Market Microstructure
Auctions

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What is the common feature of the following things?

- Flowers
- Diamonds
- Artworks
- Wine
- Company Subsidiaries
- Houses
- Electricity
- Treasury Bills
- Common shares
- Copyrights
- Drilling rights for minerals
- UMTS licenses
- Access to railroad interconnection points
- Women

All this is sold, or has been sold in the past, through

Why auctions?

- Auctions are used in several sectors of the economic activity.
- A huge volume of economic transaction is conducted through auctions.
- Auctions are a simple and well-defined and intensively studied economic environment.
- The logic of competitive bidding is at the core of many financial transactions.

Core questions in auction theory

- How should buyers bid in the auction?
  FMM ex: what is the limit order price?

- What is the seller expected revenue?
  FMM ex: what is a market order expected profit?

- How can we relate the selling price with
  - Auction mechanism, (ex market mechanism)
  - Number of bidders, (ex liquidity of the market)
  - Amount and nature of asymmetry of information?
    (Information efficiency of the market)

- How should the seller choose the auction mechanism?
  FMM ex: Financial Intermediation, market regulation, competition among markets.

The seller's problem

- You own a valuable good.
- You know there are $N > 1$ potential buyers for this good.
- You do not know exactly how much each potential buyer values the good:

$$Pr[\tilde{V}_i < z] = F_i(z)$$

where $\tilde{V}_i$ is potential buyer *i*'s valuation for the object.

- You would like to sell
  - at the maximum possible price;
  - quickly;
  - in a transparent way;

HOW?

Solution 1: posted price

The seller calls a non-negotiable price and hopes somebody willing to pay that price will show-up.

- Y: Seller valuation for the object
- $1 - G(p)$: Probability that there is at least one buyer that values the object at least p.

$$\max_p (p - Y)(1 - G(p))$$

$$\text{f.o.c.: } 1 - G(p) = G'(p)(p - Y)$$

Drawbacks:

- Tie-break-rule: several buyers
- Waiting cost: no buyer

Solution 2: auction

Definition

An **auction** is a bidding mechanism, described by a set of auction rules that specify

- How the winner is determined.
- How much the winner and other bidders have to pay to the seller.

Advanages:

- Speed of sale;
- Information revelation about buyer's valuations;
- Equal chances to all potential buyers;
- Prevent dishonest dealing between the seller's agent and buyers.

Some standard auction formats

- **First price sealed bid auction:** The bidders simultaneously submit sealed bids. The highest bidder wins the object and pays his bid.

  Applications: Divestitures, market makers competition in decentralized markets, drilling rights for minerals, telecom licenses, antiques.

- **Second price sealed bid auction:** The bidders simultaneously submit sealed bids. The highest bidder wins the object and pays a price equal to the second highest bid.

Some standard auction formats

- **English auction:** The auctioneer starts the bidding at some price. The bidders proclaim successively higher bids until no bidder is willing to bid higher. The bidder who submitted the final bid wins and pays a price equal to his bid.

  Applications: Mergers, market makers competition in centralized markets, Artworks, used cars, houses, radio communication licenses, Internet auctions.

- **Dutch auction** The price continuously decreases on a "wheel" in front of the bidders until one yells "stop!". That bidder wins and pays the price at where the wheel stopped.

Some other auction formats

- **Japanese auction**: The price continuously increases on a "wheel" in front of the bidders until all bidders but one leave the room. The last bidder in the room wins the object and pays the price at where the wheel stopped.
- **E-bay auction**: The bidders submit sealed bids during a bidding period. During the bidding period bidders observe the second highest bid. The highest bidder wins the object and pays the second highest bid.
- **All-Pay auction**: The bidders proclaim successively higher bids until no bidder is willing to bid higher. The bidder who submitted the final bid wins and every bidder pays a price equal to his last bid.
- **Uniform price auction**: Bidders submit demand functions. The good is sold at a price such that the demand is equal to the supply.
- **Survival auction**: Rounds of sealed bids. At each round the lowest bidder exits the auction. The minimum bid for the

Formal description of an auction mechanism

1. A set N of bidders;
2. For each bidder i a set of available actions X_i;

$$X := \times_{i \in N} X_i$$

3. Allocation rule:
   - Winning function: Probability with which each bidder wins given $x \in X$

$$Q : X \to \Delta N$$

   - Payment function: Cash transfer to each bidder given $x \in X$

$$P : X \to \Delta \mathbf{R}^N$$

Examples

Let $x \in X$,
$x_{-i} := \{x_1, x_2, \dots, x_{i-1}, x_{i+1}, x_n\}$,
Indicator function: $\mathbf{1}_{\{a\}}$ equals 1 if a is true and 0 otherwise.

- First price auction: $X_i = \mathbf{R}^+$, $Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -Q_i(x)x_i$.
- Second price auction: $X_i = \mathbf{R}^+$, $Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -Q_i(x)\max(x_{-i})$.
- All-pay auction: $X_i = \mathbf{R}^+$, $Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -x_i$.
- Survival auction: $X_i = (R^+)^{n-1}$, $Q_i(x) = \mathbf{1}_{\{x_i^j > \min(x^j), \forall j < n-1\}} \mathbf{1}_{\{x_i^{n-1} = \max(x^{n-1})\}}$, $P_i(b) = -Q_i(b)\max(x_{-i}^{n-1})$.

Auction as a Bayesian game

We will model bidders competition as a **non-cooperative game with incomplete information.**

Bidders' payoff functions:

- There are N risk neutral bidders
- Let V_i be bidder i's valuation for the object
- If bidder i wins the object, pays p, his ex-post payoff is

$$V_i - p$$

Private value or common value?

- **Private value framework**: Each bidder's valuation is independent from the other bidders' valuations.
  - Each bidder knows how much he/she values the object,
  - V_i does not depend on what the other bidders know.

- **Common value framework**: The object has the same value to all bidders.

$$V_i = V, \forall i$$

  However, bidders might differ in the information about this value.

Independent Private Value

Assumptions:

1. Private value framework:
   - Each bidder i exactly knows V_i
   - V_i does not depend on what the other bidders know
2. Identically independently distributed valuations. For any $j \neq i$, bidder j believes that

$$\Pr[\tilde{V}_i < z] = F(z)$$

   $\tilde{V}_i \in [0, 1]$

3. The bidders are risk neutral: if bidder i wins the object and pays p , then his ex-post payoff is

$$V_i - p$$

Probability preliminaries

- Let $\{\tilde{V}_i\}_{i=1,\dots,N}$ be N random variable i.i.d. with cumulative distribution $F(.)$ and density $f(.) := F'(.)$
- Let $\tilde{V}^{(1)}$ and $\tilde{V}^{(2)}$ be the highest and the second highest element of $\{\tilde{V}_i\}_{i=1,\dots,N}$
- Let $F^{(1,N)}$ and $F^{(2,N)}$ be the cumulative distribution functions of $\tilde{V}^{(1)}$ and $\tilde{V}^{(2)}$, respectively.

Then:

$$F^{(1,N)}(z) = F(z)^N \quad (1)$$

$$f^{(1,N)}(z) = Nf(z)F(z)^{N-1} \quad (2)$$

$$F^{(2,N)}(z) = F(z)^N + NF(z)^{N-1}(1-F(z)) \quad (3)$$

$$f^{(2,N)}(z) = N(N-1)f(z)F(z)^{N-2}(1-F(z)) \quad (4)$$

Bidder's strategies and expected payoffs

Fix the auction format (X, Q, P).

- If bidder i choses action $x_i \in X_i$ and the other bidders' action profile is x_{-i}, then bidder i's payoff is

$$V_i Q_i(x_i, x_{-i}) + P_i(x_i, x_{-i})$$

- A bidder's (pure) strategy σ_i maps a bidder's valuation V_i into an action:

$$\sigma_i : [0, 1] \to X_i$$

- If bidder i chooses action $x \in X_i$ and the others' strategies are σ_{-i}, then bidder i's expected payoff is

$$V_i E[Q_i(x, \sigma_{-i}(\tilde{V}_{-i}))] + E[P_i(x, \sigma_{-i}(\tilde{V}_{-i}))]$$

Bayesain Nash Equilibrium

Definition

A **Bayesian Nash equilibrium** specifies a bidding strategy $\sigma_i^*(.)$ for each bidder i, such that each bidder would be maximizing his own expected payoff given his valuation and the other players' strategies.

$$\sigma_i^*(V_i) \in \arg\max_{x \in X_i} V_i E[Q_i(x, \sigma_{-i}^*(\tilde{V}_{-i}))] + E[P_i(x, \sigma_{-i}^*(\tilde{V}_{-i}))]$$

In a symmetric framework, a symmetric equilibrium satisfies:

$$\sigma_i^* = \sigma^*, \forall i$$

Strategic Equivalence 1

Proposition

The Dutch auction and the first price auction are strategically equivalent.

Proof:

1. First price auction: $X_i = \mathbf{R}^+$, $Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -Q_i(x)x_i$
2. Dutch auction: $X_i = \mathbf{R}^+$, $Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -Q_i(x)x_i$
3. The information of a bidder when he sets his bid, and conditional on winning, is the same in the two auctions.

An equilibrium strategy profile of the Dutch auction is an equilibrium if and only if it is an equilibrium of the first price auction.

Strategic Equivalence 2

Proposition

Under Assumptions 1-3 the second price sealed bid auction and the Japanese auction (and the survival auction) are strategically equivalent.

Proof:

1. Second price auction: $X_i = \mathbf{R}^+$, $Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -Q_i(x)\min(x_{-i})$
2. Japanese auction: $X_i = \mathbf{R}^+$, $Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -Q_i(x)\min(x_{-i})$
3. The relevant information of a bidder when he sets his bid, and conditional on winning, is the same in the two auctions.

In the independent private value framework an equilibrium strategy profile of the Japanese auction is an equilibrium if and

Strategic Equivalence 3

Proposition

Every Bayesian Nash equilibrium of the Japanese auction induces a Bayesian Nash equilibrium of the English auction.

Proof:

1. Let $\sigma_i^*(V_i)$ be the equilibrium exiting times in the JA for bidder i.
2. In the EA, all bidders bidding the standing high bid plus an arbitrarily small bid increment in each round and stopping to bid according to these exiting times constitutes an (arbitrarily close) Bayesan Nash equilibrium of the EA.

Equilibrium of the Second Price Auction

- Fix bidder i.
- Let $\tilde{z} \geq 0$ be bidder i's competitors' highest bid.
- Bidder i believes that $\Pr[\tilde{z} < z] = G(z)$
- Bidder i expected payoff from bidding x is

$$\int_0^x (V_i - z) dG(z)$$

Equilibrium of the Second Price Auction

Proposition

*(**Truthtelling equilibrium**) Under Assumptions 1-3, in the second price sealed bid auction, bidding your own valuation is a weakly dominant strategy for all bidders. The strategy profile*

$$\sigma_i(V_i) = V_i, \forall i$$

is a Bayesian Nash equilibrium in undominated strategies of the second price sealed bid auction.

Equilibrium of the Second Price Auction

Corollary

In the truth telling equilibrium of the second price auction:

1. *The winner of the object is the bidder with the highest valuation.*
2. *The ex-ante expected payoff for a bidder with valuation V is*
$$\int_0^V F(z)^{N-1} dz$$
3. *The seller's expected revenue is*
$$E[\tilde{V}^{(2,N)}] = N \int_0^1 \left(z - \frac{1-F(z)}{f(z)} \right) F(z)^{N-1} f(z) dz$$

Derivation of symmetric equilibrium

Consider an auction format where

- $B = \mathbf{R}$,
- $Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$,
- $P_i(x) = P(x_i, \max(x_{-i}))$

Consider a Symmetric equilibrium such that

- $\sigma_i = \sigma, \forall i$
- $\sigma : [0, 1] \to \mathbf{R}$,
- σ is increasing and differentiable

Then bidder i chooses to behave like a bidder of type w then his expected payoff is :

$$\Pi(V_i, w) := V_i G(w) - \int_0^1 P(\sigma(w), \sigma(z)) dG(z)$$

where $G(z) = F(z)^{N-1}$

Derivation of symmetric equilibrium: First order condition

$$\left.\frac{\partial \Pi(V, w)}{\partial w}\right|_{w=V} = 0$$

This typically provides a differential equation in $\sigma(.)$ that can be solved imposing the condition

$$\sigma(0) = 0$$

Derivation of symmetric equilibrium: Second order condition

Quasi-concavity of the objective function:

$$\frac{\partial \Pi(V,w)}{\partial w} > 0 \quad \text{for } w < v$$
$$\frac{\partial \Pi(V,w)}{\partial w} < 0 \quad \text{for } w > v$$

Thus, it is sufficient to show that

$$\frac{\partial^2 \Pi(V,w)}{\partial w \partial V} > 0$$

Quasi-concavity and comparative statics

Remark:
Let $x_i(V) = \sigma(w)$ where w solve

$$\max_{w \in [0,1]} \Pi(V, w) := VG(w) - \int_0^1 P(\sigma(w), \sigma(z)) dG(z)$$

Let κ be a parameter of the model and suppose

$$\frac{\partial^2 \Pi(V, w)}{\partial w \partial \kappa} > 0 (resp. < 0)$$

Then $x_i(V)$ is an increasing (resp. decreasing) function of κ.

Symmetric equilibrium of the first price auction

$$\Pi(V,w)=\int_0^w (V-\sigma(w))dG(z)=(V-\sigma(w))G(w)$$

First order condition:

$$\left.\frac{\partial\Pi(V,w)}{\partial w}\right|_{w=V}=(V-\sigma(V))g(V)-\sigma'(V)G(V)=0$$

Thus,

$$\sigma(V)g(V)+\sigma'(V)G(V)=Vg(V)$$

with $\sigma(0)=0$ one has

$$\sigma(v)=\int_0^V z\frac{g(z)}{G(V)}dz=E\left[\tilde{V}^{(1,N-1)}|\tilde{V}^{(1,N-1)}\leq V\right]$$

Symmetric equilibrium of the first price auction

Second order condition

$$\frac{\partial^2 \Pi(V, w)}{\partial x \partial V} = g(w) > 0$$

Revenue Equivalence Theorem

Theorem

Under assumptions 1-3 given any auction mechanism.
If in equilibrium:

1. *the bidder who has got the highest valuation for the object is certain to win the object;*
2. *A bidder who values the object at its lowest possible level has an expected payoff of 0.*

Then:

1. *The expected profit for a bidder with valuation* V *is* $\int_0^V F(z)^{N-1} dz$
2. *The revenue generated for the seller is the expected value of the object to the second highest evaluator:*

$$E[\tilde{V}^{(2,N)}] = N \int_0^1 \left(z - \frac{1 - F(z)}{f(z)} \right) F(z)^{N-1} f(z) dz$$

Revenue Equivalence Theorem: Proof 1/4

Take any equilibrium end consider a bidder i of type V.
Let

- Equilibrium probability that bidder i wins:

$$Q_i^*(V) := E\left[Q_i(\sigma_i^*(V), \sigma_{-i}^*(\tilde{V}_{-i})\right]$$

- Equilibrium expected payment to bidder i:

$$P_i^*(V) := E\left[P_i(\sigma_i^*(V), \sigma_{-i}^*(\tilde{V}_{-i})\right]$$

- Equilibrium expected payoff for bidder i:

$$\Pi_i(V) := VQ_i^*(V) + P_i^*(V)$$

Revenue Equivalence Theorem: Proof 2/4

Revelation principle:

1. Suppose bidder *i* choses to behave as if his type was w, then his payoff would be $VQ_i^*(w) + P_i^*(w)$.
2. In equilibrium it must be that $\max_w VQ_i^*(w) + P_i^*(w) = \Pi_i(V)$.
3. First order condition gives:

$$\left.\frac{\partial VQ_i^*(w) + P_i^*(w)}{\partial w}\right|_{w=V} = 0 \Rightarrow VQ^{*\prime}_i(V) + P^{*\prime}_i(V) = 0$$

4. Differentiating $\Pi_i(V)$

$$\Pi_i'(V) = \underbrace{VQ^{*\prime}_i(V) + P^{*\prime}_i(V)}_{=0} + Q_i^*(V) = Q_i^*(V)$$

5. Hence

$$\Pi_i(V) = \Pi_i(0) + \int_0^V Q_i^*(z)dz$$

Revenue Equivalence Theorem: Proof 3/4

The theorem hypothesis are

1. the bidder who has got the highest valuation for the object is certain to win the object:

$$Q_i^*(V) = F(V)^{N-1}, \forall i$$

2. bidders who values the object at its lowest possible level has an expected payoff of 0:

$$\Pi_i(0) = 0, \forall i$$

Hence

$$\Pi_i(V) = \Pi_i(0) + \int_0^V Q_i^*(z)dz = \int_0^V F(z)^{N-1}dz$$

Revenue Equivalence Theorem: Proof 4/4

Bidder i expected payment to the seller is

$$-P_i^*(V) = VQ_i^*(V) - \Pi_i(V) = Q_i^*(V)V - \int_0^V F(z)^{n-1}dz$$

Bidder i ex-ante expected payment to the seller is

$$\begin{aligned}
-\int_0^1 P_i^*(v)f(v)dv = \int_0^1 vQ_i^*(v)f(v)dv &- \int_0^1\int_0^v Q(z)f(v)dzdv \\
= \int_0^1 vQ_i^*(v)f(v)dv - \int_0^1\int_z^1 Q_i^*(z)f(v)dvdz &= \int_0^1 Q_i^*(z)zf(z)dz - \int_0^1 Q_i^*(z)(1-F(z))dz \\
= \int_0^1 Q_i^*(z)\left(z - \frac{1-F(z)}{f(z)}\right)f(z)dz &= \int_0^1 \left(z - \frac{1-F(z)}{f(z)}\right)f(z)F(z)^{N-1}dz
\end{aligned}$$

Considering that there are N bidders, the seller's expected revenue is

$$N\int_0^1 \left(z - \frac{1-F(z)}{f(z)}\right)f(z)F(z)^{N-1}dz$$

Implications, caveat, and use of the Revenue Equivalence Theorem

- In the independent private value framework, bidders and sellers are indifferent among the different auction mechanisms.
- This applies only in equilibria where the hypotheses are met, however auctions might have other equilibria that do not satisfies the r.e.t. hypothesis.
- The Revenue Equivalence Theorem can be used to derive equilibria.

Deriving Equilibria: First Price Auction

Consider an equilibrium satisfying the RET Hypothesis. Then:

1. The equilibrium probability that Bidder *i* wins: $Q_i^*(V) = F(V)^{N-1}$
2. Bidder *i*'s equilibrium payoff : $\Pi_i(v) = VQ_i^*(V) + P_i^*(V) = \int_0^V F(z)^{N-1}dz$
3. Bidder *i*'s expected payment:
   $-P_i^*(v) = VQ_i^*(V) - \Pi_i(v) = VF(V)^{N-1} - \int_0^V F(z)^{N-1}dz$
4. In a FPA $-P_i^*(V) = \sigma^{FPA}(V)F(V)^{N-1}$
5. Equations 2 and 3 give

$$\sigma^{FPA}(V) = V - \int_0^V \left(\frac{F(z)}{F(V)}\right)^{N-1} dz = E\left[\tilde{V}^{(1,N-1)} | \tilde{V}^{(1,N-1)} \leq V\right]$$

Remarks:

- In a first price auction bidders bid less then, their valuation.
- When N goes to infinity competition increases and the underbidding goes to 0.

Deriving Equilibria: All-Pay Auction

Consider an equilibrium satisfying the theorem Hypothesis. Then:

1. Bidder i's expected payment: $-P_i^*(v) = VF(V)^{N-1} - \int_0^V F(z)^{N-1}dz$
2. In a APA $-P_i^*(V) = \sigma^{APA}(V)$
3. Equations 1 and 2 give

$$\sigma^{APA}(V) = VF(V)^{N-1} - \int_0^V F(z)^{N-1}dz = \sigma^{FPA}(V)F(V)^{N-1}$$

Remarks:

- In an all-pay auction bidders bid less then what they bid in a FPA.
- When N goes to infinity competition increases, the probability of winning decreases, and bids go to 0.

Reserve Price

Definition

A **reserve price**, denoted r, is the lower bound of acceptable bids.

- If r is positive, then all bidders with valuation $V < r$ will not bid. Hence

$$Q_i^*(V) = F(V)^{N-1}\mathbf{1}_{\{V\geq r\}}$$

$$\Pi_i(V) = \left(VQ_i^*(V) + P_i^*(V)\right)\mathbf{1}_{\{V\geq r\}}$$

$$\Pi_i(V) = \left(\int_r^V F(z)^{N-1}dz\right)\mathbf{1}_{\{V\geq r\}}$$

Bidding in a FPA with reserve price

- If $V_i < r$, then do not bid.
- If $V_i \geq r$, then

$$\begin{aligned} -P_i^*(V) &= \sigma(V)F(z)^{N-1} = VQ_i^*(V) - \Pi_i(V) \\ Q_i^*(V) &= F(V)^{N-1} \\ \Pi_i(V) &= \int_r^V F(z)^{N-1} dz \end{aligned}$$

Hence

$$\sigma(V) = V - \int_r^V \left(\frac{F(z)}{F(V)} \right)^{N-1} dz$$

Remark: When r increases, the ex ante probability of bidding decreases but the bids of those who bid increase.

Optimal Reserve Price

Suppose the seller values Y the object.

What is the reserve price maximizing the seller's expected payoff?

1. Expected payment from bidder i of type V:

$$-P_i^*(V) = (VQ_i^*(V) - \Pi_i(V))\,\mathbf{1}_{\{V \geq r\}} = \left(VQ_i^*(V) - \int_r^V F(z)^{N-1}dz\right)\mathbf{1}_{\{V \geq r\}}$$

2. Ex-ante expected revenue from bidder i

$$\begin{aligned} -\int_0^1 P_i^*(v)f(v)dv &= \int_r^1 vQ_i^*(v)f(v)dv - \int_r^1\int_r^v F(z)^{N-1}f(v)dzdv \\ &= \int_r^1 vQ_i^*(v)f(v)dv - \int_r^1\int_z^1 F(z)^{N-1}f(v)dvdz \\ &= \int_r^1 vQ_i^*(v)f(v)dv - \int_r^1 Q_i^*(v)(1-F(v))dv \end{aligned}$$

3. Seller's expected revenue:

$$N\left(\int_r^1 vQ_i^*(v)f(v)dv - \int_r^1 Q_i^*(v)(1-F(v))dv\right)$$

Optimal Reserve Price

Seller's expected payoff

$$\Pi^S(r) = N\left(\int_r^1 vQ_i^*(v)f(v)dv - \int_r^1 Q_i^*(v)(1-F(v))dv\right) + YF(r)^n$$

First order condition:

$$NQ_i^*(r)(1-F(r)+(Y-r)f(r)) = 0 \Rightarrow 1-F(r^*) = f(r^*)(r^*-Y)$$

Observe that

- $r^* > Y$ because $\left.\frac{\partial \Pi^S(r)}{\partial r}\right|_{r=Y} = NQ_i^*(Y)(1-F(Y)) > 0$
- r^* equals the price a monopoly would post if facing a single buyer.

FPA with entry fee and reserve price

- An entry fee, denote by c, is an amount a bidder must pay in order to submit a bid.
- A reserve price, denote by r, is the lower bound of acceptable bids.

If c and/or r are positive, then there is $\underline{V} \geq 0$ such that all bidders with valuation $V < \underline{V}$ will not bid. Hence, for $V \geq \underline{V}$:

$$\begin{aligned}\Pi_i(V) &= VQ_i^*(V) + P_i^*(V) = \int_{\underline{V}}^{V} F(z)^{N-1} dz \\ P_i^*(V) &= VF(V)^{N-1} - \int_{\underline{V}}^{V} F(z)^{N-1} dz = Q_i^*(V)\sigma(V) + c\end{aligned}$$

Thus,

$$\sigma(V) = V - \int_{\underline{V}}^{V} \left(\frac{F(z)}{F(V)}\right)^{N-1} dz - \frac{c}{F(V)^{N-1}}$$

and $\underline{V} \geq 0$ solves

$$\sigma(\underline{V}) = \underline{V} - \frac{c}{F(\underline{V})^{N-1}} = r$$

Summary

- Definition of standard auction formats: FPA, SPA, EA, JA, APA, SA
- Strategic equivalences.
- Common Value vs. Private Value.
- Within PV framework:
  - Equilibrium of SPA.
  - Symmetric equilibrium.
  - Revenue Equivalence Theorem.
  - Equilibrium of FPA, APA.
  - Optimal reserve price.
  - Reserve price and entry fees.

First order stochastic dominance

Let denote with F and G the c.d.f of random variables $\tilde{x}$ and $\tilde{y}$, respectively.

Definition

c.d.f F **first order stochastically dominates** c.d.f. G, iff

$$\forall x \in \mathbb{R}, F(x) \leq G(x)$$

Theorem

Take any increasing differentiable function $\gamma : \mathbb{R} \rightarrow \mathbb{R}$, then

$$E[\gamma(\tilde{x})] \geq E[\gamma(\tilde{y})]$$

Hazard rate dominance

Definition

- Given a continuous differentiable c.d.f F. Let define the **hazard rate** as the function
$$\lambda_F(x) = \frac{f(x)}{1 - F(x)}.$$
- We say that F dominates G in terms of the hazard rate if for any real number x one has
$$\lambda_F(x) \leq \lambda_G(x)$$

Theorem

If F hazard rate dominates G, then F first order stochastically dominates G.

Hazard rate dominance

Definition

- Given a continuous differentiable c.d.f F. Let define the **reverse hazard rate** as the function
$$\sigma_F(x) = \frac{f(x)}{F(x)}.$$
- We say that F dominates G in terms of the reverse hazard rate if for any real number x one has
$$\sigma_F(x) \geq \sigma_G(x)$$

Theorem

If F reverse hazard rate dominates G, then F first order stochastically dominates G.

Likelihood ratio dominance

Definition

The c.d.f. F is said to dominates c.d.f. G **in terms of the likelihood ratio** if for any $x < y$ one has

$$\frac{f(x)}{g(x)} \leq \frac{f(y)}{g(y)} \tag{5}$$

or equivalently $f(x)/g(x)$ is non-decreasing in x.

Theorem

If F likelihood ration dominates G, then F hazard-rate and reverse-hazard-rate dominates G.

Affiliated random variables

Let $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$.

Definition

- The *component-wise maximum* of x and y is

$$x \vee y = \{\max(x_1, y_1), \max(x_2, y_2), \ldots, \max(x_N, y_N)\}$$

- The *component-wise minimum* of x and y is

$$x \wedge y = \{\min(x_1, y_1), \min(x_2, y_2), \ldots, \min(x_N, y_N)\}$$

- Consider the random variables $\tilde{x}_1 \tilde{x}_2 \ldots, \tilde{x}_N$. Let $f : D \to \mathbb{R}^+$ be the joint density function. The variables $\tilde{x}_1, \tilde{x}_2 \ldots, \tilde{x}_N$ are said to be **affiliated** if for all $x, y \in D$

$$f(x \vee y) f(x \wedge y) \geq f(x) f(y)$$

Affiliated random variables: some properties

Proposition

Let $\tilde{x} = \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N\}$ be affiliated random variables, then

1. *If $\tilde{x}$ and $\tilde{y}$ are affiliated then for any $x' \geq x$, on has that $F(y|x')$ dominates $F(y|x)$ in terms of likelihood ratio.*
2. *$E[\tilde{x}_i|\tilde{x}_j = x_j]$ is an increasing function of x_j.*
3. *If γ is an increasing function from D to $\mathbb{R}$, then*

$$E[\gamma(\tilde{x})|\tilde{x}_1 \leq x_1, \tilde{x}_2 \leq x_2 \dots, \tilde{x}_N \leq x_n]$$

   Is an increasing function of $x_1, x_2, \dots, x_N$

4. *Let $b_1(\cdot), b_2(\cdot), \dots, b_N(\cdot)$ strictly increasing function. Then $b_1(\tilde{x}_1), b_2(\tilde{x}_2), \dots, b_N(\tilde{x}_N)$ are affiliated random variables.*
5. *Fix $\tilde{x}_1$ and let $\tilde{y}_1, \tilde{y}_2, \tilde{y}_{N-1}$ denote the highest, second highest and so on up to the $(N-1)$-th highest realization of $\tilde{x}_2, \tilde{x}_2, \dots, \tilde{x}_N$. Then $\tilde{x}_1, \tilde{y}_1, \tilde{y}_2, \tilde{y}_{N-1}$ are affiliated random variables.*

Milgrom Weber Econometrica (1982)

1. There are N bidders in an auction.
2. There are N random variables $\tilde{x}_1$, $\tilde{x}_2$, ..., $\tilde{x}_N$ drawn from the same interval $[0, 1]$. These random variables are affiliated.
3. Each bidder i privately observes $\tilde{x}_i$ but does not observe the realization of the other random variables.
4. A bidder actual valuation for the object is

$$v_i = u(x_i, x_{-i}) \tag{6}$$

$x_{-i} = \{\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_{i+1}, \ldots, \tilde{x}_N\}$ and $u(\cdot)$ satisfies:

- $u(\cdot)$ is bounded nondecreasing in all its arguments and twice continuously differentiable.
- $u(\cdot)$ is symmetric in the last $N-1$ components.
- $u(0, 0) = 0$

Some examples for $u(.)$

$$
\begin{aligned}
u(x_i, x_{-i}) &= \alpha x_i + \beta \sum_{j \neq i} x_j \\
u(x_i, x_{-i}) &= x_i^{\alpha} \left(\Pi_{j \neq i} x_j \right)^{\beta} \\
u(x_i, x_{-i}) &= \exp[\alpha x_i] \beta \max_{j \neq i} x_j
\end{aligned}
$$

with $\alpha, \beta > 0$.

My expected valuation given the highest of my competitors' type

Let

- $$\tilde{Y}_1 = \max_{j \neq i} \tilde{x}_j$$
-
- $$G(x|x_i) := Pr[\tilde{Y}_1 \leq x|\tilde{x}_i = x_i]$$
  $$g(x|x_i) := \frac{\partial G(x|x_i)}{\partial x}$$
- $$v(x,y) = E\left[u(\tilde{x}_i, \tilde{x}_{-i})|\tilde{x}_i = x, \tilde{Y}_1 = y\right]$$

Symmetric equilibrium of the second price auction

Proposition

In a symmetric equilibrium of a second price auction:

$$\beta^{II}(x) = v(x, x)$$

Symmetric equilibrium of the second price auction

Proof:

$$\underbrace{\Pi(x,z)}_{\text{expected payoff fo a type } x \text{ bidder bidding } \beta^{II}(z)} = \int_0^z (v(x,y) - \beta^{II}(y))g(y|x)dy$$

$$= \int_0^z (v(x,y) - v(y,y))g(y|x)dy$$

f.o.c.

$$\frac{\partial \Pi(x,z)}{\partial z} = (v(x,z) - v(z,z))g(z|x)|_{z=x} = 0$$

s.o.c.

$$\frac{\partial^2 \Pi(x,z)}{\partial x \partial z} = \frac{\partial v(x,z)}{\partial x} > 0$$

Symmetric equilibrium of the first price auction

Proposition

In a symmetric equilibrium of a first price auction:

$$\beta^I(x) = \int_0^x v(y,y)dL(y|x)$$

where

$$L(y|x) = \exp\left(-\int_y^x \frac{g(t|t)}{G(t|t)}dt\right)$$

Symmetric equilibrium of the first price auction

Proof 1/7

$$\overbrace{\Pi(x,z)}^{\text{expected payoff fo a type } x \text{ bidder bidding } \beta^I(z)} = \int_0^z (v(x,y) - \beta^I(z))g(y|x)dy$$
$$= \int_0^z v(x,y)g(y|x)dy - \beta^I(z)G(z|x)$$

f.o.c.: $\frac{\partial \Pi(x,z)}{\partial z}|_{z=x} = 0$

$$\beta^{I\prime}(x) + \beta^I(x)\frac{g(x|x)}{G(x|x)} = v(x,x)\frac{g(x|x)}{G(x|x)} \quad (7)$$

multiplying both sides of (7) by a function $\mu(x)$ satisfying $\mu'(x) = \mu(x)\frac{g(x|x)}{G(x|x)}$:

$$\mu(x)\beta^{I\prime}(x) + \beta^I(x)\mu'(x) = v(x,x)\frac{g(x|x)}{G(x|x)}\mu(x)$$

Symmetric equilibrium of the first price auction

Proof 2/7

$$\mu(x)\beta^{I\prime}(x) + \beta^{I}(x)\mu'(x) = v(x,x)\frac{g(x|x)}{G(x|x)}\mu(x)$$

integrating both sides for $z \in [0, x]$

$$\left[\mu(z)\beta^{I}(z)\right]_0^x = \int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\mu(y)dy$$

using $\beta^{I}(0) = 0$

$$\beta^{I}(x) = \frac{\int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\mu(y)dy}{\mu(x)}$$

Symmetric equilibrium of the first price auction

Proof 3/7

$$\beta^I(x) = \frac{\int_0^x v(y,y) \frac{g(y|y)}{G(y|y)} \mu(y) dy}{\mu(x)}$$

Note that

$$\mu(x) = \mu(0) \exp\left(\int_0^x \frac{g(z|z)}{G(y|z)} dz\right) \Rightarrow \mu'(x) = \mu(x) \frac{g(x|x)}{G(x|x)}$$

$$\beta^I(x) = \frac{\int_0^x v(y,y) \overbrace{\frac{g(y|y)}{G(y|y)} \mu(y)}^{\mu'(y)=\mu(0)\frac{g(y|y)}{G(y|y)} \exp\left(\int_0^y \frac{g(z|z)}{G(y|z)} dz\right)} dy}{\underbrace{\mu(x)}_{\mu(0) \exp\left(\int_0^x \frac{g(z|z)}{G(y|z)} dz\right)}} =$$

$$= \int_0^x v(y,y) \frac{g(y|y)}{G(y|y)} \exp\left(-\int_y^x \frac{g(z|z)}{G(z|z)} dz\right) dy$$

Symmetric equilibrium of the first price auction

Proof 4/7

$$\beta^I(x) = \int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\exp\left(-\int_y^x \frac{g(z|z)}{G(z|z)}dz\right)dy$$

We set

$$L(y|x) = \exp\left(-\int_y^x \frac{g(t|t)}{G(t|t)}dt\right)$$

$$\Rightarrow \beta^I(x) = \int_0^x v(y,y)dL(y|x)$$

Symmetric equilibrium of the first price auction

Proof 5/7

Second order condition:

$$\left.\frac{\partial \Pi(x,z)}{\partial z}\right|_{z<x} > 0$$

$$\left.\frac{\partial \Pi(x,z)}{\partial z}\right|_{z>x} < 0$$

$$\frac{\partial \Pi(x,z)}{\partial z} = G(z|x)\left[(v(x,z) - \beta^I(z))\frac{g(z|x)}{G(z|x)} - \beta^{I\prime}(z)\right]$$

Symmetric equilibrium of the first price auction

Proof 6/7

Second order condition:

$$\left.\frac{\partial \Pi(z,x)}{\partial z}\right|_{z<x} > 0$$

$$\frac{\partial \Pi(z,x)}{\partial z} = G(z|x)\left[\underbrace{(v(x,z)}_{>v(z,z)} - \beta'(z))\underbrace{\frac{g(z|x)}{G(z|x)}}_{>\frac{g(z|z)}{G(z|z)}}\beta''(z)\right]$$

$$> G(z|x)\underbrace{\left[(v(z,z) - \beta'(z))\frac{g(z|z)}{G(z|z)} - \beta''(z)\right]}_{=0,\text{ because of f.o.c.}}$$

Symmetric equilibrium of the first price auction

Proof 7/7

Second order condition:

$$\left.\frac{\partial \Pi(z,x)}{\partial z}\right|_{z>x} < 0$$

$$\frac{\partial \Pi(z,x)}{\partial z} = G(z|x)\left[(\underbrace{v(x,z)}_{<v(z,z)} - \beta^{I}(z))\underbrace{\frac{g(z|x)}{G(z|x)}}_{<\frac{g(z|z)}{G(z|z)}}\beta^{I\prime}(z)\right]$$

$$< G(z|x)\underbrace{\left[(v(z,z) - \beta^{I}(z))\frac{g(z|z)}{G(z|z)} - \beta^{I\prime}(z)\right]}_{=0}$$

Japanese auction

1. All bidders are in the same room.
2. The auctioneer starts with a price of 0 and gradually and continuously increases the price.
3. When a bidder deems that the price reached a level that is too high for him/her, he or she exits the room.
4. Bidders who exit are not allow to come back in the room.
5. As soon as there is only one remaining bidder in the room, the auctioneer stops increasing the price, the bidder left is the winner and pays that price.

Symmetric equilibrium of the English and the Japanese auction

Preliminaries: Let

$$\begin{aligned} J(x,x) &:= u(x,x,\ldots,x) \\ J(x,(x,x_1)) &:= u(x,x,\ldots,x,x_1) \\ J(x,(x,x_1,x_2)) &:= u(x,x,\ldots,x,x_1,x_2) \\ &\ldots \\ J(x,(x,x_1,x_2,\ldots,x_m)) &:= u(x,x,\ldots,x,x_1,x_2,\ldots,x_m) \end{aligned}$$

Remark: $J(\cdot)$ is strictly increasing. In particular it is invertible in x.
Let $x_m(p,x_1,x_2,\cdots,x_{m-1})$ be the x such that

$$J(x,(x,x_1,x_2,\ldots,x_m)) = p$$

Symmetric equilibrium of the English and the Japanese auction

Proposition

The following strategy form an equilibrium of the Japanese auction: For bidder of type x

- *As long as no bidder exits, stay until the price reaches* $J(x, x)$*, and then exit.*
- *If the first bidder exited at price* p_1*, stay until the price reaches* $J(x, (x, x[1]))$*, and then exit. Where* $J(x[1], x[1]) = p$*.*
- *If the first bidder exited at price* p_1 *and the second at price* p_2*, stay until the price reaches* $J(x, (x, x[1], x[2]))$*, and then exit. Where* $J(x'', (x'', x') = p_2$*.*
- ...
- *When there only are two bidders and the other exited at time* $p_1, p_2, \ldots p_{N-2}$*, stay until the price reaches* $J(x, (x, x[1], \ldots, x[N_2])$

Proof:

- If all other bidders follow this strategy bidder i can deuce the type of exiting bidders. That is $x[m] = \tilde{x}_m$ i.e. the signal of the m-th bidder to exit.
- If all bidders follows this strategy, the highest type wins and get a payoff of

$$u(x, x_N, x_{N-1}, \ldots x_2, x_1) - \underbrace{u(x_{N-1}, x_N, x_{N-1}, \ldots x_2, x_1)}_{\text{Price the winners pays}} \quad \underbrace{> 0}_{\text{Because } x \geq X_{N-1}} \tag{8}$$

- If a bidder deviates either
  - Has not the highest type and wins: and get a payoff

$$u(x, x_N, x_{N-1}, \ldots x_2, x_1) - \underbrace{u(x_{N-1}, x_N, x_{N-1}, \ldots x_2, x_1)}_{\text{Price the winners pays}} \quad \underbrace{< 0}_{\text{Because } x < X_{N-1}}$$

  - Has not the highest type and does not win: and get a payoff 0.
  - Has the highest type and wins: and get a payoff (8)
  - Has the highest type and does not win: and get a payoff 0