Market Microstructure
Auctions

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What is the common feature of the following things?

- Flowers
- Diamonds
- Artworks
- Wine
- Company subsidiaries
- Houses
- Electricity
- Treasury bills
- Common shares
- Copyrights
- Drilling rights for minerals
- UMTS licenses
- Access to railroad interconnection points

All of these are sold, or have been sold in the past, through auctions.

Why auctions?

- Auctions are used in several sectors of economic activity.
- A huge volume of economic transactions is conducted through auctions.
- Auctions provide a simple, well-defined, and intensively studied economic environment.
- The logic of competitive bidding is at the core of many financial transactions.

Core Questions in Auction Theory

- How should buyers bid in the auction? Example: What is the limit order price?
- What is the seller's expected revenue? Example: What is a market order's expected profit?
- How can we relate the selling price to:
  - Auction mechanisms (e.g., market mechanisms)
  - Number of bidders (e.g., market liquidity)
  - The amount and nature of asymmetry in information (e.g., market efficiency)
- How should the seller choose the auction mechanism? Example: Financial intermediation, market regulation, competition among markets.

The Seller's Problem

- You own a valuable good.
- You know there are $N > 1$ potential buyers for this good.
- You do not know exactly how much each potential buyer values the good:

$$Pr[\tilde{V}_i < z] = F_i(z)$$

where $\tilde{V}_i$ is potential buyer *i*'s valuation for the object.

- You want to sell:
  - At the maximum possible price
  - Quickly
  - In a transparent way

How?

Solution 1: Posted Price

The seller announces a non-negotiable price and hopes that someone willing to pay that price will appear.

- Y: Seller's valuation of the object.
- $1 - G(p)$: Probability that at least one buyer values the object at p or more.

$$\max_p (p - Y)(1 - G(p))$$

First-order condition:

$$1 - G(p) = G'(p)(p - Y)$$

Drawbacks:

- Tie-break rule: multiple buyers
- Waiting cost: no buyer

Solution 2: Auction

Definition

An **auction** is a bidding mechanism defined by a set of rules specifying:

- How the winner is determined
- How much the winner and other bidders must pay

Advantages:

- Speed of sale
- Information revelation about buyers' valuations
- Equal chances for all potential buyers
- Prevents dishonest dealings between the seller's agent and buyers

Some Standard Auction Formats

- **First-price sealed bid auction:** Bidders submit sealed bids simultaneously. The highest bidder wins and pays their bid. **Applications**: Divestitures, market-making

  competition in decentralized markets, mineral rights, telecom licenses, antiques.

- **Second-price sealed bid auction:** Bidders submit sealed bids simultaneously. The highest bidder wins and pays the second-highest bid.

Some Standard Auction Formats

- **English auction:** The auctioneer starts with a low price. Bidders successively bid higher amounts until no one is willing to bid more. The highest bidder wins and pays their bid. **Applications**: Mergers, centralized market-making

  competition, artworks, used cars, houses, radio communication licenses, Internet auctions.

- **Dutch auction:** The auctioneer starts with a high price and gradually lowers it. The first bidder to accept the price wins and pays that amount.

Some Other Auction Formats

- **Japanese auction**: The price continuously increases on a "wheel" in front of the bidders until all but one bidder leaves the room. The last bidder remaining wins the object and pays the price at which the wheel stopped.
- **E-bay auction**: Bidders submit sealed bids during a bidding period. Throughout the period, bidders observe the second-highest bid. The highest bidder wins the object and pays the second-highest bid.
- **All-Pay auction**: Bidders submit increasing bids until no one is willing to bid higher. The highest bidder wins the object and all bidders pay the amount of their last bid.
- **Uniform price auction**: Bidders submit demand functions. The good is sold at a price where demand equals supply.
- **Survival auction**: The auction consists of multiple rounds of sealed bids. At each round, the lowest bidder exits, and

Formal Description of an Auction Mechanism

1. A set N of bidders.
2. For each bidder i, a set of available actions X_i:

$$X := \times_{i \in N} X_i$$

3. Allocation rule:
   - Winning function: Probability of each bidder winning given $x \in X$:

$$Q : X \to \Delta N$$

   - Payment function: Cash transfer to each bidder given $x \in X$:

$$P : X \to \Delta \mathbb{R}^N$$

Examples

Let $x \in X$, and define:

$$x_{-i} := \{x_1, x_2, \ldots, x_{i-1}, x_{i+1}, x_N\}$$

Indicator function: $\mathbf{1}_{\{a\}}$ equals 1 if a is true and 0 otherwise.

- **First-price auction**: $X_i = \mathbb{R}^+, \quad Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -Q_i(x)x_i$.
- **Second-price auction**: $X_i = \mathbb{R}^+, \quad Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -Q_i(x)\max(x_{-i})$.
- **All-pay auction**: $X_i = \mathbb{R}^+, \quad Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -x_i$.
- **Survival auction**: $X_i = (\mathbb{R}^+)^{N-1}$, $Q_i(x) = \mathbf{1}_{\{x_i^j > \min(x^j), \forall j < N-1\}} \mathbf{1}_{\{x_i^{N-1} = \max(x^{N-1})\}}$, $P_i(x) = -Q_i(x)\max(x_{-i}^{N-1})$.

Auction as a Bayesian Game

Bidders compete in a **non-cooperative game with incomplete information.**

- There are N risk-neutral bidders.
- Let V_i be bidder i's valuation for the object.
- If bidder i wins the object and pays p, his ex-post payoff is:

$$V_i - p$$

Private Value or Common Value?

- **Private value framework**: Each bidder's valuation is independent of others.
  - Each bidder knows how much they value the object.
  - V_i does not depend on other bidders' information.
- **Common value framework**: The object has the same value for all bidders, but they may have different information about it.

$$V_i = V, \forall i$$

- **Interdependent value framework**: Each bidder's valuation depends on both private and common components, leading to correlation:

$$V_i \neq V_j, \quad \text{Cov}(V_i, V_j) \neq 0, \forall i, j$$

Independent Private Value

Assumptions:

1. Private value framework:
   - Each bidder i knows exactly V_i.
   - V_i does not depend on what the other bidders know.
2. Independently and identically distributed (i.i.d.) valuations. For any $j \neq i$, bidder j believes that:

$$\Pr[\tilde{V}_i < z] = F(z)$$

   where $\tilde{V}_i \in [0, 1]$.

3. The bidders are risk-neutral: if bidder i wins the object and pays p, then his ex-post payoff is:

$$V_i - p.$$

Probability Preliminaries

- Let $\{\tilde{V}_i\}_{i=1,\ldots,N}$ be N i.i.d. random variables with cumulative distribution $F(.)$ and density $f(.) = F'(.)$.
- Let $\tilde{V}^{(1,N)}$ and $\tilde{V}^{(2,N)}$ be the highest and the second-highest elements of $\{\tilde{V}_i\}_{i=1,\ldots,N}$.
- Let $F^{(1,N)}$ and $F^{(2,N)}$ be the cumulative distribution functions of $\tilde{V}^{(1,N)}$ and $\tilde{V}^{(2,N)}$, respectively.

Then:

$$F^{(1,N)}(z) = F(z)^N, \tag{1}$$

$$f^{(1,N)}(z) = Nf(z)F(z)^{N-1}, \tag{2}$$

$$F^{(2,N)}(z) = F(z)^N + NF(z)^{N-1}(1 - F(z)), \tag{3}$$

$$f^{(2,N)}(z) = N(N-1)f(z)F(z)^{N-2}(1 - F(z)). \tag{4}$$

Bidder's Strategies and Expected Payoffs

Fix the auction format (X, Q, P).

- If bidder i chooses action $x_i \in X_i$ and the other bidders' action profile is x_{-i}, then bidder i's payoff is:
$$V_i Q_i(x_i, x_{-i}) + P_i(x_i, x_{-i}).$$
- A bidder's (pure) strategy b_i maps a bidder's valuation V_i into an action:
$$b_i : [0, 1] \to X_i.$$
- If bidder i chooses action $x \in X_i$ and the others' strategies are b_{-i}, then bidder i's expected payoff is:
$$V_i E[Q_i(x, b_{-i}(\tilde{V}_{-i}))] + E[P_i(x, b_{-i}(\tilde{V}_{-i}))].$$

Bayesian Nash Equilibrium

Definition

A **Bayesian Nash equilibrium** specifies a bidding strategy $b_i^*(.)$ for each bidder i, such that each bidder maximizes their own expected payoff given their valuation and the other players' strategies:

$$b_i^*(V_i) \in \arg\max_{x \in X_i} V_i E[Q_i(x, b_{-i}^*(\tilde{V}_{-i}))] + E[P_i(x, b_{-i}^*(\tilde{V}_{-i}))].$$

In a symmetric framework, a symmetric equilibrium satisfies:

$$b_i^*(\cdot) = b^*(\cdot), \forall i.$$

Strategic Equivalence 1

Proposition

The Dutch auction and the first-price auction are strategically equivalent.

Proof:

1. First-price auction: $X_i = \mathbf{R}^+$, $Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -Q_i(x)x_i$.
2. Dutch auction: $X_i = \mathbf{R}^+$, $Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -Q_i(x)x_i$.
3. The information available to a bidder when placing a bid, and conditional on winning, is the same in both auctions.

An equilibrium strategy profile of the Dutch auction is an equilibrium if and only if it is an equilibrium of the first-price auction.

Strategic Equivalence 2

Proposition

Under Assumptions 1-3, the second-price sealed bid auction and the Japanese auction (as well as the survival auction) are strategically equivalent.

Proof:

1. Second-price auction: $X_i = \mathbf{R}^+$, $Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -Q_i(x)\min(x_{-i})$.
2. Japanese auction: $X_i = \mathbf{R}^+$, $Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$, $P_i(x) = -Q_i(x)\min(x_{-i})$.
3. The relevant information available to a bidder when setting a bid, and conditional on winning, is the same in both auctions.

In the independent private value framework, an equilibrium strategy profile of the Japanese auction is an equilibrium if and

Strategic Equivalence 3

Proposition

Every Bayesian Nash equilibrium of the Japanese auction induces a Bayesian Nash equilibrium of the English auction.

Proof:

1. Let $b_i^*(V_i)$ be the equilibrium exiting times in the Japanese auction for bidder i.
2. In the English auction, all bidders placing bids equal to the standing high bid plus an arbitrarily small bid increment in each round, and stopping bidding according to these exiting times, constitutes an (arbitrarily close) Bayesian Nash equilibrium of the English auction.

Equilibrium of the Second-Price Auction

- Fix bidder i.
- Let $\tilde{z} \geq 0$ be bidder i's competitors' highest bid.
- Bidder i believes that $\Pr[\tilde{z} < z] = G(z)$.
- Bidder i's expected payoff from bidding x is

$$\int_0^x (V_i - z)dG(z)$$

.

Equilibrium of the Second-Price Auction

Proposition

*(**Truth-telling equilibrium**) Under Assumptions 1-3, in the second-price sealed bid auction, bidding one's own valuation is a weakly dominant strategy for all bidders.*
The strategy profile:

$$b_i(V_i) = V_i, \forall i$$

is a Bayesian Nash equilibrium in undominated strategies of the second-price sealed bid auction.

Equilibrium of the Second Price Auction

Corollary

In the truth-telling equilibrium of the second price auction:

1. *The winner of the object is the bidder with the highest valuation.*
2. *The ex-ante expected payoff for a bidder with valuation V is:*

$$\int_0^V F(z)^{N-1} dz$$

3. *The seller's expected revenue is:*

$$E[\tilde{V}^{(2,N)}] = N \int_0^1 \left(z - \frac{1 - F(z)}{f(z)} \right) F(z)^{N-1} f(z) dz$$

Derivation of Symmetric Equilibrium

Consider an auction format where:

- $B = \mathbf{R}$,
- $Q_i(x) = \mathbf{1}_{\{x_i = \max(x)\}}$,
- $P_i(x) = P(x_i, \max(x_{-i}))$.

Consider a symmetric equilibrium such that:

- $b_i = b, \forall i$,
- $b : [0,1] \to \mathbf{R}$,
- b is increasing and differentiable.

Then, if bidder i chooses to behave like a bidder of type w, his expected payoff is:

$$\Pi(V_i, w) := V_i G(w) - \int_0^1 P(b(w), b(z)) dG(z)$$

where $G(z) = F(z)^{N-1}$.

Derivation of Symmetric Equilibrium: First Order Condition

$$\left.\frac{\partial \Pi(V, w)}{\partial w}\right|_{w=V} = 0$$

This typically provides a differential equation in $b(.)$ that can be solved by imposing the condition:

$$b(0) = 0$$

Derivation of Symmetric Equilibrium: Second Order Condition

Quasi-concavity of the objective function:

$$\frac{\partial \Pi(V, w)}{\partial w} > 0 \quad \text{for } w < V,$$
$$\frac{\partial \Pi(V, w)}{\partial w} < 0 \quad \text{for } w > V.$$

Thus, it is sufficient to show that:

$$\frac{\partial^2 \Pi(V, w)}{\partial w \partial V} > 0$$

Quasi-Concavity and Comparative Statics

Remark:
Let $x_i(V) = b(w)$ where w solves:

$$\max_{w \in [0,1]} \Pi(V, w) := VG(w) - \int_0^1 P(b(w), b(z))dG(z)$$

Let κ be a parameter of the model and suppose:

$$\frac{\partial^2 \Pi(V, w)}{\partial w \partial \kappa} > 0 \text{ (resp. } < 0)$$

Then $x_i(V)$ is an increasing (resp. decreasing) function of κ.

Symmetric equilibrium of the first price auction

$$\Pi(V,w) = \int_0^w (V - b(w))dG(z) = (V - b(w))G(w)$$

First order condition:

$$\left.\frac{\partial \Pi(V,w)}{\partial w}\right|_{w=V} = (V - b(V))g(V) - b'(V)G(V) = 0$$

Thus,

$$b(V)g(V) + b'(V)G(V) = Vg(V)$$

with $b(0) = 0$ one has

$$b(v) = \int_0^V z\frac{g(z)}{G(V)}dz = E\left[\tilde{V}^{(1,N-1)}|\tilde{V}^{(1,N-1)} \leq V\right] = V - \int_0^V \left(\frac{F(z)}{F(V)}\right)^{N-1}$$

Symmetric equilibrium of the first price auction

Second order condition:

$$\frac{\partial^2 \Pi(V, w)}{\partial x \partial V} = g(w) > 0$$

Revenue Equivalence Theorem

Theorem

Under assumptions 1-3, given any auction mechanism:
If in equilibrium:

1. *The bidder who has the highest valuation for the object is certain to win the object.*
2. *A bidder who values the object at its lowest possible level has an expected payoff of 0.*

Then:

1. *The expected profit for a bidder with valuation* V *is* $\int_0^V F(z)^{N-1}dz$.
2. *The revenue generated for the seller is the expected value of the object to the second highest evaluator:*

$$E[\tilde{V}^{(2,N)}] = N\int_0^1 \left(z - \frac{1-F(z)}{f(z)}\right) F(z)^{N-1} f(z)dz$$

.

Revenue Equivalence Theorem: Proof 1/4

Take any equilibrium and consider a bidder i of type V.
Let:

- Equilibrium probability that bidder i wins:

$$Q_i^*(V) := E\left[Q_i(b_i^*(V), b_{-i}^*(\tilde{V}_{-i}))\right]$$

- Equilibrium expected payment to bidder i:

$$P_i^*(V) := E\left[P_i(b_i^*(V), b_{-i}^*(\tilde{V}_{-i}))\right]$$

- Equilibrium expected payoff for bidder i:

$$\Pi_i(V) := VQ_i^*(V) + P_i^*(V)$$

Revenue Equivalence Theorem: Proof 2/4

Revelation principle:

1. Suppose bidder i chooses to behave as if his type was w, then his payoff would be $VQ_i^*(w) + P_i^*(w)$
2. In equilibrium, it must be that $\max_w VQ_i^*(w) + P_i^*(w) = \Pi_i(V)$
3. First order condition gives:

$$\left.\frac{\partial VQ_i^*(w) + P_i^*(w)}{\partial w}\right|_{w=V} = 0 \Rightarrow VQ^{*\prime}_i(V) + P^{*\prime}_i(V) = 0$$

.

4. Differentiating $\Pi_i(V)$:

$$\Pi_i'(V) = \underbrace{VQ^{*\prime}_i(V) + P^{*\prime}_i(V)}_{=0} + Q_i^*(V) = Q_i^*(V)$$

.

5. Hence:

$$\Pi_i(V) = \Pi_i(0) + \int_0^V Q_i^*(z)dz$$

Revenue Equivalence Theorem: Proof 3/4

The theorem's hypotheses are:

1. The bidder who has the highest valuation for the object is certain to win the object:
$$Q_i^*(V) = F(V)^{N-1}, \forall i$$
2. Bidders who value the object at its lowest possible level have an expected payoff of 0:
$$\Pi_i(0) = 0, \forall i$$

Hence:

$$\Pi_i(V) = \Pi_i(0) + \int_0^V Q_i^*(z)dz = \int_0^V F(z)^{N-1}dz$$

Revenue Equivalence Theorem: Proof 4/4

Bidder i expected payment to the seller is

$$-P_i^*(V) = VQ_i^*(V) - \Pi_i(V) = Q_i^*(V)V - \int_0^V F(z)^{n-1}dz$$

Bidder i ex-ante expected payment to the seller is

$$\begin{aligned} -\int_0^1 P_i^*(v)f(v)dv = \int_0^1 vQ_i^*(v)f(v)dv &- \int_0^1\int_0^v Q(z)f(v)dzdv \\ = \int_0^1 vQ_i^*(v)f(v)dv - \int_0^1\int_z^1 Q_i^*(z)f(v)dvdz &= \int_0^1 Q_i^*(z)zf(z)dz - \int_0^1 Q_i^*(z)(1-F(z))dz \\ = \int_0^1 Q_i^*(z)\left(z - \frac{1-F(z)}{f(z)}\right)f(z)dz &= \int_0^1 \left(z - \frac{1-F(z)}{f(z)}\right)f(z)F(z)^{N-1}dz \end{aligned}$$

Considering that there are N bidders, the seller's expected revenue is

$$N\int_0^1 \left(z - \frac{1-F(z)}{f(z)}\right)f(z)F(z)^{N-1}dz$$

Implications, Caveats, and Use of the Revenue Equivalence Theorem

- In the independent private value framework, bidders and sellers are indifferent among different auction mechanisms.
- This applies only in equilibria where the hypotheses are met. However, auctions might have other equilibria that do not satisfy the Revenue Equivalence Theorem (RET) hypothesis.
- The Revenue Equivalence Theorem can be used to derive equilibria.

Deriving Equilibria: First Price Auction

Consider an equilibrium satisfying the RET Hypothesis. Then:

1. The equilibrium probability that Bidder *i* wins: $Q_i^*(V) = F(V)^{N-1}$.
2. Bidder *i*'s equilibrium payoff: $\Pi_i(v) = VQ_i^*(V) + P_i^*(V) = \int_0^V F(z)^{N-1} dz$.
3. Bidder *i*'s expected payment: $-P_i^*(v) = VF(V)^{N-1} - \int_0^V F(z)^{N-1} dz$.
4. In an FPA: $-P_i^*(V) = b^{FPA}(V)F(V)^{N-1}$.
5. Equations 2 and 3 give:

$$b^{FPA}(V) = V - \int_0^V \left(\frac{F(z)}{F(V)}\right)^{N-1} dz = E[\tilde{V}^{(1,N-1)} | \tilde{V}^{(1,N-1)} \leq V]$$

.

Remarks:

- In a first price auction, bidders bid less than their valuation.
- When N increases to infinity, competition rises, and underbidding diminishes to zero.

Deriving Equilibria: All-Pay Auction

Consider an equilibrium satisfying the theorem hypothesis. Then:

1. Bidder *i*'s expected payment: $-P_i^*(v) = VF(V)^{N-1} - \int_0^V F(z)^{N-1}dz$.
2. In an APA: $-P_i^*(V) = b^{APA}(V)$.
3. Equations 1 and 2 give:

$$b^{APA}(V) = VF(V)^{N-1} - \int_0^V F(z)^{N-1}dz = b^{FPA}(V)F(V)^{N-1}$$

.

Remarks:

- In an all-pay auction, bidders bid less than in an FPA.
- As N increases to infinity, competition rises, the probability of winning decreases, and bids approach zero.

Reserve Price

Definition

A **reserve price**, denoted r, is the lower bound of acceptable bids.

- If r is positive, then all bidders with valuation $V < r$ will not bid. Hence:

$$
\begin{aligned}
Q_i^*(V) &= F(V)^{N-1}\mathbf{1}_{\{V \geq r\}} \\
\Pi_i(V) &= \left(\int_r^V F(z)^{N-1} dz\right)\mathbf{1}_{\{V \geq r\}}.
\end{aligned}
$$

Optimal Reserve Price

Suppose the seller values Y the object. What is the reserve price maximizing the seller's expected payoff?

1. Expected payment from bidder i of type V:

$$-P_i^*(V) = \left(VQ_i^*(V) - \int_r^V F(z)^{N-1} dz \right) \mathbf{1}_{\{V \geq r\}}$$

.

2. Seller's expected revenue:

$$N \left(\int_r^1 vQ_i^*(v) f(v) dv - \int_r^1 Q_i^*(v)(1 - F(v)) dv \right)$$

.

Summary

- Definition of standard auction formats: FPA, SPA, EA, JA, APA, SA.
- Strategic equivalences.
- Common Value vs. Private Value.
- Within PV framework:
  - Equilibrium of SPA.
  - Symmetric equilibrium.
  - Revenue Equivalence Theorem.
  - Equilibrium of FPA, APA.
  - Optimal reserve price.
  - Reserve price and entry fees.

Bidding in a FPA with reserve price

- If $V_i < r$, then do not bid.
- If $V_i \geq r$, then

$$
\begin{aligned}
-P_i^*(V) &= \sigma(V)F(z)^{N-1} = VQ_i^*(V) - \Pi_i(V) \\
Q_i^*(V) &= F(V)^{N-1} \\
\Pi_i(V) &= \int_r^V F(z)^{N-1} dz
\end{aligned}
$$

Hence

$$
\sigma(V) = V - \int_r^V \left(\frac{F(z)}{F(V)}\right)^{N-1} dz
$$

Remark: When r increases, the ex ante probability of bidding decreases but the bids of those who bid increase.

Optimal Reserve Price

Suppose the seller values Y the object.

What is the reserve price maximizing the seller's expected payoff?

1. Expected payment from bidder i of type V:

$$-P_i^*(V) = (VQ_i^*(V) - \Pi_i(V))\,\mathbf{1}_{\{V\geq r\}} = \left(VQ_i^*(V) - \int_r^V F(z)^{N-1}dz\right)\mathbf{1}_{\{V\geq r\}}$$

2. Ex-ante expected revenue from bidder i

$$\begin{aligned} -\int_0^1 P_i^*(v)f(v)dv &= \int_r^1 vQ_i^*(v)f(v)dv - \int_r^1\int_r^v F(z)^{N-1}f(v)dzdv \\ &= \int_r^1 vQ_i^*(v)f(v)dv - \int_r^1\int_z^1 F(z)^{N-1}f(v)dvdz \\ &= \int_r^1 vQ_i^*(v)f(v)dv - \int_r^1 Q_i^*(v)(1-F(v))dv \end{aligned}$$

3. Seller's expected revenue:

$$N\left(\int_r^1 vQ_i^*(v)f(v)dv - \int_r^1 Q_i^*(v)(1-F(v))dv\right)$$

Optimal Reserve Price

Seller's expected payoff

$$\Pi^S(r) = N\left(\int_r^1 vQ_i^*(v)f(v)dv - \int_r^1 Q_i^*(v)(1-F(v))dv\right) + YF(r)^N$$

First order condition:

$$NQ_i^*(r)(1-F(r)+(Y-r)f(r)) = 0 \Rightarrow 1-F(r^*) = f(r^*)(r^*-Y)$$

Observe that

- $r^* > Y$ because $\left.\frac{\partial \Pi^S(r)}{\partial r}\right|_{r=Y} = NQ_i^*(Y)(1-F(Y)) > 0$
- r^* equal the price a monopoly would post if facing a single buyer.

FPA with entry fee and reserve price

- An entry fee, denote by c, is an amount a bidder must pay in order to submit a bid.
- A reserve price, denote by r, is the lower bound of acceptable bids.

If c and/or r are positive, then there is $\underline{V} \geq 0$ such that all bidders with valuation $V < \underline{V}$ will not bid. Hence, for $V \geq \underline{V}$:

$$\begin{aligned}\Pi_i(V) &= VQ_i^*(V) + P_i^*(V) = \int_{\underline{V}}^{V} F(z)^{N-1} dz \\ -P_i^*(V) &= VF(V)^{N-1} - \int_{\underline{V}}^{V} F(z)^{N-1} dz = Q_i^*(V)\sigma(V) + c\end{aligned}$$

Thus,

$$\sigma(V) = V - \int_{\underline{V}}^{V} \left(\frac{F(z)}{F(V)}\right)^{N-1} dz - \frac{c}{F(V)^{N-1}}$$

and $\underline{V} \geq 0$ solves

$$\sigma(\underline{V}) = \underline{V} - \frac{c}{F(\underline{V})^{N-1}} = r$$

Summary

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- Strategic equivalences.
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First order stochastic dominance

Let denote with F and G the c.d.f of random variables $\tilde{x}$ and $\tilde{y}$, respectively.

Definition

c.d.f F **first order stochastically dominates** c.d.f. G, iff

$$\forall x \in \mathbb{R}, F(x) \leq G(x)$$

Theorem

Take any increasing differentiable function $\gamma : \mathbb{R} \to \mathbb{R}$, *then*

$$E[\gamma(\tilde{x})] \geq E[\gamma(\tilde{y})]$$

Hazard rate dominance

Definition

- Given a continuous differentiable c.d.f F. Let define the **hazard rate** as the function
$$\lambda_F(x) = \frac{f(x)}{1 - F(x)}.$$
- We say that F dominates G in terms of the hazard rate if for any real number x one has
$$\lambda_F(x) \leq \lambda_G(x)$$

Theorem

If F hazard rate dominates G, then F first order stochastically dominates G.

Hazard rate dominance

Definition

- Given a continuous differentiable c.d.f F. Let define the **reverse hazard rate** as the function
$$b_F(x) = \frac{f(x)}{F(x)}.$$
- We say that F dominates G in terms of the reverse hazard rate if for any real number x one has
$$b_F(x) \geq b_G(x)$$

Theorem

If F reverse hazard rate dominates G, then F first order stochastically dominates G.

Likelihood ratio dominance

Definition

The c.d.f. F is said to dominates c.d.f. G **in terms of the likelihood ratio** if for any $x < y$ one has

$$\frac{f(x)}{g(x)} \leq \frac{f(y)}{g(y)} \tag{5}$$

or equivalently $f(x)/g(x)$ is non-decreasing in x.

Theorem

If F likelihood ration dominates G, then F hazard-rate and reverse-hazard-rate dominates G.

Affiliated random variables

Let $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$.

Definition

- The *component-wise maximum* of x and y is

$$x \vee y = \{\max(x_1, y_1), \max(x_2, y_2), \ldots, \max(x_N, y_N)\}$$

- The *component-wise minimum* of x and y is

$$x \wedge y = \{\min(x_1, y_1), \min(x_2, y_2), \ldots, \min(x_N, y_N)\}$$

- Consider the random variables $\tilde{x}_1 \tilde{x}_2 \ldots, \tilde{x}_N$. Let $f : D \to \mathbb{R}^+$ be the joint density function. The variables $\tilde{x}_1, \tilde{x}_2 \ldots, \tilde{x}_N$ are said to be **affiliated** if for all $x, y \in D$

$$f(x \vee y) f(x \wedge y) \geq f(x) f(y)$$

Affiliated random variables: some properties

Proposition

Let $\tilde{x} = \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N\}$ *be affiliated random variables, then*

1. *If* $\tilde{x}$ *and* $\tilde{y}$ *are affiliated then for any* $x' \geq x$, *on has that* $F(y|x')$ *dominates* $F(y|x)$ *in terms of likelihood ratio.*
2. $E[\tilde{x}_i|\tilde{x}_j = x_j]$ *is an increasing function of* x_j.
3. *If* γ *is an increasing function from* D *to* $\mathbb{R}$, *then*

$$E[\gamma(\tilde{x})|\tilde{x}_1 \leq x_1, \tilde{x}_2 \leq x_2 \dots, \tilde{x}_N \leq x_n]$$

   Is an increasing function of $x_1, x_2, \dots, x_N$

4. *Let* $b_1(\cdot), b_2(\cdot), \dots, b_N(\cdot)$ *strictly increasing function. Then* $b_1(\tilde{x}_1), b_2(\tilde{x}_2), \dots, b_N(\tilde{x}_N)$ *are affiliated random variables.*
5. *Fix* $\tilde{x}_1$ *and let* $\tilde{x}^{(1,N)}, \tilde{x}^{(2,N)}, \tilde{x}^{(N-1,N)}$ *denote the highest, second highest and so on up to the* $(N-1)$*-th highest realization of* $\tilde{x}_2, \tilde{x}_2, \dots, \tilde{x}_N$. *Then* $\tilde{x}^{(1,N)}, \tilde{x}^{(2,N)}, \tilde{x}^{(N-1,N)}$ *are affiliated random variables.*

Milgrom Weber Econometrica (1982)

1. There are N bidders in an auction.
2. There are N random variables $\tilde{x}_1$, $\tilde{x}_2$, ..., $\tilde{x}_N$ drawn from the same interval $[0, 1]$. These random variables are affiliated.
3. Each bidder i privately observes $\tilde{x}_i$ but does not observe the realization of the other random variables.
4. A bidder actual valuation for the object is

$$v_i = u(x_i, x_{-i}) \tag{6}$$

$x_{-i} = \{\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_{i+1}, \ldots, \tilde{x}_N\}$ and $u(\cdot)$ satisfies:

- $u(\cdot)$ is bounded nondecreasing in all its arguments and twice continuously differentiable.
- $u(\cdot)$ is symmetric in the last $N-1$ components.
- $u(0, 0) = 0$

Some examples for $u(.)$

$$u(x_i, x_{-i}) = \alpha x_i + \beta \sum_{j \neq i} x_j$$

$$u(x_i, x_{-i}) = x_i^{\alpha} \left(\Pi_{j \neq i} x_j\right)^{\beta}$$

$$u(x_i, x_{-i}) = \exp[\alpha x_i] \beta \max_{j \neq i} x_j$$

with $\alpha, \beta > 0$.

My expected valuation given the highest of my competitors' type

Let

$$\tilde{Y}_1 = \max_{j \neq i} \tilde{x}_j$$

$$G(x|x_i) := Pr[\tilde{Y}_1 \leq x | \tilde{x}_i = x_i]$$

$$g(x|x_i) := \frac{\partial G(x|x_i)}{\partial x}$$

$$v(x, y) = E\left[u(\tilde{x}_i, \tilde{x}_{-i}) | \tilde{x}_i = x, \tilde{Y}_1 = y\right]$$

Symmetric equilibrium of the second price auction

Proposition

In a symmetric equilibrium of a second price auction:

$$\beta^{II}(x) = v(x, x)$$

Symmetric equilibrium of the second price auction

Proof:

Symmetric equilibrium of the second price auction

Proof:

$$\underbrace{\Pi(x,z)}_{\text{expected payoff fo a type } x \text{ bidder bidding } \beta^{II}(z)} = \int_0^z (v(x,y) - \beta^{II}(y))g(y|x)dy$$

$$= \int_0^z (v(x,y) - v(y,y))g(y|x)dy$$

Symmetric equilibrium of the second price auction

Proof:

$$\underbrace{\Pi(x,z)}_{\text{expected payoff fo a type } x \text{ bidder bidding } \beta^{II}(z)} = \int_0^z (v(x,y) - \beta^{II}(y))g(y|x)dy$$

$$= \int_0^z (v(x,y) - v(y,y))g(y|x)dy$$

f.o.c.

$$\frac{\partial \Pi(x,z)}{\partial z} = (v(x,z) - v(z,z))g(z|x)|_{z=x} = 0$$

Symmetric equilibrium of the second price auction

Proof:

$$\underbrace{\Pi(x,z)}_{\text{expected payoff fo a type } x \text{ bidder bidding } \beta^{II}(z)} = \int_0^z (v(x,y) - \beta^{II}(y))g(y|x)dy$$

$$= \int_0^z (v(x,y) - v(y,y))g(y|x)dy$$

f.o.c.

$$\frac{\partial \Pi(x,z)}{\partial z} = (v(x,z) - v(z,z))g(z|x)|_{z=x} = 0$$

s.o.c.

$$\frac{\partial^2 \Pi(x,z)}{\partial x \partial z} = \frac{\partial v(x,z)}{\partial x} > 0$$

Symmetric equilibrium of the first price auction

Proposition

In a symmetric equilibrium of a first price auction:

$$\beta^I(x) = \int_0^x v(y,y)dL(y|x)$$

where

$$L(y|x) = \exp\left(-\int_y^x \frac{g(t|t)}{G(t|t)}dt\right)$$

Symmetric equilibrium of the first price auction

Proof 1/7

Symmetric equilibrium of the first price auction

Proof 1/7

$$\overbrace{\Pi(x,z)}^{\text{expected payoff fo a type } x \text{ bidder bidding } \beta^I(z)} =$$

Symmetric equilibrium of the first price auction

Proof 1/7

$$\overbrace{\Pi(x,z)}^{\text{expected payoff fo a type } x \text{ bidder bidding } \beta^I(z)} = \int_0^z (v(x,y) - \beta^I(z))g(y|x)dy$$
$$= \int_0^z v(x,y)g(y|x)dy - \beta^I(z)G(z|x)$$

Symmetric equilibrium of the first price auction

Proof 1/7

$$\overbrace{\Pi(x,z)}^{\text{expected payoff fo a type } x \text{ bidder bidding } \beta^I(z)} = \int_0^z (v(x,y) - \beta^I(z))g(y|x)dy$$
$$= \int_0^z v(x,y)g(y|x)dy - \beta^I(z)G(z|x)$$

f.o.c.: $\frac{\partial \Pi(x,z)}{\partial z}|_{z=x} = 0$

Symmetric equilibrium of the first price auction

Proof 1/7

$$\overbrace{\Pi(x,z)}^{\text{expected payoff fo a type } x \text{ bidder bidding } \beta^I(z)} = \int_0^z (v(x,y) - \beta^I(z))g(y|x)dy$$

$$= \int_0^z v(x,y)g(y|x)dy - \beta^I(z)G(z|x)$$

f.o.c.: $\frac{\partial \Pi(x,z)}{\partial z}|_{z=x} = 0$

$$\beta^{I\prime}(x) + \beta^I(x)\frac{g(x|x)}{G(x|x)} = v(x,x)\frac{g(x|x)}{G(x|x)} \tag{7}$$

Symmetric equilibrium of the first price auction

Proof 1/7

$$\overbrace{\Pi(x,z)}^{\text{expected payoff fo a type } x \text{ bidder bidding } \beta^I(z)} = \int_0^z (v(x,y) - \beta^I(z))g(y|x)dy$$
$$= \int_0^z v(x,y)g(y|x)dy - \beta^I(z)G(z|x)$$

f.o.c.: $\frac{\partial \Pi(x,z)}{\partial z}|_{z=x} = 0$

$$\beta^{I\prime}(x) + \beta^I(x)\frac{g(x|x)}{G(x|x)} = v(x,x)\frac{g(x|x)}{G(x|x)} \qquad (7)$$

multiplying both sides of (7) by a function $\mu(x)$ satisfying $\mu'(x) = \mu(x)\frac{g(x|x)}{G(x|x)}$:

Symmetric equilibrium of the first price auction

Proof 1/7

$$\overbrace{\Pi(x,z)}^{\text{expected payoff fo a type } x \text{ bidder bidding } \beta^I(z)} = \int_0^z (v(x,y) - \beta^I(z))g(y|x)dy$$
$$= \int_0^z v(x,y)g(y|x)dy - \beta^I(z)G(z|x)$$

f.o.c.: $\frac{\partial \Pi(x,z)}{\partial z}|_{z=x} = 0$

$$\beta^{I\prime}(x) + \beta^I(x)\frac{g(x|x)}{G(x|x)} = v(x,x)\frac{g(x|x)}{G(x|x)} \qquad (7)$$

multiplying both sides of (7) by a function $\mu(x)$ satisfying $\mu'(x) = \mu(x)\frac{g(x|x)}{G(x|x)}$:

$$\mu(x)\beta^{I\prime}(x) + \beta^I(x)\mu'(x) = v(x,x)\frac{g(x|x)}{G(x|x)}\mu(x)$$

Symmetric equilibrium of the first price auction

Proof 2/7

$$\mu(x)\beta^{I\prime}(x) + \beta^{I}(x)\mu'(x) = v(x,x)\frac{g(x|x)}{G(x|x)}\mu(x)$$

Symmetric equilibrium of the first price auction

Proof 2/7

$$\mu(x)\beta^{I\prime}(x) + \beta^{I}(x)\mu'(x) = v(x,x)\frac{g(x|x)}{G(x|x)}\mu(x)$$

integrating both sides for $z \in [0, x]$

Symmetric equilibrium of the first price auction

Proof 2/7

$$\mu(x)\beta^{I\prime}(x) + \beta^{I}(x)\mu'(x) = v(x,x)\frac{g(x|x)}{G(x|x)}\mu(x)$$

integrating both sides for $z \in [0, x]$

$$\left[\mu(z)\beta^{I}(z)\right]_0^x = \int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\mu(y)dy$$

Symmetric equilibrium of the first price auction

Proof 2/7

$$\mu(x)\beta^{I\prime}(x) + \beta^{I}(x)\mu'(x) = v(x,x)\frac{g(x|x)}{G(x|x)}\mu(x)$$

integrating both sides for $z \in [0,x]$

$$\left[\mu(z)\beta^{I}(z)\right]_0^x = \int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\mu(y)dy$$

using $\beta^{I}(0) = 0$

Symmetric equilibrium of the first price auction

Proof 2/7

$$\mu(x)\beta^{I\prime}(x) + \beta^{I}(x)\mu'(x) = v(x,x)\frac{g(x|x)}{G(x|x)}\mu(x)$$

integrating both sides for $z \in [0, x]$

$$\left[\mu(z)\beta^{I}(z)\right]_0^x = \int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\mu(y)dy$$

using $\beta^{I}(0) = 0$

$$\beta^{I}(x) = \frac{\int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\mu(y)dy}{\mu(x)}$$

Symmetric equilibrium of the first price auction

Proof 3/7

$$\beta^I(x) = \frac{\int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\mu(y)dy}{\mu(x)}$$

Symmetric equilibrium of the first price auction

Proof 3/7

$$\beta^I(x) = \frac{\int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\mu(y)dy}{\mu(x)}$$

Note that

$$\mu(x) = \mu(0)\exp\left(\int_0^x \frac{g(z|z)}{G(y|z)}dz\right) \Rightarrow \mu'(x) = \mu(x)\frac{g(x|x)}{G(x|x)}$$

Symmetric equilibrium of the first price auction

Proof 3/7

$$\beta^I(x) = \frac{\int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\mu(y)dy}{\mu(x)}$$

Note that

$$\mu(x) = \mu(0)\exp\left(\int_0^x \frac{g(z|z)}{G(y|z)}dz\right) \Rightarrow \mu'(x) = \mu(x)\frac{g(x|x)}{G(x|x)}$$

$$\beta^I(x) = \frac{\int_0^x v(y,y) \overbrace{\frac{g(y|y)}{G(y|y)}\mu(y)}^{\mu'(y)=\mu(0)\frac{g(y|y)}{G(y|y)}\exp\left(\int_0^y \frac{g(z|z)}{G(y|z)}dz\right)} dy}{\underbrace{\mu(x)}_{\mu(0)\exp\left(\int_0^x \frac{g(z|z)}{G(y|z)}dz\right)}} =$$

$$= \int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\exp\left(-\int_y^x \frac{g(z|z)}{G(z|z)}dz\right)dy$$

Symmetric equilibrium of the first price auction

Proof 4/7

$$\beta^I(x) = \int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}\exp\left(-\int_y^x \frac{g(z|z)}{G(z|z)}dz\right)dy$$

We set

$$L(y|x) = \exp\left(-\int_y^x \frac{g(t|t)}{G(t|t)}dt\right)$$

$$\Rightarrow \beta^I(x) = \int_0^x v(y,y)dL(y|x)$$

Symmetric equilibrium of the first price auction

Proof 5/7

Second order condition:

$$\left.\frac{\partial \Pi(x,z)}{\partial z}\right|_{z<x} > 0$$

$$\left.\frac{\partial \Pi(x,z)}{\partial z}\right|_{z>x} < 0$$

$$\frac{\partial \Pi(x,z)}{\partial z} = G(z|x)\left[(v(x,z)-\beta^I(z))\frac{g(z|x)}{G(z|x)} - \beta^{I\prime}(z)\right]$$

Symmetric equilibrium of the first price auction

Proof 6/7

Second order condition:

$$\left.\frac{\partial \Pi(z,x)}{\partial z}\right|_{z<x} > 0$$

$$\frac{\partial \Pi(z,x)}{\partial z} = G(z|x)\left[\underbrace{(v(x,z)}_{>v(z,z)} - \beta^{I}(z))\underbrace{\frac{g(z|x)}{G(z|x)}}_{>\frac{g(z|z)}{G(z|z)}}\beta^{I\prime}(z)\right]$$

$$> G(z|x)\underbrace{\left[(v(z,z) - \beta^{I}(z))\frac{g(z|z)}{G(z|z)} - \beta^{I\prime}(z)\right]}_{=0,\text{ because of f.o.c.}}$$

Symmetric equilibrium of the first price auction

Proof 7/7

Second order condition:

$$\left.\frac{\partial \Pi(z,x)}{\partial z}\right|_{z>x} < 0$$

$$\frac{\partial \Pi(z,x)}{\partial z} = G(z|x)\left[\underbrace{(v(x,z)}_{<v(z,z)} - \beta^I(z))\underbrace{\frac{g(z|x)}{G(z|x)}}_{<\frac{g(z|z)}{G(z|z)}}\beta^{I\prime}(z)\right]$$

$$< G(z|x)\underbrace{\left[(v(z,z)-\beta^I(z))\frac{g(z|z)}{G(z|z)} - \beta^{I\prime}(z)\right]}_{=0}$$

Japanese auction

1. All bidders are in the same room.
2. The auctioneer starts with a price of 0 and gradually and continuously increases the price.
3. When a bidder deems that the price reached a level that is too high for him/her, he or she exits the room.
4. Bidders who exit are not allow to come back in the room.
5. As soon as there is only one remaining bidder in the room, the auctioneer stops increasing the price, the bidder left is the winner and pays that price.

Symmetric equilibrium of the English and the Japanese auction

Preliminaries: Let

$$\begin{aligned}
J(x, x) &:= u(x, x, \ldots, x) \\
J(x, (x, x_1)) &:= u(x, x, \ldots, x, x_1) \\
J(x, (x, x_1, x_2)) &:= u(x, x, \ldots, x, x_1, x_2) \\
&\ldots \\
J(x, (x, x_1, x_2, \ldots, x_m)) &:= u(x, x, \ldots, x, x_1, x_2, \ldots, x_m)
\end{aligned}$$

Remark: $J(\cdot)$ is strictly increasing. In particular it is invertible in x.
Let $x_m(p, x_1, x_2, \cdots, x_{m-1})$ be the x such that

$$J(x, (x, x_1, x_2, \ldots, x_m)) = p$$

Symmetric equilibrium of the English and the Japanese auction

Proposition

The following strategy form an equilibrium of the Japanese auction: For bidder of type x

- *As long as no bidder exits, stay until the price reaches* $J(x, x)$*, and then exit.*

Symmetric equilibrium of the English and the Japanese auction

Proposition

The following strategy form an equilibrium of the Japanese auction: For bidder of type x

- *As long as no bidder exits, stay until the price reaches* $J(x, x)$*, and then exit.*
- *If the first bidder exited at price* p_1*, stay until the price reaches* $J(x, (x, x[1]))$*, and then exit. Where* $J(x[1], x[1]) = p$*.*

Symmetric equilibrium of the English and the Japanese auction

Proposition

The following strategy form an equilibrium of the Japanese auction: For bidder of type x

- *As long as no bidder exits, stay until the price reaches* $J(x, x)$*, and then exit.*
- *If the first bidder exited at price* p_1*, stay until the price reaches* $J(x, (x, x[1]))$*, and then exit. Where* $J(x[1], x[1]) = p$*.*
- *If the first bidder exited at price* p_1 *and the second at price* p_2*, stay until the price reaches* $J(x, (x, x[1], x[2]))$*, and then exit. Where* $J(x'', (x'', x')) = p_2$*.*

Symmetric equilibrium of the English and the Japanese auction

Proposition

The following strategy form an equilibrium of the Japanese auction: For bidder of type x

- *As long as no bidder exits, stay until the price reaches* $J(x, x)$*, and then exit.*
- *If the first bidder exited at price* p_1*, stay until the price reaches* $J(x, (x, x[1]))$*, and then exit. Where* $J(x[1], x[1]) = p$*.*
- *If the first bidder exited at price* p_1 *and the second at price* p_2*, stay until the price reaches* $J(x, (x, x[1], x[2]))$*, and then exit. Where* $J(x'', (x'', x') = p_2$*.*
- ...
- *When there only are two bidders and the other exited at time* $p_1, p_2, \ldots p_{N-2}$*, stay until the price reaches* $J(x, (x, x[1], \ldots, x[N_2])$

Proof:

- If all other bidders follow this strategy bidder i can deuce the type of exiting bidders. That is $x[m] = \tilde{x}_m$ i.e. the signal of the m-th bidder to exit.
- If all bidders follows this strategy, the highest type wins and get a payoff of

$$u(x, x_N, x_{N-1}, \dots x_2, x_1) - \underbrace{u(x_{N-1}, x_N, x_{N-1}, \dots x_2, x_1)}_{\text{Price the winners pays}} \quad \underbrace{> 0}_{\text{Because } x \geq X_{N-1}} \tag{8}$$

- If a bidder deviates either
  - Has not the highest type and wins: and get a payoff

$$u(x, x_N, x_{N-1}, \dots x_2, x_1) - \underbrace{u(x_{N-1}, x_N, x_{N-1}, \dots x_2, x_1)}_{\text{Price the winners pays}} \quad \underbrace{< 0}_{\text{Because } x < X_{N-1}}$$

  - Has not the highest type and does not win: and get a payoff 0.
  - Has the highest type and wins: and get a payoff (8)
  - Has the highest type and does not win: and get a payoff 0