Financial Markets

HEC Paris – Fall 2024

Part 3: Forward and Futures

Overview

1. **Forward and futures contract basics** $\Longleftarrow$

   - Uses for these contracts

   - Definition of a forward contract

   - Payoff of a forward contract

   - Forwards versus futures

   - Counterparty risk

2. Using forwards and futures

   - Hedging

   - Speculation

3. Valuation

   - No-arbitrage pricing

   - Conducting arbitrage with forward contracts

Forward Contracts

Definition

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- The size (q) of the contract: The amount of the underlying asset that will be exchanged.
- The forward price (F): the delivery price that will be paid at maturity in exchange for each unit of the underlying asset.

Forward Contracts

Example

- Underlying asset: Lead
- Size: 200 Tons
- Maturity T = December 25th 2014
- Forward price F = 500 Eu/ton

If today two agents sign a forward contract, then **at maturity** T:

- One party has the obligation to buy the underlying asset at the forward price F that has been fixed at the moment of the contract signature.

  This party has a long position in the contract.

- The other party has the obligation to deliver the underlying asset.

  This party has a short position in the contract.

- At the time the parties sign the forward contract, there is no flow of money or underlying asset.

Forward Contracts

Example

- Underlying asset: Corn
- Size: 200 Tons
- Maturity T = December 25th 2024
- Forward price F = 500 Eu/ton

Forward Contracts

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- Forward price F = 500 Eu/ton

- **Today:** The parties sign the FWD contract but there is no transfer of cash nor of corn.
- **On Dec 25th 2024**:
  - The party with the LONG POSITION in the contract pays $500 \times 200 = 100000$ Euros to the party with the SHORT POSITION in the FWD.
  - The party with the SHORT POSITION in the contract Delivers 200 tons of corn to the party with the Long POSITION in the FWD.

Hedging

Farmer plants corn in May 2024 and plans to sell the harvest in August:

> Farmer's revenues may not cover costs if the spot (immediate transactions or cash) market corn price decreases between May & August 2024:

$\downarrow$ Spot (cash) market corn prices in ¢/bushel

> (for corn: 1 bushel = 0.0254 metric tons $\Leftrightarrow$ 1 metric ton $\approx$ 39.4 bushels)

Hedging

Consider Nestlé's problem: it plans to buy large amounts of corn in August 2024 to produce breakfast cereals.

Nestlé also bears a risk because the corn (input) prices can increase.

If the two parties could agree to buy/sell at a pre-defined future date an agreed-upon quantity of corn of a certain quality (e.g., corn type, its humidity-level) at a price that they determine when they enter into the said contract (say, May 2024), then they would have created a Forward contract.

The alternative is for the supplier and the buyer of corn to get into to similar transactions using standardized futures contracts that trade on organized derivatives markets.

Speculation

Soyabean prices hit four-year high as nerves rise over food inflation

Dry weather disrupts supply of agricultural commodities just as governments fill reserves

Emiko Terazono in London NOVEMBER 6 2020

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Soyabean prices have surged

CBOT soyabeans ($ per bushel)

The broad rally in food prices during the second half of 2020 has attracted hedge funds and other speculative buyers. The latest data from the US Commodity Futures Trading Commission showed that in the week that ended October 27, speculators held record net “long” positions in agricultural commodity futures and options.

Forward contracts: the underlying asset

- Forward (and futures) contracts are called derivative securities because their price and payoff are derived from the price of another asset (the underlying asset)

- Forward (as well as futures) contracts can be written on any type of underlying asset:

  - Financial securities: fixed income securities (typically government bonds), stock market indices, individual stocks

  - Commodities (energy, metals, agricultural products)

  - Currencies (foreign exchange)

  - Temperature indices, CO2 emissions, ...

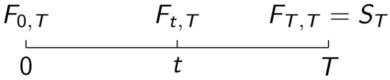
Forward price

- The delivery price agreed upon at the creation of the contract is such that both parties agree to enter into the contract for free

  ⇒ *The value of a long position (or a short position) in a forward contract when it is created is zero*

- The delivery price of a forward contract created at date t with maturity date T is called the forward price (or futures price for a futures contract) and is denoted by $F_{t,T}$ (or simply F_t if there is no ambiguity on the maturity date)

- What must $F_{T,T}$ be equal to?
  Two investors agree to trade the underlying asset now $(t = T)$ at price $F_{T,T}$. This is equivalent to trading on the spot market at price S_T. Therefore $F_{T,T} = S_T$

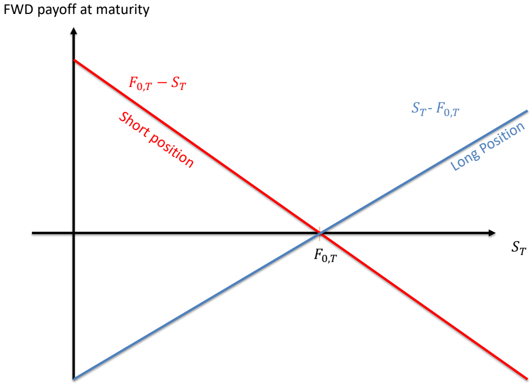


  ⇒ This is the convergence property of forward and futures prices.

Forward payoff at maturity (T)

- Payoff of a long forward position at maturity: $= S_T - F_{0,T}$
  - Spot price of the underlying asset at maturity: S_T
  - Delivery price: $F_{0,T}$
- Payoff of the short position at maturity is the mirror image of the payoff to the long position $= F_{0,T} - S_T = -(S_T - F_{0,T})$

Forward payoff at maturity



Example

On May 1st, the farm and Nestlé agree to exchange 10,000 bushels of corn at 7 \$/bushel on September 1st

Underlying asset	corn
Contract size	10,000 bushels
Long position	Nestlé
Short position	the farm
Maturity	September 1st
Delivery price	7 \$/bushel

Question 1 What are payoffs to the farm and Nestlé if the spot (cash market) price of corn is 6 \$/bushel on September 1st?

	May 1st (t=0)	September 1st (t=T)
Long position (Nestlé)	0	$(6-7)\times 10,000 = -1\times 10,000 = -10,000$
Short position (the farm)	0	$(7-6)\times 10,000 = +10,000$

Cash vs. physical settlement

Physical settlement The underlying asset is physically delivered in exchange for the payment of $F_{0,T}$, i.e., at date $t = T$ the long (short) position on the forward or the futures pays (gets paid) $F_{0,T}$ and receives (delivers) the underlying asset under the conditions pre-specified in the forward or futures contract:

If actual delivery is not desired, either side can choose to close the existing forward or futures position by taking its exact opposite (conduct a reverse trade) on the same contract thus realizing the gain or the loss on the position: but this is harder to do for forward contracts for many of which there are no active secondary markets.

Ex: if you speculate on wheat prices using wheat futures, you will not take actual delivery of wheat but reverse your position on the same wheat futures

Cash settlement The losing party pays out the difference between $F_{0,T}$ and S_T in cash, the underlying asset is not delivered:

Ex: stock index futures

Cash vs. physical settlement

- Payoffs are identical under cash and physical settlement
- Example (farm and Nestlé cont'd): $F_{0,T} = 7\ \$/bushel$, $S_T = 6\ \$/bushel$

Question 1 What is the payoff at maturity for the farm if the contract specifies cash delivery for 10,000 bushels?

The farm receives directly the difference
$(F_{0,T} - S_T) \times 10,000 = 70,000 - 60,000 = +10,000$

Question 2 What is the payoff at maturity for the farm if the contract specifies physical delivery for 10,000 bushels?

The farmer receives $F_{0,T} \times 10,000 = 70,000$ and delivers corn worth $S_T \times 10,000 = 60,000$
$\Rightarrow$ Payoff $70,000 - 60,000 = 10,000$

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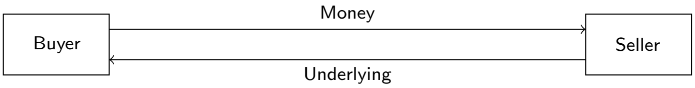
- The size (q) of the contract: The amount of the underlying asset that will be exchanged.
- The place of delivery (for commodities).
- The futures price (F): the delivery price that will be paid at maturity in exchange for each unit of the underlying asset.

Forwards vs. futures

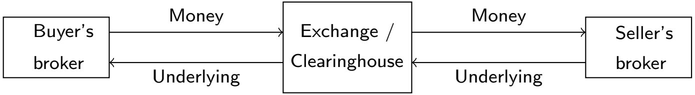
- Forwards trade *over-the-counter* (in the OTC market)

  Ex: call your bank and buy forward contract for delivery of 123 kg of frozen orange juice in Jouy-en-Josas on Thursday, December 14th



- Futures are standardized contracts that trade on organized exchanges

  Ex: buy one futures contract on ICE for delivery of 15,000 pounds of frozen orange juice



Liquidity

- Forwards are typically illiquid (see below for exceptions):
  - One needs to find a counterparty to enter into a forward contract
  - Forward contracts are typically *"tailor-made"* to the specific needs of the counterparties
  - As a result, it is difficult to reverse one's position in a forward
  - Liquid forward contracts: US Treasury forwards and FX (currency exchange rate) forward contracts are the exceptions thanks to standard contract sizes & maturity dates, and large international banks' involvement)

- Futures are liquid
  - They are easily traded on futures exchanges in standardized formats
  - To be exchange-traded, futures are standardized by the nature & quality of the underlying commodity or asset, size of the contract, maturity date, and place of delivery
  - Futures are the *"prêt-à-porter"* of delayed transaction contracts: standardization reduces the flexibility of the contracts offered, but it increases their liquidity as trades concentrate on few contracts

Some history on futures

- First commonly traded futures contracts go back to mid-1890s, with the first modern organized futures exchanges (with a central clearinghouse mandating clearing of trades) dating back to mid-1920s.
- Grain (corn, wheat, etc) forward contracts that regularly traded in secondary markets in the US Midwest during mid-1800s, eventually turned into standardized futures contracts. Agricultural product futures remained the main type of futures contracts that were available until 1970s
- New futures contracts on financial assets were introduced following the break-down of the Bretton Woods Agreement and the higher volatility that followed in the financial markets: FX-futures, interest rate (government bond) futures, and stock index futures contracts were created to allow traders to hedge as well as speculate on these dimensions
- Similarly, over the past two decades, the need to hedge against the effects of climate change has led past to the creation of new futures contracts on temperature, water, rainfall, snowfall, etc
- Newly introduced contracts remain in existence so long as there is economically meaningful trading that is conducted in them: such contracts disappear when there is little interest in using them.

Climate change related futures contracts

Chicago Mercantile Exchange (CME) introduced:

- in 1999 Temperature (Heating Degree Day and Cooling Degree Day) futures and options on these futures

- in 2006 Snowfall futures and options on these futures

- in 2007 Hurricane Index futures and options on these futures

- in 2010 Rainfall futures and options on these futures

  But, in 2014 all of the above contracts, except HDD and CDD (Temperature) futures and options, were delisted due to no trading!

- in 2020 CME decided to introduce Water futures and options on these futures based on the NASDAQ Veles California Water Index (NQH2O), but as of today, there is very little to no trading in these futures

Temperature futures contracts

Futures based on Heating Degree Days (HDD) and Cooling Degree Days (CDD), which are indicators of energy need for heating or cooling a building, based on temperature-thresholds:

- HDD = Max(0, 65 °F - daily average temperature)
- CDD = Max(0, daily average temperature - 65 °F)

The futures value is based on the daily differences in temperature with respect to 65 °F in the US (18 °C elsewhere) times $20 HDD or CDD in the US (€20 in Europe, £20 in the UK)

As a simplified example, if the daily average temperature at Heathrow Airport in London were 3 °C each day of the month, then the value of the futures contract would be: Max(0, 18 °C - 3 °C) $\times$ 30 days $\times$ £20 = £9000

Who would use these contracts: any business that is likely to incur higher heating or cooling costs, or to experience loss of business due to lower or higher than expected temperatures.

Example: S&P500 ESG Index Futures

- Suppose that you are bearish on ESG stocks in the US for some reason (say, because you believe that a major green-washing scandal will break between now and December 2023).
  - You could short (i.e., sell) the E-mini S&P 500 ESG Futures contract
  - The S&P 500 ESG Index is a market-value-weighted stock index that includes 320 or so stocks that are retained after applying ESG-based exclusion criteria to the 500 stocks that are in the S&P 500 Index.
- Futures contract on the E-mini S&P 500 ESG Index
  - Traded on the Chicago Mercantile Exchange (CME)
  - Underlying asset: US stock market index S&P 500 ESG
  - Contract size: \$ 500 $\times$ S&P 500 ESG Index
  - Maturity dates: third Friday of every March, June, September, December
  - Cash settlement based on the maturity date opening value of the S&P 500 ESG Index

Example: S&P500 ESG Index Futures

Q: You sell 10 E-mini S&P 500 ESG futures contracts with Dec 2023 maturity. What will be your cash flow at maturity (T=Dec 2023) if the futures price today at t=0 is $F_{0,T}$ = \$401, and S&P 500 ESG Index ends up at 350 at T?

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Q: You sell 10 E-mini S&P 500 ESG futures contracts with Dec 2023 maturity. What will be your cash flow at maturity (T=Dec 2023) if the futures price today at t=0 is $F_{0,T} = \$401$, and S&P 500 ESG Index ends up at 350 at T?

$$\$500 \times (F_{0,T} - S_T) \times 10 = \$500 \times (401 - 350) \times 10 = \$255,000$$

Counterparty risk

- Forward contracts carry counterparty risk: the other party might fail to meet its contractual obligation to buy/sell the underlying asset at t=T
- That's why forward contracts are created between large corporations and/or financial institutions (and not individual investors)
- In the case of futures, which are traded on markets, the associated clearinghouse (i) acts as the counterparty of both the buyer and the seller, and (ii) sets up margin requirements to minimize the counterparty risk.
- Moreover, brokers and dealers involed in the futures market estabish their own margin requirements to further protect themselves from losses that their clients might expose them to due adverse price movements (see the next slide for an example of a broker that didn't adjust the margin requirements quick enough to protect itself from its clients' losses generated by adverse price movements).
- As a result, counterparty-related futures contract failures are very rare.

Margins accounts & daily settlement

- How to eliminate counterparty risk? $\rightarrow$ Margin accounts & daily settlement
- Both the Buyer and Seller have to put up "good faith" margin money (post a "performance bond") up-front to be able to trade in the futures.
- Since (i) price movements will benefit one side of the trade at the expense of the other, and (ii) price movements cannot be predicted, the exchange will ask both parties to put down the same initial margin amount.
- Idea: Profits and losses are realized daily rather than waiting until maturity

  $\Rightarrow$ *Futures contract positions are valued every day (i.e., marked-to-market)*
- If the money remaining in the margin account falls below the maintenance margin level defined for the contract, the account will get a margin call.
- In case of margin call, the account-holder can either (i) replenish the margin account to a pre-defined margin (typically initial margin) level, or (ii) will be forced to close the account before the exchange starts losing money.
- Always used for futures contracts, and sometimes for forward contracts

Example of commodity price volatility

- Silver prices in the spot (cash) market ($/Troy ounce)
- One Troy ounce is defined as 31.1 grams (comes from French city of "Troyes" + Latin "uncia" [1/12th of the Roman pound])

Example

- Silver futures for T=December 15th delivery have a contract size of 5,000 Troy ounces (oz) of silver/contract. The contract prices, quoted in $ per Troy ounce of silver, are given until day t=3:
  $F_{0,T} = 20.00\$/oz, F_{1,T} = 19.00\$/oz, F_{2,T} = 18.50\$/oz,\text{and} F_{3,T} = 17.60\$/oz.$

- If the initial margin requirement is 20,000 $/contract, how much should the buyer of 100 contracts post as initial margin (initial performance bond)? What about the seller of 100 contracts? Fill the table in the next slide.

- If the maintenance (variation) margin is 10,000$/contract, would either long or short position get a margin call? If yes, on which day?

- Would your answer change if the initial margin were to be 20% of the position's value, and the maintenance margin were to be 10% of the position's value on that day? To answer, recreate the table.

Example (continued)

Initial Margin= $\$20,000$ per contract $\times$ 100 contracts $= 2,000,000$ \$

Maintenance Margin= $\$10,000$ per contract $\times$ 100 contracts $= 1,000,000$ \$

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Day (t)	$F_{t,T}$ (\$)	Long Position Payoff (\$)	Short Position Payoff (\$)	Margin for Long (\$)	Margin for Short (\$)	Maintenance Margin (\$)
0	20.00			2,000,000	2,000,000	1,000,000

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1	19.00	$(F_{1,T} - F_{0,T}) \cdot 5,000 \cdot 100$	$(F_{0,T} - F_{1,T}) \cdot 5,000 \cdot 100$	2,000,000	2,000,000	1,000,000

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1	19.00	$(F_{1,T} - F_{0,T}) \cdot 5,000 \cdot 100$ = (19.00-20.00)$\cdot 5,000 \cdot 100$	$(F_{0,T} - F_{1,T}) \cdot 5,000 \cdot 100$ = (20.00-19.00)$\cdot 5,000 \cdot 100$	2,000,000	2,000,000	1,000,000

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2	18.50	$(F_{2,T} - F_{1,T}) \cdot 5,000 \cdot 100$	$(F_{1,T} - F_{2,T}) \cdot 5,000 \cdot 100$	1,500,000	2,500,000	1,000,000

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2	18.50	$(F_{2,T} - F_{1,T}) \cdot 5,000 \cdot 100$ = (18.50-19.00)$\cdot$5,000 $\cdot$ 100 = -0.50$\cdot$5,000 $\cdot$ 100	$(F_{1,T} - F_{2,T}) \cdot 5,000 \cdot 100$ = (19.00-18.50)$\cdot$5,000 $\cdot$ 100 = +0.50$\cdot$5,000 $\cdot$ 100	1,500,000	2,500,000	1,000,000

Example (continued)

Initial Margin= $\$20,000$ per contract $\times$ 100 contracts $= 2,000,000$ \$

Maintenance Margin= $\$10,000$ per contract $\times$ 100 contracts $= 1,000,000$ \$

Day (t)	$F_{t,T}$ (\$)	Long Position Payoff (\$)	Short Position Payoff (\$)	Margin for Long (\$)	Margin for Short (\$)	Maintenance Margin (\$)
0	20.00			2,000,000	2,000,000	1,000,000
1	19.00	$(F_{1,T} - F_{0,T}) \cdot 5,000 \cdot 100$ $=$ (19.00-20.00)$\cdot 5,000 \cdot 100$ $=$ - 1.00$\cdot 5,000 \cdot 100$ $=$ - 500,000	$(F_{0,T} - F_{1,T}) \cdot 5,000 \cdot 100$ $=$ (20.00-19.00)$\cdot 5,000 \cdot 100$ $= +$ 1.00$\cdot 5,000 \cdot 100$ $= +$ 500,000	2,000,000 - 500,000 $=$ 1,500,000	2,000,000 $+$ 500,000 $=$ 2,500,000	1,000,000
2	18.50	$(F_{2,T} - F_{1,T}) \cdot 5,000 \cdot 100$ $=$ (18.50-19.00)$\cdot 5,000 \cdot 100$ $=$ -0.50$\cdot 5,000 \cdot 100$ $=$ -250,000	$(F_{1,T} - F_{2,T}) \cdot 5,000 \cdot 100$ $=$ (19.00-18.50)$\cdot 5,000 \cdot 100$ $=$ +0.50$\cdot 5,000 \cdot 100$ $=$ +250,000	1,500,000 - 250,000	2,500,000 $+$ 250,000	1,000,000

Example (continued)

Initial Margin= \$20, 000 per contract $\times$ 100 contracts = 2, 000, 000 \$

Maintenance Margin= \$10, 000 per contract $\times$ 100 contracts = 1, 000, 000 \$

Day (t)	$F_{t,T}$ (\$)	Long Position Payoff (\$)	Short Position Payoff (\$)	Margin for Long (\$)	Margin for Short (\$)	Maintenance Margin (\$)
0	20.00			2,000,000	2,000,000	1,000,000
1	19.00	$(F_{1,T} - F_{0,T}) \cdot 5,000 \cdot 100$ = (19.00-20.00)·5, 000 · 100 = - 1.00·5, 000 · 100 = - 500,000	$(F_{0,T} - F_{1,T}) \cdot 5,000 \cdot 100$ = (20.00-19.00)·5, 000 · 100 = + 1.00·5, 000 · 100 = + 500,000	2,000,000 - 500,000 = 1,500,000	2,000,000 + 500,000 = 2,500,000	1,000,000
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Example (continued)

Initial Margin= \$20,000 per contract $\times$ 100 contracts = 2,000,000 \$

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3	17.60	$(F_{3,T} - F_{2,T}) \cdot 5,000 \cdot 100$ = (17.60-18.50)·5,000 · 100 = -0.90·5,000 · 100 = -450,000	$(F_{2,T} - F_{3,T}) \cdot 5,000 \cdot 100$ = (18.50-17.60)·5,000 · 100 = +0.90·5,000 · 100 = +450,000	1,250,000 - 450,000 = 800,000	2,750,000 + 450,000 = 3,200,000	1,000,000

Example (continued)

Initial Margin= \$20, 000 per contract $\times$ 100 contracts = 2, 000, 000 \$

Maintenance Margin= \$10, 000 per contract $\times$ 100 contracts = 1, 000, 000 \$

Day (t)	$F_{t,T}$ (\$)	Long Position Payoff (\$)	Short Position Payoff (\$)	Margin for Long (\$)	Margin for Short (\$)	Maintenance Margin (\$)
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Total		$(F_{3,T} - F_{0,T}) \cdot 5,000 \cdot 100$ = - 1,200,000	$(F_{0,T} - F_{3,T}) \cdot 5,000 \cdot 100$ = + 1,200,000	- 1,200,000	+ 1,200,000	

Margin accounts and daily settlement

Interactive Brokers takes $88m hit from crude futures collapse

Shares in brokerage firm drop overnight after it discloses payments to clearing houses

Philip Stafford in London APRIL 22 2020

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Shares in Interactive Brokers fell as much as 10 per cent in early trading on Wednesday after the US broker took an $88m loss from the collapse in value of short-term oil futures contracts.

The Greenwich, Connecticut-based broker, founded by electronic trading pioneer and billionaire Thomas Peterffy, said late on Tuesday that several of its customers had been caught on the wrong side of Monday's plunge, forcing it to step in and pay the margin calls owed to clearing houses.

Margin accounts and daily settlement

- Initial and maintenance margins are set by the clearinghouse depending on the riskiness of the underlying

- Advantage of daily settlement: minimizes the risk of default on a position
  - Typically eliminates it if margins requirements are changed if the prices of the underlying asset become volatile ⇒ see next slide
  - So long as clearinghouses do not default!
  - The default of a clearinghouse is a very rare event: Caisse de Liquidation in 1974 (*"The Failure of a Clearinghouse: Empirical Evidence"*, Guillaume Vuillemey)

- Disadvantages of daily settlement: compared to a forward, in the case of a futures contract the cash outflows occur before maturity

Overview

1. Forward and futures contract basics

   - Uses for these contracts

   - Definition of a forward contract

   - Payoff of a forward contract

   - Forwards versus futures

   - Counterparty risk

2. **Using forwards and futures** $\Longleftarrow$

   - Hedging

   - Speculation

3. Valuation

   - No-arbitrage pricing

   - Conducting arbitrage with forward contracts

Hedging

A wheat farmer will sell his crop in T=6 months, at which time the spot price of wheat will be $\widetilde{S}_T$ (the tilde on top of S_T denotes the fact that t=T spot (cash) market price of the crop is a random number as of t=0):

Should the farmer take long or short positions on wheat futures with maturity 6 months to hedge against fluctuations in wheat prices?

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Short position to fix the price now

The crop is 1,000 bushels. The size of a wheat contract is 10 bushels.

How many futures should the farmer trade to be perfectly hedged?

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Short 100 futures	$(F_{0,T} - \widetilde{F}_{T,T}) \times 10 \times 100$ $= (F_{0,T} - \widetilde{S}_T) \times 10 \times 100$

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	CF in T=6 months
Wheat	$+\ \widetilde{S}_T \times 1,000$
Short 100 futures	$(\mathrm{F}_{0,T} - \widetilde{F}_{T,T}) \times 10 \times 100$ $= (\mathrm{F}_{0,T} - \widetilde{S}_T) \times 10 \times 100$
Total	$1,000 \times F$

What to do if perfect hedges are not possible

- The assets that need to be hedged may differ from the underlying assets of the futures contracts

  → *Use futures on assets with highly-correlated returns to do a cross hedge:*

  For ex., if you want to hedge your crop of rye (whose futures are not available) use wheat futures instead: suppose that the each wheat futures is on 10 bushels of wheat, and if the returns on rye and wheat have a correlation of 0.80, then for each 10 bushels of rye you will need $1 \div 0.80 = 1.25$ wheat contracts.

- Contract maturities may differ from the desired hedging maturity

  → *Use the futures with the closest maturity date:*

  For ex., if you want to hedge your silver position for March 1^{st}, but there is no silver futures contract matures on that day. As a result, you will have to use the silver futures that matures on March 15^{th}.

  But, this will expose you to what's called the basis risk since $F_{t,T} \neq S_t$ when t=March 1 and T=March 15. That is, futures price on March 1^{st} will not converge to spot price on March 1^{st} because t=March 1 is not the maturity date of the futures (which is T=March 15). In contrast, $F_{T,T} = S_T$ on T=March 15 because the latter is the silver futures' maturity date.

Speculation

Why speculate using futures rather than in the spot market? Two main reasons:

First, for commodities, you don't need to store the underlying asset if you take a long position, or to have it in inventory if you take a short position

Second, futures position returns are leveraged (i.e., magnified) compared to spot returns (as you only tie-up the initial margin money, i.e., a fraction of $F_{0,T}$).

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Example: Suppose you buy stock X in the spot market at $S_0 = €100$

What is your return if $S_1 = €110$ one week later?

$$HPR_{1week} = \frac{\text{payoff} - \text{investment}}{\text{investment}} = \frac{S_1 - S_0}{S_0} = \frac{10}{100} = 10\%$$

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Now, suppose you buy a futures contract on stock X maturing in one week at $F_{0,1week} = €100$ (assume that the yield curve is flat at 0%). The initial margin requirement for this contract is 5% F_0

What is your return if $S_1 = €110$ one week later?

$$HPR_{week} = \frac{S_1 - F_{0,1week}}{0.05 \times F_{0,1week}} = \frac{10}{5} = +2 = +200\%$$

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Be careful - leverage effects works in both directions: What if $S_1 = €90$?

$$HPR_{1week} = \frac{S_1 - F_{0,1week}}{0.05 \times F_{0,1week}} = \frac{-10}{5} = -2 = -200\%$$

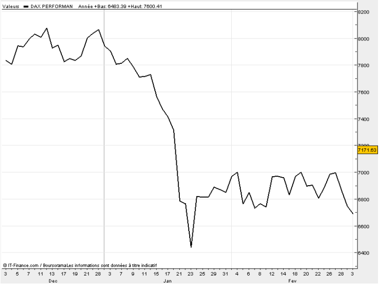
Speculation

- Speculating with futures can be (very!) risky because of leverage
- Example: Jérôme Kerviel at Société Générale, 2008

Underlying stock index	Contract size	No. of long positions	Spot price Jan 1st	Notional	Spot price Jan 18	Profits
EuroStoxx	10	743,000	4,330	32 bn €	4,000	−2.5 bn €
DAX	25	100,000	7,950	20 bn €	7,400	−1.4 bn €
Total						−3.9 bn €

Eurostoxx 50

DAX 30



Overview

1. Forward and futures contract basics
   - Uses for these contracts
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3. **Valuation** $\Longleftarrow$
   - No-arbitrage pricing
   - Conducting arbitrage with forward contracts

Valuation

- How to find the forward/futures prices $F_{0,T}$ (or more generally $F_{t,T}$)?
- By arbitrage pricing!
- From this point onward in the slides, we will assume that forwards and futures are equivalent
- We will consider three cases:

1. Contract on a financial security that pays no dividend before maturity date T of the forward,
2. Contract on a financial security that pays dividends before maturity date T of the forward,
3. Contract on a commodity that includes a cost of carry (say, cost of storage).

- You want to hold in 1 period one share of stock that does not pay any dividend → there are two ways to proceed:

  1. Buy forward: this costs you 0 today and $F_{0,1}$ in one period
  2. Buy the "Synthetic forward" (through a "cash-and-carry" trade): borrow S_0 to buy the stock today and repay the loan in one period → this costs you 0 today and $S_0(1 + r_f)$ in one period:

	CFs at t=0	**CFs at T=1 year**
Long forward position	0	$+\widetilde{S}_1 - F_{0,1}$

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Long synthetic-forward		
Buy 1 stock X	$-S_0$	$+\widetilde{S}_1$

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Long synthetic-forward		
Buy 1 stock X	$-S_0$	$+\widetilde{S}_1$
Borrow S_0 at risk-free rate	$+S_0$	$-S_0 \times (1+r_f)^1$

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Buy 1 stock X	$-S_0$	$+\widetilde{S}_1$
Borrow S_0 at risk-free rate	$+S_0$	$-S_0 \times (1+r_f)^1$
Synthetic-forward total	0	$+\widetilde{S}_1 - S_0 \times (1+r_f)^1$

To prevent arbitrage, the cost should be the same in each case: $F_{0,1} = S_0(1+r_f)$

The spot-forward parity

- The spot-forward parity (a *no-arbitrage condition*) generalizes to any T:

$$\boxed{F_{0,T} = S_0(1 + r_f)^T}$$

(we assume here that the yield curve is flat; otherwise r_f should be r_T)

- If the spot-forward parity does not hold, arbitrage opportunities exist, so long as arbitrage profits are larger than the associated transaction costs (note that in this class, unless otherwise indicated, we assume that there are no transaction costs).

Example 1: Implementing an arbitrage trade

Stock X currently sells for $S_0 = 80$ euros in the spot market. X does not pay any dividends. The risk-free rate is 0.5% per month.

Question 1 Find the no-arbitrage price of the 6-month forward on X?

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$$F_{0,T}^{no-arb} = S_0(1 + r_f)^T = 80 \times (1 + 0.005)^6 = €82.43$$

Question 2 Suppose that the market price of the forward is €82. Find an arbitrage strategy and write the arbitrage table:

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Trades engaged at t=0	**CFs at t=0**	**CFs at T=6 months**
Buy forward in the market	0	$\widetilde{S}_{6m} - 82$

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Sell (i.e., short) synthetic-forward		
Short-sell 1 stock X	+80	$-\widetilde{S}_{6m}$

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Buy forward in the market	0	$\widetilde{S}_{6m} - 82$
Sell (i.e., short) synthetic-forward		
Short-sell 1 stock X	+80	$-\widetilde{S}_{6m}$
Invest €80 at r_f for 6 mos.	−80	$80 \times 1.005^6 = 82.43$

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$$F_{0,T}^{no-arb} = S_0(1 + r_f)^T = 80 \times (1 + 0.005)^6 = €82.43$$

Question 2 Suppose that the market price of the forward is €82. Find an arbitrage strategy and write the arbitrage table:

Trades engaged at t=0	**CFs at t=0**	**CFs at T=6 months**
Buy forward in the market	0	$\widetilde{S}_{6m} - 82$
Sell (i.e., short) synthetic-forward		
Short-sell 1 stock X	$+80$	$-\widetilde{S}_{6m}$
Invest €80 at r_f for 6 mos.	-80	$80 \times 1.005^6 = 82.43$
Total for synthetic-forward sold	0	$+82.43 - \widetilde{S}_{6m}$

Example 1: Implementing an arbitrage trade

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Invest €80 at r_f for 6 mos.	-80	$80 \times 1.005^6 = 82.43$
Total for synthetic-forward sold	0	$+82.43 - \widetilde{S}_{6m}$
Arbitrage grand total	0	$+0.43$

Example 1: Implementing an arbitrage trade

Question 2 Alternative solution to obtain the arbitrage profits at date t=0:

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Sell (i.e., short) synthetic-forward		
Short-sell 1 share of X	$+80$	$-\widetilde{S}_{6m}$
Invest $\frac{82}{1.005^6} = 79.58$ at r_f	-79.58	$79.58 \times 1.005^6 = 82$

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Sell (i.e., short) synthetic-forward		
Short-sell 1 share of X	$+80$	$-\widetilde{S}_{6m}$
Invest $\frac{82}{1.005^6} = 79.58$ at r_f	-79.58	$79.58 \times 1.005^6 = 82$
Total for short synthetic forward	$+0.42$	$+82 - \widetilde{S}_{6m}$

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Short-sell 1 share of X	+80	$-\widetilde{S}_{6m}$
Invest $\frac{82}{1.005^6} = 79.58$ at r_f	−79.58	$79.58 \times 1.005^6 = 82$
Total for short synthetic forward	+0.42	$+82 - \widetilde{S}_{6m}$
Arbitrage grand total	+0.42	+0

Note that: $0.42 = 0.43/(1.005^6)$

Example 1b (newly added):

Question 3 The no-arbitrage price of the forward remains €82.43 as above. But suppose now that the market price of the forward is instead 84 €/share. How can you do arbitrage in this case?

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Trades engaged at t=0	**CFs at t=0**	**CFs at T=6 months**
Sell (short) forward in the market	0	$84 - \widetilde{S}_{6m}$

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Question 3 The no-arbitrage price of the forward remains €82.43 as above. But suppose now that the market price of the forward is instead 84 €/share. How can you do arbitrage in this case?

Trades engaged at t=0	**CFs at t=0**	**CFs at T=6 months**
Sell (short) forward in the market	0	$84 - \widetilde{S}_{6m}$
Buy (long) the synthetic-forward		
Buy 1 stock X	-80	$+\widetilde{S}_{6m}$

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Buy (long) the synthetic-forward		
Buy 1 stock X	-80	$+\widetilde{S}_{6m}$
Borrow €80 at r_f for 6 mos.	$+80$	$-80 \times 1.005^6 = -82.43$

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Buy (long) the synthetic-forward		
Buy 1 stock X	-80	$+\widetilde{S}_{6m}$
Borrow €80 at r_f for 6 mos.	$+80$	$-80 \times 1.005^6 = -82.43$
Total for synthetic-forward sold	0	$+\widetilde{S}_{6m} - 82.43$
Arbitrage grand total	0	$+1.57$

Example 2: Valuing a forward position

On January 1st, the spot price of stock XYZ is €40. XYZ does not pay any dividend. The yield curve is flat at 4% per year.

Question 1 What is the price of a delivery in one-year forward contract (i.e., with maturity on Dec 31st when T=1 year) on 1 share of XYZ?

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$$F_{0,1} = S_0(1 + r_f)^T = 40 \times 1.04^1 = €41.6$$

Six months later, on July 1st when $t = 0.5$, the stock price of XYZ is €45. The term structure of interest rates has not changed (i.e., it is flat at 4%):

Question 2 What is, on July 1st (t=0.5), the forward price for a new forward contract on 1 share of XYZ with the same maturity date as the previous contract (i.e., with maturity on Dec 31st, i.e., T=1)?

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$$F_{0.5,1} = S_{0.5}(1 + r_f)^{0.5} = 45 \times 1.04^{0.5} = €45.9$$

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You have taken a long position in the first contract, i.e., the contract that was initiated on January 1st:

Question 3 What was the value of this long position on January 1st?

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You have taken a long position in the first contract, i.e., the contract that was initiated on January 1st:

Question 3 What was the value of this long position on January 1st? **Zero**

Now, we want to determine the value of the long position on July 1st:

Question 4 How can you take actions so that you realize all your gains/losses on July 1st & have zero cash flows at all future dates?

	CF Jul 1st (t=0.5)	CF Dec 31st (T=1)
Long contract initiated on Jan 1st		$\widetilde{S}_1 - F_{0,1}$

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	CF Jul 1st (t=0.5)	CF Dec 31st (T=1)
Long contract initiated on Jan 1st		$\widetilde{S}_1 - F_{0,1}$
Close out the position:		
Short contract initiated on July 1st	0	$F_{0.5,1} - \widetilde{S}_1$

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You have taken a long position in the first contract, i.e., the contract that was initiated on January 1st:

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Long contract initiated on Jan 1st		$\widetilde{S}_1 - F_{0,1}$
Close out the position:		
Short contract initiated on July 1st	0	$F_{0.5,1} - \widetilde{S}_1$
Borrow $\frac{F_{0.5,1} - F_{0,1}}{(1+r_f)^{0.5}}$ euros	$\frac{F_{0.5,1} - F_{0,1}}{(1+r_f)^{0.5}}$	$-(F_{0.5,1} - F_{0,1})$

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Long contract initiated on Jan 1st		$\widetilde{S}_1 - F_{0,1}$
Close out the position:		
Short contract initiated on July 1st	0	$F_{0.5,1} - \widetilde{S}_1$
Borrow $\frac{F_{0.5,1}-F_{0,1}}{(1+r_f)^{0.5}}$ euros	$\frac{F_{0.5,1}-F_{0,1}}{(1+r_f)^{0.5}}$	$-(F_{0.5,1} - F_{0,1})$
Total	$\frac{45.9-41.6}{1.04^{0.5}} = 4.2$	0

Question 5 What is, on July 1st, the value of a long position in the first contract, i.e. the $F_{0,1}$ contract?

Example 2: Valuing a forward position

You have taken a long position in the first contract, i.e., the contract that was initiated on January 1st:

Question 3 What was the value of this long position on January 1st? **Zero**

Now, we want to determine the value of the long position on July 1st:

Question 4 How can you take actions so that you realize all your gains/losses on July 1st & have zero cash flows at all future dates?

	CF Jul 1st (t=0.5)	CF Dec 31st (T=1)
Long contract initiated on Jan 1st		$\widetilde{S}_1 - F_{0,1}$
Close out the position:		
Short contract initiated on July 1st	0	$F_{0.5,1} - \widetilde{S}_1$
Borrow $\frac{F_{0.5,1}-F_{0,1}}{(1+r_f)^{0.5}}$ euros	$\frac{F_{0.5,1}-F_{0,1}}{(1+r_f)^{0.5}}$	$-(F_{0.5,1} - F_{0,1})$
Total	$\frac{45.9-41.6}{1.04^{0.5}} = 4.2$	0

Question 5 What is, on July 1st, the value of a long position in the first contract, i.e. the $F_{0,1}$ contract? **€4.2**

Spot-forward parity with dividends

- Underlying asset pays a dividend or coupon D_t at $t < T$. Yield curve flat at r_f:

	CF at 0	CF at t	CF at T
Long the forward	0	0	$\widetilde{S}_T - F_{0,T}$

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	CF at 0	CF at t	CF at T
Long the forward	0	0	$\widetilde{S}_T - F_{0,T}$
Long the synthetic-forward			
Buy the underlying	$-S_0$	$+D_t$	$\widetilde{S}_T$

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	CF at 0	CF at t	CF at T
Long the forward	0	0	$\widetilde{S}_T - F_{0,T}$
Long the synthetic-forward			
Buy the underlying	$-S_0$	$+D_t$	$\widetilde{S}_T$
Borrow S_0	$+S_0$		$-S_0(1+r_f)^T$

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	CF at 0	CF at t	CF at T
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Long the synthetic-forward			
Buy the underlying	$-S_0$	$+D_t$	$\widetilde{S}_T$
Borrow S_0	$+S_0$		$-S_0(1+r_f)^T$
Reinvest D_t between t and T		$-D_t$	$D_t(1+r_f)^{(T-t)}$

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Buy the underlying	$-S_0$	$+D_t$	$\widetilde{S}_T$
Borrow S_0	$+S_0$		$-S_0(1+r_f)^T$
Reinvest D_t between t and T		$-D_t$	$D_t(1+r_f)^{(T-t)}$
	0	0	$\widetilde{S}_T - \left(\begin{array}{l} S_0(1+r_f)^T \\ -D_t(1+r_f)^{(T-t)} \end{array} \right)$

- In order to prevent arbitrage: $\mathrm{F}_{0,T} = S_0(1+r_f)^T - D_t(1+r_f)^{(T-t)}$
- More generally, with multiple dividends or coupons:

$$\boxed{F_{0,T} = S_0(1+r_f)^T - \sum_t D_t(1+r_f)^{(T-t)}}$$

Example

Stock Y pays an annual dividend of €5 euros/share. It sells for $S_0 = €100$ today (this year's dividend has just been paid). Tield curve is flat at 6% per year.

Question 1 What is the no-arbitrage price $F_{0,2}$ of the forward contract on Y with delivery in 2 years (right after the 2nd dividend payment)?

$$F_{0,2}^{no-arb} = S_0(1+r_f)^T - D_1(1+r_f)^{T-1} - D_2(1+r_f)^{T-2}$$

$$F_{0,2}^{no-arb} = 100 \times 1.06^2 - 5 \times 1.06 - 5 = €102.06$$

Question 2 What will you do if $F_{0,2}^{Market} = 104$ €/share?

	t=0	t=1	t=2
Sell forward in the market	0	0	$+F_{0,2} - \widetilde{S}_2 = 104 - \widetilde{S}_2$

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Long synthetic-forward			
Buy Y at $-S_0 = -100$	-100	$+D_1 = +5$	$+5 + \widetilde{S}_2$

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Buy Y at $-S_0 = -100$	-100	$+D_1 = +5$	$+5 + \widetilde{S}_2$
Borrow €100	$+100$		-100×1.06^2

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Stock Y pays an annual dividend of €5 euros/share. It sells for $S_0 = €100$ today (this year's dividend has just been paid). Tield curve is flat at 6% per year.

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<u>Long synthetic-forward</u>			
Buy Y at $-S_0 = -100$	-100	$+D_1 = +5$	$+5 + \widetilde{S}_2$
Borrow €100	$+100$		-100×1.06^2
Reinvest D_1 for 1 year		$-D_1 = -5$	$+5 \times 1.06$

Example

Stock Y pays an annual dividend of €5 euros/share. It sells for $S_0 = €100$ today (this year's dividend has just been paid). Tield curve is flat at 6% per year.

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Question 2 What will you do if $F_{0,2}^{Market} = 104$ €/share?

	t=0	t=1	t=2
Sell forward in the market	0	0	$+F_{0,2} - \widetilde{S}_2 = 104 - \widetilde{S}_2$
Long synthetic-forward			
Buy Y at $-S_0 = -100$	-100	$+D_1 = +5$	$+5 + \widetilde{S}_2$
Borrow €100	$+100$		-100×1.06^2
Reinvest D_1 for 1 year		$-D_1 = -5$	$+5 \times 1.06$
Synthetic-forward total	0	0	$+\widetilde{S}_2 - 102.06$

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Sell forward in the market	0	0	$+F_{0,2} - \widetilde{S}_2 = 104 - \widetilde{S}_2$
Long synthetic-forward			
Buy Y at $-S_0 = -100$	-100	$+D_1 = +5$	$+5 + \widetilde{S}_2$
Borrow €100	$+100$		-100×1.06^2
Reinvest D_1 for 1 year		$-D_1 = -5$	$+5 \times 1.06$
Synthetic-forward total	0	0	$+\widetilde{S}_2 - 102.06$
Arbitrage grand total	0	0	$104 - 102.06 = 1.94$

Example (continued)

Alternative solution to recover the arbitrage profits at date t=0:

	t=0	t=1 year	t=2 year
Sell overpriced forward	0	0	$+F_{0,2} - \widetilde{S}_2 = 104 - \widetilde{S}_2$

Example (continued)

Alternative solution to recover the arbitrage profits at date t=0:

	t=0	t=1 year	t=2 year
Sell overpriced forward	0	0	$+F_{0,2} - \widetilde{S}_2 = 104 - \widetilde{S}_2$
Long synthetic forward			
Buy Y at $-S_0 = -100$	-100.00	$+D_1 = +5$	$+5 + \widetilde{S}_2$
Borrow 101.73 for 2 years	$+101.73$		$-101.73 \times 1.06^2 = -114.30$
Reinvest D_1 for 1 year		$-D_1 = -5$	$+5 \times 1.06$

Example (continued)

Alternative solution to recover the arbitrage profits at date t=0:

	t=0	t=1 year	t=2 year
Sell overpriced forward	0	0	$+F_{0,2}-\widetilde{S}_2 = 104-\widetilde{S}_2$
Long synthetic forward			
Buy Y at $-S_0 = -100$	-100.00	$+D_1 = +5$	$+5+\widetilde{S}_2$
Borrow 101.73 for 2 years	$+101.73$		$-101.73 \times 1.06^2 = -114.30$
Reinvest D_1 for 1 year		$-D_1 = -5$	$+5 \times 1.06$
Synthetic-forward total	1.73	0	$+\widetilde{S}_2 - 104$

Example (continued)

Alternative solution to recover the arbitrage profits at date t=0:

	t=0	t=1 year	t=2 year
Sell overpriced forward	0	0	$+F_{0,2} - \widetilde{S}_2 = 104 - \widetilde{S}_2$
Long synthetic forward			
Buy Y at $-S_0 = -100$	-100.00	$+D_1 = +5$	$+5 + \widetilde{S}_2$
Borrow 101.73 for 2 years	$+101.73$		$-101.73 \times 1.06^2 = -114.30$
Reinvest D_1 for 1 year		$-D_1 = -5$	$+5 \times 1.06$
Synthetic-forward total	1.73	0	$+\widetilde{S}_2 - 104$
Arbitrage grand total	1.73	0	0

Note that: $101.73 = (104 + 5 + 5 \times 1.06) \div 1.06^2$

and that: $1.73 = 1.94 \div 1.06^2$

Commodity forwards

- What's special about commodities? $\rightarrow$ cost-of-carry $=$ storage costs+insurance
- Suppose that to store a commodity, you need to pay up-front a cost of carry (CC_0) for the whole period (until T). What happens to the spot-forward parity?

	CF at 0	CF at T
Long forward	0	$\widetilde{S}_T - F_{0,T}$
Long "synthetic" forward		
Buy commodity	$-S_0 - CC_0$	$\widetilde{S}_T$
Borrow $S_0 + CC_0$	$+S_0 + CC_0$	$-(S_0 + CC_0)(1 + r_f)^T$
	0	$\widetilde{S}_T - (S_0 + CC_0)(1 + r_f)^T$

- In order to prevent arbitrage, we must have

$$\boxed{F_{0,T} = (S_0 + CC_0)(1 + r_f)^T}$$

Commodity forwards

- How could we make arbitrage profits if the forward contract on the commodity is over-priced ($F_{0,T}^{Market} > F_{0,T}^{No-arbitrage} = (S_0 + CC_0)(1 + r_f)^T$)?

	CF at 0	CF at T
Short forward contract	0	$-(\widetilde{S}_T - F_{0,T})$
<u>Long Synthetic forward</u>		
Buy commodity at t=0	$-S_0 - CC_0$	$+\widetilde{S}_T$
Borrow $S_0 + CC_0$	$S_0 + CC_0$	$-(S_0 + CC_0)(1 + r_f)^T$
Total	0	$F_{0,T} - (S_0 + CC_0)(1 + r_f)^T > 0$

- If forward is underpriced, i.e., $F_{0,T}^{Market} < F_{0,T}^{No-arbitrage}(S_0 + CC_0)(1 + r_f)^T$?
- Arbitrage is difficult because if you sell short the commodity, you will not receive the costs of carry (i.e., storage + insurance costs)
- In fact, the answer depends on the type of commodity we are considering

Commodity Forwards

- If speculators hold inventories of the commodity, they can sell it, save the storage cost, invest the proceeds in the risk-free asset, and at the same time buy the underpriced forward

- Relevant case for investment-purpose commodities, like precious metals

- In this case, the no-arbitrage condition is $F_{0,T} = (S_0 + CC_0)(1 + r_f)^T$

- If speculators do not hold inventories of the commodity, they cannot conduct the arbitrage trade:

- Relevant case for consumption goods, like oil, cotton, etc.

- In this case, the no-arbitrage condition is $F_{0,T}^{Market} \leq F_{0,T}^{No-arbitrage} = (S_0 + CC_0)(1 + r_f)^T$

Example 1: Consumption good

The Brent crude oil forward for 3-month delivery is selling for $F_{0,3m} = \$52$ per barrel. The spot Brent crude sells for $S_0 = \$50$ per barrel today. The annual risk-free rate is $r_f = 5\%$. Speculators don't have inventories of oil

Question 1 Is there an arbitrage opportunity if the cost of carry is $CC_0 = 2.5\$$ per barrel to be paid today?

$$(S_0 + CC_0)(1 + r_f)^T = (50 + 2.5) \times 1.05^{0.25} = 53.14 > F_{0,3m}^{Market}$$

so there is no arbitrage opportunity because can't sell oil if no inventory

Question 2 What if $CC_0 = 1\$$?

$$(S_0 + CC_0)(1 + r_f)^T = (50 + 1) \times 1.05^{0.25} = 51.62 < F_{0,3m}^{Market}$$

arbitrage opportunity:

Sell oil forward	0	$52 - \widetilde{S}_{3m}$
Buy Synth-forward		
buy oil on spot	-50	$\widetilde{S}_{3m}$
pay storage	-1	
borrow	51	-51.62
arbitrage total	0	0.38

Example 2: Investment-purpose commodity

Today's spot price of silver is €6/ounce. Storage costs are 20 cents per ounce per year, paid at the end of the year. The risk-free rate is 5% per year. Speculators have inventories of silver

Question 1 What is the no-arbitrage forward price for 1 ounce of silver to be delivered in 1 year?

$F_{0,1y} = S_0(1 + r_f)^T + CC_T = 6 \times 1.05 + 0.2 = €6.5$

Question 2 Find an arbitrage strategy and write the arbitrage table if $F_{0,1y} =$ €6.4

	CF at t=0	CF at t=1
Long 1 forward contract	0	$\widetilde{S}_1 - 6.4$
Short 1 synthetic-forward		
Sell 1 ounce of silver	6	$-\widetilde{S}_1$
Save on inventory cost	0	0.2
Lend €6	-6	$6 \times 1.05 = 6.3$
Short synth-forward total	0	$6.5 - \widetilde{S}_1$
Arbitrage grand total	0	0.1