Financial Markets
4: Options

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Overview

1. Options basics
   - Payoff
2. Using options
   - Hedging
   - Speculation
   - Creating new payoffs
3. Valuation
   - Put-call parity
   - Arbitrage bounds
   - Binomial model

Guaranteed capital

Today you buy a stock index at $S_0 = Eu\ 1,000$.
You will sell the index in T years at $\tilde{S}_T$.

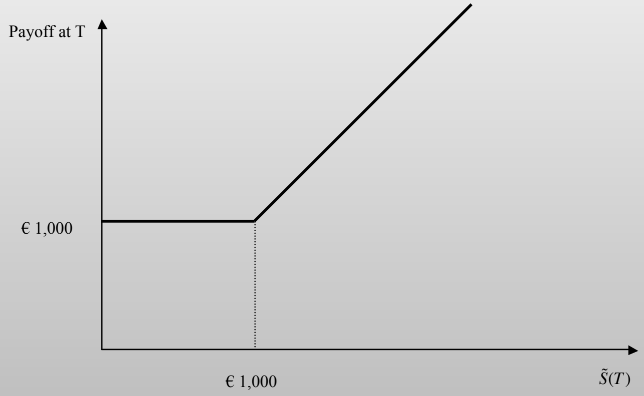
You would like to make a profit from an increase in the stock index price (i.e., when $\tilde{S}_T > S_0$).

But you do not want to lose money from a decline in the price (i.e., when $\tilde{S}_T < S_0$).

You would like to buy an insurance that pays

$$1,000 - \tilde{S}_T \text{ iff } 1,000 > \tilde{S}_T$$

Guaranteed capital



Call option

- A call option gives its holder the right, but not the obligation, to buy
  - a specified asset: the underlying asset
  - on, or before, a specified date: the maturity (or expiry) T
  - at a specified price: the strike price (or exercise price) K
- The two parties in an option contract play asymmetric roles
  - The call option holder/option buyer/long position **decides whether to exercise** the option or not
  - The call option writer/option seller/short position has the **obligation** to sell the underlying asset **only if** the holder exercises the option

Put option

- A put option gives its holder the right, but not the obligation, to sell the underlying asset on, or before, the maturity T at the strike price K
  - The put option holder/option buyer/long position decides whether to **exercise** the option or not
  - The put option writer/option seller/short position has the **obligation** to buy the underlying asset **only if** the holder exercises the option

Types of options

- There exists a lot of different types of options:

European It can be exercised at maturity only

American It can be exercised any time before maturity

Bermuda It can be exercised on a set of predetermined dates

Asian The strike price payoff is based on the average of the underlying price during the life of the option

etc.

- NB: These names have nothing to do with geography

Summary of main different types of options

From the perspective of the option owner:

	Call	Put
European	Right to buy at maturity	Right to sell at maturity
American	Right to buy until maturity	Right to sell until maturity

When is it worth exercising a call option?

You own an European call option on 1 share of Apple with
Strike price is K = USD 500,
Maturity is T = 1 year,
In 1 year, you have the right to buy one shares of Apple at price USD 500.

Scenario 1: $S_{1Y} = USD\ 400$

Payoff from not exercising the call option	0
Payoff from exercising the call option	$400 - 500 < 0$

Scenario 2: $S_{1Y} = USD\ 600$

Payoff from not exercising the call option	0
Payoff from exercising the call option	$600 - 500 > 0$

When is it worth exercising a call option?

General rule: The owner of an European call option with maturity T and strike price K will exercise the option if and only if

$$S_T \geq K$$

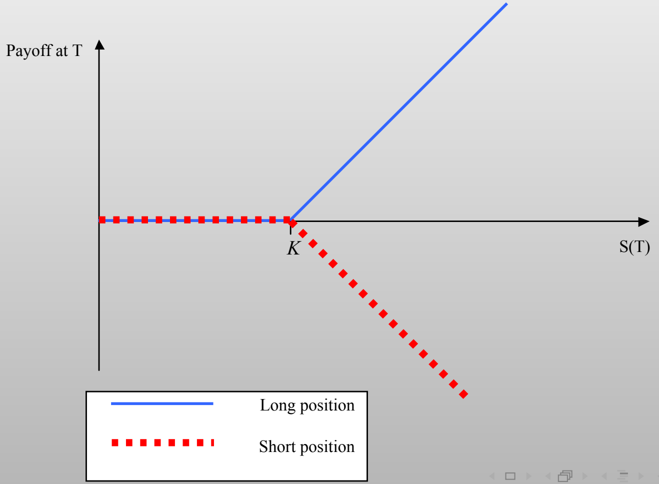
At time T, the payoff from a long position in this option is

$$max\{S_T - K, 0\}$$

At time T, the payoff from a short position in this option is

$$-max\{S_T - K, 0\} = min\{K - S_T, 0\}$$

Payoffs from a call option



When is it worth exercising a put option?

You own an European put option on 1 share of Apple with
Strike price is K = USD 500,
Maturity is T = 1 year,
In 1 year, you have the right to sell one shares of Apple at price USD 500.

Scenario 1: S_{1Y} = *USD* 400

Payoff from not exercising the put option	0
Payoff from exercising the put option	$500 - 400 > 0$

Scenario 2: S_{1Y} = *USD* 600

Payoff from not exercising the put option	0
Payoff from exercising the put option	$500 - 600 < 0$

When is it worth exercising a put option?

General rule: The owner of an European put option with maturity T and strike price K will exercise the option if and only if

$$S_T \leq K$$

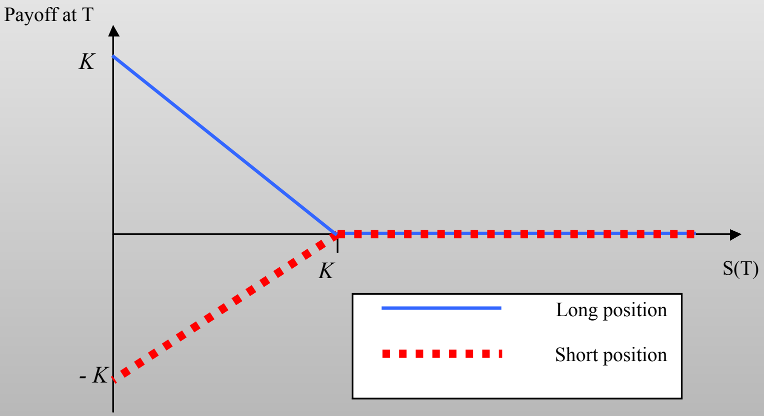
At time T, the payoff from a long position in this option is

$$max\{K - S_T, 0\}$$

At time T, the payoff from a short position in this option is

$$-max\{K - S_T, 0\} = min\{S_T - K, 0\}$$

Payoffs from a put option



Example

Q1 Consider a European call option with strike price (exercise price) $K = 80$
What are the payoffs of a long position and of a short position in this option if the underlying asset's price at expiry is $S_T =$ €105? if $S_T =$€85? if $S_T =$ €80? if $S_T =$ €60?

Spot price at expiry	105	85	80	60
Payoff of long call position at T				
Payoff of short call position at T				

Q2 Consider a European put option with strike price (exercise price) $K =$€80.
Same questions

Spot price at expiry	105	85	80	60
Payoff of long put position at T				
Payoff of short put position at T				

Payoff at maturity & Option premium

- The payoff at maturity of a European call or an American call is
$$C_T = \max\{S_T - K, 0\}$$
- The payoff at maturity of a European put or an American put is
$$P_T = \max\{K - S_T, 0\}$$
- Note that the payoff at maturity of a long position in an option is always positive or equal to zero
- Therefore, the value of an option at $t = 0$, called the premium and denoted C_0 for a call or P_0 for a put, is always positive
- The buyer of an option pays the premium C_0 or P_0 to the option seller at $t = 0$ when the option contract is created

Exercise or not?

At time $t = 0$ you pay $C_0 =$ € 10 to buy an European call option on ABC stock with strike price $K =$ € 50 and maturity T.
At time T the spot price of an ABC stock is $S_T =$€ 55.

Do you exercise the option or not?

Exercise or not?

At time $t = 0$ you pay $C_0 =$ € 10 to buy an European call option on ABC stock with strike price $K =$ € 50 and maturity T.
At time T the spot price of an ABC stock is $S_T =$€ 55.

Do you exercise the option or not?

	$t = 0$	$t = T$
Payoff from not exercising the call option	-10	0
Payoff from exercising the call option	-10	$55 - 50 > 0$

Moneyness

An option is said to be

at the money if $S = K$

$\Rightarrow$ If maturity was today, then the option holder would be indifferent between exercising the option or not

in the money if $S > K$ for a call or $S < K$ for a put

$\Rightarrow$ If maturity was today, the option holder would strictly prefer to exercise

out of the money if $S < K$ for a call or $S > K$ for a put

$\Rightarrow$ If maturity was today, the option holder would strictly prefer not to exercise the option

Options are derivatives

Just like forward/futures contracts, option contracts are derivatives that

1. exist on many types of underlying assets: individual stocks, stock indices, bonds, commodities, currencies, etc.
2. may be settled physically or in cash
3. may be traded over-the-counter or on exchanges
4. are subject to counterparty risk, unless daily settlement is put in place

   Q: Which side(s) of an option contract is subject to counterparty risk?

Example

- Euro Stoxx 50 options
    - Traded on Eurex
    - Calls and puts
    - European style
    - Underlying asset: Euro Stoxx 50 index
    - One maturity date per month
    - Many strike prices
    - Cash settlement, with margin account and marking to market to prevent counterparty risk

Using options

Options can be used to:

- Speculate
- Hedge
- Create new payoffs

Speculating with options

XYZ stock is selling for S_0 = €100. You believe the stock price will go up.

1. You buy the stock. What is your holding period return if the stock price goes up to S_1 = €105 in one month? What if S_1 = €95?
2. You buy a call option on XYZ stock with exercise price K = €100, expiring in one month, at price C_0 = €1. What is your return if S_1 = €105? if S_1 = €95?

$\Rightarrow$ Like futures, option returns are leveraged compared to spot position returns

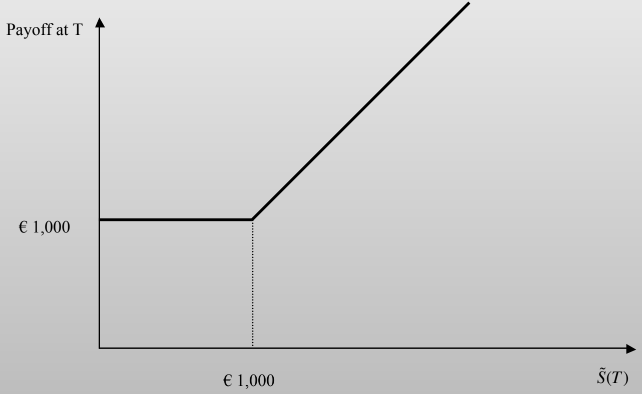
Hedging with options

You manage a pension fund holding a diversified portfolio of stocks. You want to make sure the portfolio cannot lose more than 10% of its value over the course of next year. There exist calls and puts on the stock market index with any strike price and any maturity.

- Q1 Should you buy/sell calls/puts? With which maturity and which strike price?
- Q2 Draw the payoff diagram in 1 year of the hedged portfolio (without accounting for the cost of the option)

Portfolios of options: 1) guaranteed capital

One underlying asset + 1 put option



Portfolios of options: 2) Bottom Straddle

You expect the price of the stock to change in one week but you do not know whether the price will rise or decrease. What is your trading strategy?

	Today	Time T	
		$S_T < K$	$S_T \geq K$
Long 1 put option K, T = 1 week	$-P$	$K - S_T$	0
Long 1 call option K, T = 1 week	$-C$	0	$S_T - K$
Total	$-P - C$	$K - S_T$	$S_T - K$

Portfolios of options: 3) Bottom Straddle

1 put option +1 call option with the same strike price K

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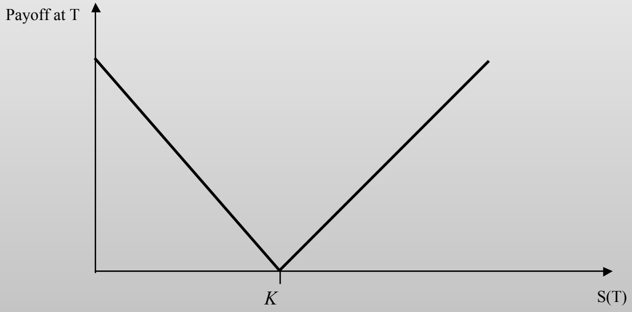


Portfolios of options: 4) Bottom vertical combination

You expect that at time T the price of the stock will change dramatically, but you do not know whether the price will increase or decrease. What is your trading strategy?

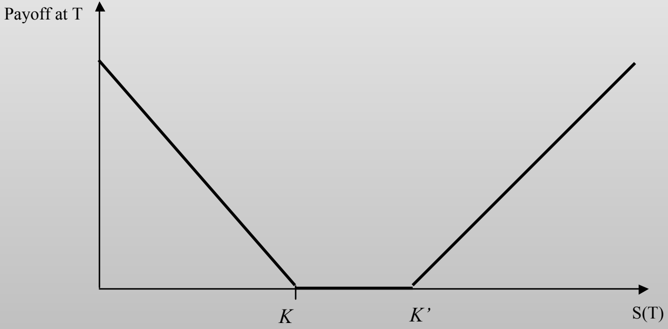
	Today	Time T		
		$S_T < K$	$K \leq S_T < K'$	$S_T \geq K'$
Long 1 put option K, T	$-P$	$K - S_T$	0	0
Long 1 call option $K' > K$, T	$-C'$	0	0	$S_T - K'$
Total	$-P - C'$	$K - S_T$	0	$S_T - K'$

Portfolios of options: 4) Bottom vertical combination

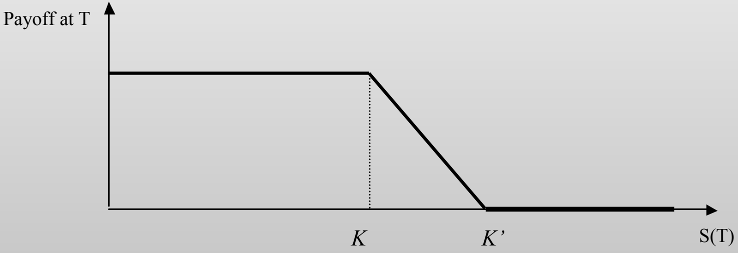
One put option with strike price K, + one call option with strike price $K' > K$

Portfolios of options: 4) Bottom vertical combination

One put option with strike price K, + one call option with strike price $K' > K$



Portfolios of options: 5) Bearish spread



How can you obtain this profile of payoff?

Portfolios of options: 5) Bearish spread

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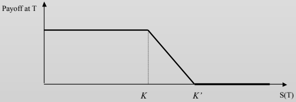

Portfolio of option 6) Bull spread

Consider the following strategy: Buy a call with strike K_1 and sell a call with $K_2 > K_1$ with same underlying asset and same T

Q1 Draw the payoff diagram at expiry

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Consider the following strategy: Buy a call with strike K_1 and sell a call with $K_2 > K_1$ with same underlying asset and same T

- Q1 Draw the payoff diagram at expiry
- Q2 What is the cash flow of the strategy at $t = 0$?
- Q3 Suppose you fund the strategy by borrowing at $t = 0$ and repaying the loan at $t = T$: draw the net profit at expiry

Portfolio of option 6) Bull spread

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Q1 Draw the payoff diagram at expiry

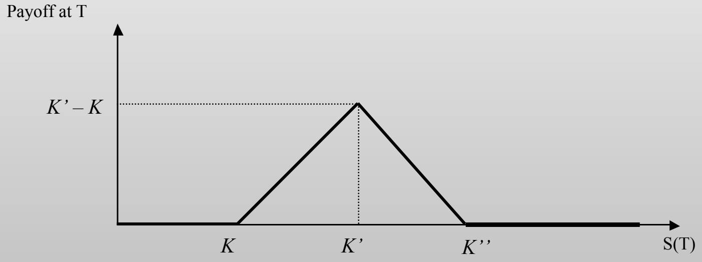
Q2 What is the cash flow of the strategy at $t = 0$?

Q3 Suppose you fund the strategy by borrowing at $t = 0$ and repaying the loan at $t = T$: draw the net profit at expiry

Q4 If $K_1 < S_0 < K_2$, what are you betting on with a bull spread?

Portfolios of options: 7) Butterfly spread

You expect that at time T the price of the stock will not change. What is your trading strategy?



With $K' = S_0$

Portfolios of options: 7) Butterfly spread

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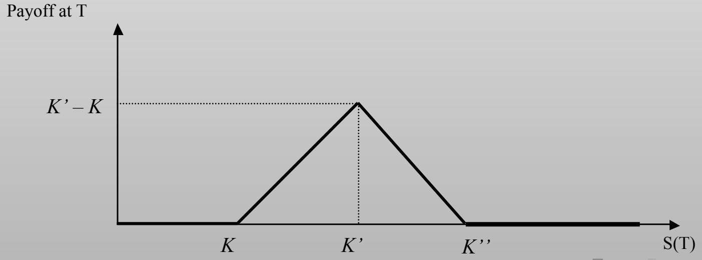
Let $K < K' < K''$ and $K'' = 2K' - K$

	Today	Time T			
		$S_T < K$	$K \leq S_T < K'$	$K' \leq S_T < K''$	$S_T \geq K''$
Long 1 Call option, K	$-C$	0	$S_T - K$	$S_T - K$	$S_T - K$
Short 2 Call options, K'	$2C'$	0	0	$2(K' - S_T)$	$2(K' - S_T)$
Long 1 Call option, K''	$-C''$	0	0	0	$S_T - K''$
Total	$2C'$-C''-C	0	$S_T - K$	$2K' - K - S_T$	0



Convertible bonds

- Consider the following convertible bond issued by company XYZ: maturity 1 year; zero-coupon; face value 100; holder has option to convert into one share of stock XYZ at maturity

Q1 Draw the payoff diagram of the convertible bond (with XYZ stock price in one year on horizontal axis)

Convertible bonds

- Consider the following convertible bond issued by company XYZ: maturity 1 year; zero-coupon; face value 100; holder has option to convert into one share of stock XYZ at maturity

Q1 Draw the payoff diagram of the convertible bond (with XYZ stock price in one year on horizontal axis)

Q2 What is the composition of a replicating portfolio of the convertible bond?

Put-Call Parity

Consider a portfolio that contains

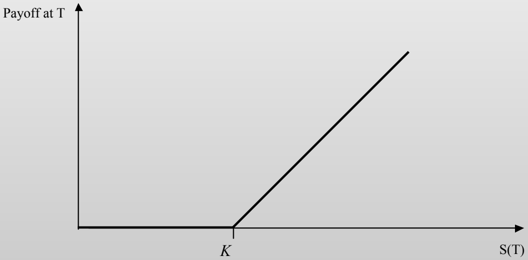
- One long position in a put option with maturity T and strike price K.
- One underlying asset.
- A borrowing of $\frac{K}{(1+r(T))^T}$ until T.

Let P be the price of the put option

	Today	Time T	
		$S_T < K$	$S_T \geq K$
Long 1 put option	$-P$	$K - S_T$	0
Long 1 underlying asset	$-S_0$	S_T	S_T
Borrowing	$\frac{K}{(1+r(T))^T}$	$-K$	$-K$
Total	$\frac{K}{(1+r(T))^T} - P - S_0$	0	$S_T - K$

Portfolios of options: 2) Put-Call Parity

1 underlying +1 put option + borrowing for $\frac{K}{(1+r(T))^T}$



Let C be the no arbitrage price of an European call option with maturity T and strike price K. Then,

$$C = -\frac{K}{(1+r(T))^T} + P + S_0$$

Options pricing

1. We derive the price of a call option (C) and we will use the put-call parity to compute the price (P) of a put option.
2. We will find a lower bound for C.
3. We will find an upper bound for C.
4. We will find the no arbitrage level of C, introducing some assumptions on the probability distribution of $\tilde{S}_T$.

Lower bound for a call option price C

Consider the put-call parity:

$$C = -\frac{K}{(1 + r(T))^T} + P + S_0 \Rightarrow P = \frac{K}{(1 + r(T))^T} + C - S_0$$

that is a put option can be replicated with a call option, investment of $\frac{K}{(1+r(T))^T}$ in ZCB with matuirty T and short selling the underlying asset.

Because $P > 0$ we must have:

$$C > \max\left\{S_0 - \frac{K}{(1 + r(T))^T}, 0\right\}$$

Lower bound for a call option price C

Example

Today, 1 share of ABC trades for $S_0 = Eu\ 60$; the 1 year interest rate is $r(1Y) = 2\%$; an European call option on 1 share of ABC, with maturity $T = 1$ year and strike price $K = Eu\ 50$ trades for $C = Eu\ 5$.

Identify an arbitrage strategy.

Lower bound for a call option price C

Example

Today, 1 share of ABC trades for $S_0 = Eu\ 60$; the 1 year interest rate is $r(1Y) = 2\%$; an European call option on 1 share of ABC, with maturity $T = 1$ year and strike price $K = Eu\ 50$ trades for $C = Eu\ 5$.

Identify an arbitrage strategy.

$$5 = C < S_0 - \frac{K}{(1 + r(T))^T} = 60 - \frac{50}{1.02} = 10.98$$

Trade	Today	Time T	
		$S_T < 50$	$S_T \geq 50$
Short sell 1 ABC	60	$-S_T$	$-S_T$
Long 1 Call option	-5	0	$S_T - 50$
Lend $\frac{50}{1.02}$	$-\frac{50}{1.02}$	50	50
Total	5.98	$50 - S_T > 0$	0

Upper bound for a call option price C

Consider the following portfolio

Trade	Today	Time T	
		$S_T < K$	$S_T \geq K$
Buy 1 underlying	$-S_0$	S_T	S_T
Short 1 Call option	C	0	$K - S_T$
Total	$C - S_0$	S_T	K

At time T this portfolio produces only strictly positive cash flows, Thus, by no arbitrage, it must result:

$$C < S_0$$

Upper bound for a call option price C

Example

Today, 1 share of ABC trades for $S_0 = Eu\ 60$; the 1 year interest rate is $r(1Y) = 2\%$; an European call option on 1 share of ABC, with maturity $T = 1$ year and strike price $K = Eu\ 4$ trades for $C = Eu\ 61$.

Identify an arbitrage strategy.

Upper bound for a call option price C

Example

Today, 1 share of ABC trades for $S_0 = Eu\ 60$; the 1 year interest rate is $r(1Y) = 2\%$; an European call option on 1 share of ABC, with maturity $T = 1$ year and strike price $K = Eu\ 4$ trades for $C = Eu\ 61$.

Identify an arbitrage strategy.

Trade	Today	Time T	
		$S_T < 4$	$S_T \geq 4$
Buy 1 ABC	-60	S_T	S_T
Short 1 Call option	61	0	$4 - S_T$
Total	1	S_T	4

Upper and lower bounds for a call option price C and for put option price P

- Call option:

$$\max\left\{S_0 - \frac{K}{(1+r(T))^T}, 0\right\} \leq C \leq S_0$$

- Put option: from the put call parity

$$P = C - S_0 + \frac{K}{(1+r(T))^T}$$

Hence, we have

$$\max\left\{\frac{K}{(1+r(T))^T} - S_0, 0\right\} \leq P \leq \frac{K}{(1+r(T))^T}$$

Some exercises

1. Today, 1 share of ABC trades for S_0 = *Eu* 60; the 1 year interest rate is $r(1Y)$ = 2%; an European put option on 1 share of ABC, with maturity T = 1 year and strike price K = *Eu* 40 trades for P = *Eu* 40. **Identify an arbitrage strategy.** Short the put option and invest P in a 1-year ZCB.

2. Prove that it is never optimal to exercise an American call option before maturity (provided the underlying asset pays no cash-flows before the maturity of the call). It is better to sell the option rather than exercise it.

3. Prove that it can be optimal to exercise an American put option before maturity.

Binomial model

- Under the assumption that the underlying asset's price next period can take only two values (a.k.a. the binomial model) we can price the option by arbitrage
- Example
  - A stock trades at S_0 = €50 and, in one year, S_1 is either €42 or €60. The risk-free rate is 5%.
  - We want to price a call option on this stock with exercise price €52 and maturity of 1 year. We proceed in three steps:

Step 1 How much is the option worth at maturity?

S_0		S_1	C_1
		€60	€8
	↗		
€50			
	↘		
		€42	0

Binomial model

Step 2 Find the replicating portfolio of the call:

n_S shares of the stock + n_B euros invested in the risk-free asset such that

$$\begin{cases} 60 \times n_S & + & 1.05 \times n_B & = & 8 \\ 42 \times n_S & + & 1.05 \times n_B & = & 0 \end{cases}$$

$\Rightarrow n_S = 0.44$ and $n_B = -17.78$: buy 0.44 shares of the stock and borrow €17.78 at $t = 0$

Step 3 Apply the Law of One Price:

$$C_0 = n_S \times S_0 + n_B = 0.44 \times 50 - 17.78 = \text{€}4.22$$

- n_S is the delta ($\triangle$) of the option: it measures the change in the option value induced by a €1 change in the price of the underlying asset
- $\triangle$ is positive $\in (0, 1)$ for calls and negative $\in (-1, 0)$ for puts

Binomial model with letters

- A stock trades at S and, in T years, S_T is either $u \times S$ or $d \times S$, where $d < (1 + r_T)^T < u$,
- The T-year risk-free rate is r_T.
- Consider an European option with maturity T and strike price K
- Let $C(T)$ denote the payoff of the option at time T
  Examples:
  - Long call $C(T) = \max\{0, u \times S_T - K\}$
  - Long put $C(T) = \max\{0, u \times K - S_T\}$
  - etc.

S_0		S_T	$C(T)$
		$u \times S$	C_u
	$\nearrow$		
S			
	$\searrow$		
		$d \times S$	C_d

Binomial model with letters

Replicating portfolio of the option composed of n_S shares of the stock + n_B euros invested a risk-free zero coupon bond with maturity T years and face value 1:

$$\begin{cases} u \times S \times n_S + (1+r_T)^T \times n_B = C_u \\ d \times S \times n_S + (1+r_T)^T \times n_B = C_d \end{cases}$$

That gives

$$\begin{cases} n_S = \frac{C_u - C_d}{(u-d)S} \\ n_B = \frac{u \times C_d - d \times C_u}{(u-d)(1+r_T)^T} \end{cases}$$

hence the current premium for the option is

$$C(0) = n_S \times S + n_B = \frac{C_u - C_d}{(u-d)} + \frac{u \times C_d - d \times C_u}{(u-d)(1+r_T)^T}$$

Risk neutral probabilities

We have seen that

$$C(0) = \frac{C_u - C_d}{(u-d)} + \frac{u \times C_d - d \times C_u}{(u-d)(1+r_T)^T}$$

Let

$$p_u := \frac{(1+r_T)^T - d}{u-d}$$

Then we can express $C(0)$ as the present value of the expected payoff of the call option when the probability of an up-movement in the underling price is p_u:

$$C(0) = \frac{p_u \times C_u + (1-p_u)C_d}{(1+r_T)^T}$$

p_u is known as the risk neutral probability.

Exercise

A stock price is currently €100. Next year, the price will be either €90 or €130. The stock pays no dividends. The risk-free rate is 10%.

Question 1 Find the value of a European put option on this stock with a strike price of €104 and a maturity of 1 year

Exercise (cont'd)

Question 2 Find the value of a European call option on this stock with a strike price of €104 and a maturity of 1 year

Question 3 Same question with an American call option

Question 4 Same question with an American put option