Financial Markets
2: Bonds

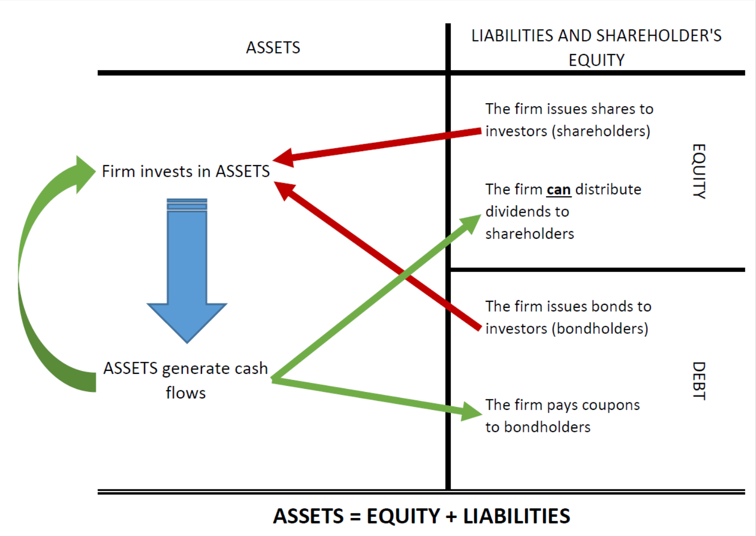
Stefano Lovo

HEC, Paris

Overview of Chapter 2: Bonds

1. Bond basics
2. Interest rates & yield-to-maturity
3. Arbitrage pricing
4. Yield curve & forward rates
5. Interest rate risk
6. Default risk

Balance sheet of a firm



What is a bond (or fixed income security)?

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Your company needs to borrow €1 000 000 during 4 years.
How?

- Borrow €1 000 0000 from a bank
- Borrow €1 000 from 1000 people

Definition

A bond is a financial asset that promises a stream of known cash flows in the future.

Example

Table: Coupon Bond

Labels	year 1	year 2	year 3	year 4
Cash-flows	€50	€50	€50	€1050

What is a bond (or fixed income security)?

- Tradable loan
  - Issuer borrows money
  - Initial purchaser on primary market lends money
  - Security can be traded on secondary market until maturity $\Rightarrow$ identity of creditors changes over time

- Name "fixed income" initially came from nature of cash flows (fixed in advance)
  - No longer true, financial innovation

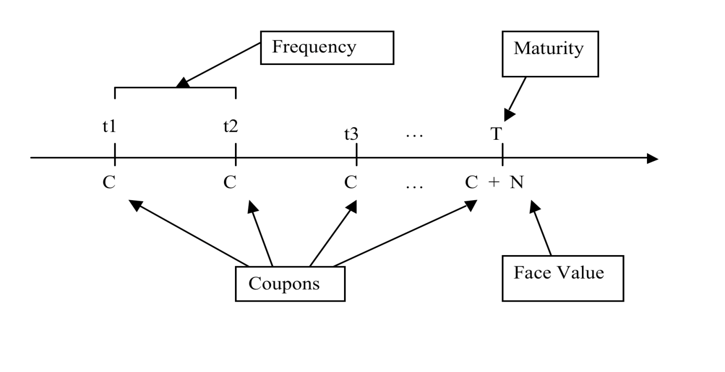
- Large spectrum of issuers

Values in trillion USD

	EU/UK	USA
Governments ("sovereign bonds")	12	18
Corporations (financial and non-financial)	15	22
Cities		
...		

The features of characterizing a bond

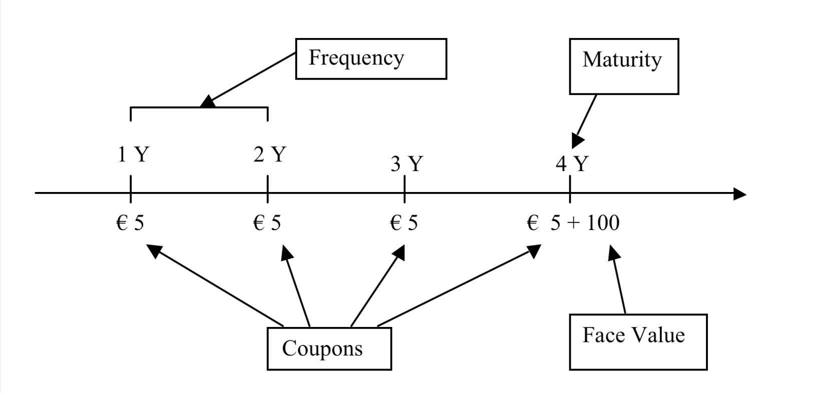
- The **Maturity** T: The date on which the last payment to the bondholder is due
- The **Coupon** C: the interest payment that is made to each bond holder at periodic dates.
- The **Face Value** N: The final payment that is made at maturity with the last coupon. (In general $N = 100$)
- The **Frequency** z with which coupons are paid (examples: once every year once every semester)



Bonds: Example 1

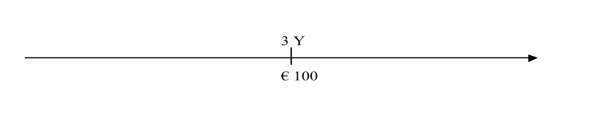
Example

Bond A has maturity 4 years, face value €100, coupon €5 frequency 1 year:

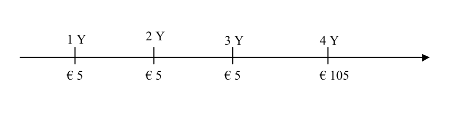


Main types of bonds

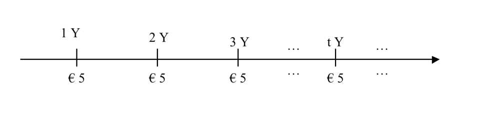
- **Zero Coupon Bond**: A bond without coupon ($C = 0$)
  Example: 3-year zero coupon bond with face value €100:



- **Coupon Bond**: A bond with a finite maturity and strictly positive coupon.
  Example: 4-year coupon bond with face value €100 and annual coupon of €5:



- **Perpetuity**: A bond with an infinite maturity and strictly positive coupon
  Example: A perpetuity with annual coupon of €5:



Quick check questions

Consider the following bonds:

Bond Name	Maturity in years	face value	Coupon	frequency
Bond A	4	€100	€8	6 months
Bond B	2.5	€1000	€50	1year
Bond Zcb	3	€200	€0	1year
Bond Pe	∞	€1000	€20	2 year

Q1 What are the cash flows of Bond A?

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Q2 What are the cash flows of Bond B?

Quick check questions

Consider the following bonds:

Bond Name	Maturity in years	face value	Coupon	frequency
Bond A	4	€100	€8	6 months
Bond B	2.5	€1000	€50	1year
Bond Zcb	3	€200	€0	1year
Bond Pe	∞	€1000	€20	2 year

Q3 What are the cash flows of Bond Zcb?

Quick check questions

Consider the following bonds:

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Bond A	4	€100	€8	6 months
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Q4 What are the cash flows of Bond Pe?

Bond yields

Consider a bond with coupon C, frequency z, face value N, maturity T, and trading for price P

- The bond's **coupon rate** is equal to the annualized coupon divided by its face value

$$i = \frac{zC}{N}$$

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- The **yield to maturity** is the discount rate y that makes the present value of the bond's cash-flows equal to its current price.

$$P = \frac{C}{(1+y)^{t_1}} + \frac{C}{(1+y)^{t_2}} + \cdots + \frac{C+N}{(1+y)^{T}}$$

where t_i is the years to wait for receiving the i-th coupon.
If $z = 1$, and T is integer, then from the annuity formula:

$$P = \frac{C}{y}\left(1 - \left(\frac{1}{1+y}\right)^{T}\right) + \frac{N}{(1+y)^{T}}$$

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Bond Name	Maturity in years	face value	Coupon	frequency	current price
Bond A	4	€100	€8	6 months	€120.5
Bond B	2.5	€1000	€50	1year	€1010
Bond Zcb	3	€200	€0	1year	€195
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Fill-in the following table:

Bond Name	coupon rate	current yield	yield to maturity
Bond A	?	?	?
Bond B	?	?	?
Bond Zcb	?	?	?
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These bonds yields are:

Bond Name	coupon rate	current yield	yield to maturity
Bond A	16%	13.28%	9.93%
Bond B	5%	4.95%	5.64%
Bond Zcb	0%	0%	0.85%
Bond Pe	1%	0.5%	0.499%

Stripping or how to create a zero-coupon bond out of a coupon bond

- Remove the coupons from the coupon bond and sell the separate parts as Separate Trading of Registered Interest and Principal of Securities (STRIPS)

Pension fund demand drives rise in Treasury 'strips' activity

US tax code changes trigger rush for retirement plans to hedge long-term liabilities

Joe Rennison in New York SEPTEMBER 6, 2018

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Pension fund demand for longer-dated US government bonds has sparked a sharp rise in Treasury "strips" activity ahead of a tax break ending this month.

With a new 21 per cent rate set to replace the old 35 per cent level later this month, pension plans have been rushing to buy strips, whereby existing 30-year Treasury bonds are separated into two types of security: one that only pays a fixed rate of interest, or coupon, over a set period, and another that is sold at a steep discount as it pays out only its principal value upon maturity.

Source

Types of bonds

- **Floating-rate bonds**: adjustable coupon depending on the level of a benchmark interest rate (example: LIBOR+1%)

- **Inflation-indexed bonds**: coupon indexed on inflation

- **Asset-Backed Securities (ABS)**: coupon payments are based on revenues of underlying assets (mortgage loans, car loans, credit card receivables, etc.); issued through securitization, make up roughly 25% of bond market

Green bonds

Green bond: The proceeds raised from a green bond issuance is restricted to finance or refinance "green" projects.

Example

In 2020 Egypt issued $750 million 5-year maturity sovereign green bond.
Proceeds were earmarked for financing

- Clean transportation
- Renewable energy
- Pollution prevention and control
- Climate change adaptation
- etc.

The logic behind green bonds

Investors might be willing to pay more for a bond when they know the money they lend to the firm will be used for “green purpose”

Example

HEC Paris is issuing two types of zero coupon bonds, let call them BH and BHG. Both bonds have maturity in 5 years and a face value of Eu 1000.

- Proceeds of BH can be used as the HEC managers consider suitable to HEC
- Proceeds of BHG will be used to improve the thermal insulation of all buildings on the campus and to install solar panels on campus buildings roofs.

If you are planning to invest into HEC bonds, which of the two bonds would you prefer?

Pros of green bonds: Lower cost of capital for the firm
Cons of green bonds: Leaves little flexibility to the management on how to invest proceeds

Sustainability-Linked Bonds

SLB: The cashflows the bond pays to the bond holder increase if the issuer does not meet a pre-specified “green” objective.

Example

In 2019 Italian energy group Enel raised 3.25 billions via SLBs

- Maturity: 2034 (15 years)
- Goal: having at least 55% of Enel installed capacity in renewable energy sources by 2021
- Coupon rate : 2.65% if the goal is reached, 2.90% if the goal is not reached

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In 2021 the goal was reached as Enel Renewable Installed Capacity Percentage was 57.2%

The logic behind SLBs

Cost of capital for the firm is reduced only if the sustainability goal is reached.

Example

HEC Paris is issuing Sustainability -Linked zero coupon bonds with maturity of 5 years

- Goal: By 2024 and until 2027, annual purchased Kw should be 15% lower than those in 2021
- Face value: Eu 1000 if the goal is reached; EU 1010 if the goal is not reached.

Pros of SLBs: Lower cost of capital for the firm, leave freedom to the management on the best way to reach the target.

Cons of SLBs: Require external audit to certify whether the goal has been reached or not.

CAT bond

CAT bond

CATastrophe bond

- Bond issued by insurance companies
- Short maturity
- High yield
- If a pre-specified catastrophe occurs, then the bond pays no more coupon and possibly no face value to bond holders.

Example

- Issuer: American Family Mutual Insurance Co.
- Issuance date: November 2010
- Maturity: November 2013
- Face value: USD 1,000
- Coupon rate 6.5%
- Trigger: losses to insurance industry from severe thunderstorms and tornadoes across the U.S. exceeds USD825 million
- Issued quantity: 100,000 CAT bonds = US 1,000,000

During the tornado season of 2011 insurance industry losses exceeded USD 825 million $\Rightarrow$ Investors were never paid the face value, AFMI could use the US 1,000,000 to recover part of the payments it made the clients it insured against tornados.

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Pros:

- For issuers: (insurance companies) transfer to investors part of the catastrophe risk.
- For investors: High yield in case of not too big catastrophe/ Climate risk not too correlated with market risk (i.e. low beta) (??)

Cons:

- For issuers: (insurance companies) high cost of capital in case of little catastrophe.
- For investors: Highly risky investment/ Big enough catastrophe can affect the whole economy and hence been positive correlated with climate risk (i.e. high beta).

Hybrid bonds

- Callable bonds: The bond issuer has the right to buy back ("called") the bond at (typically) par value plus one coupon payment after a certain number of years.
- Convertible bonds: the bondholder has the right to convert the bond for a predetermined number of shares of the company's stock after an initial waiting period. The bondholder will convert only if it is profitable to do so.

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Which one has the higher price on the market?

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Q3 A firm has issued a convertible bond and a non-convertible bond, both with same maturity, same face value, and same price.
Which one has the higher coupon?

Hybrid bonds

Tesla Stock's Surge Puts Convertible Bonds in the Money

By Randall Forsyth • Aug. 7, 2018 2:45 p.m. ET

The surge in Tesla's (TSLA) shares after CEO Elon Musk's tweet about going private has had huge, salutary effect on its key convertible bonds and its financing position. By pushing the common stock's price above the $360 level, his tweet has put a $900 million convert issue above its conversion price, which would effectively let the electric-auto maker pay off that obligation in stock instead of cash.

Tesla has a $920 million 0.25% convertible issue due Feb. 27, 2019, which would be convertible into common at a price of $359.8676 per share. If the stock trades below that price at the conversion price, the holder would be better off taking cash. That would put pressure on Tesla while it continues to burn cash, as of the second quarter. But by being able to redeem the debt in stock would obviate the obligation to come up with over $900 million in cash.

Source

Bond trading

- Bonds typically trade in over-the-counter (OTC), and nowadays electronic, platforms

Robin Wigglesworth and **Joe Rennison** in New York AUGUST 16, 2017 3

Goldman Sachs has expanded its algorithmic corporate bond trading programme, more than trebling the number of securities it quotes since last summer to more than 7,000 — and is now eyeing an expansion into areas such as junk bonds later this year.

It comes as both banks and investors, such as hedge funds and asset managers, are focusing on automating smaller-size trades, in a bid to cut costs and free up dealers for larger transactions.

The bank's algorithm scrapes publicly-available pricing data for thousands of bonds to automatically generate firm, tradable prices for investors. Earlier this year it broke into the ranks of the top-three dealers on MarketAxess in US investment grade odd-lots — defined as smaller slivers of debt below $1m, according to Goldman Sachs.

Source

Yield to maturity of zero coupon bonds

Theorem

The yield to maturity of a **zero coupon bond** *with face value N, maturity in T years and current price of P is*

$$r_T = \left(\frac{N}{P}\right)^{1/T} - 1$$

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Example: Consider a 4-year maturity ZCB with face value N =€ 100 and current price P=€ 88.85. Then, r_4 is:

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$$r_4 = \left(\frac{100}{88.85}\right)^{\frac{1}{4}} - 1 = 3\%$$

Poll: Yield to maturity of ZCB: more examples

Bond name	Maturity	Face value	Coupon	Price
ZCB1	1 year	€100	0	€98.04
ZCB2	2 year	€100	0	€96.11
ZCB3	3 year	€100	0	€88.90
ZCB4	4 year	€100	0	€88.85

Then

$$r_1 =$$

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Then

$$
\begin{aligned}
r_1 &= \left(\frac{100}{98.04}\right) - 1 = 2\% \\
r_2 &= \left(\frac{100}{96.11}\right)^{\frac{1}{2}} - 1 = 2\% \\
r_3 &= \left(\frac{100}{88.90}\right)^{\frac{1}{3}} - 1 = 4\% \\
r_4 &= \left(\frac{100}{88.85}\right)^{\frac{1}{4}} - 1 = 3\%
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Yield to maturity of zero coupon bonds: few remarks

Remarks:

- r_T is also called the T-year interest rate or T-year spot rate or T-year zero-coupon rate.

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- **Interpretation:** If I buy for price P a ZCB of maturity T year and keep it until maturity, it is as if I was investing for T years an amount P in a bank account at an effective annual rate r_T.
- The price of a T-year maturity ZCB with face value N is equal to

$$P = \frac{N}{(1 + r_T)^T}$$

More examples

Example 4. A coupon bond has a face value of €100, an annual coupon, a maturity of 10 years, a yield to maturity of 5%, and a price of €115.443. What is the bond's coupon?

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Example 5. A coupon bond has a face value of €100, an annual coupon of €4, a yield of 3%, and a price of €103.717. What is the bond's maturity?

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$$P_0 = \frac{C}{y} + \frac{N - \frac{C}{y}}{(1+y)^T} \Leftrightarrow (1+y)^T = \frac{N - \frac{C}{y}}{P_0 - \frac{C}{y}} \Leftrightarrow T = \log\left(\frac{N - \frac{C}{y}}{P_0 - \frac{C}{y}}\right) / \log(1+y) = 4 \text{ years}$$

Example 6. A coupon bond has a face value of €100,000, coupons of €2,000 that are received *semi-annually*, a maturity of 5 years, and a yield to maturity of 5%. What is the bond's price?

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Use the formula $P_0 = \frac{C}{y}\left[1 - \frac{1}{(1+y)^T}\right] + \frac{N}{(1+y)^T}$ at the semi-annual frequency, i.e.,

where C = €2,000 are semi-annual coupons, $T = 10$ semesters, and y is the semi-annually compounded bond yield. Since $(1 + \text{semi-annually compounded yield})^2$

$y = 1.05^{0.5} - 1 = 2.47\%$. Finally, $P_0 = \frac{2{,}000}{0.0247}\left[1 - \frac{1}{1.0247^{10}}\right] + \frac{100{,}000}{1.0247^{10}} = €95{,}880$

What is the equilibrium price a bond?

Bond name	Maturity	Face value	Coupon	frequency	Price
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Bond A	4 year	€1000	100	1 year	**??**

Replicating portfolio and law of one price strategy

Definition

A replicating portfolio of a given financial asset A is a portfolio R composed of other assets such that portfolio R generates **exactly the same cashflow in exactly the same circumstances** as asset A.

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Definition

An arbitrage portfolio is a portfolio that has 0 or negative cost when you buy it, and generate positive cash-flows.

Applying the law of one price to price a bond

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ZCB4	4 year	€100	0	-	€88.85
Bond A	4 year	€1000	100	1 year	**??**

Q1 What are bond A's cashflows?

Time	Bond A
Year 1	€100
Year 2	€100
Year 3	€100
Year 4	€1100

Applying the law of one price to price a bond

Bond name	Maturity	Face value	Coupon	frequency	Price
ZCB1	1 year	€100	0	-	€98.04
ZCB2	2 year	€100	0	-	€96.11
ZCB3	3 year	€100	0	-	€88.90
ZCB4	4 year	€100	0	-	€88.85
Bond A	4 year	€1000	100	1 year	**??**

Q2 How many bonds ZCB1, ZCB2, ZCB3, and ZCB4 should I buy or sell to replicate Bond A's cashflows?

Applying the law of one price to price a bond

Bond name	Maturity	Face value	Coupon	frequency	Price
ZCB1	1 year	€100	0	-	€98.04
ZCB2	2 year	€100	0	-	€96.11
ZCB3	3 year	€100	0	-	€88.90
ZCB4	4 year	€100	0	-	€88.85
Bond A	4 year	€1000	100	1 year	**??**

Q2 How many bonds ZCB1, ZCB2, ZCB3, and ZCB4 should I buy or sell to replicate Bond A's cashflows?

Time	Bond A	ZCB1	ZCB2	ZCB3	ZCB4
Year 1	€100	€100	0	0	0
Year 2	€100	0	€100	0	0
Year 3	€100	0	0	€100	0
Year 4	€1100	0	0	0	€100

Applying the law of one price to price a bond

Bond name	Maturity	Face value	Coupon	frequency	Price
ZCB1	1 year	€100	0	-	€98.04
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Bond A	4 year	€1000	100	1 year	**??**

Q2 How many bonds ZCB1, ZCB2, ZCB3, and ZCB4 should I buy or sell to replicate Bond A's cashflows?

Time	Bond A	ZCB1	ZCB2	ZCB3	ZCB4
Year 1	€100	€100	0	0	0
Year 2	€100	0	€100	0	0
Year 3	€100	0	0	€100	0
Year 4	€1100	0	0	0	€100

Bond A's replicating portfolio :

	ZCB1	ZCB2	ZCB3	ZCB4
Bond A =	1	1	1	11

Applying the law of one price to price a bond

Bond name	Maturity	Face value	Coupon	frequency	Price
ZCB1	1 year	€100	0	-	€98.04
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ZCB3	3 year	€100	0	-	€88.90
ZCB4	4 year	€100	0	-	€88.85
Bond A	4 year	€1000	100	1 year	**??**

Q2 What is the composition of portfolio that replicates Bond A's cashflows?

Time	Bond A	ZCB1	ZCB2	ZCB3	ZCB4
Year 1	€100 =	€100 $\times y_{ZCB1}$			
Year 2	€100 =		€100 $\times y_{ZCB2}$		
Year 3	€100 =			€100 $\times y_{ZCB3}$	
Year 4	€1100=				€100 $\times y_{ZCB4}$

Bond A replicating portfolio :

	ZCB1	ZCB2	ZCB3	ZCB4
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Applying the law of one price to price a bond

Bond name	Maturity	Face value	Coupon	frequency	Price	yiled
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Bond A	4 year	€1000	100	1 year	??	

Q3 What is the no arbitrage price of Bond A?

Applying the law of one price to price a bond

Bond name	Maturity	Face value	Coupon	frequency	Price	yiled
ZCB1	1 year	€100	0	-	€98.04	2%
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ZCB4	4 year	€100	0	-	€88.85	3%
Bond A	4 year	€1000	100	1 year	**??**	

Q3 What is the no arbitrage price of Bond A?

Bond A's replicating portfolio =

ZCB 1	ZCB 2	ZCB 3	ZCB 4
1	1	1	11

$$P_A = P_R = 1 \times 98.04$$

Applying the law of one price to price a bond

Bond name	Maturity	Face value	Coupon	frequency	Price	yiled
ZCB1	1 year	€100	0	-	€98.04	2%
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Q3 What is the no arbitrage price of Bond A?

Bond A's replicating portfolio =

ZCB 1	ZCB 2	ZCB 3	ZCB 4
1	1	1	11

$$P_A = P_R = 1 \times 98.04 + 1 \times 96.11$$

Applying the law of one price to price a bond

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ZCB1	1 year	€100	0	-	€98.04	2%
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ZCB 1	ZCB 2	ZCB 3	ZCB 4
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$$P_A = P_R = 1 \times 98.04 + 1 \times 96.11 + 1 \times 88.90 + 11 \times 88.85$$

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Q3 What is the no arbitrage price of Bond A?

Bond A's replicating portfolio =

ZCB 1	ZCB 2	ZCB 3	ZCB 4
1	1	1	11

$$P_A = P_R = 1 \times 98.04 + 1 \times 96.11 + 1 \times 88.90 + 11 \times 88.85 = €1260.40$$

Arbitrage portfolio

1. Suppose that you can buy and sell bond A for P_A = €1280. Build an arbitrage portfolio.

Arbitrage portfolio

1. Suppose that you can buy and sell bond A for $P_A = €1280$. Build an arbitrage portfolio.

2. Suppose that you can buy and sell bond A for $P_A = €1200$. Build an arbitrage portfolio.

Law of one price and zero rates

Bond name	Maturity	Face value	Coupon	frequency	Price	Yield
ZCB1	1 year	€100	0	-	€98.04	2%
ZCB2	2 year	€100	0	-	€96.11	2%
ZCB3	3 year	€100	0	-	€88.90	4%
ZCB4	4 year	€100	0	-	€88.85	3%
Bond A	4 year	€1000	100	1 year	€1260.40	

Bond A's cashflows=

year 1	year 2	year 3	year 4
€100	€100	€100	€1100

$$P_A = 1 \times 98.04 + 1 \times 96.11 + 1 \times 88.90 + 11 \times 88.85 = 1260.40$$

Recalling that for a ZCB $P = \frac{N}{(1+r_T)^T}$, then

Law of one price and zero rates

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Bond A	4 year	€1000	100	1 year	€1260.40	

Bond A's cashflows=

year 1	year 2	year 3	year 4
€100	€100	€100	€1100

$$P_A = 1 \times 98.04 + 1 \times 96.11 + 1 \times 88.90 + 11 \times 88.85 = 1260.40$$

Recalling that for a ZCB $P = \frac{N}{(1+r_T)^T}$, then

$$P_A = 1 \times \frac{100}{(1.02)^1} + 1 \times \frac{100}{(1.02)^2} + 1 \times \frac{100}{(1.04)^3} + 11 \times \frac{100}{(1.03)^4} = 1260.40$$

Law of one price and zero rates

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ZCB4	4 year	€100	0	-	€88.85	3%
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Bond A's cashflows=

year 1	year 2	year 3	year 4
€100	€100	€100	€1100

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year 1	year 2	year 3	year 4
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Law of one price and zero rates

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year 1	year 2	year 3	year 4
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Law of one price and zero-rates

Theorem

- *Consider a bond paying cashflows at dates $t_1, t_2, \dots t_T$.*
- *Let C_{t_i} be the cashflow that the bond pays at date t_i, $i = 1, 2, \dots T$.*
- *Let r_{t_i} be the t_i year zero-coupon rate.*

Then the no-arbitrage price of the bond is

$$P = \frac{C_{t_1}}{(1 + r_{t_1})^{t_1}} + \frac{C_{t_2}}{(1 + r_{t_2})^{t_2}} + \cdots + \frac{C_T}{(1 + r_T)^T}$$

Term structure of interest rate (or yield curve)

Definition

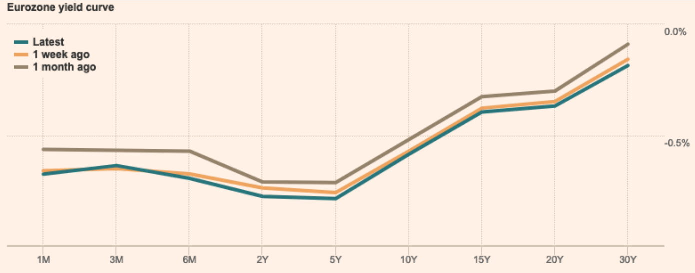
The term structure of interest rate (or yield curve) is the relation between the maturity of zero coupon bonds and their yield to maturity:

$$\{\dots r_{0.5}, \dots r_1, r_2, \dots r_T, \dots\}$$

Definition

The term structure of interest rate (or yield curve) is said to be flat if r_T does not vary with t.

Term structure of interest rate: real examples



October 2022

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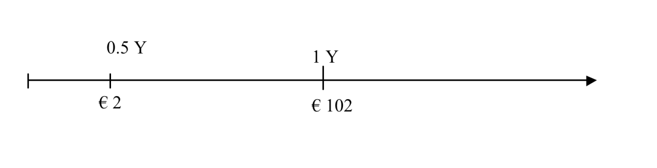
Bond pricing using the term structure: examples

Example

Consider a coupon bond with $C = Eu\ 2$, $N = Eu\ 100$, $T = 1$ year and coupons paid each semester.



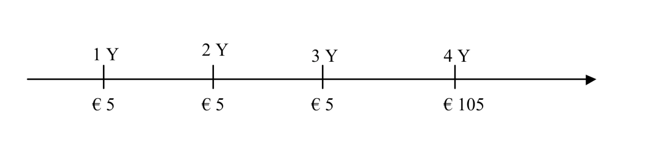
The term structure is $r_{6\ Month} = 3\%$, $r_{1\ Year} = 4\%$, $r_{2\ Year} = 5\%$. Then

$$P_{Bond} = \frac{2}{1.03^{0.5}} + \frac{102}{1.04} = 100.048$$

Bond pricing using the term structure: examples

Example

Consider a coupon bond with C = Eu 5, N = Eu 100, T = 4 year and annual coupons.



The term structure is $r_{6Month} = 3\%$, $r_{1Year} = 2\%$, $r_{2Year} = 2\%$, $r_{3Year} = 4\%$, $r_{4Year} = 3\%$. Then

$$P_{Bond} = \frac{5}{1.02} + \frac{5}{1.02^2} + \frac{5}{1.04^3} + \frac{105}{1.03^4} = 107.44$$

Consider a bond with maturity 3 years coupon rate 2% face value Eu 100 and frequency 1 year.

The term structure of interest rate is

$$r_1 = 4\%, r_2 = 3\%, r_3 = 5\%$$

1. What are the cash flows paid by this bond?
2. What is the price of this bond? Eu 91.92

Using the yield curve to price bonds

ZCB	Maturity	Face value	Price	r_T
ZCB1	1 year	€100	€98.04	2%
ZCB2	2 year	€100	€96.11	2%
ZCB3	3 year	€100	€88.90	4%
ZCB4	4 year	€100	€88.85	3%

What are the prices of the following bonds?

Bond name	Maturity	Face value	Coupon	frequency	Price
Bond 1	1 year	€200	0	-	

Using the yield curve to price bonds

ZCB	Maturity	Face value	Price	r_T
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Using the yield curve to price bonds

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What are the prices of the following bonds?

Bond name	Maturity	Face value	Coupon	frequency	Price
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Using the yield curve to price bonds

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What are the prices of the following bonds?

Bond name	Maturity	Face value	Coupon	frequency	Price
Bond 1	1 year	€200	0	-	€196.0
Bond 2	2 year	€1000	50	1 year	€1058.25

Using the yield curve to price bonds

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ZCB1	1 year	€100	€98.04	2%
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Using the yield curve to price bonds

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What are the prices of the following bonds?

Bond name	Maturity	Face value	Coupon	frequency	Price
Bond 1	1 year	€200	0	-	€196.0
Bond 2	2 year	€1000	50	1 year	€1058.25
Bond 3	3 year	€50	1	1 year	€47.28

Using the yield curve to price bonds

ZCB	Maturity	Face value	Price	r_T
ZCB1	1 year	€100	€98.04	2%
ZCB2	2 year	€100	€96.11	2%
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Bond 3	3 year	€50	1	1 year	€47.28
Bond 4	4 year	€100	10	2 year	

Using the yield curve to price bonds

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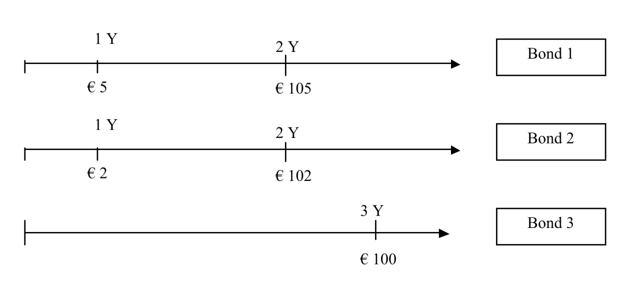
Bond 3	3 year	€50	1	1 year	€47.28
Bond 4	4 year	€100	10	2 year	€107.35

Term structure from coupon bond

It is possible to obtain the yield curve using the no-arbitrage prices of coupon bonds.

Example

	C	N	T	frequency	Price
Bond 1	5	Eu 100	2 Y	annual	Eu 103.78
Bond 2	2	Eu 100	2 Y	annual	Eu 98.07
Bond 3	0	Eu 100	3 Y	–	Eu 86.38



Term structure from coupon bond

Example

By no arbitrage the prices of bonds 1, 2 and 3 satisfy:

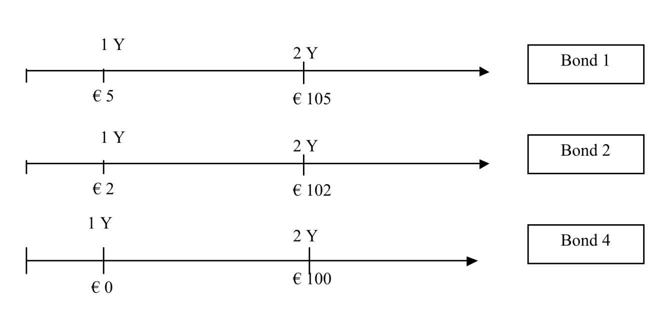
$$\begin{aligned}
103.78 &= \frac{5}{1+r_1} + \frac{105}{(1+r_2)^2} \\
98.07 &= \frac{2}{1+r_1} + \frac{102}{(1+r_2)^2} \\
86.38 &= \frac{100}{(1+r_3)^3}
\end{aligned}$$

By solving this system of three equations in r_1, r_2 and r_3, we have

$$\begin{aligned}
r_1 &= 4\% \\
r_2 &= 3\% \\
r_3 &= 5\%
\end{aligned}$$

Suppose you have the following information:

	C	N	T	frequency	Price
Bond 1	5	Eu 100	2 Y	annual	Eu 103.78
Bond 2	2	Eu 100	2 Y	annual	Eu 98.07
Bond 4	0	Eu 100	2 Y	–	Eu 90.38



Identify an arbitrage strategy

1. Build a portfolio R composed of Bond 1 and Bond 2 that replicates Bond 4.
2. Find the price of portfolio R and compare it to the price of Bond 4.
3. Buy the cheapest between portfolio R and Bond 4 and short sell the more expensive.

Money for nothing : Step 1

Build a portfolio R composed of Bond 1 and Bond 2 that replicates Bond 4.

$$\textit{Year } 1: \quad 0 = y_1 * 5 + y_2 * 2$$
$$\textit{Year } 2: \quad 100 = y_1 * 105 + y_2 * 102$$

By solving this system in y_1 and y_2, we obtain that Bond 4 can be replicated by a portfolio R:

- $y_1 = -2/3 \Rightarrow$ short sell 2/3 of Bond 1.
- $y_2 = 5/3 \Rightarrow$ buy 5/3 of Bond 2.

Cash flow of portfolio R:

Label	Cash-flow
Year 1	$0 = -\frac{2}{3}5 + \frac{5}{3}2$
year 2	$100 = -\frac{2}{3}105 + \frac{5}{3}102$

Find the price of portfolio R and compare it to the price of Bond 4.

$$
\begin{aligned}
P_R &= y_1 * P_1 + y_2 * P_2 \\
&= -\frac{2}{3}103.78 + \frac{5}{3}98.07 = 94.26 \\
&> 90.38 = P_4
\end{aligned}
$$

The replicating portfolio R is more expensive than the replicated asset, Bond 4

Money for nothing : Step 3

Buy the cheapest between portfolio R and Bond 4 and short sell the more expensive.

Because $P_R > P_4$, I should buy Bond 4 and short sell its replicating portfolio

Trade	Today	Year 1	Year 2
Buy Bond 4	-90.38	0	100
Buy $\frac{2}{3}$ of Bond 1	$-\frac{2}{3}103.78$	$\frac{2}{3}5$	$\frac{2}{3}105$
Sell $\frac{5}{3}$ of Bond 2	$\frac{5}{3}98.07$	$-\frac{5}{3}2$	$-\frac{5}{3}102$
	Eu **3.88**	Eu **0**	Eu **0**

Forward rates

Definition

A forward interest rate is the rate implied by the current term structure over a given period in the future.

Example: The one-year interest rate is $r_{1Y} = 3\%$ and the 2-year interest rate is $r_{2Y} = 4\%$.

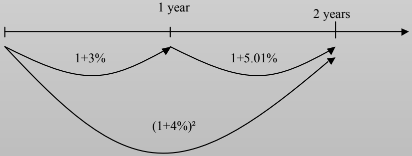
Forward rates

Definition

A forward interest rate is the rate implied by the current term structure over a given period in the future.

Example: The one-year interest rate is $r_{1Y} = 3\%$ and the 2-year interest rate is $r_{2Y} = 4\%$. What is the annual interest rate between year 1 and 2?

$$(1+3\%)(1+r_{1Y\to 2Y}) = (1+4\%)^2 \Rightarrow r_{1Y\to 2Y} = 5.01\%$$



Forward rates

Theorem

Given any two dates $t_1 < t_2$ and a term structure, the forward rate from t_1 to t_2 is equal to

$$r_{t_1 \to t_2} = \left(\frac{(1 + r_{t_2})^{t_2}}{(1 + r_{t_1})^{t_1}} \right)^{\frac{1}{t_2 - t_1}} - 1$$

Proof: Note that the forward rate $r_{t_1 \to t_2}$ must satisfy

$$(1 + r_{t_2})^{t_2} = (1 + r_{t_1})^{t_1} (1 + r_{t_1 \to t_2})^{t_2 - t_1}$$

Forward rates: example

$r_{1Y} =$	2%
$r_{2Y} =$	2%
$r_{3Y} =$	4%
$r_{4Y} =$	3%

Then

$$
\begin{aligned}
r_{1Y\to 2Y} &= \frac{1.02^2}{1.02} - 1 = 2\% \\
r_{2Y\to 3Y} &= \frac{1.04^3}{1.02^2} - 1 = 8\% \\
r_{3Y\to 4Y} &= \frac{1.03^4}{1.04^3} - 1 = 0.06\%
\end{aligned}
$$

What is $r_{2Y\to 4Y}$? 4.01%

Forward rates: interpretations

- If agents are risk neutral the forward rate from years t to $t + n$ corresponds to what investors expect to be in t years the n-year interest rate.
- The t-year interest rate can be interpreted as the composition of the first $t - 1$ years forward rates:

$$(1 + r_T)^t = \sqcap_{i=0}^{t-1}(1 + r_{i\rightarrow i+1}))$$

- If agents are risk neutral then a decreasing (increasing),term structure indicates agents expects short term interest rate to decrease (resp. increase).

Forward rates: use

Any investor can lock in the forward rate implied by a term structure.

Can I do something today to make sure that between year T and year $T' > T$, I will certainly invest at rate $r_{T\to T'}$?

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Example

The one-year interest rate is $r_{1Y} = 3\%$ and the 2-year interest rate is $r_{2Y} = 4\%$. Investors can both borrow and lend at these rates. The forward rate from year 1 to year 2 is

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Trade	today	Year 1	year 2
Borrow Eu 100 for 1 year	+100	-103	0
Invest Eu 100 for 2 years=	-100	0	$100 * 1.04^2$
	0	**-103**	**108.16**

Note that $108.16 = 103 * (1 + r_{1Y \to 2Y}) = 103 * 1.0501$

What does the shape of the term structure imply?

- Upward sloping: normal times

  This is how commercial banks make money: borrow short term, lend long term

- Steeply upward sloping: usually forecast economic expansion

  Why?

- Flat or inverted: rare but a sign of trouble

  2006 yield curves

Recap on the yield curve

1. For any date t, it provides r_T: the yield to maturity of a zero coupon bond with maturity t.
2. r_T is an annual rate even if $t \neq 1 year$
3. It can be used to find the no arbitrage price of any bond:
    1. Discount each bond's cashflows using the r_T corresponding to the time t of the cashflow
    2. The bond's price is equal to the sum of its discounted cashflows
4. Its shape is related to investors expectation on the evolution of the macroeconomy
5. It can be used to compute forward rates.

Risks of a bond portfolio

1. Interest rate risk
   - Fluctuations of market price of bond due to changes in yield curve
   - Can be hedged with portfolio immunization (see later) or interest rate derivatives (see forward rates)
   - I care about it if I want to resell the bond before maturity
2. Default risk
   - Risk that whatever is owed by the issuer is not paid
   - Gives rise to default premium
   - Can be hedged with credit derivatives (not covered in the course but see problem 3 in problem set on bonds)
   - I care about it if a plan to hold the bond until maturity

Interest rate risk

Definition

Interest rate risk is the risk that the bond price changes due to fluctuations in market interest rates (even if there is no risk of default)

Exmple: You hold a bond that matures in 2 years, makes annual coupon payments, whose yield curve is currently flat at 5% and is trading at par $P_0 = N = \$1000$

Q0 What are the cash-flows that this bond will pay? Answ: 50 in 1Y, 1050 in 2Y

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- Q3 What if interest rates increase by 1 percentage point (and the yield curve remains flat) between today and next year? Answ: 4%
- Q4 What if they decrease by 1 percentage point? Answ: 6%

Duration

- Definition: The duration (D) of a bond is the sensitivity (or elasticity) of its price to interest rates changes:

$$\boxed{\frac{\triangle P_0}{P_0} \simeq -D\frac{\triangle y}{1+y}}$$

  - If the yield changes by $\triangle y$, the % change in the bond price is approximately $-D\frac{\triangle y}{1+y}$
  - The definition has a $\simeq$ because it is an approximation valid for small $\triangle y$
  - The larger a bond's duration, the "less" I know about the resale price of the bond if there is a change in the interest rate.

Duration

- How to calculate the duration?
  - Formula: D = weighted-average maturity of cash flows

$$\boxed{D = \frac{1 \times \frac{C}{1+y} + 2 \times \frac{C}{(1+y)^2} + \ldots + T \times \frac{C+N}{(1+y)^T}}{P_0}}$$

  (detailed calculation on the next slide)

  - Intuition: longer duration when cash flows further away in the future
  - D is in years
  - The duration of a portfolio of bonds is the weighted average of the durations of the bonds in the portfolio

Proof of the duration formula

The definition of duration is

$$D = -\frac{\frac{dP_0}{P_0}}{\frac{d(1+y)}{1+y}} = -\frac{dP_0}{dy} \times \frac{1+y}{P_0} \tag{1}$$

The bond price is equal

$$P_0 = \frac{C}{1+y} + \frac{C}{(1+y)^2} + \ldots + \frac{(C+N)}{(1+y)^T}$$

which we differentiate with respect to y

$$\frac{dP_0}{dy} = -\frac{C}{(1+y)^2} - \frac{2C}{(1+y)^3} - \ldots - \frac{T(C+N)}{(1+y)^{T+1}} \tag{2}$$

Replacing (2) into (1), we obtain

$$D = \frac{1 \times \frac{C}{1+y} + 2 \times \frac{C}{(1+y)^2} + \ldots + T \times \frac{C+N}{(1+y)^T}}{P_0}$$

Duration: Example

	Coupon	Face value	Maturity	Frequency	P
Bond 1	Eu 2	Eu 100	2 Y	annual	Eu 98.09
Bond 2	Eu 2	Eu 100	3 Y	annual	Eu 97.17

$r_{1Y} = r_{2Y} = r_{3Y} = 3\%$ For Bond 1:

$$P_1 = \frac{2}{1.03} + \frac{102}{1.03^2} = 1.94 + 96.15 = 98.09$$

$$\omega_1 = \frac{1.94}{98.09} = 0.0198 \text{ and } \omega_2 = \frac{96.15}{98.09} = 0.9802$$

$$D_1 = 1 * 0.0198 + 2 * 0.9802 = 1.98$$

For Bond 2:

$$D_2 = 1\frac{2}{1.03}\frac{1}{97.17} + 2\frac{2}{1.03^2}\frac{1}{97.17} + 3\frac{102}{1.03^3}\frac{1}{97.17} = 2.94$$

Duration: Example

Now suppose that the yield curve becomes
$r_{1Y} = r_{2Y} = r_{3Y} = 4\%$.
The new prices of Bond 1 and Bond 2 are:

$$
\begin{aligned}
P_1' &= \frac{2}{1.04} + \frac{102}{1.04^2} = 96.23 \\
P_2' &= \frac{2}{1.04} + \frac{2}{1.04^2} + \frac{102}{1.04^3} = 94.45
\end{aligned}
$$

The percentage changes in the prices are:
For Bond 1:

$$\frac{96.23 - 98.09}{98.09} = -1.89\%$$

For Bond 2:

$$\frac{94.45 - 97.17}{97.17} = -2.8\%$$

Exercise 1

Q 1 What is the duration of a 3-year zero-coupon bond?

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$P_0 = 1136.16$, $D = 2.75$ years

Q 3 If the yield curve shifts upward by 1 percentage point (i.e., +100 basis points), what will be the % change in the bond's price?

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$P_0 = 1136.16$, $D = 2.75$ years

Q 3 If the yield curve shifts upward by 1 percentage point (i.e., +100 basis points), what will be the % change in the bond's price?
-2.57%

Q 4 If interest rates decrease by 0.5% (i.e., -50 basis points) , what will be the €-change in the bond's price?

Exercise 1

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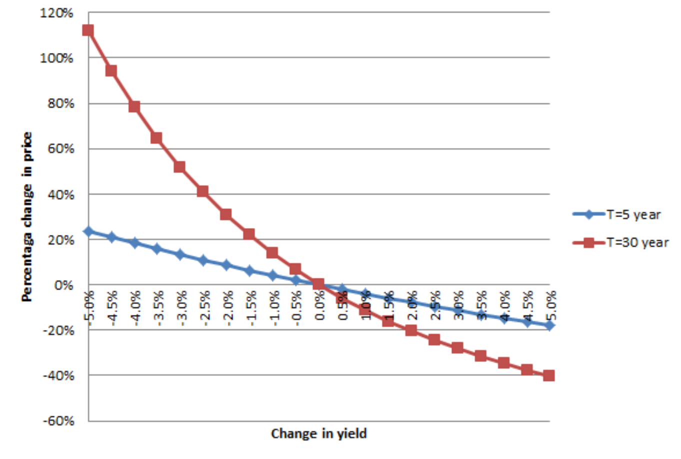
-2.57%

Q 4 If interest rates decrease by 0.5% (i.e., -50 basis points) , what will be the €-change in the bond's price? €15.03, Approx. €15.64

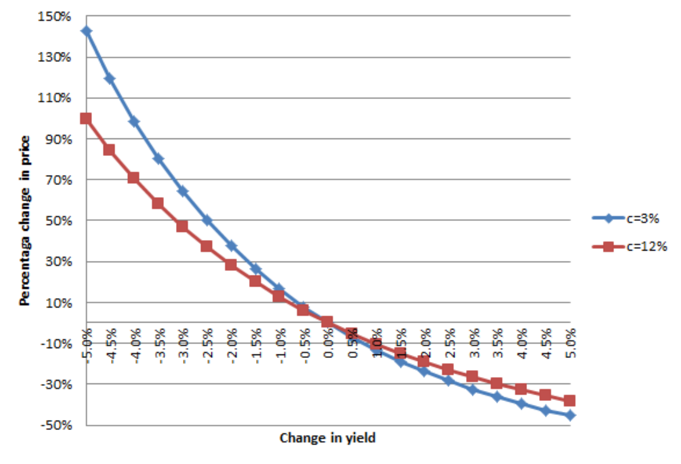
Factors influencing interest rate risk

- The longer the time to maturity (T) of a bond, the *more* its value will be affected by a change in interest rates



Factors influencing interest rate risk

- The larger the coupon (C) of a bond, the *less* its value will be affected by a change in interest rates



Exercise 2

Q 1 Rank the following bonds in order of descending duration

Bond	Coupon rate (%)	Time to maturity (years)	Yield to maturity (%)
A	15	20	10
B	15	15	10
C	8	20	10
D	0	20	10

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Bond	Coupon rate (%)	Time to maturity (years)	Yield to maturity (%)
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B	15	15	10
C	8	20	10
D	0	20	10

$$D(D) > D(C) > D(A) > D(B)$$

Q 2 Consider a long-short portfolio composed of a €1 million long position in Bond A and €1 million short position in Bond C (thus, the initial value of the portfolio is €0).How is the value of the portfolio affected by a rise in interest rates?

Duration of a portfolio

- Consider a portfolio of n different bonds.
- Let x_i be the **weight** of bond i in the portfolio, i.e., $\sum_i^n x_i = 1$

Then, the duration of the portfolio is

$$D_p = x_1 D_1 + x_2 D_2 + \cdots + D_n x_n$$

where D_i is the duration of bond i.

Proof

Consider a portfolio composed of Bonds $\{B1, B2, \dots B_M\}$. Let N_i be the number of bond B_i in the portfolio, and P_i be the price of one bond B_i. Then the total value of the portfolio is

$$P = N_1 P_1 + N_2 P_2 + \dots N_M P_M$$

and the weight of bond B_i in the portfolio is

$$x_i = \frac{N_i P_i}{P}$$

Let $\{t_1, t_2, \dots t_T\}$ be the time at which bonds in the portfolio pay cash-flow and $C_{i,t}$ be the cashflow paid by bond B_i at time t. (note that $C_{i,t}$ can be nil).
The duration of the portfolio is equal to

$$D_P = \frac{\sum_{\tau=1}^{\tau=T} t_\tau \sum_{i=1}^{i=M} \frac{N_i C_{i,\tau}}{(1+r_{t_\tau})^{t\tau}}}{P} = \frac{\sum_{i=1}^{i=M} \sum_{\tau=1}^{\tau=T} t_\tau \frac{N_i C_{i,\tau}}{(1+r_{t_\tau})^{t\tau}}}{P} = \sum_{i=1}^{i=M} \sum_{\tau=1}^{\tau=T} t_\tau \frac{N_i C_{i,\tau}}{P(1+r_{t_\tau})^{t\tau}}$$

$$= \sum_{i=1}^{i=M} \sum_{\tau=1}^{\tau=T} t_\tau \frac{x_i C_{i,\tau}}{P_i(1+r_{t_\tau})^{t\tau}} = \sum_{i=1}^{i=M} x_i \sum_{\tau=1}^{\tau=T} t_\tau \frac{C_{i,\tau}}{P_i(1+r_{t_\tau})^{t\tau}} = \sum_{i=1}^{i=M} x_i D_i$$

Q.E.D.

Example of portfolio immunization

Bond A is a ZCB with maturity 1 year and Bond B is a ZCB with maturity 30 years.
If you invest $x_A > 0$ of your wealth in Bond A, what should be the weight of Bond B so that your portfolio has zero duration? (assume your portfolio has no other bonds)

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Bond A is a ZCB with maturity 1 year and Bond B is a ZCB with maturity 30 years.
If you invest $x_A > 0$ of your wealth in Bond A, what should be the weight of Bond B so that your portfolio has zero duration? (assume your portfolio has no other bonds)

$$D_p = x_A \times 1 + x_B \times 30 + (1 - x_A - x_B) \times 0 = 0$$

$$\Rightarrow$$

$$x_B = -x_A \frac{1}{30}$$

Default risk

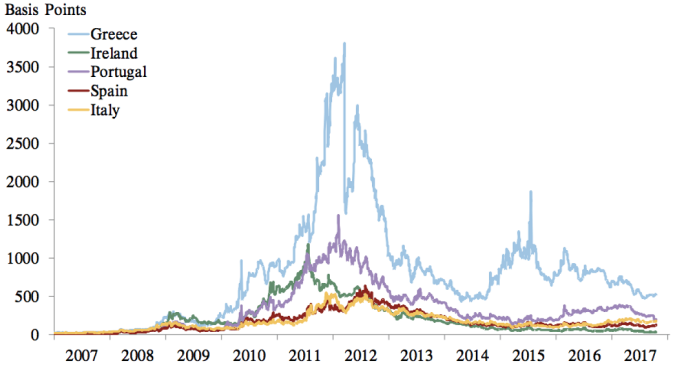
- Default risk (also called credit risk) is the risk that the issuer defaults on its obligation
  - Corporates and sovereigns are subject to default risk (General Motors 2009, Greece 2011, Banca Monte dei Paschi 2017, etc.)
- Credit ratings by rating agencies evaluate the likelihood of default

	S&P	Moody's	Fitch
	AAA	Aaa	AAA
Investment grade	AA	Aa	AA
	A	A	A
	BBB	Baa	BBB
	BB	Ba	BB
	B	B	B
High-yield	CCC	Caa	CCC
	CC	Ca	
	C		
In default	D	C	D

Default risk: spread

- Default risk is reflected in a higher bond yield through the default premium
- These days the German Treasury Bill is considered the safest bond.
- A bond spread is the difference between the bond yield to maturity and the yield to maturity of the German Treasury Bill

- Government bonds are not necessarily default-risk-free.
  - 10-year sovereign bond spreads above risk-free rate (= German rate)



NOTE: The chart shows the spread, or difference, in interest rates between 10-year government bonds for various countries and German 10-year government bonds.

Default risk: high-yield bonds

- High-yield bonds are also called “junk bonds”

can be “original junk issues”

Altice returns to bond market with $3bn sale

Robert Smith JULY 17, 2018

8

Altice’s French unit completed a nearly $3bn high-yield debt sale on Tuesday, raising junk bonds for the first time since concerns around the cable group’s debt pile spooked investors at the end of last year.

The group raised $1.75bn of dollar bonds at 8.125 per cent yield and €1bn of euro bonds at 5.875 per cent yield, the highest yields its French unit has been charged in both markets since it was created out of the merger of Numericable and SFR in 2014.

Source

or “fallen angels”

Bond investors wary of threat from potential ‘fallen angels’

Analysts expect more companies to complete slide from investment grade to junk

Eric Platt in New York JULY 25, 2016

1

Investors in the most highly rated US corporate bonds have enjoyed a buoyant 2016, with the Barclays index returning nearly 9 per cent. However, the market now faces an unsettling threat from a potential new wave of fallen angels — companies that first sold debt with investment grade status but have since been downgraded to junk.

The list of companies now on the brink of junk includes watchmaker Fossil Group, which suffered a 9 per cent drop in sales in its first quarter, and internet security company Symantec after it agreed to purchase Blue Coat for $4.65bn with $2.8bn of new debt. They join multinationals such as Rémy Cointreau, LG Electronics and miner Goldcorp sitting on the edge of speculative rating territory.

Source

Compare the 10-year interest rates of:

GE-IT: What can you conclude?

GE-IT: The *realized* return on the Italian bond will be higher than that on the German bond: true or false?

The *expected* return on the Italian bond is higher than that on the German bond: true or false?

FR-US: What can you conclude?

Default risk: examples

Consider the following Zero-coupon bonds

	Rating	N	T	yield to maturity	Price
Bond 1	AAA	Eu 100	1 Y	5.2%	Eu 95
Bond 2	BB	Eu 100	1 Y	11.11%	Eu 90
Bond 3	BB	Eu 100	2 Y	12%	Eu 79.71
Bond 4	AAA	Eu 100	2 Y	5.5%	Eu 89.86

Remark: Keeping fixed the features of a bond, the higher its default risk the lower its market price, the higher its yield to maturity.

What are the term structure for AAA bonds and for BB bonds?

Maturity	term structure for AAA	term structure for BB
1 Year	5.2%	11.11%
2 Year	5.5%	12%

Default risk: examples

Maturity	term structure for AAA	term structure for BB
1 Year	5.2%	11.11%
2 Year	5.5%	12%

What is the no-arbitrage price of a bond with the following properties?
$C = Eu\ 2$
$N = Eu\ 100$
$T = 2\ years$
Frequency = annual
Rating: BB

$$P = \frac{2}{1.1111} + \frac{102}{1.12^2} = Eu\ 83.11$$

Default risk: arbitrage and pricing

As a general rule, the bonds I can use to build a replicating portfolio must be as similar as possible to the bond I want to price.

- Bonds denominated in the same currency
- Bonds with the same rating
- Issued by firms with the same characteristics (ex. sector of activity, size, country etc.)

Default risk: exercise

	T	C	yield to maturity	Rating
Bond 4	4 year	Eu 5	5%	AAA
Bond 3	3 year	Eu 3	4%	AAA
Bond 2	2 year	Eu 0	3%	AAA
Bond 1	1 year	Eu 2	2%	AAA
Bond 1bis	1 year	Eu 2	10%	B

Suppose that coupons are paid every year and that the face value is Eu 100 for each bond.

1. What are the prices of these bonds?
   $P1bis = 92.72$, $P1 = 100$, $P2 = 94.26$, $P3 = 97.22$, $P4 = 100$.
2. What is the term structure of interest rate for AAA Bonds?
   $r_{1Y} = 2\%$, $r_{2Y} = 3\%$, $r_{3Y} = 4.04\%$, $r_{4Y} = 5.13\%$.
3. Suppose that after six months, the term structure is flat at 2%. What will be the price of Bond 3? 103.91

Conclusion

By now you should know:

- What a bond is and what cash-flows it generates.
- What the yield curve is.
- How to price a bond using the no-arbitrage method and/or the yield curve.
- Why bonds are risky and how to measure default and interest rate risks.