Financial Markets **HEC Paris, Fall 2024**

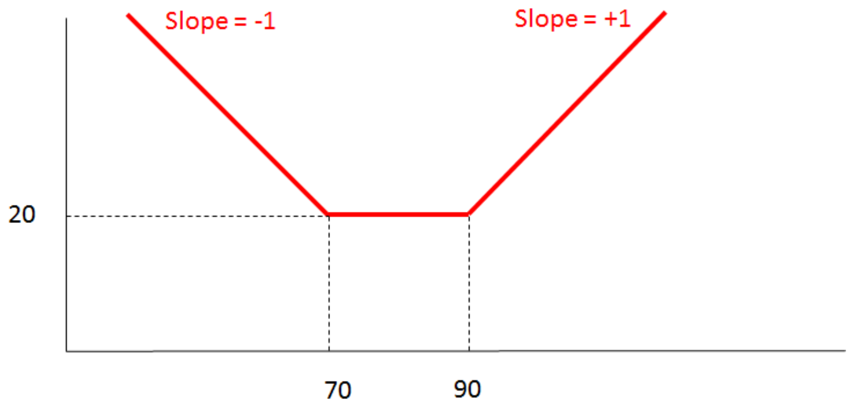
Problem Set 4.c on Options

Problem 1:

Stock XYZ, which will not pay any dividend over the next year, currently trades at a price of $S_{XYZ}(0)$ = €80. The risk-free interest rate is 5% per year. Consider these European options on stock XYZ:

	Expiration	Strike price (exercise price)
Call A	1 year	€ 70
Put A	1 year	€ 70
Call B	1 year	€ 90
Put B	1 year	€ 90

a) Find a portfolio of options that generate the following combined cash flow at t=1 year as a function of the spot price of stock XYZ at this date:



b) Assume in this question only that, at date t=0, the price of Call A is €10 and the price of Put A is €5. Find an arbitrage strategy. [Write the arbitrage table with the positions taken at t=0 and the cash flows at t=0 and t=1 year.]

In questions c-d-e-f) you will assume that, at t=1 year, XYZ stock will be trading at either $S_{XYZ}(1Y)$=€60 or $S_{XYZ}(1Y)$=€100 per share. Remember that Stock XYZ trades at a price of $S_{XYZ}(0)$=€80 per share at t=0.

c) What is the no-arbitrage price at date t=0 of Put B?

d) What is the no-arbitrage price at date t=0 of Call B?

e) What can you say about the no-arbitrage price of an American put option on stock XYZ with expiry in 1 year and strike price €90 (i.e., the same characteristics as Put B except that it is American instead of European)?

f) What can you say about the no-arbitrage price of an American call option on stock XYZ with expiry in 1 year and strike price €90 (i.e., the same characteristics as Call B except that it is American instead of European)?

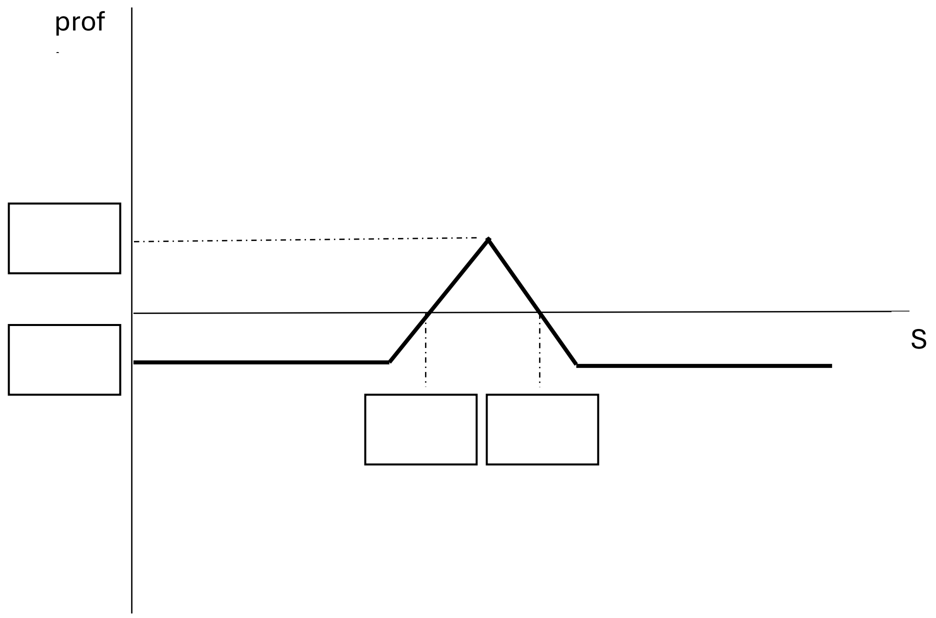
Problem 2:

Stock X does not pay dividends and trades today (t=0) at price €45 per share. The risk-free yield curve is flat at 0% per year. Consider the following European call options on stock X with expiry in 1 year (t=1):

	Strike (exercise) price	Price (premium) at t=0
Call option "C40"	€40	€14
Call option "C45"	€45	€10
Call option "C50"	€50	€7

a) Find the no-arbitrage price (premium) today (t=0) of a European put option on stock X with expiry of 1 year and strike price of 50 €

b) Find a combination of the above call options that has a net profit (that is, payoff after having fully accounted for the option premiums paid or received at t=0) at expiry (t=1) as a function of stock price of X at expiry (t=1) represented on the profit diagram:



c) Fill in the numbers on x and y axes that correspond to the combination of call options. Do this on the graph in the previous page by filling in the four boxes with the correct numbers.

d) What are you betting on if you implement the option strategy of the profit diagram of question b)? (The answer should be no longer than one line).

e) You will now assume that the price of stock X in one year (t=1) will be either €37 or €57. Using the binomial model, what is today (t=0) the no-arbitrage price (premium) of a European put option on Stock X with a strike price of €41 and a maturity of 1 year?

f) Consider an American put option with the same characteristics (same underlying asset, same maturity, same strike price) as the European put option of question e. Would you exercise this American put option today (at t=0)? You can answer (i) yes, (ii) no, or (iii) it depends/not enough information to answer. You need to justify your answer in 2 lines maximum.

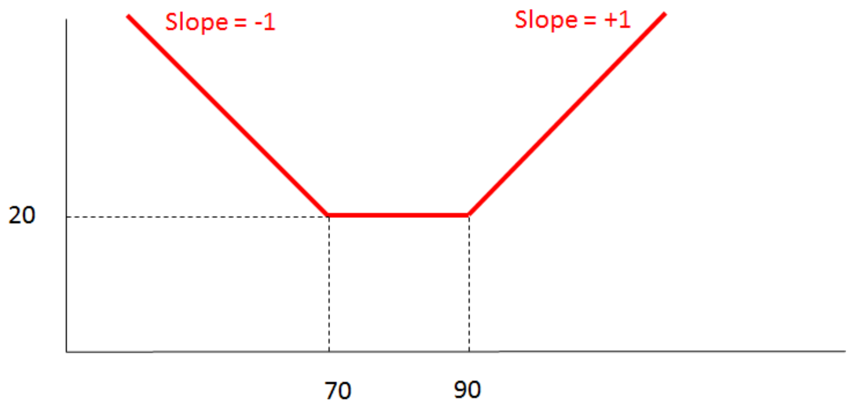
Problem Set 4.c on Options - Solutions

Problem 1:

Stock XYZ, which will not pay any dividend over the next year, currently trades at a price of $S_{XYZ}(0)$ = €80. The risk-free interest rate is 5% per year. Consider these European options on stock XYZ:

	Expiration	Strike price (exercise price)
Call A	1 year	€ 70
Put A	1 year	€ 70
Call B	1 year	€ 90
Put B	1 year	€ 90

a) Find a portfolio of options that generate the following combined cash flow at t=1 year as a function of the spot price of stock XYZ at this date:



Buy 1 Call A + Buy 1 Put B

b) Assume in this question only that, at date t=0, the price of Call A is €10 and the price of Put A is €5. Find an arbitrage strategy. [Write the arbitrage table with the positions taken at t=0 and the cash flows at t=0 and t=1 year.]

$C_0^{No\text{-}arbitrage} = P_0 + S_0 - X/(1+r_F)^1 = 5 + 80 - 70/1.05 = 18.33$ €/share > $C_0 = 10$ €/share

		CF @ T = 1 year	
Transactions @ t=0	CF @ t = 0	$S_1 < K = 70$	$S_1 > K = 70$
Buy Call @ 10 €/share	$-C_0 = 10$	0	$S_1 - K = S_1 - 70$
Sell the Synthetic Call			
Sell the Put	$+P_0 = +5$	$-(K - S_1) = S_1 - 70$	0
Short-sell the Stock	$+S_0 = +80$	$-S_1$	$-S_1$
Lend 70/1.05 @ 5%	– 70/1.05 = – 66.67	+70	+70
Replicating p/f total	+18.33	0	$70 - S_1$
Arbitrage total	+ 8.33	0	0

Alternative solution:

$P_0^{No\text{-}arbitrage} = C_0 + X/(1+r_F)^1 - S_0 = 10 + 70/1.05 - 80 = -3.33$ €/share < $P_0 = 5$ €/share

Note that $P_0^{No\text{-}arbitrage} = -3.33$ €/share < 0, i.e. outside of arbitrage bound.

		CF @ T = 1 year	
Transactions @ t=0	CF @ t = 0	$S_1 < K = 70$	$S_1 > K = 70$
Sell Put at 5 €/share	$+ P_0 = + 5$	$-(K - S_1) = S_1 - 70$	0
Buy the Synthetic Put			
Buy the Call at 10 €	$- C_0 = - 10$	0	$S_1 - 70$
Short-sell the Stock	$+ S_0 = + 80$	$- S_1$	$- S_1$
Lend 70/1.05 @ 5%	$- 70/1.05 = - 66.67$	+70	+70
Replicating p/f total	+ 3.33	$70 - S_1$	0
Arbitrage total	+ 8.33	0	0

In questions c-d-e-f) you will assume that, at t=1 year, XYZ stock will be trading at either $S_{XYZ}(1Y)$ = €60 or $S_{XYZ}(1Y)$ = €100 per share. Remember that Stock XYZ trades at a price of $S_{XYZ}(0)$ = €80 per share at t=0.

c) What is the no-arbitrage price at date t=0 of Put B?

Binomial Option Pricing with Replicating Portfolios:

$P_1^u = \text{Max}\{K - S_1^u, 0\} = \text{Max}\{90 - 100, 0\} = 0 = n_S \times S_1^u + n_B \times (1+r_F)^1 = n_S \times 100 + n_B \times 1.05$

$P_1^d = \text{Max}\{K - S_1^d, 0\} = \text{Max}\{90 - 60, 0\} = 30 = n_S \times S_1^u + n_B \times (1+r_F)^1 = n_S \times 60 + n_B \times 1.05$

Taking the difference:

$0 = + n_S \times 100 + n_B \times 1.05$

$\underline{- 30 = - n_S \times 60 - n_B \times 1.05}$

$- 30 = + n_S \times 40$ → $n_S = - 0.75$ → $n_B = + 71.43$

→ $P_0 = n_S \times S_0 + n_B \times 1 = - 0.75 \times 80 + 71.43 = 11.43$ €/share

d) What is the no-arbitrage price at date t=0 of Call B?

Put-call parity → $C_0 = P_0 + S_0 - K/(1+r) = 11.43 + 80 - 90/1.05 = 5.71$ €/share

e) What can you say about the no-arbitrage price of an American put option on stock XYZ with expiry in 1 year and strike price €90 (i.e., the same characteristics as Put B except that it is American instead of European)?

The no-arbitrage price of an American Put on stock XYZ will be greater than or equal to the price of the European Put B (€11.42), as it may be beneficial to exercise the put early rather than waiting.

f) What can you say about the no-arbitrage price of an American call option on stock XYZ with expiry in 1 year and strike price €90 (i.e., the same characteristics as Call B except that it is American instead of European)?

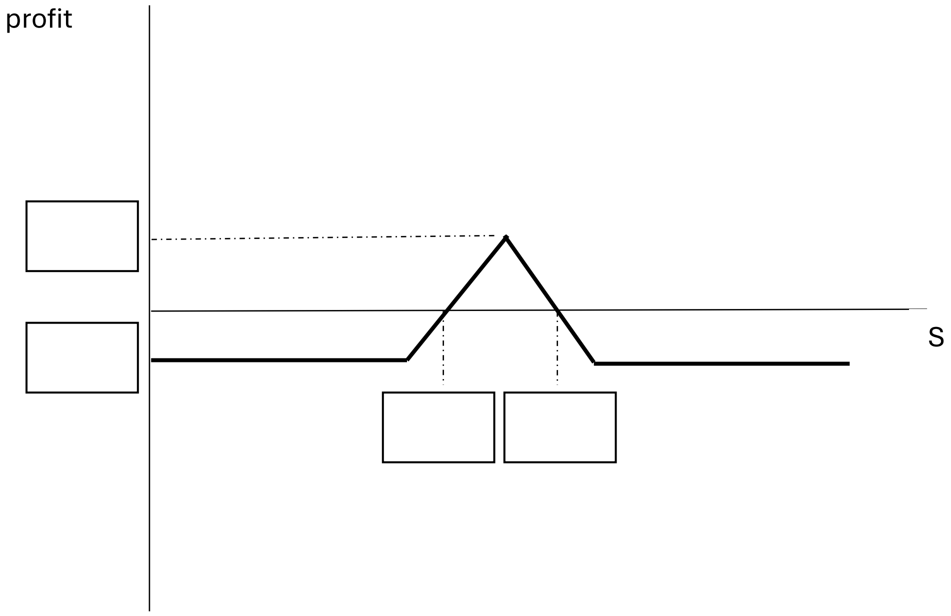
American call's price is equal to the price of European Call B (€5.71) if the underlying stock doesn't pay dividends.

<u>Problem 2</u>:

a) Find the no-arbitrage price (premium) today (t=0) of a European put option on stock X with expiry of 1 year and strike price of 50 €

$P_0 = 7 - 45 + 50 = 12$

b) Find a combination of the above call options that has a net profit (that is, payoff after having fully accounted for the option premiums paid or received at t=0) at expiry (t=1) as a function of stock price of X at expiry (t=1) represented on the profit diagram:



Long 1 Call option "C40" & Short 2 Call options "C45" & Long 1 Call option "C50"

c) Fill in the numbers on x and y axes that correspond to the combination of call options. Do this on the graph in the previous page by filling in the four boxes with the correct numbers.

Y axis top = 4; Y axis bottom = – 1 ; X axis left = 41; X axis right = 49

d) What are you betting on if you implement the option strategy of the profit diagram of question b)? (The answer should be no longer than one line).

That stock X does not change much in price in the coming year.

Alternative: That volatility is low.

Alternative: That the market over-estimates volatility.

Etc.

e) You will now assume that the price of stock X in one year (t=1) will be either €37 or €57. Using the binomial model, what is today (t=0) the no-arbitrage price (premium) of a European put option on Stock X with a strike price of €41 and a maturity of 1 year?

Solution with Binomial Option Pricing with Replicating Portfolios:

$57 \times n_s + n_b = 0$

$37 \times n_s + n_b = 4$

$n_s = -4 / 20 = -0.2$

$n_b = +11.4$

$P_0 = -0.2 \times 45 + 11.4 = 2.4$ €/share

Alternative solution using Binomial Option Pricing with state prices:

$u = S_1^u/S_0 = 57/45 = 1.2667$

$d = S_1^d/S_0 = 37/45 = 0.8222$

$$\pi_u = \frac{1 + r_F - d}{(1 + r_F)(u - d)} = \frac{1 + 0 - 0.8222}{(1 + 0)(1.2667 - 0.8222)} = 0.4$$

$$\pi_d = \frac{u - (1 + r_F)}{(1 + r_F)(u - d)} = \frac{1.2667 - (1 + 0)}{(1 + 0)(1.2667 - 0.8222)} = 0.6$$

$$P_0 = \pi_u Max\{K - S^u, 0\} + \pi_d Max\{K - S^d, 0\}$$

$$P_0 = \pi_u Max\{41 - 57{,}0\} + \pi_d Max\{41 - 37{,}0\} = 0.4 \times 0 + 0.6 \times 4 = 2.40$$

f) Consider an American put option with the same characteristics (same underlying asset, same maturity, same strike price) as the European put option of question e. Would you exercise this American put option today (at t=0)? You can answer (i) yes, (ii) no, or (iii) it depends/not enough information to answer. You need to justify your answer in 2 lines maximum.

Do not exercise, because out of the money. (Alternative justification: when interest rate is zero, the time value of a put is positive, so it is never optimal to exercise an American put before maturity.)