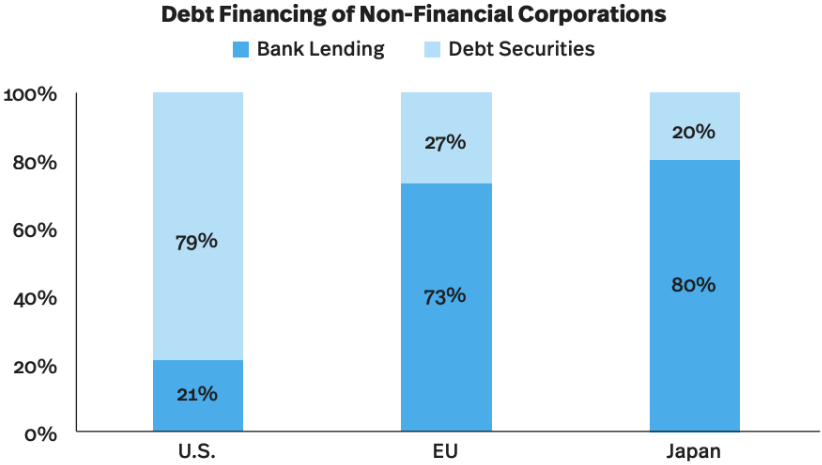
Financial Markets

HEC Paris – Fall 2025

Part 2: Bonds

What is a bond (a.k.a., a fixed-income security)?

- Non-financial corporations rely differently on bond issuance across regions:

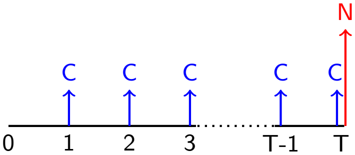


What is a bond (a.k.a., a fixed-income security)?

- "Tradable loan"
  - Issuer borrows money on the primary market
  - Initial purchaser on the primary market lends money
  - Bonds can be traded on secondary market until maturity $\Rightarrow$ bondholders can change over time

A typical bond's cash flows

- *Promised* cash flows of a bond



The features of a bond

- The **Maturity** T: The date on which the last payment to the bondholder is due
- The **Face Value** N: The final payment that is made at maturity with the last coupon. Typically a round number (e.g., 1.000 €, 10.000 \$, 100.000.000 ¥, ...)
- The **Coupon** C: the interest payment that is made to each bond holder at periodic dates.
- The **Frequency** z with which coupons are paid (examples: once every year, once every semester)
- The **Coupon rate:** $\frac{C}{N}$ typically in %.

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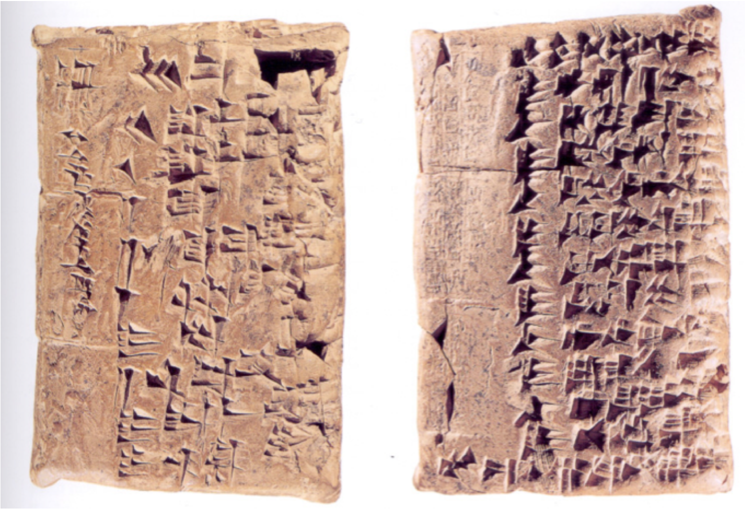
1. What are the cashflow of a bond with maturity on December 31st 2030, face value $N =$ Euro 1000, coupon $C =$Euro 20 and frequency $z =$ once per year?
2. What is this bond coupon rate? Ans.: $C/N = 2\%$

A bit of history

4000 years old Babylonian debt contracts (source: The Origins of Value, Goetzmann and Rouwenhorst, 2005)



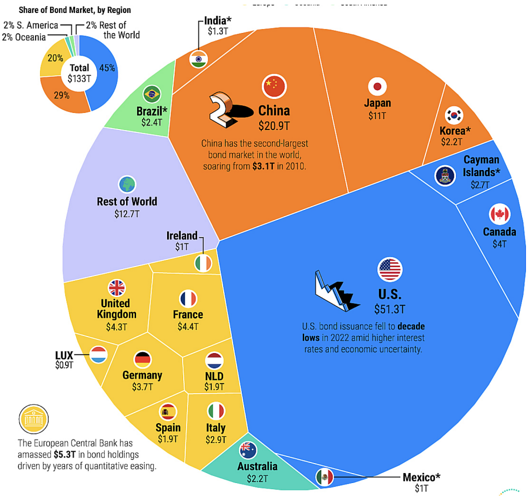
Bond markets size (today)

As of 2022 global bond market is 133 trillion $ (table from Visual Capitalist based on BIS data):

Bond Market Rank	Country / Region	Total Debt Outstanding	Share of Total Bond Market
1	🇺🇸 U.S.	$51.3T	39%
2	🇨🇳 China	$20.9T	16%
3	🇯🇵 Japan	$11.0T	8%
4	🇫🇷 France	$4.4T	3%
5	🇬🇧 United Kingdom	$4.3T	3%
6	🇨🇦 Canada	$4.0T	3%
7	🇩🇪 Germany	$3.7T	3%
8	🇮🇹 Italy	$2.9T	2%
9	🇰🇾 Cayman Islands*	$2.7T	2%
10	🇧🇷 Brazil*	$2.4T	2%

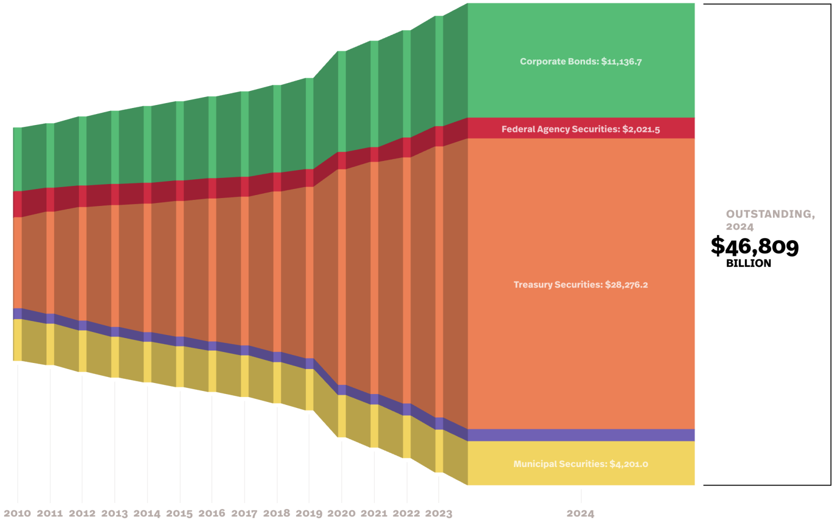
What is a bond (a.k.a., a fixed-income security)?

- Figure from Visual Capitalist based on BIS data:



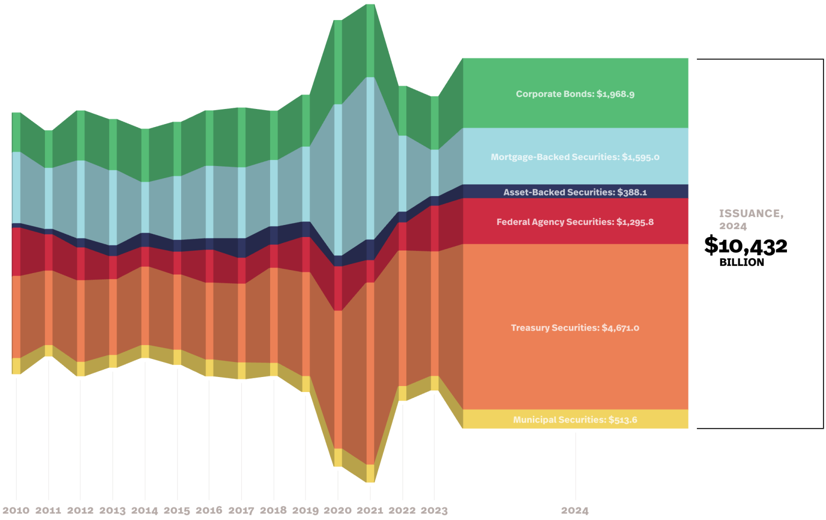
What is a bond (a.k.a., a fixed-income security)?

- The name "fixed income" refers to the fact that the formula determining the cash flows is fixed (true even in the case of floating-rate instruments).
- US bond market: breakdown of outstanding amounts (source: SIFMA)



What is a bond (a.k.a., a fixed-income security)?

- Financial innovation: Mortgage or Asset Backed Securities (MBS or ABS)
- US bond market: breakdown of new issuances (source: SIFMA)

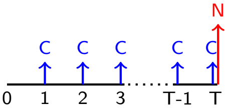


Overview

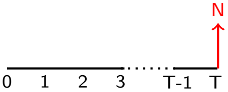
1. **Bond Basics** $\Longleftarrow$
2. Valuation & Term Structure of Interest Rates
3. Arbitrage Pricing
4. Forward Interest Rates
5. Default Risk
6. Interest Rate Risk

Types of bonds

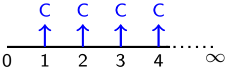
- Coupon bonds



- Zero-coupon bonds (or zeros, or discount bonds)



- Perpetuities



Coupon bonds

Coupon-bonds typically have maturities up to 30 years, though governments and companies are known to issue very long term bonds (up to 100 years in maturity)

Austria sells record largest €3.5bn century bond

Latest indication of hot investor demand for very long-dated debt

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SEPTEMBER 12, 2017 by **Kate Allen**

Austria has sold €3.5bn of 100-year debt in the largest century bond to hit the markets to date, the latest indication of hot investor demand for very long-dated debt.

Bids from potential investors reached €11.4bn, dealmakers said.

The deal — which was priced at a yield of 2.112 per cent — is the eurozone's first syndicated century bond. Belgium and Ireland each raised €100m in century bonds last year but those deals were private placements rather than open market offerings.

Their issuance is part of a wider trend: governments around the world sold a record $63.5bn of debt with ultra-long maturities in 2016, according to figures from data provider Dealogic.

Stripping or how to create a zero-coupon bond out of a coupon-paying bond

- Remove the coupons from the coupon bond and sell the separate parts as Separate Trading of Registered Interest and Principal of Securities (STRIPS) – this is an example of securitization

Pension fund demand drives rise in Treasury 'strips' activity

US tax code changes trigger rush for retirement plans to hedge long-term liabilities

Joe Rennison in New York SEPTEMBER 6, 2018 0

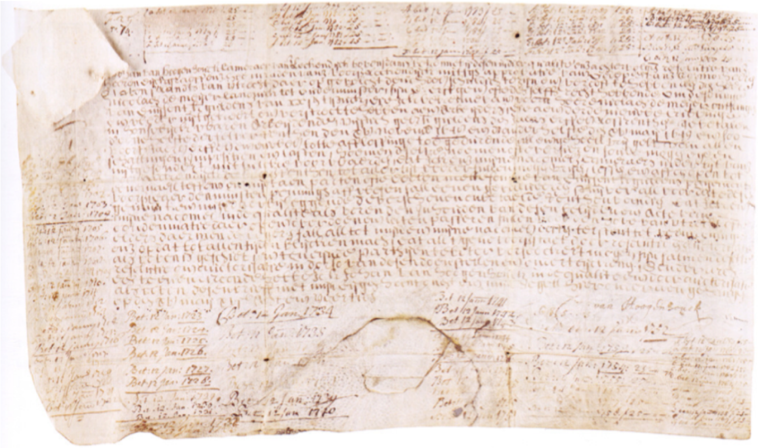
Pension fund demand for longer-dated US government bonds has sparked a sharp rise in Treasury "strips" activity ahead of a tax break ending this month.

With a new 21 per cent rate set to replace the old 35 per cent level later this month, pension plans have been rushing to buy strips, whereby existing 30-year Treasury bonds are separated into two types of security: one that only pays a fixed rate of interest, or coupon, over a set period, and another that is sold at a steep discount as it pays out only its principal value upon maturity.

- In France, the resulting zero-coupon bonds are called OATs Démembrés

Perpetuities (Consols)

An early example of a perpetual bond: 5% perpetual bond of 1000 guilders issued by the Water Board of Leckdijk Bovendams on May 15, 1648 (source: The Origins of Value, Goetzmann and Rouwenhorst, 2005)



Perpetuities (Consols)

Corporations can also issue perpetual bonds

April 10, 2013 7:05 pm

Trafigura raises $500m with perpetual bond

By Javier Blas in London and Jack Farchy in Santiago

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Trafigura, one of the world's largest commodities trading houses, has launched its first perpetual bond, tapping the public capital market in a further sign of change in the way trading titans finance themselves.

The trading house, which last year moved its incorporation from Geneva to Singapore, raised $500m with its bond, up from an initial target of $300m. The note, which was five times subscribed, will yield a 7.65 per cent coupon.

Floating-rate bonds

- Adjustable coupon depending on the level of a benchmark interest rate (example: SOFR+35bps, or SONIA+50bps, or EONIA+1%)*

Credit Suisse becomes first bank to issue debt tied to Sofr

Joe Rennison in New York AUGUST 21 2018

Credit Suisse has become the first bank to issue debt tied to the new US interest rate chosen to replace the London Interbank Offered Rate (Libor), selling a $100m six-month certificate of deposit on Monday.

The bank is the third institution overall to issue debt tied to the Secured Overnight Financing Rate (Sofr), following a $6bn floating rate note from mortgage agency Fannie Mae and a $1bn floating rate bond from the World Bank.

The issuance comes as the financial industry attempts to move away from Libor, following a rigging scandal and a decline in transaction volumes in the underlying market.

The Sofr rate, drawn from transactions in the repo market, has been chosen by an industry group and set up by the Federal Reserve to shepherd a move away from Libor. Corporate issuance is seen by the industry as an important step in aiding adoption of the new rate.

The Credit Suisse debt priced at 35 basis points above Sofr, according to people with knowledge of the sale. Sofr, which resets daily, stood at 1.90 per cent on Friday — the most recent rate available from the Federal Reserve Bank of New York, which publishes it.

* SOFR:Secured Overnight Financing Rate, SONIA: Sterling Overnight Index Average; EONIA: Euro Overnight Index Average

Inflation-indexed bonds

- Initially coupon-indexed or capital (principal)-indexed on past inflation, nowadays all cash flows are indexed on past inflation.
- Issued by UK (since 1981), Canada (1993), USA (1997), France (1998), ...
- In the US they are called TIPS (Treasury Inflation Protected Securities)

No. [redacted] STATE of MASSACHUSETTS BAY.

£454

The First Day of January, A. D. 1780.

In Behalf of the State of Massachusetts-Bay, I the Subscriber do hereby promise and oblige Myself and Successors in the Office of Treasurer of said State, to pay unto John Roper or to his Order, the Sum of Four hundred & Fifty four pounds on or before the First Day of March, in the Year of our Lord One Thousand Seven Hundred and Eighty four with Interest at Six per Cent. per Annum: Both Principal and Interest to be paid in the then current Money of said State, in a greater or less Sum, according as Five Bushels of CORN, Sixty-eight Pounds and four-seventh Parts of a Pound of BEEF, Ten Pounds of SHEEPS WOOL, and Sixteen Pounds of SOLE LEATHER shall then cost, more or less than One Hundred and Thirty Pounds current Money, at the then current Prices of said Articles—This Sum being Thirty two Times and an Half what the same Quantities of the same Articles would cost at the Prices affixed to them in a Law of this State made in the Year of our Lord One Thousand Seven Hundred and Seventy-seven, intitled, "An Act to prevent Monopoly and Oppression." The current Prices of said Articles, and the consequent Value of every Pound of the Sum herein promised, to be determined agreeable to a LAW of this State, intitled, "An ACT to provide for the Security and Payment of the Balances that may appear to be due by Virtue of a Resolution of the General Assembly of the Sixth of February One Thousand Seven Hundred and Seventy-nine, to this State's Quota of the CONTINENTAL ARMY, agreeable to the Recommendation of CONGRESS, and for Supplying the Treasury with a Sum of Money for that Purpose." No John Roper

Committee

Witness my Hand

Treasurer

Asset-Backed Securities (ABS)

- Coupons are based on revenues of underlying assets (mortgages, car loans, credit card receivables, etc) - another example of securitization:

Government reforms unleash India's securitisation market

Issuers bypass bond market and sell collateralised debts

© Reuters

Aliya Ram in New Delhi JUNE 8 2017 3

India's structured debt market is gaining renewed momentum following the introduction of a tax reprieve for unlisted debt securities and new rules that pave the way for foreign portfolio managers to invest in the sector.

Financial institutions are issuing a range of securitised instruments that typically involve the pooling of small loans to farmers, small businesses, mortgages and car loans.

These asset-backed bonds are rated and then sold at auction in sizes ranging from $2m to $155m. They largely comprise asset and mortgage-backed securities.

The government has pushed for such auctions as they shift loans off the balance sheets of non-bank financiers and thereby facilitate greater lending to the country's farmers and small businesses.

Hybrid bonds

- Convertible bonds: the bondholder has the right to convert the bond for a predetermined number of shares of the company's stock after an initial waiting period. The bondholder will convert only if it is profitable to do so.

Q1 A convertible bond is trading at €900. It can be converted into 100 shares of the company's stock. The stock is trading €10.
Would you convert?

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Would you convert?
If converted: $100 \times €10 = €1{,}000 > €900 \Rightarrow$ should convert

Q2 A firm has issued a convertible bond as well as a non-convertible bond, both with same maturity, same priority in payment (same "seniority"), and same face value.
Which one has the higher market price?

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Q2 A firm has issued a convertible bond as well as a non-convertible bond, both with same maturity, same priority in payment (same "seniority"), and same face value.
Which one has the higher market price?
If same coupon, then convertible bond would be more valuable. Can have same price only if non-convertible bond has higher coupon.

Hybrid bonds

Bayer launches convertible bond to help fund Monsanto deal

fastFT

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NOVEMBER 15, 2016 by **Thomas Hale**

Bayer, the German chemicals company, has launched the largest ever mandatory convertible bond from a European issuer, as part of the funding for its $66bn acquisition of US biotechnology company Monsanto.

The €4bn convertible bond, which is expected to pay a coupon of 5.125-5.625 per cent, had fully covered its books late on Tuesday, according to bankers familiar with the deal, **writes Thomas Hale in London.**

The instruments will mandatorily convert to equity in three years' time, meaning investors are effectively making a play on the future share price of the company.

While investors have full downside exposure, the structure of the deal effectively delays the point at which they make money on the upside.

Hybrid bonds

- Callable bonds: the issuer has the right to buy back the bond at a pre-determined price during the bond's life

Q1 A firm has issued a callable bond and a non-callable bond, both with same maturity, same face value, and same coupon.
Which one has the higher price?

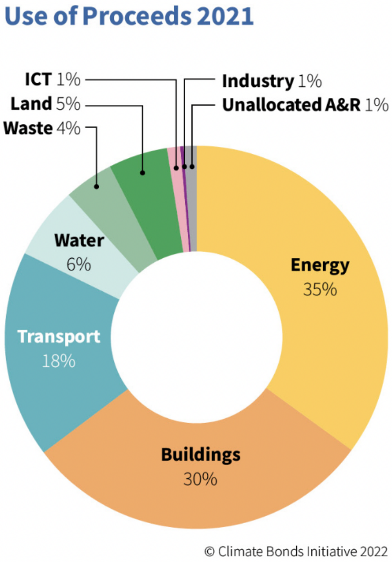
Hybrid bonds

- Callable bonds: the issuer has the right to buy back the bond at a pre-determined price during the bond's life

Q1 A firm has issued a callable bond and a non-callable bond, both with same maturity, same face value, and same coupon.
Which one has the higher price?
The non-callable bond.

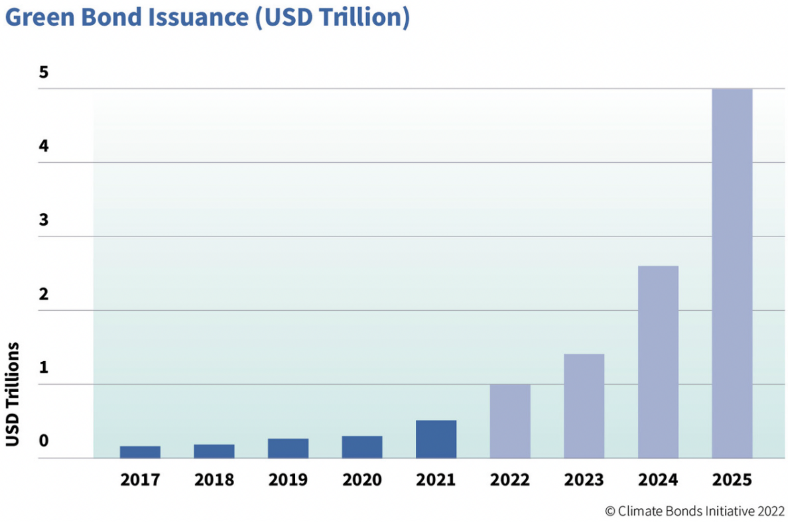
Green bonds

- The proceeds raised from a green bond issuance is restricted to finance or refinance “green” projects:



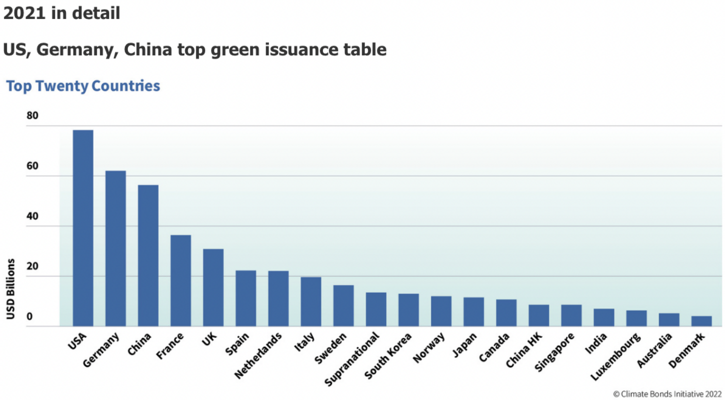
Green bonds

- Green bond issuances are growing rapidly:



Green bonds

- Biggest green bond issuing countries:



- Pros of green bonds: Lower cost of capital for the issuing government or firm as these days demand by ESG-investors outstrips supply by ESG-issuers
- Cons of green bonds: Leaves little flexibility to the management on how to invest the funds raised by the green bonds

Sustainability-linked Green bonds

- The cashflows that the bond promises to pay to the bond-holders increase if the issuer does not meet a pre-specified "green" objective.
- In 2019 Italian energy group Enel raised 3.25 billion via Sustainability-Linked Bonds
  - Maturity: 2034 (15 years)
  - Goal: having at least 55% of Enel installed capacity in renewable energy sources by 2021
  - Coupon rate : 2.65% if the goal is reached, 2.90% if the goal is not reached

Catastrophe bonds, a.k.a cat bonds

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Catastrophe bonds

- Bond issued by insurance companies, typically with a short maturity and a high yield
- If a pre-specified catastrophe occurs, then the bond pays no more coupon and possibly pays a fraction of the face value to bondholders.
- Example: American Family Mutual Insurance Co.
  - Issuance date: November 2010
  - Maturity: November 2013
  - Face value: USD 1,000
  - Coupon rate: 6.5%
  - Trigger: losses to insurance industry from severe thunderstorms and tornadoes across the U.S. exceed 825 million USD
  - Issued quantity: 100,000 CAT bonds = US 1,000,000
  - During the tornado season of 2011 insurance industry losses exceeded USD 825 million $\rightarrow$ Investors were never paid the face value, AFMI could use the US 1,000,000 to recover part of the payments it made to the clients it insured against tornados.

Catastrophe bonds

- A securitized form of insurance
- Key feature is how the "trigger" is set on event-linked cat-bonds: World Bank's "pandemic bonds" were too slow to make payments to countries that needed them as the trigger didn't kick in until April 2020.
- Pros of "cat" bonds:
  - For issuers: (insurance companies) transfer to investors part of the catastrophe risk.
  - For investors: High-yield earned in case catastrophe/climate risk-related event is not materialized through a bond that is not too correlated with the market risk (i.e. low beta, at least for the moment!)
- Cons of "cat" bonds:
  - For issuers: (insurance companies) high cost of capital in case of "little" catastrophe.
  - For investors: Highly risky investment – big enough catastrophe can affect the whole economy and hence generate positive correlation with the market (i.e. high beta).

Bond trading

- Bonds typically trade in over-the-counter (OTC), which are nowadays electronic platforms

Robin Wigglesworth and **Joe Rennison** in New York AUGUST 16, 2017 3

Goldman Sachs has expanded its algorithmic corporate bond trading programme, more than trebling the number of securities it quotes since last summer to more than 7,000 — and is now eyeing an expansion into areas such as junk bonds later this year.

It comes as both banks and investors, such as hedge funds and asset managers, are focusing on automating smaller-size trades, in a bid to cut costs and free up dealers for larger transactions.

The bank's algorithm scrapes publicly-available pricing data for thousands of bonds to automatically generate firm, tradable prices for investors. Earlier this year it broke into the ranks of the top-three dealers on MarketAxess in US investment grade odd-lots — defined as smaller slivers of debt below $1m, according to Goldman Sachs.

Source

Risks of a bond or bond portfolio

- Default risk
  - Risk that what is owed by the issuer is not paid
  - Gives rise to default premium
  - Can be hedged with credit derivatives
- Interest rate risk
  - Fluctuations of market price of bond due to changes in interest rates
  - Can be hedged with portfolio immunization or interest rate derivatives

Overview

1. Bond Basics
2. **Valuation & Term Structure of Interest Rates** $\Longleftarrow$
3. Arbitrage Pricing
4. Forward Interest Rates
5. Default Risk
6. Interest Rate Risk

The yield-to-maturity of a bond

Definition

The **yield-to-maturity** is the discount rate y that makes the present value of the bond's cash-flows equal to its current price.

$$B_0 = \frac{C}{(1+y)^{t_1}} + \frac{C}{(1+y)^{t_2}} + \cdots + \frac{C+N}{(1+y)^T}$$

where B_0 is the current price of the bond.

Question: What is the yield-to-maturity of a 2-year zero-coupon bond with face value Eur 100 and current price Eur 95? 2.60%

Yield-to-maturity of zero-coupon bonds

Theorem

The yield-to-maturity of a zero-coupon bond with face value N, maturity in T years and current price of Z_0 is

$$r_T = \left(\frac{N}{Z_0}\right)^{1/T} - 1$$

Example

Consider a 4-year ZCB with face value $N = Eur\ 100$ and current price $P_0 = Eur\ 88.85$. Then

$$r_4 = \left(\frac{100}{88.85}\right)^{\frac{1}{4}} - 1 = 3\%$$

Interpretation: when I invest my money for 4 years in ZCB I receive an annual effective interest rate of 3%.

More examples on the yield-to-maturity of ZCB

- Z1 is a zero-coupon bond with maturity 1 year and face value $N1 =$ Eur 20. Its current price in Euros is $Z1_0 = 19.61$
  What is the yield to maturity of Z1?

More examples on the yield-to-maturity of ZCB

- Z1 is a zero-coupon bond with maturity 1 year and face value $N1 =$ Eur 20. Its current price in Euros is $Z1_0 = 19.61$
  What is the yield to maturity of Z1?
  Ans: 1.99%

- Z2 is a zero-coupon bond with maturity 2 years and face value $N2 =$ Eur 1020. Its current price in Euros is $Z2_0 = 961.45$
  What is the yield to maturity of Z2?

More examples on the yield-to-maturity of ZCB

- Z1 is a zero-coupon bond with maturity 1 year and face value $N1 =$ Eur 20. Its current price in Euros is $Z1_0 = 19.61$
  What is the yield to maturity of Z1?
  Ans: 1.99%

- Z2 is a zero-coupon bond with maturity 2 years and face value $N2 =$ Eur 1020. Its current price in Euros is $Z2_0 = 961.45$
  What is the yield to maturity of Z2?
  Ans: 3%

Term structure of interest rates (a.k.a., the yield curve)

- The term structure of interest rates (or yield curve) is the relation between the yield-to-maturity of zero-coupon bonds (r_t) and their maturity (t).
- They are deduced from the market price of zero coupon bonds $r_t = \left(\frac{N}{Z}\right)^{1/t} - 1$, but can also be derived from coupon bonds prices (more on this later on)
- In fact, there are several yield curves, one for each currency, and a separate yield curves exist for each level of default risk for a given currency (however, in the financial media most yield curves are drawn for govt. bonds)
- Why? $r_t =$ risk-free rate, which depends on currency
  $+$ default premium, which depends on risk of default
- Examples:
  - Latest yield curve of AAA-rated Euro area government bonds
  - Latest USA yield curve
  - Latest German versus French yield curves

How much should an investor pay for a bond?

Consider a bond B with maturity in $T = 2$ years, face value $N =$ Eur 1000, coupon $C =$ Eur 20 and frequency once per year.

- What are the cash flows that bond B will pay to its owner? Ans: Euro 20 in one year and Euro 1020 in two years
- What are the cashflows of a zero-coupon Z1 bond with maturity 1 year and face value Euro 20?

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- What are the cashflows of a zero-coupon Z1 bond with maturity 1 year and face value Euro 20? Ans: Euro 20 in one year

- What are the cash flows of a zero-coupon Z2 bond with maturity 2 years and face value Euro 1020?

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- What are the cashflows of a zero-coupon Z1 bond with maturity 1 year and face value Euro 20? Ans: Euro 20 in one year
- What are the cash flows of a zero-coupon Z2 bond with maturity 2 years and face value Euro 1020? Ans: Euro 1020 in two years
- Suppose today you can buy or short-sell Z1 for a price $Z1_0 =$ Euro 19.61. And you can buy short sell Z2 for a price $Z2_0 =$ Euro 961.45. What do you think is the fair price of bond B?

How much should an investor pay for a bond?

Consider a bond B with maturity in $T = 2$ years, face value $N =$ Eur 1000, coupon $C =$ Eur 20 and frequency once per year.

- What are the cash flows that bond B will pay to its owner? Ans: Euro 20 in one year and Euro 1020 in two years
- What are the cashflows of a zero-coupon Z1 bond with maturity 1 year and face value Euro 20? Ans: Euro 20 in one year
- What are the cash flows of a zero-coupon Z2 bond with maturity 2 years and face value Euro 1020? Ans: Euro 1020 in two years
- Suppose today you can buy or short-sell Z1 for a price $Z1_0 =$ Euro 19.61. And you can buy short sell Z2 for a price $Z2_0 =$ Euro 961.45.
  What do you think is the fair price of bond B?
  Ans: $B_0 = 19.61 + 961.45 = 981.06$

Arbitrage strategies: Money for nothing

- Suppose today that you can buy or short sell:
  - bond B for 1000 Euros.
  - Bond $Z1$ for Euros 19.61
  - Bond $Z2$ for Euros 961.45

  Can you become rich? If yes, how?

- Suppose today that you can buy or short sell:
  - bond B for 900 Euros.
  - Bond $Z1$ for Euros 19.61
  - Bond $Z2$ for Euros 961.45

  Can you become rich? If yes, how?

Pricing formula

Recall the definition of yield-to-maturity of a ZCB

$$r_T = \left(\frac{N}{Z_0}\right)^{1/T} - 1 \Rightarrow Z_0 = \frac{N}{(1+r_T)^T}$$

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We showed that

$$B_0 = 981.06 = Z1_0 + Z2_0 = 19.61 + 961.45$$

That is:

$$B_0 = \frac{N1}{1+r_1} + \frac{N2}{(1+r_2)^2} = \frac{20}{1+r_1} + \frac{1020}{(1+r_2)^2} = \frac{C}{1+r_1} + \frac{C+NB}{(1+r_2)^2}$$

Bond price & interest rates

- Bond's price is equal to the discounted value of its promised cash flows
- For a coupon-paying bond with next coupon due in exactly one year:

$$B_0 = \frac{C}{1+r_1} + \frac{C}{(1+r_2)^2} + \ldots + \frac{C}{(1+r_{T-1})^{T-1}} + \frac{C+N}{(1+r_T)^T}$$

where r_t is the interest rate corresponding to time t in the yield curve.

- Alternatively, we can re-write this equation in terms of discount factors (d), i.e., present value of one euro to be gotten in t years:

$$B_0 = C \times d_1 + C \times d_2 + \ldots + C \times d_{T-1} + (C+N) \times d_T$$

where, $d_t = 1/(1+r_t)^t$

- For a zero-coupon bond:

$$Z_0 = \frac{N}{(1+r_T)^T} = N \times d_T$$

- r_T is also called the T-year interest rate

Term structure of interest rates (a.k.a., the yield curve)

- Remember that the equilibrium price of a coupon-paying bond, whose next annual coupon payment is exactly one year away, is:

$$B_0 = \frac{C}{1+r_1} + \frac{C}{(1+r_2)^2} + \ldots + \frac{C}{(1+r_{T-1})^{T-1}} + \frac{C+N}{(1+r_T)^T}$$

- For example, for a sovereign bond with no default-risk, r_1, r_2, ..., r_{T-1}, and r_T correspond to the $1-$, $2-$, ..., $(T-1)-$, and $T-$year interest rates on that sovereign issuer's risk-free yield curve.

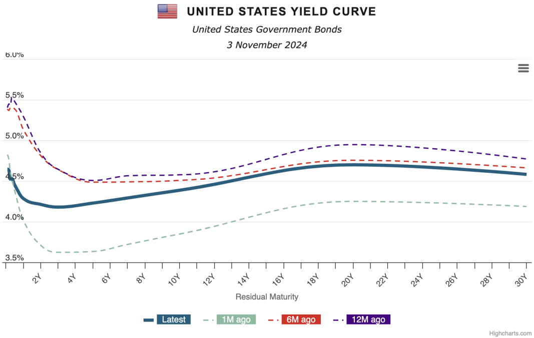
- These rates are easy to find on the Term Structure of Interest Rates Table that corresponds to that particular Yield Curve (see the next slide).

U.S. Term Structure & Yield Curve on October 28, 2023

US Treasury Securities-based yield curve

Residual Maturity	Last
1 month	4.646%
2 months	4.631%
3 months	4.522%
4 months	4.530%
6 months	4.452%
1 year	4.284%
2 years	4.212%
3 years	4.179%
5 years	4.228%
7 years	4.291%
10 years	4.386%
20 years	4.699%
30 years	4.581%

Last Update: 3 Nov 2024, 0:15 GMT-



More on the Yield-to-maturity

- Equivalently, we could write the price of a bond in terms of its bond-specific yield-to-maturity (also known as YTM or yield).
- A bond's yield-to-maturity, which is specific to that particular bond, is the discount rate y such that the bond's price equals the present value of its cash flows discounted at rate y:

$$\boxed{B_0 = \frac{C}{1+y} + \frac{C}{(1+y)^2} + \ldots + \frac{C}{(1+y)^{T-1}} + \frac{C+N}{(1+y)^T}}$$

- The YTM (yield) of a bond is the annual return rate an investor makes if 1) he/she keeps the bond until maturity, and 2) the bond issuer does not default.
- The above formula can be re-written using the annuity formula:

$$\boxed{B_0 = \frac{C}{y}\left[1 - \left(\frac{1}{1+y}\right)^T\right] + \frac{N}{(1+y)^T}}$$

Exercise 1

A government bond with exactly 2-years until maturity, $C = 150$ € and $N = 1000$ €. The government bond term structure has the discount rates $r_1 = 2\%$ and $r_2 = 4\%$.

Question 1 Can its yield be equal to $y = 1.5\%$?

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Impossible: The yield must be a weighted-average of $r_1 = 2\%$ and $r_2 = 4\%$

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Yes:

$$B_0 = \frac{C}{1+r_1} + \frac{C+N}{(1+r_2)^2} = \frac{C}{1+y} + \frac{C+N}{(1+y)^2}$$

$$B_0 = \frac{150}{1+0.02} + \frac{150+1,000}{(1+0.04)^2} = \frac{150}{1+0.0387} + \frac{150+1,000}{(1+0.0387)^2} = 1,210.3 \text{ €}$$

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Only if $C = 0$, i.e., if the bond is a 2−year "zero-coupon bond".

Exercise 2

A coupon bond has a face value of $N = €1,000$, an annual coupon frequency (next coupon is exactly one year away from today) with a coupon rate of 8%, and a maturity of exactly 10 years. Discount rates on this bond are constant across maturities $r_1 = r_2 = \ldots = 6\%$ (i.e., the yield curve is flat).

Question 1 What is the bond's yield-to-maturity?

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$$r_1 = r_2 = ... = r_{10} = 6\% \Rightarrow$$

$$B_0 = \frac{80}{1+r_1} + \frac{80}{(1+r_2)^2} + \ldots + \frac{80}{(1+r_9)^9} + \frac{80+1000}{(1+r_{10})^{10}} \Rightarrow \text{y=6\%}$$

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$$B_0 = \frac{80}{0.06}\left[1 - \frac{1}{1.06^{10}}\right] + \frac{1{,}000}{1.06^{10}} = 1,147.20\ €$$

Question 3 What if discount rates r_t are now all equal to 8%?

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$$y = 8\% \Rightarrow B_0 = 1,000 \text{ €} = N$$

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A coupon bond has a face value of $N =$ €1,000, an annual coupon frequency (next coupon is exactly one year away from today) with a coupon rate of 8%, and a maturity of exactly 10 years. Discount rates on this bond are constant across maturities $r_1 = r_2 = \ldots = 6\%$ (i.e., the yield curve is flat).

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$$B_0 = \frac{80}{0.10}\left[1 - \frac{1}{1.10^{10}}\right] + \frac{1{,}000}{1.10^{10}} = 877.11 \text{ €}$$

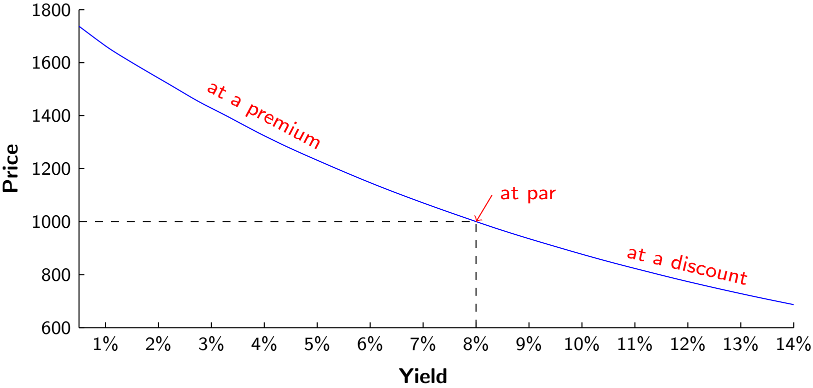
Lesson from Exercise 2

- The bond price and the bond yield are inversely related
  If the next coupon date is in exactly one year, then

$$\text{yield} < \text{coupon rate} \Leftrightarrow \text{price} > \text{face value: bond is selling “at a premium”}$$

$$\text{yield} = \text{coupon rate} \Leftrightarrow \text{price} = \text{face value: bond is selling “at par”}$$

$$\text{yield} > \text{coupon rate} \Leftrightarrow \text{price} < \text{face value: bond is selling “at a discount”}$$



Exercise 3

Consider a bond with face value €100, annual coupon rate of 5% with next annual coupon exactly one-year away, and maturity in 3 years. The 1-year, 2-year and 3-year discount rates on the bond are $r_1 = 1.0\%$, $r_2 = 1.5\%$ and $r_3 = 2.0\%$.

Question 1 What is the bond's price? 108.75

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$$B_0 = \frac{5}{1+0.010} + \frac{5}{(1+0.015)^2} + \frac{5+100}{(1+0.020)^3} = \text{€}108.75$$

Question 2 What is the bond's yield (i.e., yield-to-maturity?) $y = 1.97\%$

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y such that $\frac{5}{1+y} + \frac{5}{(1+y)^2} + \frac{105}{(1+y)^3} = 108.75.$

Use a computer, or MS Excel's Solver, or a financial calculator to find $y = 1.97\%$

More Exercises

Exercise 4 A coupon bond has a face value of €100, an annual coupon, a maturity of 10 years, a yield of 5%, and a price of €115.443. What is this bond's coupon? €7

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$$B_0 = \frac{C}{y}\left[1 - \frac{1}{(1+y)^T}\right] + \frac{N}{(1+y)^T} \Leftrightarrow C = y\frac{B_0 - \frac{N}{(1+y)^T}}{1 - \frac{1}{(1+y)^T}} = €7$$

Exercise 5 A coupon bond has a face value of €100, an annual coupon of €4, a yield of 3%, and a price of €103.717.
What is this bond's maturity? 4 years

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$$B_0 = \frac{C}{y} + \frac{N - \frac{C}{y}}{(1+y)^T} \Leftrightarrow (1+y)^T = \frac{N - \frac{C}{y}}{B_0 - \frac{C}{y}}$$

$$\Leftrightarrow T = \log\left(\frac{N - \frac{C}{y}}{B_0 - \frac{C}{y}}\right) / \log(1+y) = 4 \text{ years}$$

Exercise 6 A coupon-bond has a face value of €100,000, coupons of €2,000 that are received *semi-annually*, a maturity of 5 years, and an *annually* compounded yield of 5%. What is this bond's price? €95,880

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$$y = 1.05^{0.5} - 1 = 2.47\%.$$

$$B_0 = \frac{2{,}000}{0.0247}\left[1 - \frac{1}{1.0247^{10}}\right] + \frac{100{,}000}{1.0247^{10}} = €95{,}880$$

Recap: Coupon rate vs Yield-to-maturity vs Market Interest rates

- The issuer typically chooses bond's T, N, and c (but *not* y):
  - In the case of fixed-rate-coupon bonds, the issuer sets the amount to be raised (i.e., par value N times the number of bonds to be sold), and commits to pay back a fixed stream of cash flows (coupons C + par [face] value N);
  - The issuer cannot choose y at issue, which is determined by bond's Initial Public Offering (IPO) process (for corporates) or auction (for govies) and depends on the currency and default risk perceived by investors;
  - The issuer sets the bond's coupon rate c equal to its required-rate-of-return y as of issuance date t=0;
  - This has the effect of equating the price of the bond at issuance (P_0) equal to its par (face) value N;
  - Thus, the firm/government borrows an amount N × number of bonds sold;
  - As the relevant yield curve changes over time, so does the price of the bond in the secondary market;
  - As a result, yield to maturity of the bond would change over time as well, and so would the actual (ex post) realized rate of return on the bond, which typically will not be equal to (ex ante) y;
  - One exception to the last bullet point is zero-coupon bonds under the condition that they are held until maturity: then $y_0 = y_T = r_T$

Overview

1. Bond Basics
2. Valuation & Term Structure of Interest Rates
3. **Arbitrage Pricing** $\Longleftarrow$
4. Forward Interest Rates
5. Default Risk
6. Interest Rate Risk

Arbitrage pricing

- We want to price bond XYZ, which is a default-risk-free bond with face value €1000, maturity 3 years, coupon rate 10% with annual payments.

  We know the prices of 1-year, 2-year, and 3-year default risk-free zeros with face value €100: €97, €92, and €86, respectively.

Question 1 Find the portfolio composed of the 1-, 2-, and 3-year zeros that replicates the cash flows of bond XYZ

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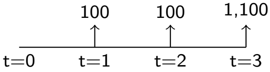

1 one-year zero + 1 two-year zero + 11 three-year zeros

- This is the replicating portfolio for bond XYZ

Arbitrage pricing

Question 2 Then, what should be the price of bond XYZ?

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- This is the Law of One Price: "If two sets of securities have exactly the same risk, same cash flows at the exactly same future dates, then their prices should exactly be the same today."

Question 3 What would you do if the *market price* of XYZ is €1150?

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short-sell bond XYZ	+1150	-100	-100	-1,100

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buy 1×1-yr zero	-97	+100	0	0
buy 1×2-yr zero	-92	0	+100	0
buy 11×3-yr zero	-11×86	0	0	+11×100
Total	+15	0	0	0

Arbitrage pricing

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– how much money did you tie up initially?

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zero (so long as all of the trades are done at the same split second)

– what is the maximum amount of profits you can make?

Arbitrage pricing

Question 4 In doing these trades *simultaneously*:

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zero, since you use no money of your own by doing all of the above trades simultaneously (if you don't do simultaneous trades, you would expose yourself to the risk that prices can change and you might not only make less money, but could actually lose money)

– what is your maximum loss?

zero (so long as all of the trades are done at the same split second)

– what is the maximum amount of profits you can make?

unlimited, because you can repeat this trade many times (as long as prices don't adjust)

- When the law of one price is not satisfied, there is an arbitrage opportunity: a costless portfolio that cannot result in a loss and has a non-zero probability of a positive profit – i.e., misalignment in security prices lead to risk-free arbitrage

Question 5 What if the price of XYZ is €1000?

Arbitrage pricing

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Question 5 What if the price of XYZ is €1000?

XYZ is too cheap compared to the replicating portfolio's value: buy XYZ and short-sell the replicating portfolio

Arbitrage in financial markets

- In practice, the *Law of One Price* almost always perfectly holds
  - Because arbitrage transactions increase (decrease) the price of the relatively cheaper (expensive) securities, the arbitrage opportunity would dissipate as arbitrageurs take advantage of mispricings across securities that have the same risk and that can be combined into portfolios that can replicate another's CFs (this is especially true with algorithmic trading).
  - So, when you see what looks line an arbitrage opportunity:

    ...it may actually be risky arbitrage, if the risk of default is not the same across securities (in the above examples, default risk was zero across the board)

    ...there may be transaction costs, which make the trade unprofitable if they are higher than the initially perceived arbitrage profits

    ...it might be a real arbitrage opportunity: be fast, it will disappear quickly: especially so in the era of algorithmic trading!

- The Law of One Price does **not necessarily** imply that the market is informationally efficient
  - It might so happen that two securities with the same cash flows may have the same price, yet this price might not reflect economic fundamentals

Arbitrage pricing without zeros

- Goal: Price Bond 4

	Maturity	Face value	Coupon	Price
Bond 1	1 year	€ 100	none	€ 98.04
Bond 2	3 years	€1,000	€ 50	€ 927.90
Bond 3	3 years	€ 1,000	€ 40	€ 901.08
Bond 4	3 years	€ 1,000	€ 100	?

- Method:

Step 1 Find the replicating portfolio of Bond 4

Step 2 Apply the law of one price

Arbitrage pricing without zeros

Step 1: Find Bond 4's replicating portfolio

- It is composed of n_1 numbers of Bonds 1; n_2 of Bonds 2; and n_3 of Bonds 3
- n_1, n_2, and n_3 are such that it replicates Bond 4's cash flows at

t=1: $100 \times n_1 + 50 \times n_2 + 40 \times n_3 = 100$

Arbitrage pricing without zeros

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t=3: $0 \times n_1 + 1050 \times n_2 + 1040 \times n_3 = 1100$

Arbitrage pricing without zeros

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t=3: $0 \times n_1 + 1050 \times n_2 + 1040 \times n_3 = 1100$

Last two generate the amounts n_2 of Bond 2 and n_3 of Bond 3 in the replicating portfolio, then the first equation would give n_1 for Bond 1

Arbitrage pricing without zeros

Find n_1, n_2, and n_3

From Eq. for t=2 above: $n_3 = \dfrac{100 - 50 \times n_2}{40}$

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$$1050 \times n_2 + 1040 \times \frac{100 - 50 \times n_2}{40} = 1100 \quad \Rightarrow \quad n_2 = 6$$

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$\Rightarrow$ Replicating portfolio consists of 6 long (buying) positions in Bond 2, 5 short (selling) positions in Bond 3, and no Bond 1.

Step 2: Law of One Price

$$P_0(\text{Bond } 4) = 6 \times 927.90 - 5 \times 901.08 = 1,062 \text{ €}$$

- An alternative method for pricing bonds: determine the yield curve first, and then price the bond(s) (next exercise)

Exercise

Consider these three bonds, all with face value of €1000 and annual coupons

	Maturity	Coupon rate	Price
Bond A	1 year	2%	€ 980.77
Bond B	2 years	6%	€ 1,019.14
Bond C	3 years	5%	€ 975.03

Q1 Find the yield curve that is prevailing in the background

Alternatively, first find the discount factors d_1, d_2, d_3, and then r_1, r_2, r_3:

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Then: $d_1 = \frac{1}{1+r_1} \Rightarrow r_1 = \frac{1}{d_1} - 1 = \frac{1}{0.96154} - 1 = 0.04 = 4\%$

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Exercise (cont'd)

Q2 What is the price of a 3-year zero-coupon bond D with a face value of €100,000?

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$\frac{100,000}{1.06^3} = 83,961.93$

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Suppose the previous zero called D is trading at €83,000 in the market.

Q3 Find an arbitrage strategy: what do we do first?

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Find the composition of the portfolio that replicates zero-coupon bond D → find the numbers of Bond A, B, and C that are needed: n_A, n_B, n_C for, respectively, such that

$$\begin{cases} t = 1: & 0 & = & 1,020 n_A & + & 60 n_B & + & 50 n_C \\ t = 2: & 0 & = & & & 1,060 n_B & + & 50 n_C \\ t = 3: & 100,000 & = & & & & & 1,050 n_C \end{cases}$$

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$n_C = +95.2381$
$n_B = -4.4924$
$n_A = -4.4043$

Exercise (cont'd)

Now, we can construct the arbitrage table:

	t=0	t=1	t=2	t=3
Buy 1 zero-coupon	–83,000.00	0	0	+100,000.00

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Short-sell n_C of C	+95.2381x975.03	–95.2381 x 50	–95.2381x50	–95.2381x1,050
Repl'ting p/f total	+83,961.93	0	0	–100,000.00
Arbitrage total	+961.93	0	0	0

Exercise (cont'd)

Suppose that the price of Bond D in the market is instead 84,000 €: how can you do arbitrage now using the same replicating portfolio?

	t=0	t=1	t=2	t=3
Short-sell 1 Bond D	+84,000.00	0	0	–100,000.00

Exercise (cont'd)

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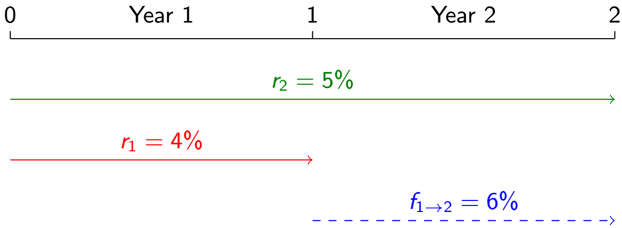
Short-sell n_B of B	+4.4924x1,019.14	–4.4924x60	–4.4924x1,060	
Buy n_C of C	–95.2381x975.03	+95.2381x50	+95.2381x50	+95.2381x1,050
Replicating p/f total	–83,961.93	0	0	+100,000.00
Arbitrage Total	+38.17	0	0	0

Overview

1. Bond Basics
2. Valuation & Term Structure of Interest Rates
3. Arbitrage Pricing
4. **Forward Interest Rates** $\Longleftarrow$
5. Default Risk
6. Interest Rate Risk

Forward rates

- Suppose the 1-year interest rate is 4% per year and the 2-year interest rate is 5% per year:



- Investing €1 in a two-year bond yields €1×1.05^2 in two years
- Notice that €$1 \times 1.05^2 =$ €$1 \times 1.04 \times 1.06$
- Thus, investing in a two-year bond locks in a return of 6% over Year 2. This rate is called the forward interest rate (or simply the forward rate) and is denoted by $f_{1\to2}$

Forward rates

- The one-year forward rate one can lock in for investing between t=1 and t=2 is determined by

$$(1 + r_2)^2 = (1 + r_1) \times (1 + f_{1\to 2}) \quad \Leftrightarrow \quad f_{1\to 2} = \frac{(1 + r_2)^2}{1 + r_1} - 1$$

- It does not mean that next year's one-year rate will be necessarily equal to $f_{1\to 2}$

    - $f_{1\to 2}$ is *today's expectations* about next year's one-year interest rate imbedded in bond prices reflecting bond market participants anticipation of future one-year rate one-year away *(for more on this, see upper level classes in finance)*

Forward rates

- More generally, the forward rate $f_{T_1 \to T_2}$ for investing between t$=T_1$ and t$=T_2$ is determined by

$$(1 + r_{T_2})^{T_2} = (1 + r_{T_1})^{T_1} \times (1 + f_{T_1 \to T_2})^{T_2 - T_1}$$

- In fact, any T-year interest rate r_T read from the yield curve (or the term structure of interest rates table) is nothing more than an average of the 1-year rate that prevails today (because the 1-year zero-coupon is sold today, so we can observe its price, and hence its yield) and the 1-year forward rates that will prevail in any year between dates t=1 and t=T:

$$(1+r_T)^T = (1+r_1)\times(1+f_{1\to 2})\times(1+f_{2\to 3})\ldots\times(1+f_{T-2\to T-1})\times(1+f_{T-1\to T})$$

- As such, r_T is a geometric average of all the 1-year interest rates r_1, $f_{1\to 2}, f_{2\to 3}, \ldots, f_{T-1\to T}$

Example

In one year from now your company will have to take out a $1000 loan with 4-year maturity. Suppose you can invest and borrow in US dollars at the risk-free rate.

Question 1 What is the per-year interest rate you can lock in today for a 4-year borrowing starting with date t=1?

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$$(1 + r_5)^5 = (1 + r_1)(1 + f_{1\to 5})^4 \Rightarrow f_{1\to 5} = \left(\frac{(1+r_5)^5}{(1+r_1)}\right)^{1/4} - 1$$

Question 2 Show the trades you can implement today to lock in this rate

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Question 2 Show the trades you can implement today to lock in this rate

I want +1000 at t=1 $\to$ at t=0 invest $\frac{1000}{1+r_1}$ in the 1-year bond

To fund this position: borrow an amount $\frac{1000}{1+r_1}$ with 5-year maturity (short-sell $\frac{1000}{1+r_1}$ of the 5-year zero)

$\to$ will have to repay $\frac{1000}{(1+r_1)} \times (1+r_5)^5 = 1000 \times \frac{(1+r_5)^5}{(1+r_1)}$ at t=5

This is equivalent to repaying $1000 \times (1+f_{1\to5})^4$ at t=5

The yield curve and the business cycle

What does the shape of the term structure imply? US yield-curves since 2000

- Upward sloping: normal times

  Remark: This is how commercial banks make money as they borrow short term (through deposits) and lend long term

- For the explanations below, remember that any r_T is a geometric average of all the 1-year interest rates, the one now observable 1-year rate and all 1-year forward rates in the future, all the way up to T:
- Steeply upward sloping: usually forecasts economic expansion – Why?

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What does the shape of the term structure imply? US yield-curves since 2000

- Upward sloping: normal times

  Remark: This is how commercial banks make money as they borrow short term (through deposits) and lend long term

- For the explanations below, remember that any r_T is a geometric average of all the 1-year interest rates, the one now observable 1-year rate and all 1-year forward rates in the future, all the way up to T:
- Steeply upward sloping: usually forecasts economic expansion – Why?
  - Steep yield curve means higher r_t's going forward. This suggests that bond market participants expect that the Central Bank will conduct a monetary policy that decreases the liquidity (cash) in the banking system in order to increase the short term interest rate in the future. As 1-year forward rates go up, so do r_t's, hence the steep yield curve.
- Inverted: rare but most likely a sign of recession ahead, as during periods that right before recessions bond market participants expect the CB to conduct a monetary policy that will flush the banking system with liquidity (so that banks will lend again, and economic activity will get the financing it needs), which will push forward interest rates to decline.

Risks of a bond (or a bond portfolio)

1. Default risk

    - Risk that the payments (C and/or N) promised at issue date are not paid
    - This possibility gives rise to default premium
    - For individual bonds the default risk can be hedged with credit derivatives if such instruments are available (not covered in the course but see problem 3 in problem set on bonds)

2. Interest rate risk

    - Even default-risk-free bonds are subjected to interest rate risk !
    - This risk is due to the fluctuations in the market prices of bonds due to changes in economy-wide interest rates (as reflected in changes in the risk-category-specific yield curve)
    - And it can be hedged with interest rate derivatives (see forward rates)

Overview

1. Bond Basics
2. Valuation & Term Structure of Interest Rates
3. Arbitrage Pricing
4. Forward Interest Rates
5. **Default Risk** $\Longleftarrow$
6. Interest Rate Risk

Default risk

- A bond's **default risk** is the risk that the issuer of the bond will not pay all the promised coupons and/or face value payments.

Default risk

- A bond's **default risk** is the risk that the issuer of the bond will not pay all the promised coupons and/or face value payments.
- The interest rate is the sum of two components:

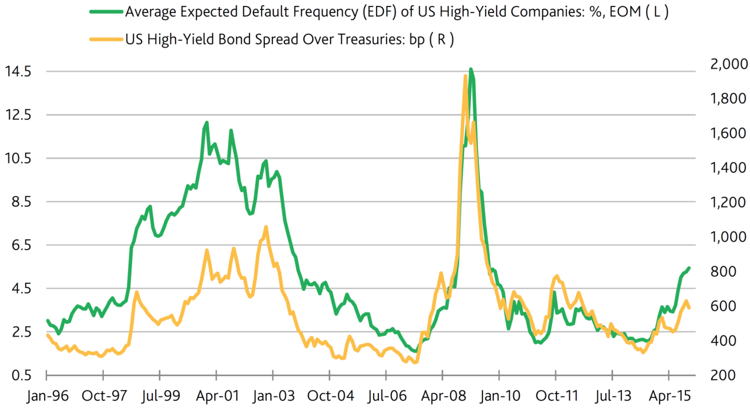
$$\boxed{\text{interest rate } (r_t) = \text{risk-free rate } (r_t^f) + \text{default premium}}$$

- The risk-free interest rate (r_t^f) is the discount rate if the promised CF is for sure
  - It is the same for all issuers in a given currency
- The default premium (or default spread) is the compensation for the risk of default
  - It depends on the probability of default of the particular issuer
- r_t, r_t^f, and default premium all depend on horizon (t)

Default risk and bond yields

Corporate bond risk is highly correlated with higher corporate default rates.

- Average 10-year corporate bond yields (right scale) and default rates (left scale):



Default risk

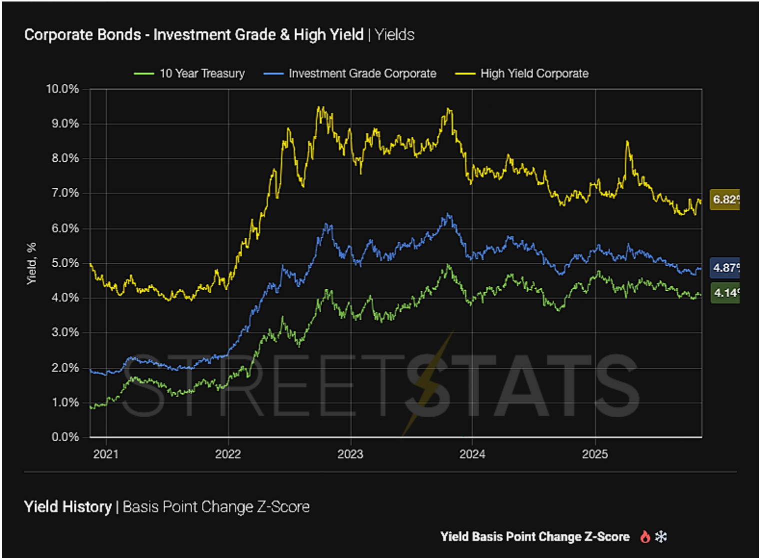
- Default (or Credit) risk: risk that the issuer defaults on bond's promised payments:
    - Corporates and certain sovereigns are subject to default risk (Argentina 2001 2014, Greece 2011, General Motors 2009, Banca Monte dei Paschi 2017, etc.)
- Credit ratings by rating agencies provide indications of the likelihood of default

	S&P	Moody's	Fitch
	AAA	Aaa	AAA
Investment	AA	Aa	AA
grade	A	A	A
	BBB	Baa	BBB
	BB	Ba	BB
	B	B	B
High-yield	CCC	Caa	CCC
	CC	Ca	
	C		
In default	D	C	D

- Default risk is reflected in higher bond yields through the default premium

Default risk and bond yields

Investors require higher returns on bonds with higher risk of default, so the spread between bonds with high and lower ratings typically rises during recessions:



Default risk and bond yields

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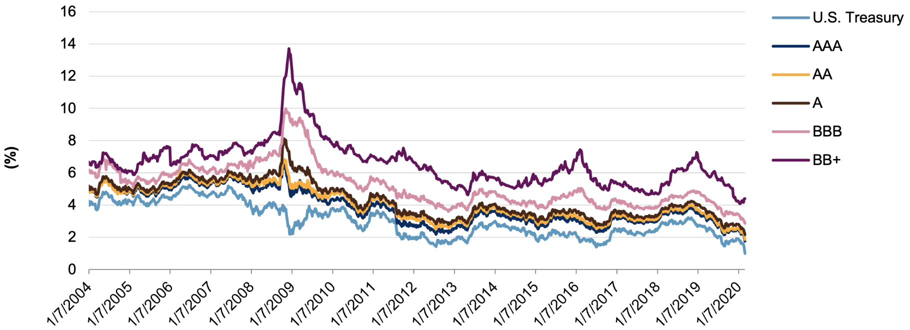
Historical Corporate Bond Yields--10-Year Maturity



Data as of March 4, 2020. S&P Global Ratings Research.

Default risk: high-yield bonds

- High-yield bonds are also called “junk bonds”

can be “original junk issues” or “fallen angels”

Altice returns to bond market with $3bn sale

Robert Smith JULY 17, 2018 8

Altice’s French unit completed a nearly $3bn high-yield debt sale on Tuesday, raising junk bonds for the first time since concerns around the cable group’s debt pile spooked investors at the end of last year.

The group raised $1.75bn of dollar bonds at 8.125 per cent yield and €1bn of euro bonds at 5.875 per cent yield, the highest yields its French unit has been charged in both markets since it was created out of the merger of Numericable and SFR in 2014.

Source

Bond investors wary of threat from potential ‘fallen angels’

Analysts expect more companies to complete slide from investment grade to junk

Eric Platt in New York JULY 25, 2016 1

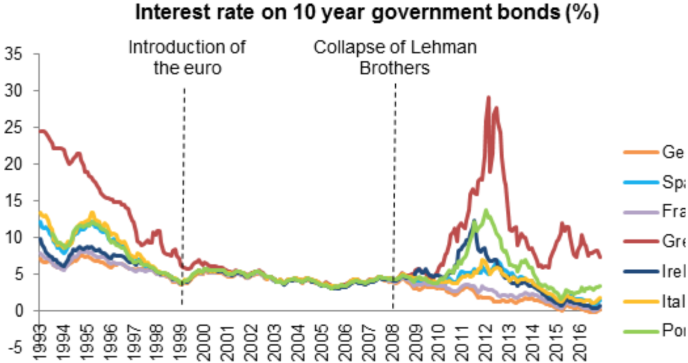
Investors in the most highly rated US corporate bonds have enjoyed a buoyant 2016, with the Barclays index returning nearly 9 per cent. However, the market now faces an unsettling threat from a potential new wave of fallen angels — companies that first sold debt with investment grade status but have since been downgraded to junk.

The list of companies now on the brink of junk includes watchmaker Fossil Group, which suffered a 9 per cent drop in sales in its first quarter, and internet security company Symantec after it agreed to purchase Blue Coat for $4.65bn with $2.8bn of new debt. They join multinationals such as Rémy Cointreau, LG Electronics and miner Goldcorp sitting on the edge of speculative rating territory.

Source

Default risk: sovereign bonds

- Government bonds are not necessarily default-risk-free:
  - 10-year European sovereign bond spreads (above risk-free German rate)



- Today government spreads

Overview

1. Bond Basics
2. Valuation & Term Structure of Interest Rates
3. Arbitrage Pricing
4. Forward Interest Rates
5. Default Risk
6. **Interest Rate Risk** $\Longleftarrow$

Interest rate risk

Exists for all bonds (including default-risk free bonds), because the yield curve shifts constantly:

The US Treasury Yield Curve since year 2000

Shifts in bond yields can be abrupt – 10-year US Treasury yields' change on 2020.11.09

Interest rate risk

You hold a bond with a face value of N=$1000 that matures in exactly 2 years, it makes annual coupon payments next of which is exactly one year away, the corresponding yield curve is currently flat at 5% and the bond is trading at par.

Q1 What is the YTM and coupon of the bond?

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Flat yield curve $\Rightarrow y = 5\%$ Bond sells at par $\Rightarrow c := \frac{C}{N} = y = 5\%$ $C = \$50$

Q2 What is your holding period return over the course of next year if interest rates do not change?

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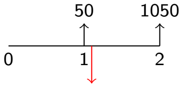
$HPR_{1year} = \frac{C + P_1 - P_0}{P_0}$ where P_1 is the price just after the next coupon payment



$P_1 = \frac{1050}{1+0.05} = \$1000 \Rightarrow HPR_1 = \frac{50+0}{1000} = 0.05 = 5\%$ per year

Q3 What if all interest rates on the yield curve increase by 1 percentage point (and the yield curve remains flat) between today and next year?

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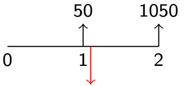

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promised CF have not changed, but return is $\searrow$ because P_1 is $\searrow$ as rates $\nearrow$

Q4 What if they decrease by 1 percentage point?

Interest rate risk

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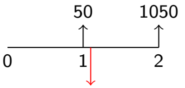

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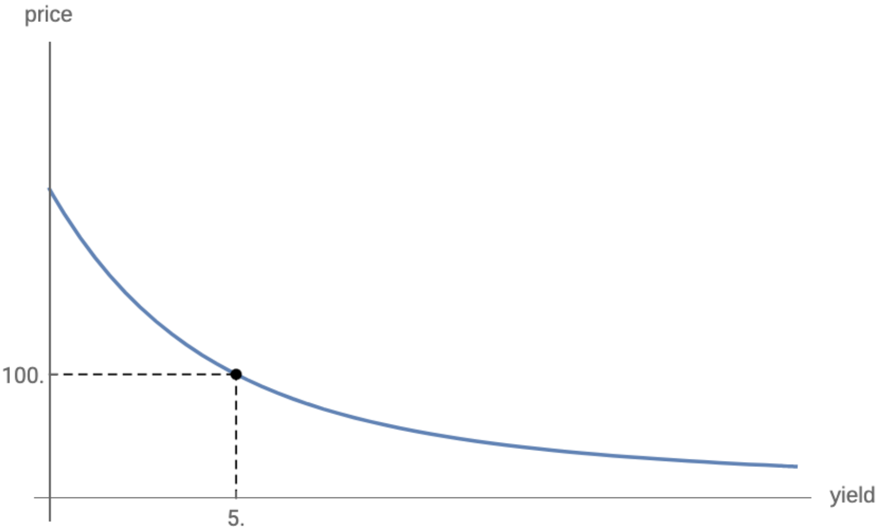
promised CF have not changed, but return is $\searrow$ because P_1 is $\searrow$ as rates $\nearrow$

Q4 What if they decrease by 1 percentage point?

$P_1 = \frac{1050}{1.04} \approx \$1,010 \Rightarrow \text{HPR} = \frac{50+10}{1000} = 6\%$ per year

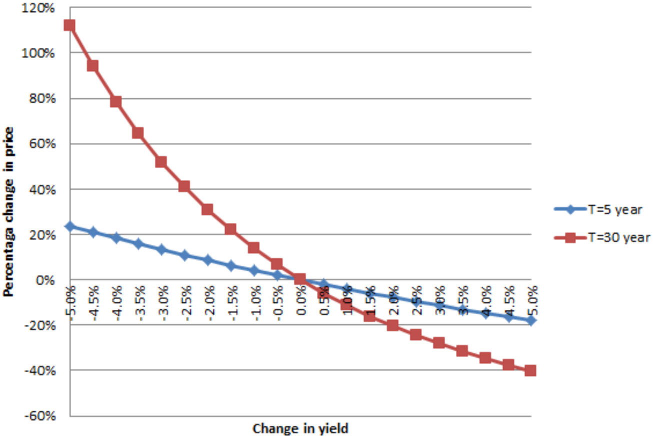
Factors influencing interest rate risk

Reminder – Bond Price-Yield Curve:



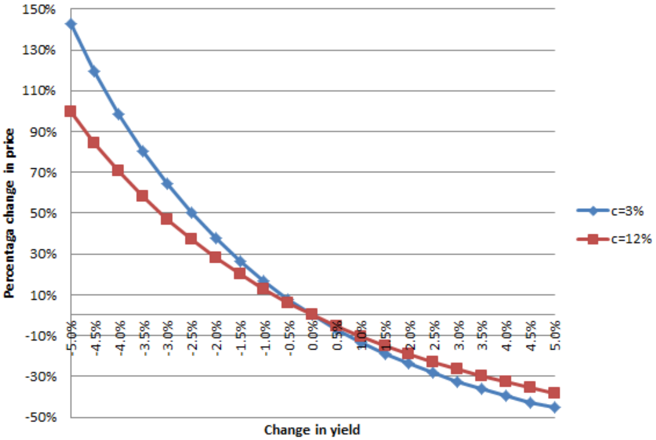
Factors influencing interest rate risk

- The longer the time to maturity (T) of a bond, the *more* its value will be affected by a change in interest rates



Factors influencing interest rate risk

- The larger the coupon (C) of a bond, the *less* its value will be affected by a change in interest rates



Duration

- The duration (D) of a bond is an indicator of the sensitivity of its price to interest rates movements:
  - If the yield changes by a small Δy, the % change in the bond price is:

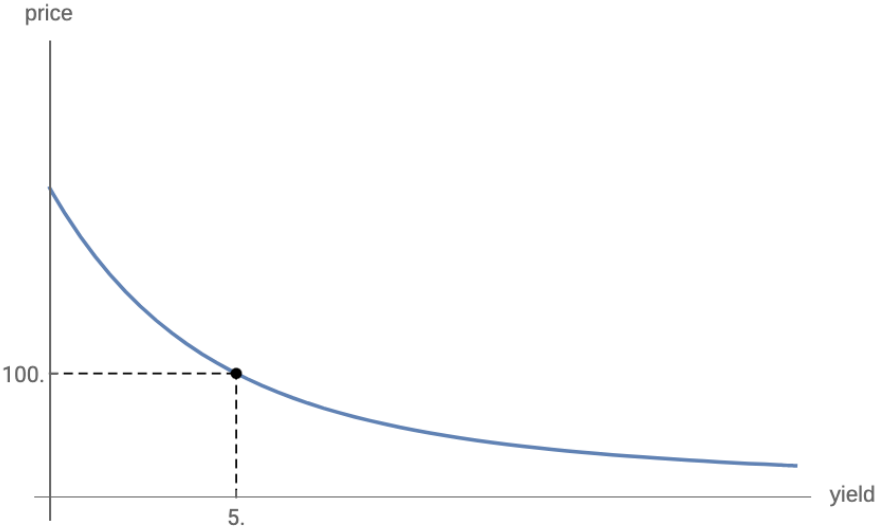
$$\boxed{\frac{\Delta P_0}{P_0} = -D\frac{\Delta y}{1+y}}$$

  - This formula *approximates the actual percentage change in the bond's price given a change in* y*: as an approximation, it works for small* Δy *around the original* y *(but the approximation doesn't work well for large* Δy *– see the next three slides).*
  - *Example: for a bond with* $D = 5$ *years, if its yield increases from 0% to 1%, then its price decreases by 5%:*

$$\frac{\Delta P_0}{P_0} = -D\frac{\Delta y}{1+y} = -5 \times \frac{(0.01 - 0.00)}{1 + 0.00} = -0.05$$

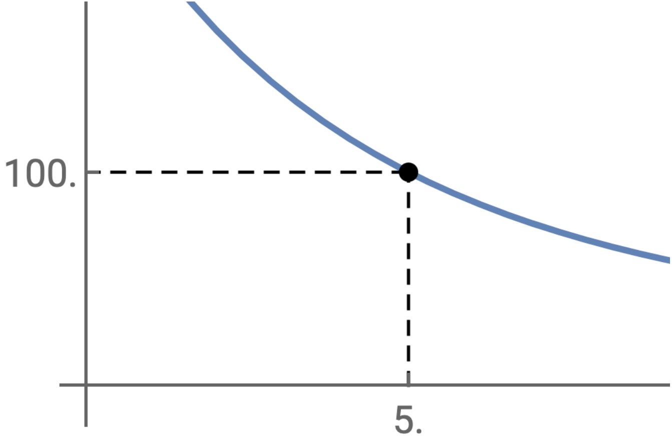
Bond price changes with the Duration approach

Reminder – Bond Price-Yield Curve:



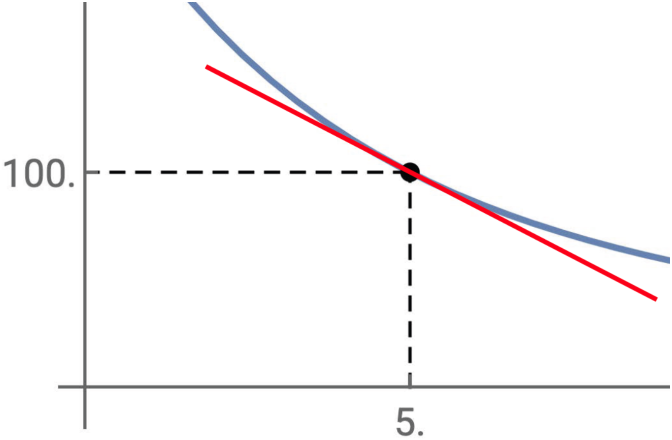
Bond price changes with the Duration approach

Magnifying the area around $y = 5\%$:



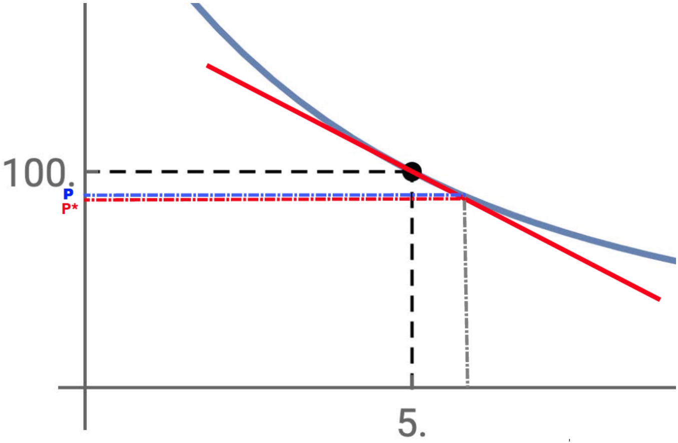
Factors influencing interest rate risk

The duration formula relies on the first-order Taylor Series expansion to approximate the bond pricing curve around a given y (here around $y = 5\%$) with a straight line (the tangent at y):



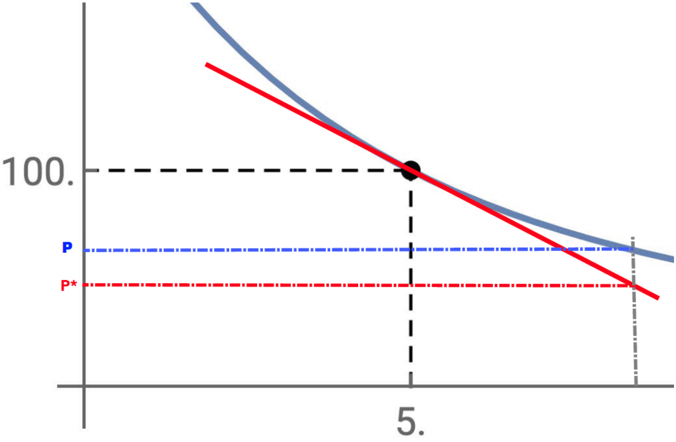
Factors influencing interest rate risk

This approximation works well for small Δy (i.e., close neighborhood of y):



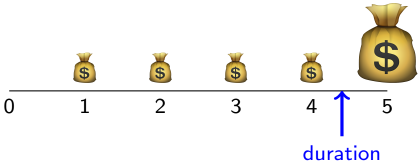
Factors influencing interest rate risk

But the approximation becomes more and more imprecise as Δy gets large (i.e., as we move away from the close vicinity of y):



Duration

- Intuition: longer duration when cash flows further away in the future



- Formula: D = weighted-average maturity of cash flows ($\Rightarrow D$ is in years)

$$\boxed{D = w_1 \times 1 + w_2 \times 2 + ... + w_{T-1} \times (T-1) + w_T \times T}$$

where

$$w_1 = \frac{\frac{C}{1+y}}{P_0}; \quad w_2 = \frac{\frac{C}{(1+y)^2}}{P_0}; \quad ... \quad w_{T-1} = \frac{\frac{C}{(1+y)^{T-1}}}{P_0}; \quad w_T = \frac{\frac{C+N}{(1+y)^T}}{P_0}$$

with:

$$\boxed{w_1 \ + \ w_2 \ + \ ... \ + \ w_{T-1} \ + \ w_T \ = \ 1}$$

Proof of the duration formula

Duration is the elasticity of the bond's price to (1+y):

$$D = -\frac{\frac{dP_0}{P_0}}{\frac{d(1+y)}{1+y}}$$

which can be re-written as (note $d(1+y) = d(y)$ since 1 is a constant):

$$D = -\frac{dP_0}{d(1+y)} \times \frac{1+y}{P_0} \quad \Rightarrow \quad D = -\frac{dP_0}{d(y)} \times \frac{1+y}{P_0} \tag{1}$$

The bond price is equal to:

$$P_0 = \frac{C}{1+y} + \frac{C}{(1+y)^2} + \ldots + \frac{(C+N)}{(1+y)^T}$$

To take the derivative of the ratio of two functions with respect to y:

$$\frac{d\left(\frac{f(y)}{g(y)}\right)}{d(y)} = \frac{f'(y)g(y) - f(y)g'(y)}{(g(y))^2}$$

Proof of the duration formula

For example, suppose that $f(y) = C$ and $g(y) = (1+y)^2$, then:

$$\frac{d\left(\frac{C}{(1+y)^2}\right)}{d(y)} = \frac{0 \times (1+y)^2 - C \times (2 \times (1+y))}{((1+y)^2)^2} = -\frac{2C \times (1+y)}{(1+y)^4} = -\frac{2C}{(1+y)^3}$$

Then, if we differentiate P_0 with respect to y, we get:

$$\frac{dP_0}{dy} = -\frac{C}{(1+y)^2} - \frac{2C}{(1+y)^3} - \ldots - \frac{T(C+N)}{(1+y)^{T+1}} \tag{2}$$

Replacing (2) into (1), we obtain the duration formula:

$$D = \frac{1 \times \frac{C}{1+y} + 2 \times \frac{C}{(1+y)^2} + \ldots + (T-1) \times \frac{C}{(1+y)^{T-1}} + T \times \frac{C+N}{(1+y)^T}}{P_0}$$

$$D = 1 \times \left(\frac{\frac{C}{1+y}}{P_0}\right) + 2 \times \left(\frac{\frac{C}{(1+y)^2}}{P_0}\right) + \ldots + (T-1) \times \left(\frac{\frac{C}{(1+y)^{T-1}}}{P_0}\right) + T \times \left(\frac{\frac{C+N}{(1+y)^T}}{P_0}\right)$$

Duration of a Portfolio of Bonds

- The **duration of a portfolio of bonds** is the weighted average of the durations of the bonds in the portfolio, with the weights being the fraction of money invested in each bond.

- For a portfolio that consists of N bonds, if we denote the fraction of the value of each bond with respect to the total value of the whole portfolio as x, then the portfolio's duration is defined as:

$$\boxed{D_{Portfolio} = x_{Bond_1} \times D_{Bond_1} + x_{Bond_2} \times D_{Bond_2} + ... + x_{Bond_N} \times D_{Bond_N}}$$

Exercise 1

Question 1 What is the duration of a 3-year zero-coupon bond?

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3 years: $C = 0 \Rightarrow \quad \mathrm{w}_1 = w_2 = 0$ and $w_3 = \frac{N}{(1+y)^3} / P_0 = 1$

Question 2 Calculate the duration of an annual coupon-paying bond with $T + 3$ years, $N = 1{,}000$ €, $y = 5\%$ and $c = 10\%$

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Question 2 Calculate the duration of an annual coupon-paying bond with $T + 3$ years, $N =$ 1,000 €, $y =$ 5% and $c =$10%

$$P_0 = \frac{100}{1.05} + \frac{100}{1.05^2} + \frac{1,100}{1.05^3} = 1,136$$

$$D = \frac{\frac{100}{1.05}}{1,136} \times 1 + \frac{\frac{100}{1.05^2}}{1,136} \times 2 + \frac{\frac{1,100}{1.05^3}}{1,136} \times 3 = 2.75 \text{ years}$$

Question 3 If the yield curve shifts upward by 1 percentage point, what's the % change in the bond's price?

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$\frac{\Delta P_0}{P_0} = -D\frac{\Delta y}{1+y} = -2.75 \times \frac{(+0.01)}{1.05} = -0.0262 = -2.62\%$

Question 4 If y decreases by 50 basis points, what's the €-change in the bond's price?

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$\Delta P_0 = -D\frac{\Delta y}{1+y} \times P_0 = -2.75 \times \frac{(-0.005)}{1.05} \times 1,136 = 14.88$ €

Exercise

Question Rank the following bonds in order of descending sensitivity to changes in interest rates

Bond	Coupon rate (%)	Time to maturity (years)	Yield to maturity (%)
A	15	20	10
B	15	15	10
C	8	20	10
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$$Bond_D > Bond_C > Bond_A > Bond_B$$