Part 3: Forward and Futures

November 14, 2022

Overview

1. Forward/futures basics
   - Payoff
   - Forwards vs. futures
   - Counterparty risk
2. Using forwards/futures
   - Hedging
   - Speculation
3. Valuation
   - Without dividends
   - With dividends
   - Commodities

Example

- Consider a farm planting corn in April and planning to sell its $180m^3$ of crop in September
  - The farm bears risk because corn price in September is uncertain
- Consider Nestlé that plans to buy corn in September to produce breakfast cereals
  - Nestlé also bears a risk because the corn price in September is uncertain
- Mutually beneficial arrangement: Agree today on price at which they will transact in September

$\Rightarrow$ *This is a forward contract if parties contract directly, a futures contract if they do it through standard contracts that trade in organized derivatives markets*

Example continued

- Today the farm commits to deliver to Nestlé, on Sept 03^{rd} 2023, 180 m^3 of corn.
- Today Nestlé commits to pay to the farm, on Sept 03^{rd} 2023, an amount of USD 700.
- On Sept 03^{rd} 2023 the farm delivers the 180 m^3 of corn to Nestlé and Nestlé pays USD 700 to the farm.

$$\Downarrow$$

- The farm is sure to sell its corn for USD 700 $\Rightarrow$

$$Corn : -180m^3; USD : +700$$

- Nestlé is sure to pay USD 700 for the corn $\Rightarrow$

$$Corn : +180m^3; USD : -700$$

Forward and futures contracts

Forward and futures contracts are agreements between two parties

- to trade a specified asset: the **underlying asset**
- a specific quantity of the asset: the **size of the contract**
- at a specified date in the future: the **maturity** (or expiration) date T
- at a specified unit price: the **delivery price** $F_{0,T}$ set at t=0
- at $t = 0$, when parties sign the contract, there is no transfer of cash nor of the underlying.

Example continued

Another way to see the deal between the farm and Nestlé.

Let S_T be the market price for $180m^3$ of corn on September 3^{rd} 2023. Today we do not know what will S_T be.

- Today the farm commits to transfer to Nesté USD, on September 3^{rd} 2023, an USD amount of

$$700 - S_T$$

- Today Nesté commits transfer to the farm an USD amount of

$$S_T - 700$$
$$\Downarrow$$

Sn September 3^{rd} 2023

- The farm sells its corn in the market for S_T and transfer $700 - S_T$ to Nestlé. $\Rightarrow$

$$Corn : -130m^3; USD : S_T + (700 - S_T) = +700$$

- Nestlé purchases $130m^3$ in the market and transfer $S_T - 700$ to the farm $\Rightarrow$

$$Corn : +130m^3; USD : -S_T + (S_T - 700) = -700$$

Forward and futures payoffs

Consider a forward/futures contract of size 1:

- The buyer of the forward/futures (who is said to hold the long position in the contract) buys the underlying asset at maturity

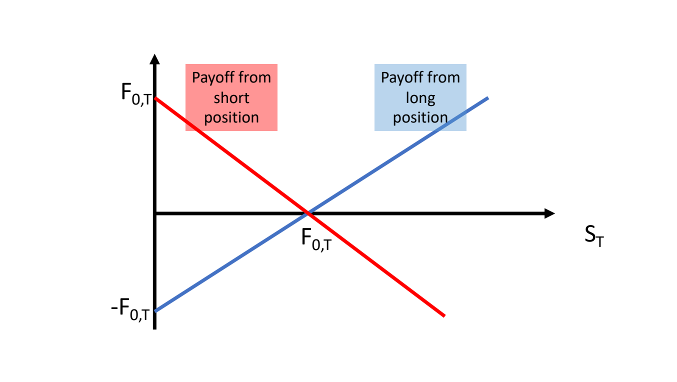
$$\boxed{\text{payoff of long position at maturity} = S_T - F_{0,T}}$$

where S_T denotes the underlying asset's spot price at maturity

- The seller of the forward/futures (who is said to hold the short position in the contract) sells the underlying asset at maturity

$$\boxed{\text{payoff of short position at maturity} = F_{0,T} - S_T}$$

Forward and futures payoffs



Forward and futures contracts

- Forward/futures contracts are called derivative securities because their price and payoff are derived from another asset (the underlying asset)
- Forward/futures contracts can be written on any type of underlying asset:
  - Financial securities: fixed income (typically government) securities, stock market indices, individual stocks
  - Commodities (energy, metals, agricultural products)
  - Currencies (foreign exchange)
  - Temperature indices, CO2 emissions, ...

Forward price

- The delivery price of a forward contract created at date t with maturity date T is called the forward price (or futures price for a futures contract) and is denoted by $F_{t,T}$ (or F_t if there is no ambiguity on the maturity date)
  - Two forward contracts with the same maturity date, created at different dates will generally have different forward prices: $F_{0,T} \neq F_{t,T}$
- How is the delivery price chosen at the creation of the contract?
  - Such that it is “fair” for both parties
  - That is, such that both parties agree to enter into the contract for free (more on this later in this course)

Example

On April 1st, the farm and Nestlé agree to exchange 10,000 bushels of corn at 4 $/bushel on September 1st

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Underlying asset	corn
Contract size	10,000 bushels
Long position	Nestlé
Short position	the farm
Maturity	September 1st
Delivery price	4 $/bushel

Q1 What the farm and Nestlé's cash flows if the spot price of corn is 3.40 $/bushel on September 1st?

	Apr 1st (t=0)	Sep 1st (t=T)
Long position (Nestlé)	?	?
Short position (the farm)	?	?

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Long position	Nestlé
Short position	the farm
Maturity	September 1st
Delivery price	4 $/bushel

Q1 What the farm and Nestlé's cash flows if the spot price of corn is 3.40 $/bushel on September 1st?

	Apr 1st (t=0)	Sep 1st (t=T)
Long position (Nestlé)	0	$(3.4 - 4) \times 10,000 = -6,000$
Short position (the farm)	0	$(4 - 3.4) \times 10,000 = +6,000$

Example

- Futures contract E-mini S&P 500
  - Traded on the Chicago Mercantile Exchange
  - Underlying asset: US stock market index S&P 500
  - Contract size: 50 $\times$ index
  - Maturity dates: every March, June, September, December
  - Futures price $F_{0,Dec.2025}$
  - Cash settlement

You buy one E-mini S&P 500 futures contract with December 2025 maturity. What will be your cash flow at maturity if the S&P 500 is at $2,900 at maturity?

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$$50 \times (2,900 - F_{0,Dec.2025}) = \$ \ldots$$

Cash vs. physical settlement

Physical settlement The underlying asset is physically delivered in exchange for the payment of F_0

Q2 (Farm and Nestlé cont'd: $F_0 = 4$, $S_T = 3.4$)What is the payoff at maturity for the farm if the contract specifies physical delivery?

Cash settlement The losing party pays out the (absolute value of) difference between F_0 and S_T in cash, the underlying asset is not delivered

Q3 What is the payoff at maturity for the farm if the contract specifies cash delivery?

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Q3 What is the payoff at maturity for the farm if the contract specifies cash delivery?

$\Rightarrow$ Payoffs are identical under cash and physical settlement

Closing-out a position

6 months ago (t = -0.5) I entered a long position into Fwd contract for 10,000 barrels of crude oil at forward price F(-0.5,T) = USD 39, maturity T = 31/12/2031.

today (t=0) I see that the crude oil forward price for delivery 31/12/2031 increased to F(0,31/12/2031)=USD 42.

I would like to close my position in order to cash-in the gain.

HOW?

Closing-out a position

Today, I can enter a short position in a Fwd contract with exactly the same properties:
Underlying asset: barrel of crude oil;
Maturity: 31/12/31;
Size 10,000 barrels;
F(0,31/12/2031)=USD 42.

What is my global payoff?

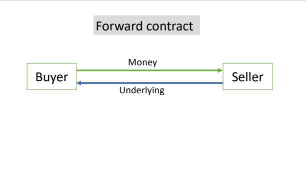
Position	Cash flow on 31/12/31
Long position I entered in t=-0.5	$10,000(\tilde{S}(T) - 39)$
Short position I enter in t=0	$10,000(42 - \tilde{S}(T))$
Overall Payoff:	**10,000(42-39)= USD 30,000**

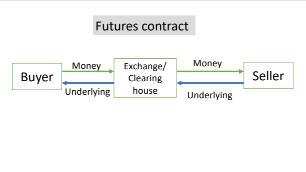
I make USD 30,000, no matter what will be the spot price of oil on 31/12/31

Forwards vs. futures

- Forwards trade *over-the-counter* (in the OTC market)
  Ex: call your bank and buy forward contract for delivery of 123 kg of frozen orange juice in Jouy-en-Josas on Thursday, December 13th

- Futures are standardized contracts that trade on exchanges
  Ex: buy one futures contract on ICE for delivery of 15,000 pounds of frozen orange juice





Forwards vs. futures

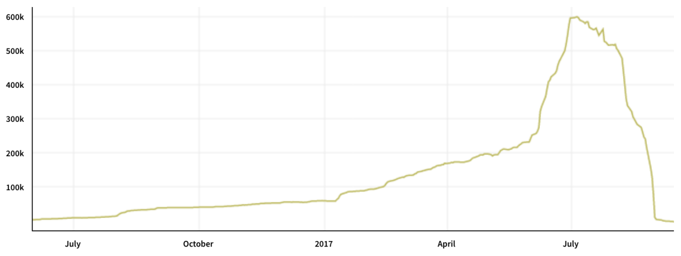
Forwards are:	Futures are:
illiquid: one needs to look for a counterparty to enter in a forward contract	liquid: futures are easily traded on trading exchanges (organized markets)
flexible: forwards can be tailored to the specific needs of the counterparties	standardized: to be exchange-traded, futures are standardized (in terms of nature of the underlying, size of the contract, maturity date, place of delivery)

How are futures contract types introduced?

- Grain (corn, wheat, etc) forward contracts that regularly traded in secondary markets in the US Midwest during mid-1800s, eventually turned into standardized futures contracts.
- First first modern organized futures exchanges (with a central clearinghouse mandating clearing of trades) dating back to mid-1920s.
- Agricultural product futures remained the main type of futures contracts that were available until 1970s
- New futures contracts on financial assets were introduced following the break-down of the Bretton Woods Agreement and the higher volatility that followed in the financial markets: FX-futures, interest rate (government bond) futures, and stock index futures contracts
- Similarly, over the past two decades, the need to hedge against the effects of climate change has led past to the creation of new futures contracts on temperature, water, rainfall, snowfall, etc
- Newly introduced contracts remain in existence so long as there is economically meaningful trading that is conducted in them: such contracts disappear when there is little interest in using them.

How are individual futures contracts created?

- For futures to be created, first the exchange has to allow trading of a particular contract with a specific expiration (maturity) date; then, new contracts will be created when buyers and sellers get into trades.
- The number of contracts outstanding, called the open interest, first increases slowly (as investors get into trades), and then collapses close to futures maturity date (as investors take reverse positions to avoid physical delivery).
- Open Interest for the CME corn futures maturing on 2017.07.14:



- Very few (less than 2%) of contracts will require actual delivery: almost all buyers (sellers) will get out of their positions by selling (buying) the very same contract.

Climate change related futures contracts

Chicago Mercantile Exchange (CME) introduced:

- in 1999 Temperature (Heating Degree Day and Cooling Degree Day) futures and options on these futures
- in 2006 Snowfall futures and options on these futures
- in 2007 Hurricane Index futures and options on these futures
- in 2010 Rainfall futures and options on these futures

But, in 2014 all of the above contracts, except HDD and CDD (Temperature) futures and options, were delisted due to no trading!

- in 2020 CME decided to introduce Water futures and options on these futures based on the NASDAQ Veles California Water Index (NQH2O), but as of today, there is very little to no trading in these futures

Temperature futures contracts

- Futures based on Heating Degree Days (HDD) and Cooling Degree Days (CDD), which are indicators of energy need for heating or cooling a building, based on temperature-thresholds:
- HDD = Max(0, 65 °F - daily average temperature)
- CDD = Max(0, daily average temperature - 65 °F)
- HDD = Max(0, 18 °C - daily average temperature)
- The futures value is based on the daily differences in temperature with respect to 65 °F in the US (18 °C elsewhere) times $20 HDD or CDD in the US (€20 in Europe, £20 in the UK)

  As a simplified example, if the daily average temperature at Heathrow Airport in London were 3 °C each day of the month, then the value of the futures contract would be: Max(0, 18 °C - 3 °C) $\times$ 30 days $\times$ £20 = £9000
- Who would use these contracts: any business that is likely to incur higher heating or cooling costs, or to experience loss of business due to lower or higher than expected temperatures.

Counterparty risk

- Forward contracts carry counterparty risk, the risk that the other party in the contract fails to meet its contractual obligation at maturity

Margins & Daily settlement

- How to eliminate counterparty risk?
  1. Buyers and sellers put up money on margin account up-front at the creation of the contract
  2. Then, profits and losses are credited/debited to both parties every day (daily settlement). This requires that the contract is valued every day, i.e., marked-to-market
  3. If your margin account falls below a limit, you receive a margin call and have to put more money in the margin account.
  4. The margin money is returned when the contracts expires or the position is closed
- Daily settlement is always used for futures contracts: a clearing house acts as the counterparty to all buyers and sellers and is in charge of calling margins

Margins & Daily settlement

- Example
  - Cash flows if you take a long position in a contract with forward price $F_0 = 1,000$ on Day 0 and close it on Day 3 under (1) settlement at maturity vs. (2) daily settlement with margin of 5%

Days	Forward price	CF with settlement at maturity (forward)	CF with daily settlement (futures)	Margin account (futures)
0	1000	0		50
1	980	0	–20	30
2	990	0	+10	40
3	1010	+10	+20	60
Total		+10	+10	net+10

- Advantage of daily settlement: no counterparty risk
- Disadvantage of daily settlement: cash outflows before maturity

Using forwards & futures

Why are these contracts useful?

- Forwards and futures can be used to hedge, that is, to insure against specific sources of risk
- Forwards and futures can also be used to speculate on financial assets or commodities

Hedging

An airline will buy 1 million barrels of kerosene in $T = 6$ months, at which time the spot price of kerosene will be S_T per barrel, a price not known today

Q1 Should the airline take a long or a short positions on kerosene futures with maturity 6 months to hedge against fluctuations in kerosene prices?

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1,000 contracts

	CF in 6 months
Buy kerosene	-1,000,000 S_T
Long 1,000 futures	1,000 x 1,000 x $(S_T - F_0)$
Total	-1,000,000 F_0

When perfect hedges are not possible

- When using standardized futures contracts
  - The assets that need to be hedged may differ from the underlying assets of the futures contracts

    For instance, if you want to hedge your crop of rye (whose futures are no longer available) use wheat futures instead

  - Contract maturities may differ from the desired hedging maturity

    In this case, use futures with the closest maturity date

- The remaining risk is called the basis risk

Speculation

Why speculate in the futures/forward market rather than in the spot market? Two main reasons:

1. For commodities, you don't need to store the underlying asset if you take a long position or to already hold it if you take a short position
2. Futures position returns are leveraged compared to spot position returns

Example: Suppose you buy stock X in the spot market at S_0 = €100
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- Now, suppose you buy a futures contract on stock X maturing in one week at $F_0 = €100$. The initial margin requirement for this contract is 5% of F_0.

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  What is your return if S_1 = €110 one week later?

$$HPR = \frac{S_1 - F_0}{0.05 \times F_0} = \frac{10}{5} = 2 = 200\%$$

- What if S_1 = €90?

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- What if S_1 = €90? −10% and −200%

Example: S&P500 ESG Index Futures

- Suppose that you are bearish on ESG stocks in the US for some reason (say, because you believe that a major green-washing scandal will break between now and December).
  - You could short (i.e., sell) the E-mini S&P 500 ESG Futures contract
  - The S&P 500 ESG Index is a market-value-weighted stock index that includes 320 or so stocks that are retained after applying ESG-based exclusion criteria to the 500 stocks that are in the S&P 500 Index.
  - Traded on the Chicago Mercantile Exchange (CME)
  - Underlying asset: US stock market index S&P 500 ESG
  - Contract size: \$ 500 $\times$ S&P 500 ESG Index
  - Maturity dates: third Friday of every March, June, September, December
  - Cash settlement based on the maturity date opening value of the S&P 500 ESG Index

Example: S&P500 ESG Index Futures

- [Q:]You sell 10 E-mini S&P 500 ESG futures contracts with Dec 2022 maturity. What will be your cash flow at maturity (T=Dec 2022) if the futures price today at t=0 is $F_{0,T}$ = \$316, and S&P 500 ESG Index ends up equaling 300 at T?

Example: S&P500 ESG Index Futures

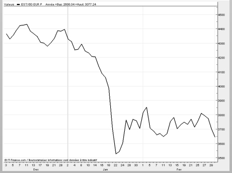
- [Q:]You sell 10 E-mini S&P 500 ESG futures contracts with Dec 2022 maturity. What will be your cash flow at maturity (T=Dec 2022) if the futures price today at t=0 is $F_{0,T}$ = \$316, and S&P 500 ESG Index ends up equaling 300 at T?
  $\$500 \times (F_{0,T} - S_T) \times 10 = \$500 \times (316 - 300) \times 10 = \$80,000$

Speculation

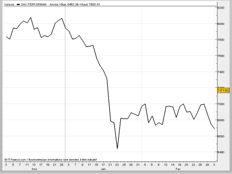
- Speculating with futures can be (very!) risky because of leverage
- Example: Jerome Kerviel at Societe Generale, 2008

Underlying stock index	Contract size	No. of long positions	Spot price Jan 1st	Notional	Spot price Jan 18	Profits
EuroStoxx	10	743,000	4,330	32 bn €	4,000	−2.5 bn €
DAX	25	100,000	7,950	20 bn €	7,400	−1.4 bn €
Total						−3.9 bn €

Eurostoxx 50



DAX 30



Valuation

- We can determine the forward/futures prices $F_{0,T}$ by arbitrage pricing
- We will consider three cases
  1. Contract on a security that pays no dividend before the maturity of the futures
  2. Contract on a security that pays dividends before the maturity of the futures
  3. Contract on a commodity that includes a cost of carry

The spot-forward parity

Theorem

Consider a forward contract with maturity T years on an underlying asset that does not pay cash flows before T years.

- *Let r_T be the T year interest rate;*
- *Let S_0 be the spot price of the underlying asset;*

Then, the (no arbitrage) forward price is:

$$F_{0,T} = S_0(1 + r_T)^T$$

Forward no-arbitrage prices

Forward no-arbitrage prices

Proof: It is possible to replicate the cash flows of a long position in the forward contract with the following portfolio R:

Trade	Today	Time T
Borrow $\frac{F_{0,T}}{(1+r_T)^T}$	$\frac{F_{0,T}}{(1+r_T)^T}$	$-F_{0,T}$
Buy spot the underlying asset	$-S_0$	$\tilde{S}_T$
Value of this portfolio	$\frac{F_{0,T}}{(1+r_T)^T} - S_0$	$\tilde{S}_T - F_{0,T}$

After T years, the value of this portfolio is equal to the payoff from a long position in the forward contract:

$$\tilde{S}_T - F_{0,T}$$

By no arbitrage, today cost of this portfolio must be equal to what you pay to enter a Fwd contract, that is 0. Hence,

$$\frac{F_{0,T}}{(1+r_T)^T} - S_0 = 0$$

Forward price

Example

The current price of one ABC share is $S_0 = Eu\ 50$. The term structure of interest rate is flat at level $r = 3\%$. ABC will not pay dividends before 1 year.

- What is the 9-month forward price for ABC?

Forward price

Example

The current price of one ABC share is $S_0 = Eu\ 50$. The term structure of interest rate is flat at level $r = 3\%$. ABC will not pay dividends before 1 year.

- What is the 9-month forward price for ABC?

$$F_{0,9M} = 50(1.03)^{3/4} = Eu\ 51.2$$

- Suppose that 6 months later, the term structure of interest rate is flat at $r = 3\%$ and the spot price of one share of ABC is unchanged: $S_{6M} = Eu\ 50$.Then the forward price of ABC for the same maturity date will be:

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$$F_{6M,9M} = S_{6M}(1 + r)^{1/4} = 50(1.03)^{1/4} = 50.37$$

Forward price

Example

Three months ago you entered into a long position for one Fwd contract on a non-dividend paying stock. The Fwd price was €19 and the Fwd maturity is in 13 months from today.
Today S_0 = €20 and the term structure is flat at $r = 3\%$.
What is the value of your long position in the Fwd contract?

Forward price

Example

Three months ago you entered into a long position for one Fwd contract on a non-dividend paying stock. The Fwd price was €19 and the Fwd maturity is in 13 months from today.
Today S_0 = €20 and the term structure is flat at $r = 3\%$.
What is the value of your long position in the Fwd contract?
Today Fwd price is

$$F_{0,13M} = 20(1.03)^{13/12} = 20.65$$

If today you close out your position, your payoff at maturity will be :

$$\underbrace{20.65}_{\text{short position fwd price}} - \underbrace{19}_{\text{long position fwd price}} = 1.65$$

As you will receive this cash flow only in 13 month, today value of your position is

$$\frac{1.65}{1.03^{13/12}} = 1.60 = S_0 - \frac{19}{(1 + r_T)^T}$$

Term structure and forward prices

Example

The 6 month forward price of a non-dividend paying stock is

$$F_{0,6M} = Eu\ 60.$$

The spot price of the stock is

$$S_0 = Eu\ 58.55$$

What is the 6 month interest rate?

Term structure and forward prices

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$$S_0 = Eu\ 58.55$$

What is the 6 month interest rate?

$$60 = 58.55(1 + r_{6M})^{0.5} \Rightarrow r_{6M} = \left(\frac{60}{58.55}\right)^{1/0.5} - 1 = 5.01\%$$

Forward price of cash-flow paying underlying

Forward price of cash-flow paying underlying

Theorem

Consider a forward contract with maturity T

- *The underlying asset pays known cash flows $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ at dates $t_1, t_2, \ldots, t_n$, with $t_1, < t_2, < \cdots < t_n < T$.*
- *Let S_0 be the spot price of the underlying asset.*
- *Let r_T be the t-year interest rate;*

Then the (no arbitrage) forward price is:

$$F_{0,T} =$$

$$\left(S_0 - \frac{I_{t_1}}{(1 + r_{t_1})^{t_1}} - \cdots - \frac{I_{t_n}}{(1 + r_{t_n})^{t_n}} \right) (1 + r_T)^T$$

$$= \left(S_0 - \sum_{i=1}^{n} \frac{I_{t_i}}{(1 + r_{t_i})^{t_i}} \right) (1 + r_T)^T$$

Forward price of cash-flow paying underlying

Proof: It is possible to replicate the cash flow of a long position in the forward contract with the following portfolio strategy:

1. Buy 1 underlying asset.
2. Borrow $\frac{F_{0,T}}{(1+r_T)^T} + \sum_{i=1}^{n} \frac{I_{t_i}}{(1+r_{t_i})^{t_i}}$.
3. At each date t_i, use the underlying asset cash flow I_{t_i} to partially reimburse your debt.

After T years the value of this portfolio is equal to the payoff from a long position in the forward contract:

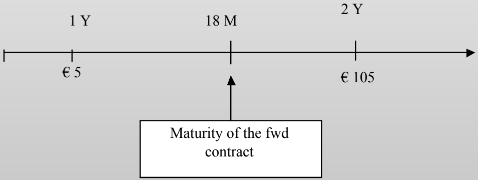
$$S_T - F_{0,T}$$

As it is costless to enter a forward contract, the cost of this replicating strategy must be zero:

$$S_0 - \frac{F_{0,T}}{(1+r_T)^T} - \sum_{i=1}^{n} \frac{I_{t_i}}{(1+r_{t_i})^{t_i}} = 0$$

Forward price of cash-flow paying underlying

Consider a bond with maturity in 2 years, annual coupon of €5 and face value €100.
The term structure is flat at 3%.
The spot price for this bond is S_0 =€103.83

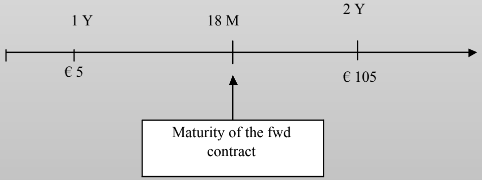


The no arbitrage 18 month forward price is

$$F_{0,18M} = \left(103.83 - \frac{5}{1.03}\right)(1.03)^{1.5} = €103.46$$

Forward price of a cash-flow-paying underlying

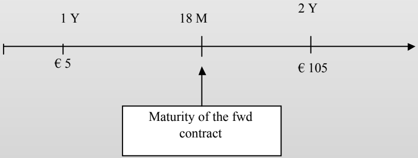
Consider a bond with maturity in 2 years, annual coupon of €5 and face value €100.
The term structure is flat at 3%.
The spot price for this bond is S_0 = €103.83
You observe a forward price of $F_{0,18M}$ =€ 104.



Identify an arbitrage strategy

Forward price of cash-flow paying underlying

$r = 3\%; S_0 = 103.83; F_{0,18M} =$ € $104.$

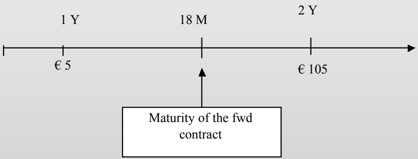


Arbitrage strategy: sell forward and replicate a long position in the forward contract

Trade	Today	1 Y	18 M
Sell forward the bond	0	0	$104 - \tilde{S}_{18M}$
Buy spot the bond	-103.83	5	$\tilde{S}_{18M}$
Borrow for 1 year $\frac{5}{1.03}$	$\frac{5}{1.03}$	-5	
Borrow for 18 months $103.83 - \frac{5}{1.03} = 98.97$	98.97	0	$-98.97 * 1.03^{1.5} = -103.46$
net cash flows	**0**	**0**	**0.54**

Forward price of cash-flow paying underlying

What if you observe a forward price $F_{0,18M}$ =€ 100?



Arbitrage strategy:buy forward and replicate a short position in the forward contract

Trade	Today	1 Y	18 M
buy forward the bond	0	0	$\tilde{S}_{18M} - 100$
short spot the bond	103.83	-5	$-\tilde{S}_{18M}$
Invest for 1 year $\frac{5}{1.03}$	$-\frac{5}{1.03}$	5	
Invest for 18 months $103.83 - \frac{5}{1.03} = 98.97$	-98.97	0	$+98.97 * 1.03^{1.5} = +103.46$
net cash flow	**0**	**0**	**3.46**

Forward price of cash-flow paying underlying

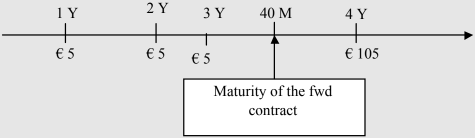
Example

Bond A: T= 4 Y, N =€100 , C = €5, z= 1 Y.

Term structure

	1 Y	2 Y	3 Y	40 M	4 Y
r_T	2%	2%	3%	3.5%	4%

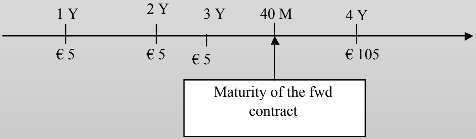
What is the 40 month forward price for bond A?



Forward price of cash-flows paying underlying

Bond A: T= 4 Y, N =€100 , C = €5, z= 1 Y.

	1 Y	2 Y	3 Y	40 M	4 Y
r_T	2%	2%	3%	3.5%	4%



$$S_0 = \frac{5}{1.02} + \frac{5}{1.02^2} + \frac{5}{1.03^3} + \frac{105}{1.04^4} = 104.04$$

$$F_{0,40M} = \left(104.04 - \frac{5}{1.02} - \frac{5}{1.02^2} - \frac{5}{1.03^3}\right) 1.035^{40/12} = 100.66$$

Forward prices of commodities

Theorem

Consider a forward contract with maturity T on a commodity whose spot price is S_0. The cost of storing the commodity for T years is C, in todays money.

Then, the (no arbitrage) forward price satisfies the following relation:

$$F_{0,T} \leq \overline{F}_{0,T} = (S_0 + C)(1 + r_T)^T$$

Forward prices of commodities

Proof:

1. It is possible to replicate the cash flows of a long position in this contract with a portfolio containing
   - A loan for $\frac{F_{0,T}}{(1+r_T)^T}$
   - 1 underlying asset

   Thus, if

   $$F_{0,T} > (S_0 + C)(1 + r_T)^T,$$

   then it is possible to build an arbitrage portfolio.

2. However, it is possible to replicate a short position **only if** you already hold the commodity.
   Thus, even if

   $$F_{0,T} < (S_0 + C)(1 + r_T)^T,$$

   it is not always possible to build an arbitrage strategy.

Forward prices of commodities

One ton of wheat: S_0= BPD 78; present value of storage cost for 6 months = 1 BPD; BPD term structure is flat at 2%.

What can you say about the no arbitrage forward price for delivery in 6 months?

Forward prices of commodities

One ton of wheat: S_0= BPD 78; present value of storage cost for 6 months = 1 BPD; BPD term structure is flat at 2%.

What can you say about the no arbitrage forward price for delivery in 6 months? The no arbitrage forward price cannot be lager than

$$\overline{F}_{0,6M} = (78 + 1)1.02^{0.5} = 79.79$$

Suppose you observe a forward price $F_{0,6M}$=BPD 80. Identify an arbitrage strategy

Forward prices of commodities

One ton of wheat: S_0= BPD 78; present value of storage cost for 6 months = 1 BPD; BPD term structure is flat at 2%.

What can you say about the no arbitrage forward price for delivery in 6 months? The no arbitrage forward price cannot be lager than

$$\overline{F}_{0,6M} = (78 + 1)1.02^{0.5} = 79.79$$

Suppose you observe a forward price $F_{0,6M}$=BPD 80. Identify an arbitrage strategy

Arbitrage: Sell forward and replicate a long position in the forward

trade	today	6 months
Borrow BPD 79 for 6 months	79	$-79 * 1.02^{0.5} = -79.79$
Buy spot 1 ton of wheat	-78	$\tilde{S}_{6M}$
Pay the storage cost	-1	
Sell forward wheat	0	$80 - \tilde{S}_{6M}$
Net cash flows	**0**	**0.21**

Forward prices of commodities

One ton of wheat: $S_0 = BPD78$; $F_{0,6M} = 70BPD$; present value of storage cost for 6 months = 1 BPD .
BPD term structure: Flat at 2%.

You already own one ton of wheat. Identify an arbitrage strategy

trade	today	6 months
Invest BPD 79 for 6 months	-79	$79 * 1.02^{0.5} = 79.79$
sell spot 1 ton of wheat	78	$-\tilde{S}_{6M}$
save the storage cost	1	
Buy forward wheat	0	$\tilde{S}_{6M} - 70$
Net cash flows	**0**	**9.79**

Commodities cannot be sold short ⇒ You can make this arbitrage only if you already hold the commodity .

Forward prices of commodities

Example

1,000 bottles of Bordeaux: $S_0 =$€ $25,000$, $F_{0,1Y} =$€ $26,000$.
Term structure: $r_{1Y} = 2\%$, $r_{2Y} = 2.5\%$.

What can you say about the 1 year storage cost for 1,000 bottles of Bordeaux?

Forward prices of commodities

Example

1,000 bottles of Bordeaux: $S_0 =€ 25,000$, $F_{0,1Y} =€ 26,000$.
Term structure: $r_{1Y} = 2\%$, $r_{2Y} = 2.5\%$.

What can you say about the 1 year storage cost for 1,000 bottles of Bordeaux?

$$F_{0,1Y} \leq (S_0 + C)(1 + r_{1Y})$$

$$\Downarrow$$

$$C \geq \frac{F_{0,1Y}}{(1 + r_{1Y})} - S_0$$

$$\Downarrow$$

$$C \geq \frac{26,000}{1.02} - 25,000 = 490.20$$

Forward prices of commodities

One ton of Nickel: $S_0 = USD\,6,350$, $F_{0,3M} = USD\,6,210$; trimester storage cost: C =USD 5.
Term structure: $r_{3M} = 3\%$, $r_{1Y} = 4\%$.

- What is the maximum level of the no-arbitrage 3-month forward price?

Forward prices of commodities

One ton of Nickel: $S_0 = USD\ 6,350$, $F_{0,3M} = USD\ 6,210$; trimester storage cost: C =USD 5.
Term structure: $r_{3M} = 3\%$, $r_{1Y} = 4\%$.

- What is the maximum level of the no-arbitrage 3-month forward price?

$$\overline{F}_{0,3M} = (6,350 + 5)(1.03)^{1/4} = 6,402.14$$

- Suppose you own one ton of Nickel as row material in your production process. If you sell it and buy it back after 3 months what is the maximum profit you can make? USD 192.14

Conclusion

- What forward and futures contracts are.
- Differences between forward and futures contracts.
- Cash&carry: How to determine the forward price for
  - Financial assets paying no cash flows before the Fwd maturity.
  - Financial assets paying cash flows before the Fwd maturity.
  - Commodities