Auctions with interdependent values: the symmetric model of Milgrom and Weber (1982)

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1 Bidders interdependent valuations

Let consider an auction for an item whose fundamental value depends on some common value component, such as for example the quality of manufacturing, and some private value component that depends on the idiosyncrasies of the bidder, for example the colour and aesthetics of the object. In situation like this, each bidder might possess information regarding its own idiosyncratic taste but also about the intrinsic quality of the object. When this happens we says that bidder's valuations are interdependent.

Milgrom and Weber (1982) modelled situations like this as follows:

1. There are N bidders.

2. There are N random variables $\tilde{x}_1$, $\tilde{x}_2$, ..., $\tilde{x}_N$ drawn from the same interval $[0, 1]$. These random variables are affiliated.

3. Each bidder i privately observes the realization x_i of a random variable $\tilde{x}_i$ but does not observe the realization of the other random variables. We interpret x_i as a private information that the bidder has received and that is helpful for the bidder to better estimate his valuation for the object.

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4. A bidder actual valuation for the object depends on the realization of all random variables. Namely, let x_i be the realization of $\tilde{x}_i$ and let $x_{-i} = \{\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_{i+1}, \ldots, \tilde{x}_N\}$ denote the vector with the realization of the other $N-1$ random variables. Then the value of the item to bidder i is equal to

$$v_i = u(x_i, x_{-i}) \tag{1}$$

where the function $u(\cdot)$ satisfies the following properties

(a) $u(\cdot)$ is bounded nondecreasing in all its variables and twice continuously differentiable.

(b) $u(\cdot)$ is symmetric in the last $N-1$ components. This means that if x_{-i} is the vector with the realizations of the $N-1$ random variables different from $\tilde{x}_i$ and x'_{-i} is a permutation of x_{-i}, i.e., x_{-i} and x_{-i} have the same numbers but in a different order, then

$$u(x_i, x_{-i}) = u(x_i, x'_{-i})$$

(c) We normalize $u(0,0) = 0$, that is if all bidder's signals realization equal 0, then the item is worth 0 to all bidders.

That is bidder i's valuation v_i is increasing in his and in the other bidder's signals. However the other bidder's signals can be interchanged without affecting bidder i's valuation, or in other words, the information of all comptetitors of i are equally relevant for i to determine his own valuation. Below some examples:

$$\begin{aligned} u(x_i, x_{-i}) &= \alpha x_i + \beta \sum_{j \neq i} x_j \\ u(x_i, x_{-i}) &= x_i^{\alpha} \left(\Pi_{j \neq i} x_j\right)^{\beta} \\ u(x_i, x_{-i}) &= \exp[\alpha x_i] \beta \max_{j \neq i} x_j \end{aligned}$$

with $\alpha, \beta > 0$.

Fix bidder i and let $\tilde{Y}_1$ denote the highest realization of the the random variables different from $\tilde{x}_i$. That is,

$$\tilde{Y}_1 = \max_{j \neq i} \tilde{x}_j \tag{2}$$

let

$$v(x, y) = E\left[u(\tilde{x}_i, \tilde{x}_{-i}) | \tilde{x}_i = x, \tilde{Y}_1 = y\right] \tag{3}$$

This is the expected value of the object to bidder i conditionally on the realization of bidder i signal being x and on the realization of the highest of other bidder's signal being y.

In what follows we will consider the equilibrium bidding strategies in different auction formats when bidders valuation are interdependent. We will assume that the random variable $\tilde{x}_1, \tilde{x}_2, \ldots, x_N$ representing the N bidders' private signals are affiliated and continuously distributed on $[0, 1]$.

2 Symmetric equilibrium of the second price sealed-bid auction

Let consider first a second price sealed bid auction. We will focus on the symmetric equilibrium that is defined by a continuously increasing and differentiable function $\beta^{II} : [0, 1] \to \mathbf{R}$, with the interpretation that if a bidder i private signal realization is $\tilde{x}_i = x$, then it is an equilibrium for this bidder to bid $\beta^{II}(x)$ in a second price auction. Then we have

Proposition 1 *In a symmetric equilibrium of a second price auction:*

$$\beta^{II}(x) = v(x, x)$$

Recall the meaning of the function $v(x, y)$ given in (3). The proposition states that after observing a private signal $\tilde{x}_i = x$, it is an equilibrium for a bidder i in a second price auction to bid an amount equal to the expected value of the object conditionally on his signal being x and the highest of his competitors signal being also equal to x.

Proof: Fix any bidder i and suppose he expects that all other bidders will bid according to the bidding function $\beta^{II}(\cdot)$. Observe that because $\beta^{II}(x) = v(x,x)$ another bidder j's bid is a continuously increasing differentiable function of that bidder's private signal realization. Becuase $\tilde{x}_j \in [0,1]$, all other bids are included between $\beta^{II}(0) = v(0,0) = 0$ and $\beta^{II}(1) = v(1,1)$. Clearly it is optimal for bidder i to bid at least $\beta^{II}(0)$ and at most $\beta^{II}(1)$. This because by bidding less than $\beta^{II}(0)$ he is certain to lose the auction, and he can achieve the same outcome by bidding $v(0,0) = 0$, whereas by bidding more than $\beta^{II}(1)$ he is certain to win the auction and pay at most $\beta^{II}(1)$, but then he can achieve the same outcome by bidding $\beta^{II}(1)$. So our bidder i has to choose a bid b between $\beta^{II}(0)$ and $\beta^{II}(1)$. Now, because $\beta^{II}(\cdot)$ is a strictly increasing function, this is equivalent to choosing a type $z \in [0,1]$ and then bid $\beta^{II}(z)$, that is to bid like another bidder with signal z would bid.

If he bids $\beta^{II}(z)$, then he wins only if the highest of his competitor bid is not larger than $\beta^{II}(z)$, that occurs if and only if the highest of his competitors' signals is not larger than z. That is, using the notation in (2), bidder i wins only if $\tilde{Y}_1 < z$. Let denote with $G(\cdot|x)$ and with $g(\cdot|x)$ the the c.d.f. and the density of of $\tilde{Y}_1$ conditional on $\tilde{x}_i = x$, respectively. That is, $G(z|x)$ is the probability that the highest of bidder i's competitors' signal is not larger than z conditional on bidder i's signal being x.

Suppose the bidder i bids $\beta^{II}(z)$, he wins and he pays $\beta^{II}(y)$, what is in this case the expected value of the object to bidder i? This is the expectation of $u(\cdot)$ conditionally on his signal being equal to x and the highest of his competitor's signal being equal to y, that is precisely $v(x,y)$ by expression (3).

We can now determine $\Pi(z,x)$ that is the expected payoff of bidder i from bidding $\beta^{II}(z)$ given that the realization of his private signal $\tilde{x}_i$ is equal to x.

$$\begin{aligned}\Pi(z,x) &= \int_0^z (v(x,y) - \beta^{II}(y))g(y|x)dy \\ &= \int_0^z (v(x,y) - v(y,y))g(y|x)dy\end{aligned}$$

Bidder i chooses the z that maximizes $\Pi(z,x)$. By taking the derivative of

$\Pi(z,x)$ with respect to z we have

$$\frac{\partial \Pi(z,x)}{\partial z} = v(x,z) - v(z,z)$$

Because $v(\cdot)$ is increasing in the two argument the above expression is positive nil for $z = x$, it is positive for $z < x$ and it is negative for $z > x$. But this implies that $\Pi(z,x)$ is maximized for $z = x$. Another way to see that $z = x$ maximize our bidders expected payoff is to verify that $\Pi(z,x)$ is quasi-concave. In facts,

$$\frac{\partial^2 \Pi(z,x)}{\partial x \partial z} = \frac{\partial v(x,z)}{\partial x} > 0$$

Thus, is if all other bidders bid according to $\beta^{II}(\cdot)$, the the best bidder i can do given his signal x is to bid precisely $b = \beta^{II}(x) = v(x,x)$. Q.E.D.

3 Symmetric equilibrium of the first price sealed-bid auction

Let consider now the symmetric equilibrium of the first price sealed bid auction. And let denote with $\beta^I : [0,1] \to \mathbf{R}$ the equilibrium bidding strategy. Then,

Proposition 2 *In a symmetric equilibrium of a first price auction:*

$$\beta^I(x) = \int_0^x v(y,y) dL(y|x)$$

where

$$L(y|x) = \exp\left(- \int_y^x \frac{g(t|t)}{G(t|t)} dt \right)$$

Proof: As for the proof for the previous proposition let denote with $\Pi(z,x)$ the expected payoff of bidder i given that all other bidders use strategy $\beta^I(\cdot)$, he has signal x and bids $\beta^I(z)$. As long as $\beta^I(\cdot)$ is a strictly increasing function, we have

$$\Pi(z,x) = \int_0^z (v(x,y) - \beta^I(z)) g(y|x) dy = \int_0^z v(x,y) g(y|x) dy - \beta^I(z) G(z|x)$$

Observe that in equilibrium the expected value of the object given that a bidder with signal x wins is $v(x,x)$ so a bidder will never bid more than $v(x,x)$ that is $v(x,x) - \beta^I(x) \geq 0$ for all x. Hence from $v(0,0) = 0$, we have $\beta^I(0) = 0$.

Let's take the derivative of $\Pi(z,x)$ with respect to z :

$$\frac{\partial \Pi(z,x)}{\partial z} = v(x,z)g(z|x) - \beta^I(z)g(z|x) - \beta^{I'}(z)G(z|x)$$

For βI to be the equilibrium bidding strategy, it must be optimal for bidder i to bid $\beta^I(x)$, and hence the first order condition $\left.\frac{\partial \Pi(z,x)}{\partial z}\right|_{z=x} = 0$ must be satisfied. This leads to the following differential equation:

$$\beta^{I'}(x) + \beta^I(x)\frac{g(x|x)}{G(x|x)} = v(x,x)\frac{g(x|x)}{G(x|x)} \tag{4}$$

That together with the initial condition $\beta^I(0) = 0$ provides the expression of β^I given in the proposition.

Namely, take the function $\mu(x)$ such that $\mu'(x) = \mu(x)\frac{g(x|x)}{G(x|x)}$. Then multiplying both sides of (4) by $\mu(x)$, we get:

$$\mu(x)\beta^{I'}(x) + \beta^I(x)\mu'(x) = v(x,x)\frac{g(x|x)}{G(x|x)}\mu(x)$$

Taking the integral on both side one has

$$\left[\mu(z)\beta^I(z)\right]_0^x = \int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\mu(y)dy$$

Considering that $\beta^I(0) = 0$ the previous inequality can be rearranged as follows

$$\beta^I(x) = \frac{\int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\mu(y)dy}{\mu(x)} \tag{5}$$

It remains to determine the function $\mu(\cdot)$. From $\mu'(x) = \mu(x)\frac{g(x|x)}{G(x|x)}$ it follows that

$$\mu(x) = \mu(0)\exp\left(\int_0^x \frac{g(z|z)}{G(y|z)}dz\right)$$

so replacing ths expression in (5) and simplifying, we get

$$\beta^I(x) = \int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\exp\left(-\int_y^x \frac{g(z|z)}{G(z|z)}dz\right)dy \tag{6}$$

Considering the definition of $L(\cdot)$ given in the proposition, the above expression is precisely β^I of the proposition.

To complete the proof that bidding $\beta^I(x)$ is the best bid for a bidder with signal x we have to show that

$$\left.\frac{\partial\Pi(z,x)}{\partial z}\right|_{z<x} > 0$$

and

$$\left.\frac{\partial\Pi(z,x)}{\partial z}\right|_{z>x} < 0$$

Let rewrite $\frac{\partial\Pi(z,x)}{\partial z}$ as follows

$$\frac{\partial\Pi(z,x)}{\partial z} = G(z|x)\left[(v(x,z)-\beta^I(z))\frac{g(z|x)}{G(z|x)} - \beta^{I\prime}(z)\right]$$

Take $z < x$, then $v(x,z) > v(z,z)$ and because of affiliation, $G(z|x)$ stochastically dominate in the sense of the reverse hazard rate $G(z|z)$. That is,

$$\frac{g(z|x)}{G(z|x)} \geq \frac{g(z|z)}{G(z|z)}.$$

Hence

$$\frac{\partial\Pi(z,x)}{\partial z} > G(z|x)\left[(v(z,z)-\beta^I(z))\frac{g(z|z)}{G(z|z)} - \beta^{I\prime}(z)\right] = 0$$

Take $z > x$, then $v(x,z) < v(z,z)$ and because of affiliation, $G(z|z)$ stochastically dominate in the sense of the reverse hazard rate $G(z|x)$, that is

$$\frac{g(z|z)}{G(z|z)} > \frac{g(z|x)}{G(z|x)}$$

Hence

$$\frac{\partial\Pi(z,x)}{\partial z} < G(z|x)\left[(v(z,z)-\beta^I(z))\frac{g(z|z)}{G(z|z)} - \beta^{I\prime}(z)\right] = 0$$

Q.E.D.

4 Symmetric equilibrium of the English auction

In first price and second price sealed-bid auctions the only action a bidder has to take is to choose his bid. Once the bidder has bid, he or she only has to wait for the outcome of the auction and, if he or she wins, pay his/her own bid in the first price auction, and the highest of his competitor's bid in the second price auction. In an English auction bidders are in the same room, see each other declaring bids and can react. So, a priori any bidder can declare any bid as a reaction to a competitor's announcement. In this section we will focus on a variant of the English auction, also known as the **Japanese auction** whose rules are as follows:

1. All bidders are in the same room.
2. The auctioneer starts with a price of 0 and gradually and continuously increases the price.
3. When a bidder deems that the price reached a level that is too high for him/her, he or she exits the room.
4. Bidders who exit are not allow to come back in the room.
5. As soon as there is only one remaining bidder in the room, the auctioneer stops increasing the price, the bidder left is the winner and pays that price.

In a Japanese auction each bidder has to choose when to leave the room. The item is sold at the price at which the one before the last bidder exited the room and the winner of the auction is the last bidder left in the room. Because bidders in the room observe the prices at which the other bidders leave, they might deduce the exiting bidders' opinion about the value of the item, and hence adapt their exiting price accordingly.

Without loss of generality, we can call time 0 the beginning of the auction and assume that at time t the auctioneer announces a price equal to t. So the bidders' exiting times correspond to their bid. Suppose that initially there are N bidders and that, by time t, $n < N$ bidders have left the room. Bidders still in the room have observed the exiting time $p_1 \leq p_2, \cdots \leq p_n \leq t$ and now each one of them has to decide a time $t^* \geq t$ at which he will exit the room if by that time no other bidder has left. In this framework a bidding strategy β^J answers the following question: given the realization x of my signal, the bidders who have exited and the time at which they have exited, at what time do I want to exit the room?

Take a bidder i and recall that if the realization of bidder i's signal is x_i, and the realization of the other bidder's signal is $x_{-i} = \{x_1, \ldots, x_{i-}, x_{i+1}, \ldots, x_N\}$, then bidder i's valuation for the object is

$$v_i = u(x_i, x_{-i})$$

Let denote with $u(x, x)$ the value of the object to bidder i if all bidders have received the same signal x as bidder i. Let $u(x, (x, x_1))$ the value of the object to bidder $i \neq 1$ if bidder 1's signal is x_1 and all other bidder's signal is identical and equal to x. We can define recursively in the same way $u(x, (x, x_1, x_2))$, $u(x, (x, x_1, x_2, x_3))$ and so on and so forth until $u(x, x_{-i})$.

We can now describe the strategy β^J in a symmetric equilibrium of the Japanese auction from the moment the auction starts until the moment the winner is determined.

1. As long as all bidders are in the room, a bidder with signal x stays if the price is below $u(x, x)$ and exits as soon as the price reaches $u(x, x)$. Two observation follows from the fact that $u(\cdot)$ is a strictly increasing function:

    (a) The first bidder to exit the room is the bidder with the lowest private signal.

    (b) By observing the time p_1 at which the first bidder exited, all bidders can deduce the exiting bidder's private signal x_1. Namely x_1 must be such that

    $$u(x_1, x_1) = p_1.$$

2. As long as only one bidder has already exited and his signal is x_1, then a bidder with signal $x > x_1$ stays if the price is below $u(x, (x, x_1))$ and exits as soon as the price reaches $u(x, (x, x_1))$. Two observation follows from the fact that $u(\cdot)$ is a strictly increasing function:

(a) The second bidder to exit the room is the bidder with the second lowest private signal.

(b) By knowing x_1 and observing the time p_2 at which the second bidder exited, all bidders can deduce that the exiting bidder's private signal x_2. Namely x_2 must be such that

$$u(x_2, (x_2, x_1)) = p_2.$$

...

n. Suppose that $n - 1 \leq N - 2$ bidders have exited and from their exiting times all bidders have deduced the exited bidders signals' $x_1 < x_2 < x_3 < \cdots < x_{n-1}$. Then a bidder in the room with signal $x > x_{n-1}$ stays if the price is below $u(x, (x, x_1, x_2, x_3, \ldots, x_{n-1}))$ and exits as soon as the price reaches $u(x, (x, x_1, x_2, x_3, \ldots, x_{n-1}))$.Two observation follows from the fact that $u(\cdot)$ is a strictly increasing function:

(a) The n-th bidder to exit the room is the bidder with the n-lowest private signal.

(b) By knowing $x_1, \ldots, x_{n-1}$ and observing the time p_n of the n-th exit, all bidders can deduce the exiting bidder's private signal x_n. Namely x_n must be such that

$$u(x_n, (x_n, x_1, x_2, x_3, \ldots, x_{n-1})) = p_n.$$

...

N-1. The wining bidder is the bidder with the highest signal and he pays the price

$$p_{N-1} = u(x_{N-1}, (x_{N-1}, x_1, x_2, x_3, \ldots, x_{N-2})) = u(x_{N-1}, (x_1, x_2, x_3, \ldots, x_{N-1})).$$

where the second equality follows from the fact that $u(\cdot)$ is symmetric in its last $N - 1$ arguments. Hence if the winner's signal is x, his or her payoff is

$$u(x, (x_1, x_2, x_3, \ldots, x_{N-1})) - u(x_{N-1}, (x_1, x_2, x_3, \ldots, x_{N-1})) > 0 \qquad (7)$$

where the inequality follows from the fact $u(\cdot)$ is strictly increasing and $x > x_{N-1}$.

Proposition 3 *The strategy β^J is a symmetric equilibrium of the Japanese auction.*

Proof: Suppose all bidders but bidder i follow the strategy β^J. If bidder i with signal x follows the strategy, then he wins only if he has the highest signal. Upon winning his payoff is given by (7) and is strictly positive because $x > x_{N-1}$. Suppose bidder i, deviates and eventually loses the auction. If he is not the bidder with the highest signal he does not gain by deviating, because he would have lost also following the equilibrium strategy. However if he is the bidder with the highest signal, by deviating he loses the strictly positive payoff (7). Suppose bidder i, deviates and eventually wins the auction. If he is the bidder with the highest signal he does not gain by this deviation, because he would have also won by following the equilibrium strategy, and would have paid the same price p_{N-1}. However if he is not the bidder with the highest signal then, it must be that $x < x_{N-1}$, but in this case his winner's payoff (7) is strictly negative because $u\cdot$ is strictly increasing. So the best bidder i can do is to follow β^J. Q.E.D.

5 Revenue ranking with interdependent values: the linkage principle

We have seen that within a private value framework the seller's expected revenue does not depend on the auction mechanism as long as the equilibrium satisfies the following two conditions. First, the bidder with the highest valuation wins the auction with certainty, and second, a bidder whose valuation is nil has an equilibrium payoff of 0. From the above analysis of the symmetric equilibria of the first price, second price and English auction with interdependent valuation, we know that in each of these equilibria:

i. The bidder with highest signal wins the auction with certainty

ii. A bidder with signal 0 has a payoff of 0.

Can we then deduce that also for the interdependent values case these different auction mechanism provide the seller with the same expected revenue? To answer this question let us make a reasoning similar to the one we used to prove the revenue equivalence theorem for the private value case.

Take an auction mechanism and suppose that it has an equilibrium satisfying properties i. and ii. Let denote with $\Pi(z, x)$ the expected payoff for a bidder with signal x if he bids as if his signal were z. Let denote with $W(z, x)$ the expected price paid by bidder i if he is the winning bidder, he is of type x but bids as if his type were z. For example, in a first price auction $W^I(z, x) = \beta^I(z)$ and in a second price $W^{II}(z, x) = \int_0^z \beta^{II}(y) \frac{g(y|x)}{G(z|x)} dy$. Property i. implies

$$\Pi(z, x) = \int_0^z v(x, y) g(y|x) dy - W(z, x) G(z|x). \tag{8}$$

Let denote

$$W_1(z, x) = \frac{\partial W(z, x)}{\partial z}$$

and

$$W_2(z, x) = \frac{\partial W(z, x)}{\partial x}.$$

In equilibrium it must be optimal for the bidder of type x to behave according to his type. That is the first order condition must hold:

$$\left. \frac{\partial \Pi(z, x)}{\partial z} \right|_{z=x} = (v(x, x) - W(x, x)) g(x|x) - W_1(x, x) G(x|x) = 0$$

Or equivalently:

$$W_1(x, x) = (v(x, x) - W(x, x)) \frac{g(x|x)}{G(x|x)}. \tag{9}$$

Now let consider two auction mechanisms, A and B. Suppose each mechanism has an equilibrium satisfying i. and ii. The question is whether in these equilibria the two mechanisms provide the same expected revenue or not. We know that because of property i. if a bidder of signal 0 wins, all bidders must have signals 0, the value of the item is $u(0, 0) = 0$, and hence because of property ii. the winning bidder pays exactly 0 no matter whether the mechanism is A or B. That is,

$$W^A(0,0) = W^B(0,0) = 0. \tag{10}$$

What if the winner is of type $x > 0$? What is in this case the difference in what the winner would pay in auction A compared to what the bidder would pay in auction B? That is, what is $\Delta(x) := W^A(x,x) - W^B(x,x)$?

We have

$$\Delta(x) = \Delta(0) + \int_0^x \Delta'(z)dz = \int_0^x \Delta'(z)dz$$

where the second equality follows from (10). Now,

$$\begin{aligned} \Delta'(x) &= \left(W_1^A(x,x) - W_1^B(x,x)\right) + \left(W_2^A(x,x) - W_2^B(x,x)\right) \\ &= \frac{g(x|x)}{G(x|x)}\Delta(x) + \left(W_2^A(x,x) - W_2^B(x,x)\right) \end{aligned}$$

where the second equality follows from (9). From this expression we deduce the following proposition.

Proposition 4 *Take two auction mechanisms A and B, and suppose each mechanism has an equilibrium such that*

1. *Property i. is satisfied.*
2. *Property ii. is satisfied.*
3. *For all $x > 0$, it results $W_2^A(x,x) \geq W_2^B(x,x)$.*

Then the seller expected equilibrium revenue in auction A is not smaller than in auction B.

The interpretation of proposition 4 and in particular of its condition 3 is known as the **linkage principle** that in words can be stated as follows:

"The more closely the winning bidder's payment is linked to his actual type (as opposed to his bid), the greater the expected revenue will be."

"The more things the winning bidder's payment depends on that are positively correlated with his type, the greater the expected revenue will be."

Let us apply Proposition 4 to compare the first and the second price sealed bid auction. For the first price auction $W^I(z,x) = \beta^I(z)$ and so $W_2^I(x,x) = W_2^I(z,x) = 0$. For the second price auction

$$W^{II}(z,x) = E[v(\tilde{Y}_1, \tilde{Y}_1)|\tilde{x}_i = x, \tilde{Y}_1 < z]$$

If signals are affiliated, then the above expression is an increasing function of x and so

$$W_2^{II}(x,x) \geq 0.$$

Thus applying the proposition we have

Corollary 1 *The seller expected revenue of the symmetric equilibrium of the second price auction is at least as large as the expected seller revenue of the symmetric equilibrium of the first price auction.*

Observe that for second price auction revenue to be strictly larger than first price auction revenue, it is necessary that $W_2^{II}(x,x) > 0$ and hence that the signals are 'strictly' affiliated. That if signals are independent then even in the interdependent framework $W_2^{II}(x,x) = 0 = W_2^I(x,x)$ and so the two auctions generate the same expected revenue.

6 Disclosing information in auctions

In this section we will use the linkage principle to understand whether publicly disclosing information about the item value can affect the seller's revenue. Consider our interdependent value framework where bidder's signals are affiliated with joint distribution f. Suppose the seller also privately observe a signal $\tilde{x}_S$ that is affiliated with the bidder's signals. From an ex-ante perspective, does the seller gain or loses by publicly disclosing his signal before the auction?

Consider an auction whose equilibrium satisfies property i. and ii. Consider first the case where the seller does not disclose his information. For any given joint distribution function of f of bidders affiliated signals, let $\beta^f(\cdot)$ be the symmetric equilibrium strategy and let $W^f(z,x)$ be the expected payment of a bidder of type x if he wins by behaving as if his type was z.

Now consider the case in which before the auction the seller disclose that his signal x_S. All bidders update their beliefs, and the resulting joint distribution of signals conditional x_S is $f(\cdot|x_S)$ and signals are still affiliated. So a symmetric equilibrium is played with strategy $\beta^{f(\cdot|x_S)}(\cdot)$ that are non decreasing in x_S. In fact because of affiliation, an increase in the realization of the seller's signals increase all bidders expected valuation for the item and hence cannot decrease their bids. Then also $Wf(\cdot|x_S)(x)$ is non-decreasing in x_S. From an ex-ante perspective, the expected payment in the auction with disclosure is

$$W^D(z,x) = E\left[W^{f(\cdot|x_S)}(z,x)\right] = \int_0^1 W^{f(\cdot|x_S)}(z,x)h(x_S|x)dx_S$$

where $h(\cdot|x)$ is the marginal distribution of the seller's signal $\tilde{x}_S$ conditional on bidder i being $x_i = x$). Now because $\tilde{x}_i$ and $\tilde{x}_S$ are affiliated for any $x' > x$ we have that $h(\cdot|x')$ first order stochastically dominates $h(\cdot|x)$. Now intuitively an increase in x affects $W^D(z,x)$ through two channels, it affects directly trough $W_2^{f(\cdot|x_S)}(z,x)$ in the same way as $W_2^f(z,x)$, but it also affect the distribution of $\tilde{x}_S$ making high realizations of $\tilde{x}_S$ more likely and hence increasing W^D. Hence we have that

$$W_2^D(z,x) \geq W_2^f(z,x)$$

Thus, applying Proposition 4 by committing to disclose his information the seller increase his ex-ante expected revenue.

Let see an example for the first price auction. We have that $W^f(z,x) = \beta^I(z)$ whereas $W^{f(\cdot|x_S)}(z,x) = \beta^I(z,x_s)$ that is increasing both in z and in x_S. Thus, whereas $W^f(z,x)$ is independent of x, $W^D(z,x) = E[\beta^I(z,\tilde{x}_s)|\tilde{x}_i = x]$ that is strictly increasing in x because $\tilde{x}_s$ and $\tilde{x}_i$ are affiliated. Thus by committing to truthfully report its signal before the auction the seller can increase his expected revenue in a first price auction. A similar argument can be developed for a second price auction and for a Japanese auction.

Reference

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