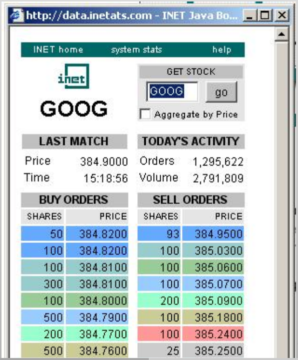
Limit Order Markets

- All market participants have the choice between submitting limit orders and market orders.
- Quotes must belong to a grid with a minimum tick.
- Cumulative depth on the bid side at price b:= maximum amount of share one can sell for at least b.
- Cumulative depth on the ask side at price a:= maximum amount of share one can buy for at most a.



A simple model of limit order market

- Impatient traders (IT): submit market orders.
  $F(Q)$: (exogenous) probability the quantity $\tilde{Q}$ demanded by IT is less than Q. ($Q > 0$ meaning trader buys)
- Patient riks neutral traders (PT): submit limit orders.
- Limit order quotes must belong to a grid P with tick ϵ:

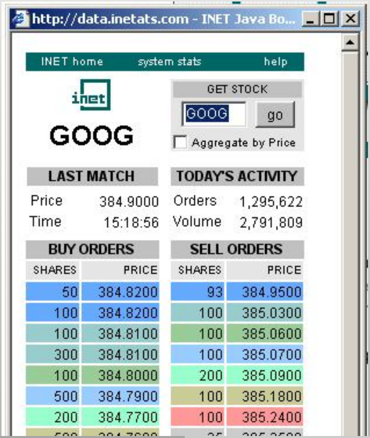
$$P = \{p_k\}_{k=0,1,...}$$

$$p_k = \epsilon k$$

- Limit order submission has a cost c.
- Symmetric information.

Timing

1. PT submit their limit orders.
2. IT submit their market orders.
3. Limit orders are executed on the basis of price priority (and time priority).
4. The asset value $\tilde{v}$ is realized.



Marginal Profit from limit Orders

Let Y_k be the cumulative depth of the bid side for bid $b = p_k$

What is the expected marginal profit for the PT offering to buy the Y_k-th share?

$$\Pi_k^{bid}(Y_k) = \Pr(\tilde{Q} \leq -Y_k)(E[\tilde{v}|\tilde{Q} \leq -Y_k] - b) - c$$

Let Y_j be the cumulative depth of the ask side for ask $a = p_j$

What is the expected marginal profit for the PT offering to sell the Y_j-th share?

$$\Pi_j^{ask}(Y_j) = \Pr(\tilde{Q} \geq Y_j)(a - E[\tilde{v}|\tilde{Q} \geq Y_j]) - c$$

Zero Profit condition

What is the equilibrium depth of the market for a given level of price if there is no asymmetric information?

- PT's marginal utility from offering to buy one extra share at price p_k is nil:

$$\Pr(\tilde{Q} \leq -Y_k) = \frac{c}{E[\tilde{v}] - p_k}$$

- PT's marginal utility from offering to sell one extra share at price p_j is nil:

$$\Pr(\tilde{Q} \geq Y_j) = \frac{c}{p_j - E[\tilde{v}]}$$

Example

Probability of a buy market order of at most Q shares = Probability of a sell market order of at most Q shares =

$$\frac{1}{2}F(Q) : [0, \infty] \to [0, 1]$$

Take bid price b on the grid such that $b < E[\tilde{v}] - 2c$, then Y_k solves

$$F(Y_k) = 1 + \frac{2c}{b - E[\tilde{v}]}$$

Fix ask price a on the grid such that $a_k > E[\tilde{v}] + 2c$, then Y_j solves

$$F(Y_j) = 1 + \frac{2c}{E[\tilde{v}] - a}$$

Limit Order Book

bid	bid quantity	bid depth	ask	ask quantity	ask depth
$b_0 = E[v]$	$bq_0 = 0$	$Y_0 = q_0$	$a_0 = E[v]$	$aq_0 = 0$	$aY_0 = aq_0$
$b_1 = E[v] - \epsilon$	bq_1	$bY_1 = bq_1 + bY_0$	$a_1 = E[v] + \epsilon$	aq_1	$aY_1 = aq_1 + aY_0$
$b_2 = E[v] - 2\epsilon$	bq_2	$bY_2 = bq_2 + bY_1$	$a_2 = E[v] + 2\epsilon$	aq_2	$aY_2 = aq_2 + aY_1$
...	...	...	...	...	...
$b_k = E[v] - k\epsilon$	bq_k	$bY_k = bq_k + bY_{k-1}$	$a_k = E[v] + k\epsilon$	aq_k	$aY_k = aq_k + aY_{k-1}$

Market order Q uniformly distributed on $[0, M]$: $F(Q) = \frac{Q}{M}$

$$\frac{bY_k}{M} = F(bY_k) = 1 + \frac{2c}{\underbrace{b_k}_{E[v]-k\epsilon} - E[\tilde{v}]} = 1 - \frac{2c}{k\epsilon} \Rightarrow bY_k = M\left(1 - \frac{2c}{k\epsilon}\right)$$

$$aY_k = M\left(1 - \frac{2c}{k\epsilon}\right)$$

Example

Implications:

- Minimum bid-ask spread is $4c$ even without asymmetric information and with infinitesimal small unit of trade.

$$0 \leq F(bY_k) = 1 - \frac{2c}{k\epsilon} \Rightarrow \varepsilon k \geq 2c$$

$$\Rightarrow \text{ bid } \leq E[v] - 2c, \text{ bid } \geq E[v] + 2c$$

- Depth decreases with limit order cost c.
- If the F first order stochastically dominates G, then depth for F is larger than depth for $G \Rightarrow$ Depth increases with average market order size.

Informed IT

- $\tilde{v} \in \{v_1, v_2\}$
- Price tick size $\epsilon \to 0$.
- round lot $q > 0$ shares.
- A fraction $1 - \mu$ of IT are liquidity traders (LIT):

$$\begin{aligned}
\Pr(\text{LIT sells } 2q) = \Pr(\text{LIT sells } 1q) &= \frac{1}{4} \\
\Pr(\text{LIT buys } 1q) = \Pr(\text{LIT buys } 2q) &= \frac{1}{4}
\end{aligned}$$

- A fraction μ of IT know $\tilde{v}$ (IIT):
  - IIT buys all that sells for less than $\tilde{v}$
  - IIT sells all that trades for more than $\tilde{v}$
- PT are risk neutral and not informed.

Informed IT: equilibrium

- PT's zero profit condition

$$\begin{aligned} \text{bid side:} \quad & \Pr(\tilde{Q} \leq -Y_k)(E[\tilde{v}|\tilde{Q} \leq -Y_k] - b) - c = 0 \\ \text{ask side:} \quad & \Pr(\tilde{Q} \geq Y_j)(a - E[\tilde{v}|\tilde{Q} \geq Y_j]) - c = 0 \end{aligned}$$

- IIT equilibrium behaviour:
  - IIT buys all that sells for less than $\tilde{v}$
  - IIT sells all that trades for more than $\tilde{v}$

Informed IT: bid side equilibrium

Let $b \in [v_1, v_2]$

$$\begin{aligned}
\Pi^{bid}(-3) &= q\left(\pi(v_2 - b) * 0 + (1-\pi)(v_1 - b)\mu\right) - c \\
\Pi^{bid}(-2) &= q\left(\pi(v_2 - b)\frac{1-\mu}{4} + (1-\pi)(v_1 - b)\left(\frac{1-\mu}{4} + \mu\right)\right) - c \\
\Pi^{bid}(-1) &= q\left(\pi(v_2 - b)\frac{1-\mu}{2} + (1-\pi)(v_1 - b)\left(\frac{1-\mu}{2} + \mu\right)\right) - c
\end{aligned}$$

Let

$$b(Y_k) = E[\tilde{v} | \tilde{Q} \leq -Y_k] - \frac{c}{q \Pr(\tilde{Q} \leq -Y_k)}$$

Then $E[v] > b(1) > b(2) > v_1 > b(3)$ and the bid side book consists of

- A buy limit order for q share at price $b(1)$
- A buy limit order for q share at price $b(2)$

Event	Probability given $\tilde{V} = V_1$	Probability given $\tilde{V} = V_2$
trader sells at least 1 unit		
trader sells at least 2 unit		
trader sells at least 3 unit		
trader buys at least 1 unit		
trader buys at least 2 unit		
trader buys at least 3 unit		

Informed IT: ask side equilibrium

Let

$$a(Y_k) = E[\tilde{v}|\tilde{Q} \geq Y_k] + \frac{c}{q\,\Pr(\tilde{Q} \geq Y_k)}$$

Then $E[v] < a(1) < a(2) < v_2 < a(3)$ and the ask side book consists of

- A sell limit orders for q share at price $a(1)$
- A sell limit orders for q share at price $a(2)$

LOB with informed IT: implications

- The depth of the LOB decreases with the informativeness of the order flow
  - $b(Y)$ is decreasing in μ
  - $a(Y)$ is increasing in μ
- The bid depth of the LOB decreases with the ex-ante uncertainty $\pi(1-\pi)$.
- The ask depth of the LOB decreases with the ex-ante uncertainty $\pi(1-\pi)$.
- Minimum bid ask spread is positive even when $c = 0$ even for arbitrarily very small size of the round lot q.