Quote Driven Market: Dynamic Models

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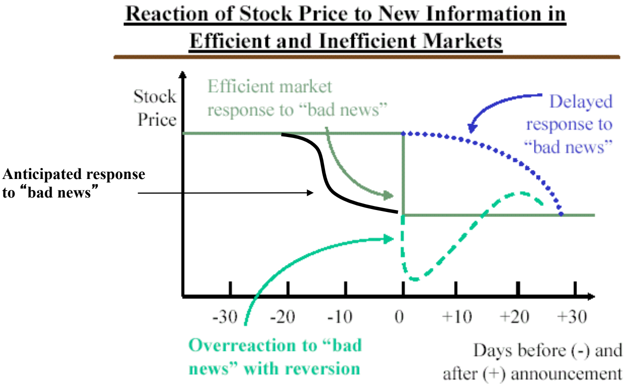
Market Informational Efficiency

- Does the price system aggregate all the pieces of information that are dispersed among investors?
- How does the trading technology affect financial markets informational efficiency?

Definition

- **Weak form efficiency**: Trading prices incorporate all past public information.
- **Semi-Strong form efficiency**: Trading prices incorporate all present and past public information.
- **Strong form efficiency**: Trading prices incorporate all public and private information available in the economy.

Detecting Informational Efficiency



Empirical Evidence

- Financial market is weak form efficient.
- Financial market is semi-strong form efficient.
- Financial market is not strong form efficient.

Dynamic Glosten and Milgrom model

- $t = 0, 1, 2, \ldots$
- At time $t = 0$ Nature determines the asset fundamental value:

$$\tilde{v} = \tilde{V} + \tilde{\varepsilon}$$

with $\tilde{V} \in \{V_1, V_2\}$, $\Pr(\tilde{V} = V_2) = \pi$, $V_1 < V_2$, $E\left[\tilde{\varepsilon}|\tilde{V}\right] = 0$, $Var(\tilde{\varepsilon}|\tilde{V}) \geq 0$.

- In every period t

1. Uninformed competitive MMs set their bid and ask quotes.
2. A trader (informed or liquidity) arrives and decides whether to buy sell or not trade

   q shares of the security.
3. All MMs observe the trading decision and update their beliefs about $\tilde{v}$.
4. The trader leaves the market.

Public beliefs

- Let h_t denote the history of trade preceding period t. This is observed by all market participants

- Let $\pi_t := \Pr(\tilde{V} = v_2 | h_t)$ denote the public belief at the beginning of period t that $\tilde{V} = v_2$.

Evolution of bid-ask spread

Theorem

$$0 \leq E[a_{t+1} - b_{t+1}] < a_t - b_t$$

Proof.

1. $a_t \in (V_1, V_2)$ and is a increasing and concave function of π_t.
2. $E[\pi_{t+1}] = \pi_t$
3. because of a_t is cocnave in $\pi_t \Rightarrow$

$$E[a_{t+1}] = E[a(\pi_{t+1})] < a(E[\pi_{t+1}]) = a(\pi_t) = a_t$$

4. Symmetric argument for the bid side.

□

Concavity of *ask* w.r.t. belief π_t

Let

$$\Pi(a,\pi) := \pi(a - V_2)D(a, V_2) + (1 - \pi_t)(a - V_1)D(a, V_1)$$

Equilibrium ask a_t solves $\Pi(a_t, \pi_t) = 0 \Rightarrow$

- $$V_1 < a(\pi_t) < V_2$$

- $$\frac{\partial a_t}{\partial \pi_t} = -\frac{\partial \Pi/\partial \pi}{\partial \Pi/\partial a} > 0$$

- $$\frac{\partial^2 a_t}{\partial \pi_t^2} = -\frac{(\partial \Pi/\partial a)(\partial \Pi^2/\partial \pi_t^2) - (\partial \Pi/\partial \pi_t)(\partial \Pi^2/\partial \pi_t \partial a)}{(\partial \Pi/\partial a)^2} = \frac{(\partial \Pi/\partial \pi_t)(\partial \Pi^2/\partial \pi_t \partial a)}{(\partial \Pi/\partial a)^2}$$

  the second equility follows from the fact that $\Pi(a, \pi_t)$ is linear in π_t.

- Because $\partial \Pi/\partial \pi_t < 0$, we have that $\frac{\partial^2 a_t}{\partial \pi_t^2} < 0$ only if

$$\frac{\partial \Pi^2}{\partial a \partial \mu} > 0$$

$$\underbrace{D(a, V_2) - D(a, V_1)}_{>0} + \underbrace{(a - V_2)\frac{\partial D(a, V_2)}{\partial a}}_{>0} > \underbrace{(a - v_1)\frac{\partial D(a, V_1)}{\partial a}}_{<0}$$

Traders

- With exogenous probability μ, time t trader is informed and receives private signal $\tilde{s} \in \{l, h\}$ with

$$\Pr(\tilde{s} = l | V_1) = \Pr(\tilde{s} = h | V_2) = r \in \left(\frac{1}{2}, 1\right)$$

- With exogenous probability $1 - \mu$ time t trader is a liquidity trader. A liquidity trader will buy or sell with probability $\frac{1}{2}$

Public and private beliefs

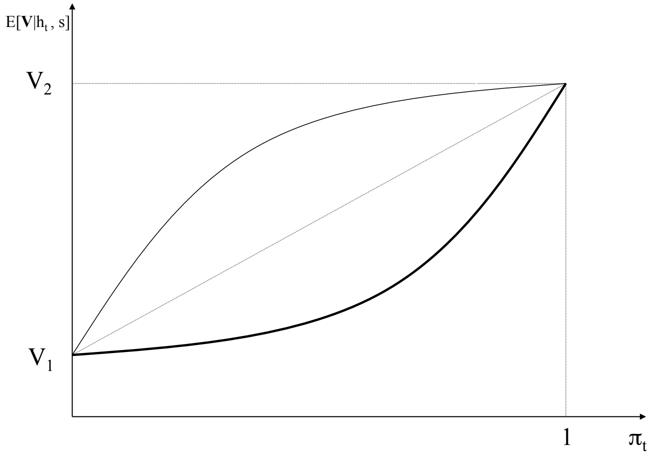
- **Public beliefs**:
  - Let h_t denote the history of trade preceding period t. This is observed by all market participants.
  - Let $\pi_t := \Pr(\tilde{V} = v_2 | h_t)$ denote the public belief at the beginning of period t that $\tilde{V} = v_2$.
- **Informed traders' beliefs**:
  Let $\pi_t^s := \Pr(\tilde{V} = v_2 | h_t, s)$ denote the belief of an informed trader who received signal $s \in \{l, h\}$ at the beginning of period t:

$$\pi_t^l = \frac{\pi_t(1-r)}{\pi_t(1-r) + (1-\pi_t)r} < \pi_t$$

$$\pi_t^h = \frac{\pi_t r}{\pi_t r + (1-\pi_t)(1-r)} > \pi_t$$

Public and private beliefs

Informed traders valuation for the asset:



What can traders and MM learn?

- Fundamental value: $\tilde{v} := \tilde{V} + \tilde{\varepsilon}$
- Informed traders only have information about $\tilde{V}$.
- No market participant has information about $\tilde{\varepsilon}$

Definition

The market is **informational efficient in the long run** if all private information is eventually revealed: $E[\tilde{V}|h_t]$ tends to $\tilde{V}$ as t goes to infinity.

A Toy Model

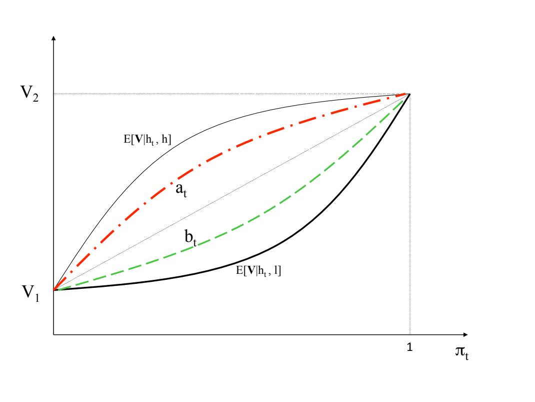
An asset whose fundamental value is $\tilde{v}$ is worth

- $\tilde{v}$ to informed traders.
- $\theta\tilde{v} + \eta$ to MMs.

Equilibrium: In every period t MMs set their bid and ask quotes at

$$\begin{aligned} a_t &= \theta E[\tilde{v}|h_t, \text{trader buys}] + \eta \\ b_t &= \theta E[\tilde{v}|h_t, \text{trader sells}] + \eta \end{aligned}$$

The Glosten and Milgrom case: $\theta = 1$ $\eta = 0$



- No matter h_t, an informed trader will buy (sell) iff $s = h$ (resp. ($s = l$).
- The statistic of the order flow is sufficient to learn market $\tilde{V}$.
- The market is efficient.

Definition

(Avery and Zemisky (1998))
An **information cascade** occurs at time t if the order flow ceases to provide information about $\tilde{V}$:

$$\Pr(\tilde{V} = V_2 | h_t, \text{trader buys}) = \pi_t$$

$$\Pr(\tilde{V} = V_2 | h_t, \text{trader sells}) = \pi_t$$

$$\Pr(\tilde{V} = V_2 | h_t, \text{no trade}) = \pi_t$$

Herd behavior

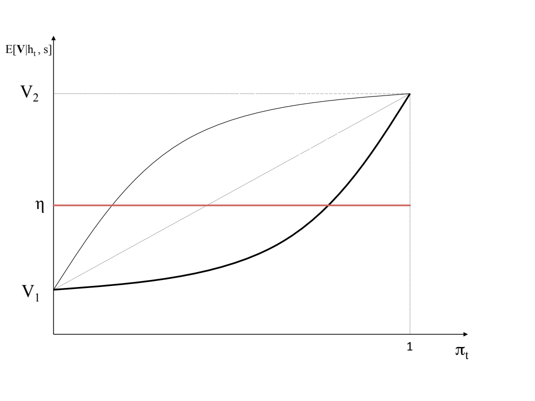
Definition

(Avery and Zemisky (1998))

- A trader engages in **buy herd behavior** if:
  1. Initially he strictly prefers not to buy.
  2. After a positive history h_t, i.e., $\pi_t > \pi$, he strictly prefers buying.

- A trader engages in **sell herd behavior** if
  1. Initially he strictly prefers not to sell.
  2. After a negative history h_t, i.e., $\pi_t < \pi$, he strictly prefers selling.

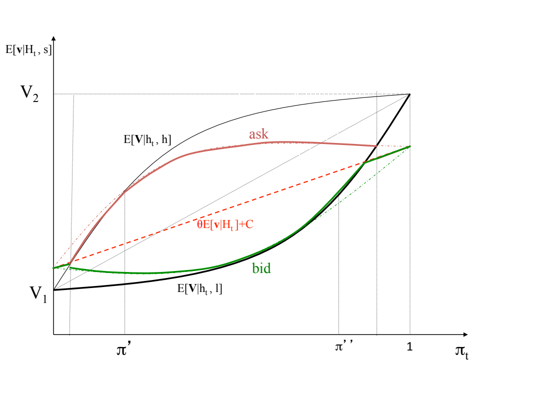
Bikhchandani, Hirshleifer and Welch (1992): $\theta = 0$ $V_1 < \eta < V_2$



- Herding eventually occurs.
- The market cannot learn $\tilde{V}$.

Price under-reaction:

$\theta \in (0, 1); \eta > \in (0, V_2 - V_1)$



- Herding eventually occurs.
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Contrarian behavior

Definition

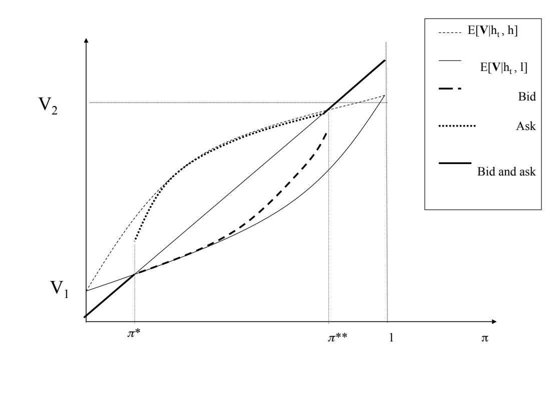
(Avery and Zemisky (1998))

- A trader engages in **buy contrarian behavior** if:
  1. Initially he strictly prefers not to buy.
  2. After a negative history h_t, i.e., $\pi_t < \pi$, he strictly prefers buying.

- A trader engages in **sell contrarian behavior** if
  1. Initially he strictly prefers not to sell.
  2. After a positive history h_t, i.e., $\pi_t > \pi$, he strictly prefers selling.

Price over-reaction:

$\theta < 0; \eta < 0$



- Contrarian behavior eventually occurs.
- The market cannot learn $\tilde{V}$.

Market efficiency with competitive MM (Decamps and Lovo, JME (2006))

Theorem

In a sequential trading set-up, if

- *MMs set quotes to make zero expected profit,*
- *Traders and MM differs in their valuation for the asset,*
- *Agents exchange discrete quantities,*

Then,

long run informational efficiency is impossible.

- $t = 0, 1, 2, \ldots$
- At time $t = 0$ Nature determines the asset fundamental value:

$$\tilde{v} = \tilde{V} + \tilde{\varepsilon}$$

  with $\tilde{V} \in \{V_1, \ldots V_n\}$, $V_i < V_{i+1}$, for any V_i: $E[\tilde{\varepsilon}|V_i] = 0$, $Var(\tilde{\varepsilon}|V_i) \geq 0$.

- Uninformed risk neutral market makers.
- Risk averse informed traders.
- Traders private signals $\tilde{s} \in \{s_1, \ldots s_m\}$, conditionally i.i.d., with

$$\Pr(\tilde{s} = s_i|\tilde{V} = V_j\} > \epsilon > 0, \forall i, j$$

  only regards $\tilde{V}$.

Trading Protocol

1. At time t a trader arrives and submits market order
$$q_t \in Q$$
2. Market makers observe q_t and compete in price to fill the order.
3. Trading occurs and time t trader leaves the market.

with

- Q is a finite and discrete set of tradeable quantities.
- $F(\theta) : \Theta \to [0,1]$ be the probability that time t trader is of type θ.
- Let u_θ denote the increasing and concave utility function of type θ trader and C_θ, I_θ its initial amount of cash and risky asset, respectively.
- A **price schedule** $P_t(q)$ defines the price at which the market order of size $q \in Q$, will be executed by market makers.

Equilibrium Concept

Definition

In **Equilibrium**

- If time t trader is of type θ and received signal s, then chooses

$$q_t =$$

$$q^*_\theta(P_t(.), h_t, s) \in \arg\max_q E[u_\theta(C_\theta + \tilde{v}(I_\theta + q) - qP_t(q))|h_t, \tilde{s}]$$

- A time t, MMs price schedule satisfies:

$$P_t(q_t) = E[\tilde{v}|h_t, q_t]$$

Non-informative Trades

Definition

Type θ trader is said to submit a **non-informative order** whenever

$$q^*_\theta(P_t(.), h_t, s) = q^*_\theta(P_t(.), h_t, s')$$

for all signals s, s'

Long run informational inefficiency

Theorem

There exists $\alpha > 0$ *such that as soon as*

$$Var[\tilde{V}|h_t] \leq \alpha$$

- *All traders submit non informative orders.*
- $P_\tau(q_\tau) = E[\tilde{v}|h_t], \forall q\tau \in Q, \tau \geq t$
- *An information cascade occurs and order flows provides no information.*

Sketch of the proof

1. **Strong past history overwhelms private imperfect signals:** Because $\Pr(\tilde{s} = s_i | \tilde{V}_j) > 0$ for all i, j, then $\forall \varepsilon > 0, \exists \alpha$ such that

$$Var[\tilde{V}|h_t] \leq \alpha \Rightarrow \max_{s_i, s_j} ||E[\tilde{V}|h_t, s_i] - E[\tilde{V}|h_t, s_j]|| < \varepsilon$$

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2. **Strong past history leads to flat pricing schedule:** Because $P_t(q) = E[\tilde{V}|h_t, q_t]$, $\forall\varepsilon, \exists\alpha$ such that

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3. **Flat pricing schedule and weak private signals leads to non-informative orders:** If for all $q \in Q$, $P_t(q) \simeq E[\tilde{V}|h_t]$, then for all s and θ, $Var[\tilde{V}|h_t] \leq \alpha$ implies

$$\arg\max_q E[u_\theta(m_\theta + \tilde{v}(I_\theta + q) - p(q)q)|h_t, s] = -I_\theta$$

Because u_θ is increasing and concave, $E[\tilde{\varepsilon}|h_t] = 0$ and $Var[\tilde{\varepsilon}|h_t] > 0$.

Herding and contraria behaviour with risk neutral agents (Park and Sabourian (Econometrica 2011))

- $\tilde{V} \in \{V_1, V_2, V_3\}$
- $\pi_0^1 = \pi_0^2 = \pi_0^3 = 1/3$
- $1 - \mu$ liquidity traders: buy, sell or do no trade with probability $1/3$.
- μ risk neutral informed traders receive private signal $s \in S := \{s_1, s_2, s_3\}$
- Non informed risk-neutral market makers set quotes at

$$a_t = E[\tilde{V}|h_t, \text{buy order}]$$

$$b_t = E[\tilde{V}|h_t, \text{sell order}]$$

Styles of private signals

Take a signal $s \in S$ then we say that:

Definition

- s is increasing if $\Pr(s|V_1) < \Pr(s|V_2) < \Pr(s|V_3)$
- s is decreasing if $\Pr(s|V_1) > \Pr(s|V_2) > \Pr(s|V_3)$
- s is U-shaped if $\Pr(s|V_1) > \Pr(s|V_2) < \Pr(s|V_3)$
- s is $\cap$-shaped if $\Pr(s|V_1) < \Pr(s|V_2) > \Pr(s|V_3)$
- s has positive biased if $\Pr(s|V_1) < \Pr(s|V_3)$
- s has negative biased if $\Pr(s|V_1) > \Pr(s|V_3)$

Information cascades are impossible

As long as there is $s \in S$ such that $E[\tilde{V}|h_t, s] \neq E[\tilde{V}|h_t]$, there are informative orders.

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$$E[\tilde{V}|h_t, s'] < E[\tilde{V}|h_t] < E[\tilde{V}|h_t, s'']$$

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3. If no informed type sells, then $b_t = E[\tilde{V}|h_t]$ but then s' would sell, hence a contradiction.

Herding or contrarian behavior are impossible if signals are monotonic

A trader with increasing (decreasing) signal will never sell (buy)

Sketch of the proof:

1. $b_t \leq E[\tilde{V}|h_t] \leq a_t$
2. Take a a buyer with decreasing signal s then

$$E[\tilde{V}|h_t, s] < E[\tilde{V}|h_t]$$

   hence he will not buy for a_t

3. Take a a buyer with increasing signal s then

$$E[\tilde{V}|h_t, s] > E[\tilde{V}|h_t]$$

   hence he will not sell for b_t

Herding or contraria behavior are possible with U shaped and ∩-shaped signals

If μ is small enough, then:

	∪-shaped	∩-shaped
positive bias	sell herding	sell contrarian
negative bias	buy herding	buy contrarian

Herding or contrarian behavior are possible with U shaped and ∩-shaped signals

Sketch of the proof: Let s be U-shaped with negative bias. We want to prove buy herding is possible.

1. For $i \in \{1, 2, 3\}$, let $\pi_t^i := Pr(\tilde{V} = V_i | h_t)$
2. U-shaped + negative bias implies $E[\tilde{V}|s] < E[\tilde{V}]$, thus type s does not buy at time 0.
3. Take $\pi_t^1 \simeq 0$, then $E[\tilde{V}|h_t] \simeq V_2\pi_t^2 + V_3\pi_t^3 > E[\tilde{V}]$
4. Because s be U-shaped, $\Pr(s|V_2) < \Pr(s|V_3)$ hence $E[\tilde{V}|h_t, s] > E[\tilde{V}|h_t]$.
5. if μ is small enough $a_t \simeq [\tilde{V}|h_t] < E[\tilde{V}|h_t, s]$ and the trader with signal s will buy.

Main findings using the standard 0-profits approach.

If market makers are equally uninformed and in perfect competition then the price at which quantity x_t is trade is :

$$p_t(x_t) = E[\tilde{V}|h_t, x_t]$$

Main findings using the standard 0-profits approach.

- Market makers make zero profit in equilibrium
- The trading price equals the expected value of the asset given all past public information
- Price volatility reflects beliefs volatility
- In a risk neutral world price eventually converge to fundamentals.

A step back...

Market Microstructure Theory: The classical approach

1. Model of an economy where agents meet over time and exchange a financial asset whose fundamental value is unknown.
2. Assume:
   - Trading protocol
   - Agents preferences
   - Structure of information asymmetry
3. Solve for a Bayesian equilibrium.
4. Derive empirical implications.

Some issues of the standard 0*-profits approach.*

Real life vs Models

1. Actual trading protocol: observable $\Rightarrow$ the model can fit it.
2. Actual agents preferences: not observable, but most theory predictions are robust to changes in risk preferences.
3. Actual information structure: not observable. **Are theory predictions robust to changes in information structure?**

Strengths and weakness of this approach.

Issues

1. Which one of the above results rely on the simplifying non-realistic assumption that all market makers share the exact same information?
2. What predictions are robust to changes in the assumptions about information asymmetries across market makers?
3. What would be a realistic assumption about asymmetries of information, given that information structures are not observable?

Real life vs models

In the real world, the structure of information....

... is not observable.

Real life vs models

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- Impossible to say whether a model's assumptions capture actual information asymmetries.
- Actual information structures are too complex to lead to tractable models.
- Microstructure theory is silent about robustness of its predictions to changes in information structure.

The belief-free approach (Hörner, Lovo, Tomala JFE 2018)

- Provide a price formation model whose predictions are robust to changes in information structure.
- Provide a set of necessary conditions that a price formation equilibrium needs to satisfy to be robust.
- Provide a set of sufficient conditions guaranteeing that a price formation equilibrium is robust.
- Keep the model as general and as tractable as possible.

How robust?

Belief-free: The same dealers' strategy profile forms a sub-game perfect equilibrium no matter the state of Nature.

A belief-free equilibrium remains an equilibrium

- No matter each dealers' information about the state of Nature and the hierarchies of beliefs.
- No matter whether dealers are fully bayesian or not.
- No matter whether dealers are ambiguity averse or not.

Main results

Dealers=Long-lived agents

Traders=Short-lived agents

1. If
   - There is room for trade
   - Dealers are patient enough

   Then there are belief-free equilibria.

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2. A strategy profile forms a belief-free equilibrium only if:
   - Over time, dealers make positive profits no matter the economy fundamentals.
   - Dealers' inventories remain bounded.
   - Stock price volatility exceeds the volatility of the Bayesian expectation of the stock fundamental value.

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   - Over time, dealers make positive profits no matter the economy fundamentals.
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   - Stock price volatility exceeds the volatility of the Bayesian expectation of the stock fundamental value.
3. If
   - A strategy profile is ε-exploring and ε-exploiting
   - dealers are patient enough

   Then the strategy forms a belief-free equilibrium.

Roadmap

- Set-up
- Necessary Conditions
- Sufficient Conditions
- Example
- Extension
- Conclusion

Illustrative example: dynamic Glosten and Milgrom

- $t = 0, 1, 2, \ldots$
- At time $t = 0$ Nature determines the state $\omega \in \Omega$
  Asset fundamental value:

$$W(\omega) = V(\omega) + \varepsilon(\omega)$$

with $\forall \omega \in \Omega$, $V \in \{V_1, V_2\}$, $V_1 < V_2$, $E[\tilde{\varepsilon}|V_i] = 0$

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- Market participants

  1. N finite risk neutral long lived MMs with No assumption regarding what each single dealer knows about ω.
  2. Infinite sequence of short lived traders
     - Mass μ trader informed of $V(\omega)$
     - Mass $(1 - \mu)/2$ liquidity traders willing to buy
     - Mass $(1 - \mu)/2$ liquidity traders willing to sell

Trading protocol, MMs' actions and traders' reaction

- Trading protocol in any given t
  1. MMs Simultaneously set their bid and ask quote
  2. A trader arrives and choose wether to buy or sell
  3. Transaction occurs between the trader and the MM setting the best quote
  4. The trader leaves the market all MMs stay for the next period trade
- MMs action profile $a \in \mathbf{R}^N \times \mathbf{R}^N$:= set of bid and ask quote set by all market makers.
  - $Bid(a)$:= highest among bids in a
  - $Ask(a)$:= lowest among the asks in a
- Trader reaction: $s \in S := \{\text{buy}, \text{sell}, \text{No trade}\}$

Stage trading round

Trades take place according to a protocol specifying:

- $Q_i(a,s)$:= transfer of asset to agent i given (a,s).
- $P_i(a,s)$:= transfer of cash to agent i given (a,s).
- $$\sum_i Q_i(a,s) = \sum_i P_i(a,s) = 0$$

Traders types and distribution of reaction reaction to MM's actions

- Set of possible trader's type: $\Theta = \{\text{informed}, \text{liquidity buyer}, \text{liquidity seller}\}$
  For all $\omega \in \Omega$:

$$\begin{aligned} z(\omega, \text{informed}) &= \mu \\ z(\omega, \text{liquidity buyer}) &= \frac{1-\mu}{2} \\ z(\omega, \text{liquidity seller}) &= \frac{1-\mu}{2} \end{aligned}$$

- $F(\omega, a, s)$:= probability of trader choosing reaction s given MMs action is a and the true state is ω
  - $F(\omega, a, \text{buy}) = \frac{1-\mu}{2} + \mu 1_{\{Ask(a) \le V(\omega)\}}$
  - $F(\omega, a, \text{sell}) = \frac{1-\mu}{2} + \mu 1_{\{Bid(a) \ge V(\omega)\}}$
  - $F(\omega, a, \text{no-trade}) = \mu 1_{\{Bid(a) \le V(\omega) \le Ask(a)\}}$

Room from trade

Assumption: Elastic Traders Demand (ETD)

The distribution of traders reactions $F : \Omega \times A \to \Delta S$ is such that:

there is $\rho > 0$ such that for any $\omega \in \Omega$.

- If $Ask(a) \leq W(\omega) + \rho$, then traders buy at price $Ask(a)$ with strictly positive probability.
- If $Bid(b) \geq W(\omega) - \rho$, then traders sell at price $Bid(b)$ with strictly positive probability.

Stage trading round: Dealers payoffs

Dealers are risk neutral:

- Dealer i's ex-post trading round payoff in state ω:

$$u_i(\omega, a, s) = W(\omega)Q_i(a, s) + P_i(a, s)$$

- Dealer i's expected trading round payoffs from $a \in A$ given ω:

$$u_i(\omega, a) = W(\omega)\sum_{s\in S} F(\omega, a, s)Q_i(a, s) + \sum_{s\in S} F(\omega, a, s)P_i(a, s)$$

Repeated game payoff

Given some action outcome $\{a^t\}_{t=1}^{\infty}$, dealer i's payoff in state ω is

$$\sum_{t=0}^{\infty}(1-\delta)\delta^t u_i(\omega, a^t)$$

where $\delta \in (0, 1)$ is the discount factor.

Repeated game strategy

- Public history $h^t = \{a^\tau, s^\tau\}_{t=1}^{t-1}$
- Dealer i's strategy: $\sigma_i : H^t \to \Delta A_i$,
- Occupation measure for $\sigma := \{\sigma_i\}_{i=1}^{n}$ given ω and h^t:

$$\mu^\sigma_{\omega,h^t}(a) := \mathbb{E}_\sigma \left[\sum_{\tau \geq t} (1-\delta)\delta^\tau 1_{\{a^\tau = a\}} \middle| \omega, h^t \right], a \in A$$

- Continuation payoff in state ω after observing history h^t when player's continuation strategy follows σ:

$$V_i(\omega, \sigma | h^t) = \sum_{a \in A} \mu^\sigma_{\omega,h^t}(a) u_i(\omega, a)$$

Sub-Game Perfect Equilibria and Belief-Free Equilibria

Definition

Sub-game perfect equilibrium: $\forall i, \forall t, \forall h_i^t$, dealer *i*'s equilibrium strategy

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$$\sigma_i \in \arg\max_{x_i \in A_i} \sum_{\omega \in \Omega} p_i^t(\omega) \mathbb{E}\left[V_i(\omega, x_i, \sigma_{-i} | h_i^t) \right]$$

where $p_i^t \in \Delta\Omega$ is dealer *i*'s belief about ω given h_i^t that is dealer *i*'s information (private + public).

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for all $\omega \in \Omega$.

What can be learned from traders' behavior?

Definition

Let $\hat{\Omega}$ be the partition over Ω induced by the function F . That is $\omega, \omega' \in \hat{\omega}$ iff $F(\omega, a) = F(\omega', a)$ for all $a \in A$.

Interpretation:

- $\hat{\Omega}$ is the information that can be statistically gathered by observing how traders react to dealers' actions.
- If two states belong the the same element $\hat{\omega} \in \hat{\Omega}$, then the distribution of traders' reaction to dealers' actions is identical in those two states.

What can be learned from traders' behavior?

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Interpretation:

- $\hat{\Omega} := \{\hat{\omega}_1, \hat{\omega}_2\}$ with
- $$\hat{\omega}_1 = \{\omega \in \Omega \text{ s.t. } V(\omega) = V_1\}$$
  $$\hat{\omega}_2 = \{\omega \in \Omega \text{ s.t. } V(\omega) = V_2\}$$

Some properties of the stage game payoff

Under assumption ETD, for any given $\hat{\omega} \in \hat{\Omega}$, all $\omega \in \hat{\omega}$:

1. There is $A^\star(\hat{\omega}) \subset A$ such that for each dealer i and $a \in A^\star(\hat{\omega})$:

$$u_i(\omega, a) > 0$$

$$A^\star(\hat{\omega}_1) = \{a \in A \text{ s.t. } bid_i = Bid(a) < V_1 < Ask(a) = ask_i, \forall i \in N\} \Rightarrow$$

$$u_i(\omega, a) = \frac{1-\mu}{2N}(ask_i - bid_i) > 0$$

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2. $\forall i$ and $\mu \in \Delta\hat{\omega}$, other dealers have $\underline{a}_{-i}(\mu) \in \Delta A_{-i}$ such that

$$\max_{a_i} \sum_{\omega \in \hat{\omega}} \mu(\omega) u_i(\omega, a_i, \underline{a}_{-i}(\mu)) \leq 0.$$

$$\underline{a}_{-i}(\mu) = \{a \text{ s.t. } Ask(a) = Bid(a) = \textstyle\sum_{\omega \in \hat{\omega}} \mu(\omega) W(\omega)\}$$

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4. There is $\{a(1)(\hat{\omega}), \ldots, a(n)(\hat{\omega})\} \in (\Delta A)^n$ such that

$$u_i(\omega, a(i)(\hat{\omega})) < u_i(\omega, a(j)(\hat{\omega})) \text{ for every } j \neq i.$$

Necessary conditions for σ to form a belief-free equilibria

Theorem

Let $\sigma : H \to \Delta A$ form a BFE, then

- σ *is measurable with respect to* $\hat{\Omega}$.
- $\forall \omega \in \Omega$, *each dealer equilibrium payoff is strictly positive.*
- $\forall \omega \in \Omega$, *each dealer average inventory is bounded.*
- *Trading price volatility does not decrease with time.*

Lemma

Let $\sigma : H \rightarrow \Delta A$ *form a BFE, then*

$$\omega, \omega' \in \hat{\omega} \Rightarrow \sigma(\omega) = \sigma(\omega')$$

Proof:

- A BFE must remain an equilibrium even when dealers have no private information.
- In this case no agent can tell apart $\omega, \omega' \in \hat{\omega}$.
- The play must be the same in ω and ω'.

Necessary conditions for belief-free equilibria

Strictly positive dealers' profits

Lemma

Let $\sigma : H \to \Delta A$ form a BFE, then $\forall \omega \in \Omega$, each dealer equilibrium payoff is strictly positive.

Proof:

- Fix an arbitrary $\omega \in \Omega$.
- A BFE must remain an equilibrium even when a dealer is almost sure the true state is ω.
- No matter the true ω, each dealer can guarantee 0 by not trading.

Necessary conditions for belief-free equilibria

Bounded dealers' inventories

Lemma

Let $\sigma : H \to \Delta A$ form a BFE,
Let $Q_i(\omega, \sigma)$ be the equilibrium level of dealer i's inventory, given ω.
Let $TV_i(\omega, \sigma)$ be the equilibrium level trading volume with dealer i, given ω.
Then there is $k > 0$ bounded such that $\forall \omega, i$,

$$\frac{|Q_i(\omega, \sigma)|}{TV_i(\omega, \sigma)} < k$$

Proof: For each dealer i, from ETD:

$$\max_{a,s}(v(\hat{\omega}) + \varepsilon(\omega))Q_i(a, s) + P_i(a, s) \leq$$

$$\overbrace{(V(\hat{\omega}) + \varepsilon(\omega) - \underbrace{(V(\hat{\omega}) - \rho)}_{\text{min purchase price}}\;)Q_i^+(a, s)}^{\text{dealers buys}}$$

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$$\overbrace{(V(\hat{\omega}) + \varepsilon(\omega) - \underbrace{(V(\hat{\omega}) - \rho)}_{\text{min purchase price}})Q_i^+(a, s)}^{\text{dealers buys}} \overbrace{-(V(\hat{\omega}) + \varepsilon(\omega) - \underbrace{(V(\hat{\omega}) - \rho)}_{\text{max selling price}})Q_i^-(a, s)}^{\text{dealers sells}}$$

$$= \varepsilon(\omega)Q_i(a, s) + \rho TV_i(a, s) \Rightarrow \min_{\omega \in \hat{\omega}} \varepsilon(\omega)Q_i(a, s) + \rho TV_i(a, s) > 0$$

Necessary conditions for belief-free equilibria

Trading price volatility does not decrease with time.

No matter the history h_t, there is k finite such that

$$|m(\omega, h_t) - V(\hat{\omega})| < k$$

where $m(\omega, h_t)$:= average discounted mid-quote computed using the equilibrium occupation measure following history h_t in state ω

Proof: If not

- the dealers' inventory explode in some states.
- Expected profit become negative for dealer that believes that state is the true one.

$\Rightarrow$ quotes must be more sensitive than Bayesian beliefs to the order flow.

Sufficient conditions for σ to form a belief-free equilibrium

Theorem

Under assumption ETD, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon < \bar{\varepsilon}$, if strategy profile σ is ε-learning and ε-exploiting, then there exists $\underline{\delta} < 1$ such that the outcome induced by σ is a belief-free equilibrium outcome, for all $\delta \in (\underline{\delta}, 1)$.

Proof: Constructive...

Exploring and exploiting

The couple (ϕ, σ) are said:

1. ε-*Exploratory* : if the true state is ω, then the market measure will frequently points at $\hat{\omega}(\omega)$, no matter π^0.
2. ε-*Exploiting* : when the market measure points at $\hat{\omega}$, dealers' actions lead all of them to make positive profits in all states $\omega \in \hat{\omega}$, i.e., $a^t \in A^\star(\hat{\omega})$.

Definition

1. The pair (ϕ, σ) is ε-*learning*, for $\varepsilon > 0$, if for any $\omega \in \Omega$ and any $\pi^0 \in \Pi$,

$$\Pr_{\omega,\sigma}\left[\liminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} 1_{\{\pi^t(\hat{\omega}(\omega))>1-\varepsilon\}} < 1-\varepsilon\right] < \varepsilon,$$

2. The pair (ϕ, σ) is ε-*exploiting*, for $\varepsilon > 0$, if for all $\hat{\omega} \in \hat{\Omega}$ and all h^t such that $\pi^t(\hat{\omega}) \geq 1-\varepsilon$, we have $\Pr_\sigma\left[a^t \in A^\star(\hat{\omega})|h^t\right] > 1-\varepsilon$.

BFE equilibrium construction: ingredients

- Market measure $\pi \in \Delta\hat{\Omega}$: probability over the possible $\hat{\omega} \in \hat{\Omega}$.
- Market measure updating rule ϕ: Market measure is only affected by the public history $h^t : \{a^t, s^t\}_{\tau=0}^{t-1}$:

$$\pi^{t+1} = \phi(\pi^t, a^t, s^t)$$

- For a given $\varepsilon > 0$, market measure is said to point at $\hat{\omega}$ at t if

$$\pi^t(\hat{\omega}) > 1 - \varepsilon$$

- On path, dealers' actions at t only depend on the π^t:

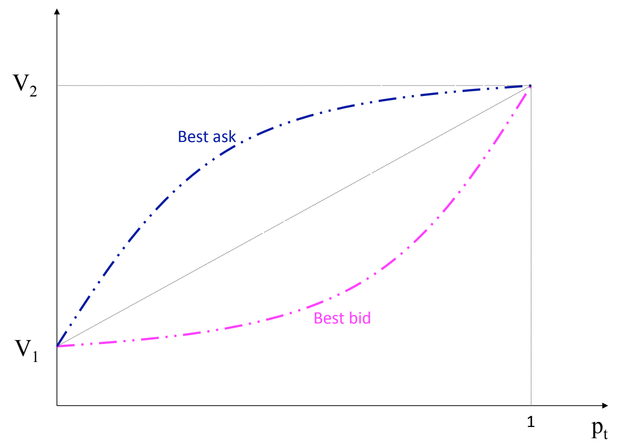
$$\sigma_i : \Delta\hat{\Omega} \rightarrow \Delta A_i$$

If we assume equally uninformed MMs with common belief $p^t := Pr(\hat{\omega} = \hat{\omega}_2 | h^t)$, then repetition of static Bertrand competition leads to

$$\begin{aligned} \alpha^t &= \alpha(p^t) := \mathbb{E}\left[\tilde{V} | h^{t-1}, s^t = buy\right] \\ \beta^t &= \beta(p^t) := \mathbb{E}\left[\tilde{V} | h^{t-1}, s^t = sell\right] \\ p^{t+1} &= \phi_B(p^t, a^t, s^t) \end{aligned}$$

where ϕ_B is the Bayesian updating and h^t is the history of trades until time t.

Illustrative Example: Canonical zero profit equilibrium



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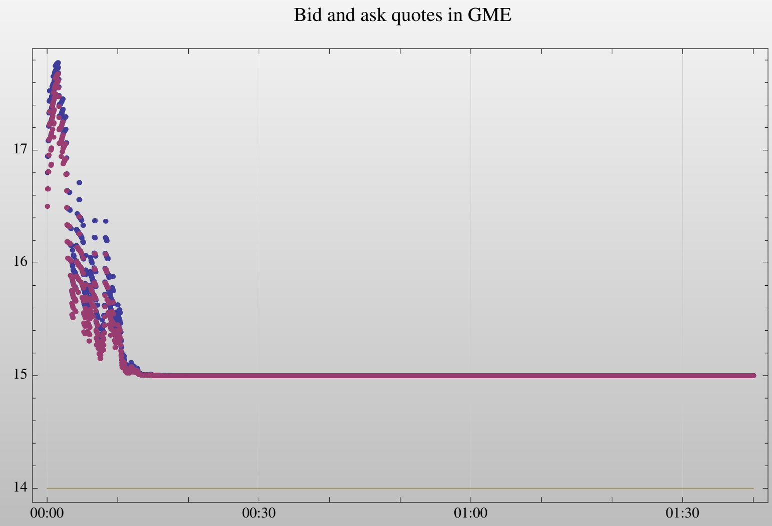


Illusrative Example: In BFE, Market measure replaces beliefs

Market measure

- Fix arbitrary $\pi^0 \in \Pi := [\varepsilon/4, 1 - \varepsilon/4]$.
- Market measure updating rule:

$$\pi^{t+1} = \phi(\pi^t, a^t, s^t) := \arg\min_{\pi \in \Pi} \left\| \pi - \phi_B(\pi^t, a^t, s^t) \right\|$$

- Bid and ask are increasing in π^t and decreasing in MMs' aggregate inventory.
- Bid-ask Spread remains bounded away from 0.

Illusrative Example: exploring and exploiting

Exploring: If $\pi^t \in [\varepsilon, 1-\varepsilon]$:

$$\begin{aligned} \alpha^t &= \alpha(\pi^t) + d - cY^t \\ \beta^t &= \beta(\pi^t) - d - cY^t \end{aligned}$$

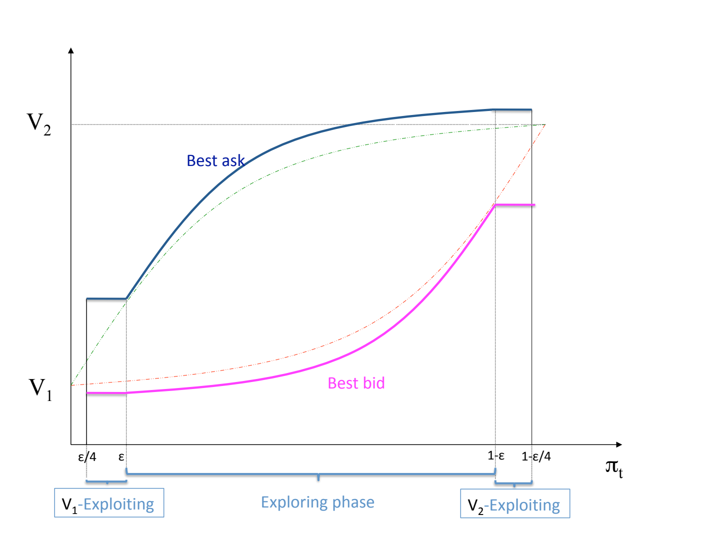
Exploiting in v_1 : If $\pi^t < \varepsilon$:

$$\begin{aligned} \alpha^t &= v_1 + d - cY^t \\ \beta^t &= v_1 - d - cY^t \end{aligned}$$

Exploiting in v_2 : If $\pi^t > 1-\varepsilon$:

$$\begin{aligned} \alpha^t &= v_2 + d - cY^t \\ \beta^t &= v_2 - d - cY^t \end{aligned}$$

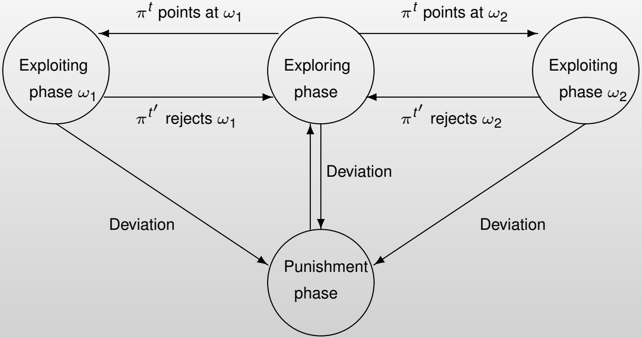
Illustrative Example: BFE



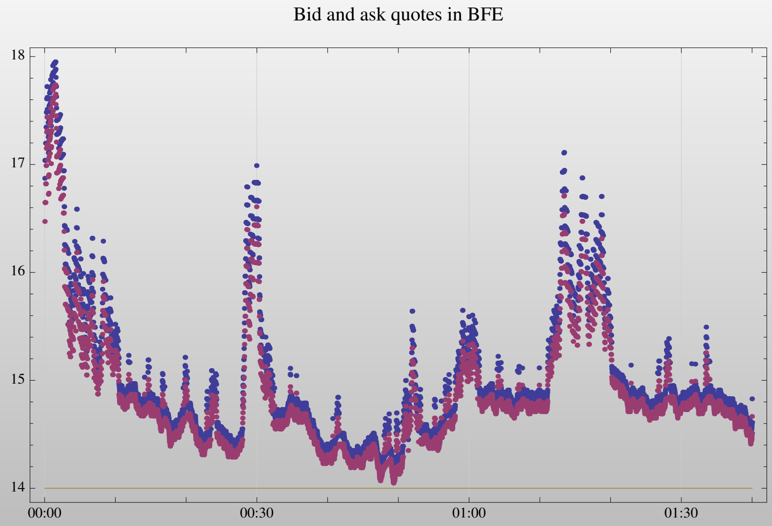
Equilibrium construction

- **Exploring phases**: dealers choose actions to induce informative traders' reactions. This moves the market measure.
- **Transition to exploiting** "$\hat{\omega}$": As soon as the market measure points at $\hat{\omega}$.
- **Exploiting phases** "$\hat{\omega}$" : dealers choose actions to make profits given $\hat{\omega}$.
- **Transition to exploring phase**: As soon as the market measure ceases pointing at a state.
- **IR constraint**:
  - All dealers get strictly positive profits.
  - Deviations lead to temporary punishment and involving non-positive profit to the deviating dealer.

BFE: Phase transitions

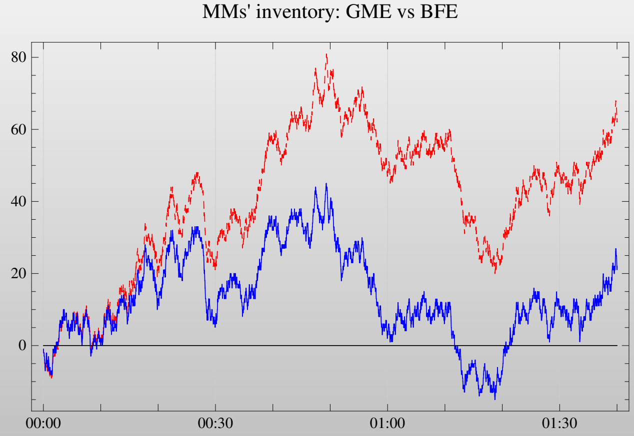


Illustrative Example: BFE



Illustrative example: Glosten and Milgrom economy

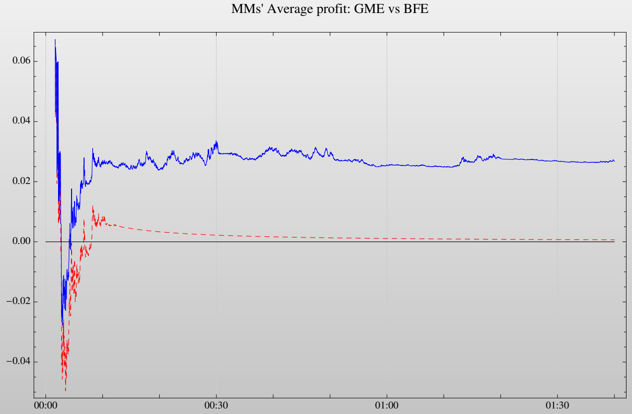
Evolution of Dealers' aggregate inventories



Canonical zero expected profit equilibrium
Belief-free equilibrium

Illustrative example: Glosten and Milgrom economy

Comparison of Dealer's realized profits



Canonical zero expected profit equilibrium
Belief-free equilibrium

Why exploring and exploiting is optimal no matter dealers beliefs?

- Why dealers do not deviate?
  - All dealers get strictly positive long term profits in all states.
  - Dealers do not deviate because the others can ensure nobody profits again (in the classical repeated-game fashion with sufficiently low discount rate).
- Why exploiting cannot last forever?
  - Dealer who disagrees with the consensus asset value must be given incentives to play along and wait for play to shift towards the asset value he believes correct.
- Why exploiting require balanced inventories.
  - Knowing $\hat{\omega}$ does not imply knowing ω and hence $W(\omega)$.
  - Profit from largely imbalanced inventory depend on $W(\omega)$ and might be negative for some beliefs about $\omega \in \hat{\omega}$.

What can we explain by applying BFE to a Glosten and Milgrom model?

- Trading volume moves prices.
  (Chordia, Roll and Subrahmanyam (2002), Boehmer and Wu (2008)), Pasquariello and Vega (2005), Evans and Lyons (2002), Fleming, Kirby and Ostdiek (2006))
- Volatility clustering: Price sensitivity to volume is larger in exploring phases than in exploiting phases.
  (Cont (2001))
- Inter-dealer market is used to rebalance/share positions taken with trades.
  (Hasbrouck and Sofianos (1993), Reiss and Werner (1998), (2005) Hansch, Naik and Viswanathan (1998), Evans and Lyons (2002))
- Collusive type equilibrium.
  (Christie and Schultz (1994), Christie, Harris and Schultz (1994), Ellis, Michaely, and O'Hara (2003))

Conclusion

- Microstructure models where long-lived, patient enough dealers interact with short-lived traders.
- Extremely robust equilibria exist under very mild conditions.
- belief-free price formation strategies require:
  1. Positive profits
  2. bounded inventories
  3. Excess price volatility
- belief-free price formation strategies can be achieved when:
  1. Dealers manage to collectively learn the value of fundamentals relevant to traders.
  2. Dealers make positive profits through intermediation.
- A single model explains some well documented stylized facts.

Summary

- Auction theory and revenue equivalence theorem.
- Inventory models
- Informed non strategic traders
- Informed strategic trader.
- Informed market makers.
- Market information efficiency and herding.
- Belief-free pricing.

THANK YOU!