The replicating portfolio contains a long position (one share) in the underlying stock and a short position (which is worth 93.45) in the riskless asset. This shows that a call can be viewed as a long position in the underlying stock and a short position in the riskless asset. Let V_{1R} be the value of this portfolio at date 1. Absence of arbitrage opportunities imposes:

$$V_{1R} = C_1.$$

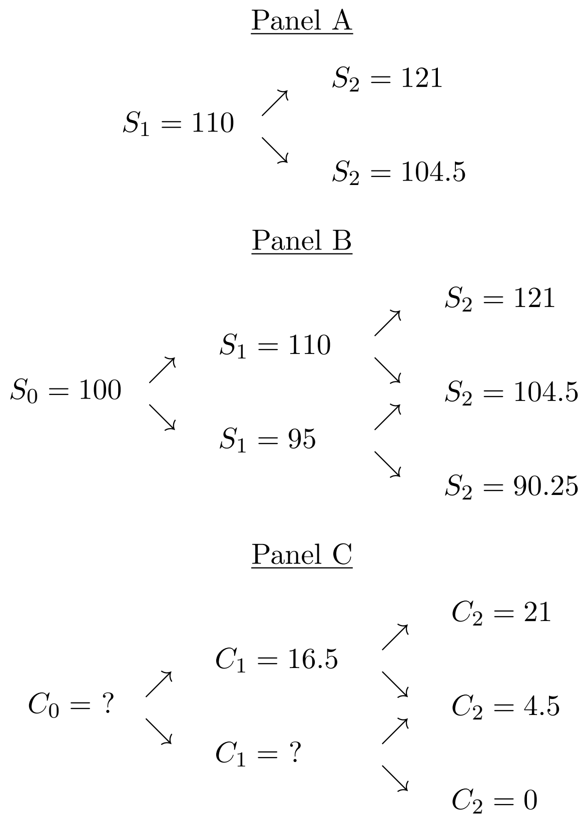
Why? As $V_{1R} = 16.54$, we deduce that the call option at date 1 is worth $C_1 = 16.54$.

Can we generalize this line of reasoning when there is more than one period before maturity? Yes. But this requires changing the composition of the replicating portfolio over time, that is to follow a **dynamic arbitrage strategy**. We show how to proceed in the case of the previous example when we have two periods of time before maturity. The possible paths for the stock price are represented by the binomial tree in Figure 7, panel B. In panel C, we represent the process which is followed by the price of the call. At time 1, we know how to compute the value of the call when the stock price is $S_1 = 110$. Using a similar reasoning, we can compute the value of the call when $S_1 = 95$. In this case the replicating portfolio is $n'_{S_1} = 0.3157$ and $n'_{B_1} = -26.64$. We deduce that $C_1 = 3.36$ when the value of the stock is $S_1 = 95$. Note that, consistent with the intuition, the price of the call increases with the stock price.

Now, we construct a portfolio, R", at date 0 which has value 16.54 if the stock price is $S_1 = 110$ at time 1 and value 3.36 if the stock price is $S_1 = 95$. Let n_{S_0} and n_{B_0} be number of shares of the underlying stock and the amount invested in the riskless asset in portfolio R''. They must satisfy:

$$\begin{cases} 16.54 = n_{S_0} 110 + n_{B_0}(1 + 7\%) \\ 3.36 = n_{S_0} 95 + n_{B_0}(1 + 7\%) \end{cases}$$

Figure 7: Dynamic Arbitrage Strategy



and we find: $n_{S_0} = 0.878$ and $n_{B_0} = -74.85$. The portfolio R" can be used to replicate the final payoffs of the European call. This requires however to change the composition of the portfolio at date 1. But by construction, this requires no additional inflows or outflows of money. The replicating portfolio is said to be **self-financed**. To see this point, suppose that at time 1, the stock price turns out to be $S_1 = 95$. Then we will sell $n_{S_0} - n'_{S_1} = 0.563$ shares at $S_1 = 95$. We receive $53.46 = 95 \times 0.563$ from this sale that we use to decrease our short position in the riskless asset to: $74.85(1 + 7\%) - 53.46 = 26.63$. In this way, we end up with the replicating portfolio that we need for the last period of time before expiration.

As R" is a self-financed replicating portfolio, its value is equal to the value of the European call. This means that:

$$C_0 = 0.878 \times 100 - 74.85 = 13.00$$

This type of dynamic replicating strategy can be used to determine the price of the call, whatever the number of periods before maturity.

Using the previous approach, we can show the following properties of the price of a European call:

1. The price of the call increases with the price of the underlying stock (S_0).
2. The price of the call decreases with the strike price K. (Why?)
3. The price of the call increases with the riskless interest rate r_f. (Why?)
4. The price of the call increases with the volatility of the stock (as measured by $u - d$).

4.3.3 A Limit Case: The Black and Scholes Formula

Let the maturity T of the option be given. What happens to the option price when the length of a subperiod (Δ) becomes smaller and smaller (goes towards zero)? In general the answer will depend on how we specify the dependence between u, d and Δ. Let assume that:

$$u = e^{\sigma\Delta}$$
$$d = e^{\sigma\Delta}$$

In this case it is possible to show (difficult!) that when Δ goes to zero, the option price at date $t = 0$ becomes:

$$C_0 = S_0 \ \Phi(d_1) - \frac{K}{(1+r_f)^T} \ \Phi(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r_f + \frac{\sigma^2}{2}T\right)}{\sigma\sqrt{R}}$$
$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r_f - \frac{\sigma^2}{2}T\right)}{\sigma\sqrt{R}}$$

and $\Phi(.)$ the cumulative Normal distribution. This formula is the celebrated **Black and Scholes formula**. The only parameter that enters the formula and which is not directly observable is σ. This parameter is called the volatility of the stock and can be estimated with statistical methods.