HEC, Paris Financial Markets S. Lovo

Formulae

1. **Stocks**

   - According to the CAPM, a company i required return= OCC $= k$, where

   $$k = r_f + \beta_i(E[r_m] - r_f)$$

   - If the company is expected to pay dividend D_t in year t from now, then today spot price of one share of the company is

   $$S_0 = \sum_{t=1}^{\infty} \frac{D_t}{(1+k)^t}$$

   - If dividends are expected to grow at a constant rate $g < k$, then

   $$S_0 = \frac{D_1}{k - g}$$

   - If in every year t the company reinvests a fraction b of its (per share) earning E_t to grow, and the reinvested capital generate a return of ROE, then

     i) $g = b \times ROE$

     ii) $E_{t+1} = E_t \times (1+g)$

     iii) $D_{t+1} = D_t \times (1+g)$

     iv) $S_0 = \frac{E_1 \times (1-b)}{k - b \times ROE}$

   - The present value of growth opportunity is equal to $S_0 - \frac{E_1}{k}$

2. **Bonds**

   Let c = bond's coupon; N = bond's face value; z = frequency of payments of the coupon in a year (ex. once per year, twice per year, etc.); S_0 = spot price of the bond; t_i = the time (in years) you have to wait to receive i-th coupon; T = bond's maturity, i.e. the time (in years) you have to wait to receive the last coupon and the face value.

   Then:

- The bond's coupon rate $= \frac{z \times c}{N}$
- The bond's current yield $= \frac{z \times c}{S_0}$
- The bond's yield to maturity is the y such that

$$S_0 = \frac{c}{(1+y)^{t_1}} + \frac{c}{(1+y)^{t_2}} + \cdots + \frac{N+c}{(1+y)^T}$$

If $t_1 = 1, t_2 = 2, \ldots, t_i = i, \ldots, T = n$, then

$$S_0 = \frac{c}{y}\left(1 - \frac{1}{(1+y)^n}\right) + \frac{N}{(1+y)^n}$$

- The yield to maturity of a zero-coupon-bond with maturity in T years from now is

$$r_t = \left(\frac{N}{S_0}\right)^{\frac{1}{T}} - 1$$

- The forward rate for investing from time t to time $t' > t$ is

$$r(t, t') = \left(\frac{(1+r_{t'})^{t'}}{(1+r_t)^t}\right)^{\frac{1}{t'-t}} - 1$$

- The no-arbitrage spot price of a bond satisfies:

$$S_0 = \frac{c}{(1+r_{t_1})^{t_1}} + \frac{c}{(1+r_{t_2})^{t_2}} + \cdots + \frac{N+c}{(1+r_T)^T}$$

- The yield curve is flat at r if for all dates t, one has $r_T = r$.
- Suppose the yield curve is flat, then a bond's duration is

$$D = \frac{\left(t_1 \frac{c}{(1+r)^{t_1}} + t_2 \frac{c}{(1+r)^{t_2}} + \cdots + T \frac{N+c}{(1+r)^T}\right)}{S_0}$$

- If a flat yield curve suddenly shift from r to $r' = r + \Delta$ then the spot price of the bond will move from S_0 to S_0', where

$$\frac{S_0' - S_0}{S_0} \simeq -D \frac{\Delta}{1+r}$$

3. **Forward contracts**

   Consider a forward contract with maturity T. Suppose that the spot price of the underlying asset is S_0 and that before T the underlying assets pays cashflows at time $t_1, t_2, \ldots, t_n < T$. Let I_{t_i} be the cashflow paid at time t_i. Then the no-arbotrage forward price is

$$F_{0,T} = \left(S_0 - \frac{I_{t_1}}{(1+r_{t_1})^{t_1}} - \frac{I_{t_2}}{(1+r_{t_2})^{t_2}} - \cdots - \frac{I_{t_n}}{(1+r_{t_n})^{t_n}} \right) (1+r_T)^T$$

   Remarks

   - If the underling assets pays no cashflows before T, then $I_{t_i} = 0, \forall t_i$, and $F_{0,T} = S_0(1+r_T)^T$.
   - If the underlying asset is a commodity then $I_{t_i} < 0$ and can be interpreted as the storage costs of the commodity.

4. **Options**

   Let $K =$ the strike price of the option, $T =$ the maturity of the option, $S_t =$ be the spot price of the underling asset at time t, $C_0 =$ the premium of a call option and $P_0 =$ the premium of a put option.

   Then:

   - at time T
     - The payoff from a long position in a Eu call option is
$$C_T = \max\{0, S_T - K\}$$
     - The payoff from a short position in a Eu call option is
$$-C_T$$
     - The payoff from a long position in a Eu put option is
$$P_T = \max\{0, K - S_T\}$$
     - The payoff from a short position in a Eu put option is
$$-P_T$$
   - Put-call parity:
$$C_0 = P_0 + S_0 - \frac{K}{(1+r_T)^T}$$
     or equivalently
$$P_0 = C_0 - S_0 + \frac{K}{(1+r_T)^T}$$

- Arbitrage bounds: If at least one of the following inequalities is NOT satisfied, then there is an arbitrage opportunity:

$$\max\left\{S_0 - \frac{K}{(1+r_T)^T}, 0\right\} < C_0 < S_0$$

$$\max\left\{\frac{K}{(1+r_T)^T} - S_0, 0\right\} < P_0 < \frac{K}{(1+r_T)^T}$$

- Binomial model and no arbitrage price .

  Suppose that at time t the spot price of the underlying asset is either $S_T = u \times S_0$ or $S_T = d \times S_0$, where $d < (1+r_T)^T < u$.

  For any given option, let C_u be the option payoff at T if $S_T = u \times S_0$, and let C_d be the option payoff at T if $S_T = d \times S_0$.

  For example if the option is an Eu call option $C_u = \max\{0, u \times S_0 - K\}$ *and* $C_d = \max\{0, d \times S_0 - K\}$.

  Then:

  - Today I can replicate the option by buying n_S underling asset and invest n_B in a ZCB with maturity T (n.b. if n_B is negative you short sell the ZCB). Where:

$$n_S = \frac{C_u - C_d}{(u-d)S}$$

$$n_B = \frac{u \times C_d - d \times C_u}{(u-d)(1+r_T)^T}$$

  - The no arbitrage premium of the option is

$$C_0 = \frac{C_u - C_d}{(u-d)} + \frac{u \times C_d - d \times C_u}{(u-d)(1+r_T)^T}$$

$$= \frac{p \times C_u + (1-p) \times C_d}{(1+r_T)^t}$$

    where

$$p = \frac{(1+r_T)^T - d}{u-d} \in (0,1)$$

    is the risk neutral probability.