Market Microstructure
Quote Driven Markets

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- **Traders**
  - **Customers**
    - Institutional Investors:
    - Individual Investors:
  - **Dealers**
- **Intermediaries**
  - **Brokers**: Match customer orders when possible; otherwise, they transmit traders' orders to the market. Brokers do not trade for their own accounts.
  - **Specialists:** (NYSE) Agents responsible for providing liquidity and maintaining orderly trading in designated securities.
  - **Market Makers:** Agents who continuously buy and sell securities at their bid and ask prices, supplying liquidity.

- **Order-Driven Markets** (or Pure Auction Markets): Investors are represented by brokers and trade directly without market-maker intermediation. (e.g., Island, Paris Bourse)
  - Call Auction Markets: Trading occurs at specific times (e.g., at market open or fixing). Investors place orders (prices and quantities), which execute at a single clearing price that maximizes trade volume.
  - Continuous Auction Markets: Investors trade against resting orders submitted earlier by other investors (or floor brokers, if applicable). (e.g., Euronext, Toronto Stock Exchange, ECNs)
- **Quote-Driven Markets**
- **Hybrid Markets**

Types of Markets

- **Order-Driven Markets**
  - Call Auction Markets
  - Continuous Auction Markets

- **Quote-Driven Markets**: Dealers post bid and ask quotes at which public investors can trade (e.g., Bond Markets MTS, FX markets, London Stock Exchange). In this case, dealers are also called market makers.

- **Hybrid Markets**: Call auction markets open to dealers and specialists (e.g., NYSE). Dealer markets where traders' limit orders are also posted (e.g., Nasdaq).

Quote-Driven Market

MB Trader ELTR

Name	Bid	Size	Name	Ask	Size
MLCO	18 1/4	10	LEHM	18 3/8	20
GSCO	18 1/4	20	MASH	18 1/2	10
PRUS	18 1/4	30	AGED	18 5/8	17
BEST	18 1/8	10	SHWD	18 5/8	10
TSCO	18 1/16	10	TSCO	18 3/4	50

- **Bid Price**: The price at which a market maker is willing to buy a given quantity of the asset.
- **Ask Price**: The price at which a market maker is willing to sell a given quantity of the asset.
- **Inside Spread**: The difference between the lowest ask price and the highest bid price.
- **Market Depth**: The maximum volume of trade available at the same price.
- **Tick Size**: The minimum allowable price increment for quotes.
- **Round Lot**: The standard trading unit.

Key Questions Addressed in This Course

- What factors influence price levels?
- Why do bid and ask prices differ?
- What determines the inside spread?
- What affects market depth?
- How can informed speculators best exploit their private information?
- What information is embedded in trading prices?
- What information is contained in trading volume?

Normal-CARA Mathematical Preliminaries

Let $\tilde{x} \sim N(m_x, \sigma_x^2)$ and $\tilde{y} \sim N(m_y, \sigma_y^2)$ with $Cov(\tilde{x}, \tilde{y}) = \sigma_{xy}$.
Then:

$$E\left[\tilde{x}|\tilde{y} = y\right] = m_x + (y - m_y)\frac{\sigma_{xy}}{\sigma_y^2} \quad (1)$$

If $\tilde{z}$ follows $f(\tilde{z}|x) \sim N(x, \sigma_z^2)$, then:

$$f(\tilde{x}|\tilde{z} = z) \sim N\left(\frac{\frac{m_x}{\sigma_x^2} + \frac{z}{\sigma_z^2}}{\frac{1}{\sigma_x^2} + \frac{1}{\sigma_z^2}}, \frac{1}{\frac{1}{\sigma_x^2} + \frac{1}{\sigma_z^2}}\right) \quad (2)$$

For a Constant Absolute Risk Aversion (CARA) utility function :

$$U(w) = -e^{-\gamma w} \quad (3)$$

where γ is the risk aversion coefficient.
Then:

$$E\left[U(x)\right] = E\left[-e^{-\gamma\tilde{x}}\right] = -e^{-\gamma\left(m_x - \gamma\frac{\sigma_x^2}{2}\right)} \quad (4)$$

- Inventory Model
- Informed Traders
  - Non-strategic informed traders (Glosten and Milgrom (1985))
  - Strategic informed traders (Kyle (1985))
- Informed Market Makers

A Simple Model of Quote-Driven Markets

- We consider a market where a risky asset (security) is exchanged for a risk-free asset (money). The fundamental value of the risky asset is represented by the random variable $\tilde{v}$.

- **Trading Rules**:

  1. Market makers simultaneously post bid and ask prices at which they are willing to buy or sell, respectively, a predefined quantity q of the security.

  2. Traders decide whether to buy or sell the risky asset (submit market orders):

     - If they want to sell, they sell q shares of the risky asset to the market maker posting the highest bid price.
     - If they want to buy, they buy q shares of the risky asset from the market maker posting the lowest ask price.

Price competition among market makers can be modeled as two first-price sealed-bid auctions within a common value framework:

- Bid prices result from bidding strategies in a first-price auction to buy the risky asset.
- Ask prices result from bidding strategies in a first-price procurement auction to sell the risky asset.

Benchmark: Risk-Neutral Market Makers with No Information Asymmetry

Lemma

If market makers are risk-neutral, then:

Ask price = Bid price = $E[\tilde{v}]$

Proof: Bertrand competition.

Implications: When market makers are risk-neutral and share a common distribution for $\tilde{v}$, the inside spread is zero, minimizing transaction costs.

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The Economy

- **Fundamental Value of the Asset:** $\tilde{v} \sim N(V, \sigma^2)$
- **Liquidity Traders:** Experience liquidity shocks requiring them to either sell q units of the risky asset or buy q units.
- **Market Makers:** Agents responsible for clearing traders' market orders. The post-trade utility function for market maker i is:

$$u(C_i + I_i\tilde{v} - q(P - \tilde{v}))$$

  - CARA utility function: $u(w) = -e^{-\gamma w}$
  - Market maker i's cash endowment: C_i
  - Market maker i's initial inventory: I_i
  - Trading price: P
  - Traded quantity: q

Definition

- The **bid reservation quote** rb_i for market maker i is the maximum price they are willing to pay for q additional units of the risky asset:

$$E[u(C_i + I_i\tilde{v})] = E[u(C_i + I_i\tilde{v} - q(rb_i - \tilde{v}))]$$

- The **ask reservation quote** ra_i for market maker i is the minimum price at which they are willing to sell q units of the risky asset:

$$E[u(C_i + I_i\tilde{v})] = E[u(C_i + I_i\tilde{v} + q(ra_i - \tilde{v}))]$$

Lemma

Given the CARA normal setup:

$$rb_i = V - \frac{\gamma\sigma^2}{2}(q + 2I_i)$$
$$ra_i = V + \frac{\gamma\sigma^2}{2}(q - 2I_i)$$

- *Reservation quotes increase with V.*
- *Reservation quotes decrease with I_i.*
- $rb_i < ra_i$.
- $ra_i - rb_i = \gamma\sigma^2 q$ *increases with γ and σ.*

Proof: Recall that for $\tilde{x} \sim N(m_x, \sigma_x^2)$, we have:

$$E[U(x)] = E\left[-e^{-\gamma\tilde{x}}\right] = -e^{-\gamma\left(m_x - \gamma\frac{\sigma_x^2}{2}\right)}$$

Fragmented / Anonymous Markets

- Each market maker (MM) knows their own initial portfolio (C_i, I_i) but not those of others.

- MMs' inventories are independently and identically distributed (i.i.d.).

Computing the Optimal Quoting Strategy

$$\begin{aligned} W_i(0) &:= C_i + I_i \tilde{v} \\ W_i(b) &:= W_i(0) + q(\tilde{v} - b). \end{aligned}$$

Note that:

$$\begin{aligned} E[u(W_i(b))] &= E[u(W_i(0))e^{-\gamma(rb_i - b)q}] \\ &\simeq E[u(W_i(0))] + E[u(W_i(0))]\gamma q(b - rb_i) \end{aligned}$$

In the bid auction, MM i sets b_i^* such that:

$$b_i^* \in \arg\max_b (E[u(W_i(b))] - E[u(W_i(0))]) \Pr(b_{-i} < b_i)$$

$$\simeq \arg\max_b (rb_i - b) \Pr(b_{-i} < b_i)$$

$$b_i^* \simeq E[\tilde{rb}_{-i}^{(1)} | \tilde{rb}_{-i}^{(1)} \leq rb_i]$$

Note that rb_i is decreasing in I_i.

Lemma

In a symmetric equilibrium, MM i sets their bid and ask quotes as:

$$b(I_i) \simeq E[rb_i(\tilde{I}_{-i}^{(l)})|\tilde{I}_{-i}^{(l)} > I_i]$$
$$a(I_i) \simeq E[ra_i(\tilde{I}_{-i}^{(h)})|\tilde{I}_{-i}^{(h)} < I_i]$$

where $\tilde{I}_{-i}^{(l)}$ *and* $\tilde{I}_{-i}^{(h)}$ *are the lowest and highest inventories among MM i's competitors, respectively.*

Empirical Implications

- The MM who wins the bid auction has the smallest inventory.
- The MM who wins the ask auction has the largest inventory.
- The larger an MM's inventory, the lower their bid and ask quotes.
- Transaction costs, measured as the inside spread:
  - Decrease with the number of MMs.
  - Increase with MMs' risk aversion γ.
  - Increase with the intrinsic asset risk σ^2.

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  - Strategic informed traders (Kyle (1985))
- Informed Intermediaries
  - Informed brokers
  - Informed market makers

Informed Non-Strategic Traders (Glosten and Milgrom JFE 1985)

The Economy: Agents from three groups trade the security for money:

- **Risk-Neutral Market Makers**: Provide liquidity to the market by trading the risky asset with other agents at their bid and ask prices.
- **Risk-Neutral Informed Speculators (Informed Traders)**: Risk-neutral traders with some private information about the liquidation value of the risky asset.
- **Liquidity (or Noise) Traders**: Participate in the market for reasons other than speculation (e.g., liquidity needs, portfolio hedging, etc.); their excess demand is exogenous and random.

The trader population consists of a proportion μ of informed traders and $(1-\mu)/2$ buyer liquidity traders and $(1-\mu)/2$ seller liquidity traders.

- **Security Fundamental Value:**

$$\tilde{v} = \tilde{V} + \tilde{\varepsilon}$$

where $\tilde{V} \in \{V_1, V_2\}$, $\Pr(\tilde{V} = V_2) = \pi$, $V_1 < V_2$, $E\left[\tilde{\varepsilon}|\tilde{V}\right] = 0$, and $Var(\tilde{\varepsilon}|\tilde{V}) \geq 0$.

- **Informed Traders' Information:**
Each informed trader receives a private signal $\tilde{s} \in \{l, h\}$, with

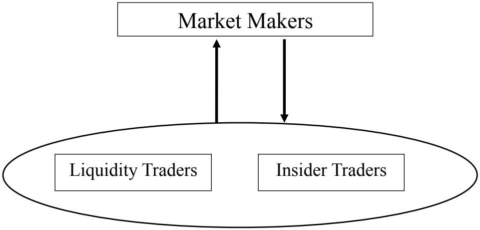
$$\Pr(\tilde{s} = l|V_1) = \Pr(\tilde{s} = h|V_2) = r \in \left(\frac{1}{2}, 1\right)$$

Signals are independent across informed traders and uncorrelated with $\tilde{\varepsilon}$.

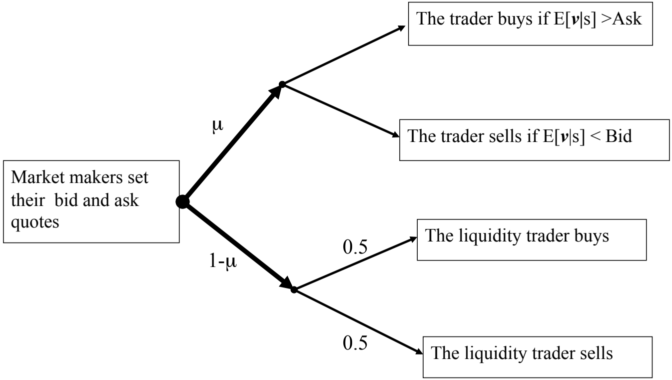
$$V_1 \leq E\left[\tilde{v}|l\right] < E\left[\tilde{v}\right] < E\left[\tilde{v}|h\right] \leq V_2$$

- Market makers have no private information about $\tilde{v}$.

1. Market makers submit bid and ask prices.
2. Informed traders and liquidity traders submit anonymous orders.



Trading Mechanism



Equilibrium

Theorem

In equilibrium, all market makers' bid and ask prices satisfy:

$$b^* = E\left[\tilde{v} | \textit{trader sells at } b^*\right] = V_1 + \pi^S(\pi, \mu, r)(V_2 - V_1) \quad (5)$$

$$a^* = E\left[\tilde{v} | \textit{trader buys at } a^*\right] = V_1 + \pi^B(\pi, \mu, r)(V_2 - V_1) \quad (6)$$

where

$$\pi^S(\pi, \mu, r) = \frac{(\mu(1-r) + (1-\mu)/2)\pi}{(1-\mu)/2 + \mu((1-r)\pi + r(1-\pi))}$$

$$\pi^B(\pi, \mu, r) = \frac{(\mu r + (1-\mu)/2)\pi}{(1-\mu)/2 + \mu(r\pi + (1-r)(1-\pi))}$$

Empirical Implications

- If $\mu > 0$ and $r > 1/2$, then the bid-ask spread is positive despite market makers being risk-neutral.
- The bid-ask spread increases with μ, the proportion of informed traders in the market.
- The bid-ask spread increases with r, the quality of informed traders' signals.
- The bid-ask spread increases with the relevance of traders' information, as measured by:

$$Var[\tilde{V}] = \pi(1-\pi)(V_2 - V_1)^2$$

A Generalization of Glosten and Milgrom

The fundamental value of the asset is given by: $\tilde{v} = \tilde{V} + \tilde{\varepsilon}$, where

$\tilde{V} \in [V_1, V_2]$ with density $f(.)$, and $E[\tilde{\varepsilon}|\tilde{V}] = 0$, with $Var(\tilde{\varepsilon}) \geq 0$

- Risk-neutral market makers.
- Informed, risk-averse investors who buy with probability $D(a, V)$ and sell with probability $S(b, V)$.
  - $D(.)$ is continuous and increasing in V and decreasing in a.
  - $S(.)$ is continuous and decreasing in V and increasing in b.

Market maker's expected payoff:

$$W^b_{MM}(b) := E\left[(\tilde{V} - b)S(b, \tilde{V})\right]$$

Bertrand competition should lead MMs to set b such that

$$E\left[(\tilde{V} - b)S(b, \tilde{V})\right] = 0$$

- Do we know this equation has a solution in b?
- What if there is more than one solution?

Existence of zero-profit equilibrium

- Supposes that there exists $\delta > 0$, such that $S(V-\delta, V) > 0$, Then

$$E\left[(\tilde{V}-b)S(b,\tilde{V})\right]\Big|_{b=V_1-\delta} > 0$$

- Note that

$$E\left[(\tilde{V}-b)S(b,\tilde{V})\right]\Big|_{b=E[\tilde{V}]} =$$

$$= \left(\int_{V_1}^{V_2} vS(E[\tilde{V}],v)f(v)dv - E[\tilde{V}]\int_{V_1}^{V_2} S(E[\tilde{V}],v)f(v)dv\right) < 0$$

because

$$E[\tilde{V}] = \int_{V_1}^{V_2} vf(v)dv > \int_{V_1}^{V_2} vf(v)\left(\frac{S(E[\tilde{V}],v)}{\int_{V_1}^{V_2} f(z)S(E[\tilde{V}],z)dz}\right)dv$$

- Hence $E\left[(\tilde{V}-b)S(b,\tilde{V})\right]$ is nil for some $b \in (V_1-\delta, E[\tilde{V}])$

Generalized GM: Equilibrium

Theorem

An equilibrium always exists. Market makers' expected profit is zero, and bid and ask prices satisfy:

$$b < E[\tilde{V}] < a$$

The spread $a - b$ *increases with* $\frac{\partial(D(a,V)-S(b,V))}{\partial V}$.

Generalized GM: Example

- Investors have a CARA utility function with risk aversion parameter γ.
- Investors hold an initial inventory I of the risky asset.
- The cumulative distribution of investor inventories is denoted by: $G(I) = \Pr(\text{Investor } i\text{'s inventory is less than } I)$.
- The fundamental value of the asset follows: $\tilde{v} = \tilde{V} + \tilde{\varepsilon}$, where

$$\tilde{V} \sim N(V, \sigma_V^2)$$
$$\tilde{\varepsilon} \sim N(0, \sigma_\varepsilon^2)$$

- Investors observe the realization of $\tilde{V}$, but not the realization of $\tilde{\varepsilon}$.

Exercise: Show that:

$$D(a, \tilde{V}) = G\left(\frac{\tilde{V} - a}{\gamma\sigma_\varepsilon^2} - \frac{1}{2}\right)$$

$$S(b, \tilde{V}) = 1 - G\left(\frac{\tilde{V} - b}{\gamma\sigma_\varepsilon^2} + \frac{1}{2}\right)$$

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- Informed Market Makers

The Economy:

Security liquidation value:

$$\tilde{v} \sim N(w, \sigma^2)$$

Three types of agents trade the security for money:

- **Noise traders**: Trade for liquidity reasons. Their aggregate market order is a random variable $\tilde{m}$, independent of $\tilde{v}$, with

$$\tilde{m} \sim N(0, \sigma_m^2)$$

- **A single informed speculator**: A risk-neutral trader who privately knows $\tilde{v}$ but not $\tilde{m}$.
- **Risk-neutral dealers**: Provide liquidity to the market. They do not observe $\tilde{v}$ or $\tilde{m}$.

Trading Protocol

1. Liquidity traders and the informed trader simultaneously choose the quantity they want to trade.
   No restrictions on trade volume.
   - m: Quantity traded by liquidity traders.
   - x: Quantity traded by the informed trader.

2. Dealers observe the aggregate order $q = x + m$ and set a price p at which they clear the market.

Informed trader's payoff:

$$\Pi_I = x(\tilde{v} - p)$$

Dealers' payoff:

$$\Pi_D = (x + m)(p - \tilde{v})$$

Equilibrium Requirements

1. Competition among dealers forces p to be set such that they earn zero expected profit:

$$E[\Pi_D] = E\left[(x+m)(p-\tilde{v})\middle|\, x+m\right] = 0 \Rightarrow$$

$$p^*(x+m) = E[\tilde{v}|x+m]$$

Remark: The conditional expectation depends on the informed trader's strategy $x^*(\cdot)$.

2. The informed trader chooses the quantity x^* to maximize expected profit:

$$x^*(v) \in \arg\max_x E\left[x(v - p^*(x+\tilde{m}))\right]$$

Remark: The optimal x^* depends on the dealers' pricing strategy $p^*(.)$.

Linear Equilibrium

Theorem

There exists a linear equilibrium where

$$x^*(v) = (v - w)\beta \quad (7)$$

$$p^*(x + m) = w + (x + m)\lambda \quad (8)$$

with $\beta := \sqrt{\frac{\sigma_m^2}{\sigma^2}}$ *and* $\lambda := \frac{1}{2}\sqrt{\frac{\sigma^2}{\sigma_m^2}}$.
The informed trader's ex-ante expected payoff is:

$$E[\Pi_I] = \frac{\sigma_m \sigma}{2}$$

The ex-ante transaction cost for trading z *is:*

$$z(\tilde{v} - E[\tilde{p}|z]) = \lambda z^2$$

- The informed trader's strategy becomes more (less) aggressive as σ_m^2 (resp. σ^2) increases.
- Price sensitivity to trading volume decreases (increases) as σ_m^2 (resp. σ^2) increases.
- The informed trader's profit increases with the amount of noise trading σ_m^2 and the precision of their signal σ^2.
- Transaction costs decrease with noise trading σ_m^2 and increase with the precision of the informed trader's signal σ^2.

Proof 1/2

- $\tilde{x} + \tilde{m} =$

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- $\tilde{x} + \tilde{m} = x^*(\tilde{v}) + \tilde{m} = \beta(\tilde{v} - w) + \tilde{m} : N(0, \beta^2\sigma^2 + \sigma_m^2)$

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- $\tilde{x} + \tilde{m} = x^*(\tilde{v}) + \tilde{m} = \beta(\tilde{v} - w) + \tilde{m} : N(0, \beta^2\sigma^2 + \sigma_m^2)$
- $Cov(\tilde{v}, \tilde{x} + \tilde{m}) =$

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- $Cov(\tilde{v}, \tilde{x} + \tilde{m}) = \beta\sigma^2$.
- Recall that: if $\tilde{y} \sim N(m_y, \sigma_y^2)$, $\tilde{z} \sim N(m_z, \sigma_z^2)$, and $Cov(\tilde{y}, \tilde{z}) = \sigma_{yz}$, then

$$E\left[\tilde{y}|\tilde{z} = z\right] = m_y + (z - m_z)\frac{\sigma_{yz}}{\sigma_z^2}$$

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1 Given the informed trader's strategy $x^*(\tilde{v}) = \beta(\tilde{v} - w)$, the dealer's price is:

$$p^*(x + m) \quad =$$

- $\tilde{x}+\tilde{m}=x^*(\tilde{v})+\tilde{m}=\beta(\tilde{v}-w)+\tilde{m}:N(0,\beta^2\sigma^2+\sigma_m^2)$
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1 Given the informed trader's strategy $x^*(\tilde{v}) = \beta(\tilde{v} - w)$, the dealer's price is:

$$\begin{aligned} p^*(x+m) &= E[\tilde{v} | \tilde{x} + \tilde{m} = x + m] \\ &= w + (x+m)\underbrace{\left(\frac{\beta\sigma^2}{\beta^2\sigma^2 + \sigma_m^2}\right)}_{\lambda} \end{aligned}$$

Proof 1/2

- $\tilde{x} + \tilde{m} = x^*(\tilde{v}) + \tilde{m} = \beta(\tilde{v} - w) + \tilde{m} : N(0, \beta^2\sigma^2 + \sigma_m^2)$
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1. The informed trader's maximization problem, given $p^*(x+m) = w + \lambda(x+m)$:

$$\max_x \quad E\left[x(v - (w + \lambda(x + \tilde{m})))\right] = \max_x x(v-w) - \lambda x^2$$

$$\Rightarrow \quad x^*(v) =$$

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$$\Rightarrow \quad x^*(v) = \underbrace{\left(\frac{1}{2\lambda}\right)}_{\beta}(v-w) = \beta(v-w)$$

2.

$$\lambda = \frac{\beta\sigma^2}{\beta^2\sigma^2 + \sigma_m^2}$$

$$\beta = \frac{1}{2\lambda}$$

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$$\Rightarrow \quad x^*(v) = \underbrace{\left(\frac{1}{2\lambda}\right)}_{\beta}(v-w) = \beta(v-w)$$

2.

$$\lambda = \frac{\beta\sigma^2}{\beta^2\sigma^2 + \sigma_m^2}$$
$$\beta = \frac{1}{2\lambda}$$

which is satisfied for:

$$\beta = \sqrt{\frac{\sigma_m^2}{\sigma^2}}$$

and

$$\lambda = \frac{1}{2}\sqrt{\frac{\sigma^2}{\sigma_m^2}}$$

Main Implications from Glosten & Milgrom and Kyle

1. Uninformed dealers compete in quotes to attract traders' market orders.

2. Traders can be either informed or uninformed about $\tilde{V}$.

3. Semi-strong form informational efficiency: The transaction price at which an order x is executed satisfies:

$$p(x) = E[\tilde{V}|x]$$

$$E[p(\tilde{x})] = E[\tilde{V}]$$

4. Price sensitivity to volume increases with:
   - Adverse selection: The probability of facing an informed trader.
   - $Var(\tilde{V})$: The significance of the informed trader's information.

5. Market makers/dealers earn zero expected profit.

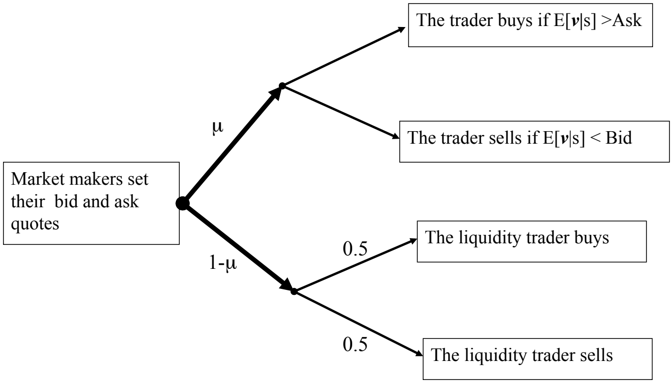
- Inventory model
- Informed traders
  - Non-strategic informed traders
  - Strategic informed traders
- Informed market makers

Economy:

- Fundamental asset value : $\tilde{v} = \tilde{V} + \tilde{\varepsilon}$, with $\tilde{V} \in \{V_1, V_2\}$, $\Pr(\tilde{V} = V_2) = \pi$, and $E[\tilde{\varepsilon}|\tilde{V}] = 0$.
- Market participants :
  - Risk-neutral market makers :
    - MM1 : A market maker who knows $\tilde{V}$.
    - MM2 : One or more uninformed market makers.
  - μ informed speculators who know $\tilde{V}$.
  - $1 - \mu$ liquidity (or noise) traders who buy or sell with probability $1/2$.

1. Market makers simultaneously post bid and ask prices at which they are willing to buy or sell a fixed institutional amount $q = 1$ of the security.
2. Traders decide whether to buy or sell the risky asset (submit market orders):
   - If they wish to sell, they sell $q = 1$ shares to the market maker offering the highest bid.
   - If they wish to buy, they buy $q = 1$ shares from the market maker quoting the lowest ask price.

Trading Mechanism



Market Makers' Payoff Functions

First-price sealed-bid and ask auctions in a common value framework with one informed bidder and adverse selection from traders.

- MM1's expected payoff for a given $\tilde{V} \in \{V_1, V_2\}$:

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$$\begin{array}{lll} (a_1 - V_1)\Pr(a_2 > a_1)\frac{1-\mu}{2} + (V_1 - b_1)\Pr(b_2 < b_1)\frac{1+\mu}{2} & , & \tilde{V} = V_1 \\ (a_1 - V_2)\Pr(a_2 > a_1)\frac{1+\mu}{2} + (V_2 - b_1)\Pr(b_2 < b_1)\frac{1-\mu}{2} & , & \tilde{V} = V_2 \end{array}$$

- MM2's expected payoff:

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- MM2's expected payoff:

$$(1-\pi)(a_2 - V_1)\Pr(a_1 > a_2|V_1)\frac{1-\mu}{2} + \pi(a_2 - V_2)\Pr(a_1 > a_2|V_2)\frac{1+\mu}{2}$$
$$+(1-\pi)(V_1 - b_2)\Pr(b_1 < b_2|V_1)\frac{1+\mu}{2} + \pi(V_2 - b_2)\Pr(b_1 < b_2|V_2)\frac{1-\mu}{2}$$

Equilibrium Quoting Strategy

Lemma

The equilibrium quoting strategy is in mixed strategies , fully revealing (MM1's quotes reveal $\tilde{V}$), and unique.

- If $\tilde{V} = V_2$, it is profitable to buy :
  1. MM1 randomizes its bid in $]V_1, b^*]$.
  2. MM1 sets $a_1 = V_2$.
- If $\tilde{V} = V_1$, it is profitable to sell :
  1. MM1 sets $b_1 = V_1$.
  2. MM1 randomizes its ask in $]a^*, V_2]$.

- MM2 does not know whether it is profitable to buy or sell:
  1. MM2 randomizes its bid in $[V_1, b^*]$.
  2. MM2 randomizes its ask in $[a^*, V_2]$.

a^* and b^* are the bid and ask prices in Glosten and Milgrom.

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