Judgment Aggregation Theory Can Entail New Social Choice Results[1]

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Abstract

Judgment (or logical) aggregation theory is logically more powerful than social choice theory and has been put to use to recover some classic results of this field. Whether it could also enrich it with genuinely new results is still controversial. To support a positive answer, we prove a social choice theorem by using the advanced nonbinary form of judgment aggregation theory developed by Dokow and Holzman (2010c). This application involves aggregating *classifications* (specifically *assignments*) instead of *preferences*, and this focus justifies shifting away from the binary framework of standard judgement aggregation theory to a more general one.

Keywords: Social choice, Judgment aggregation, Logical aggregation, Aggregation of classifications, Assignments, Nonbinary evaluations.

JEL Classification Numbers: C65, D71.

Introduction

The newly developed theory of judgment (or logical) aggregation stems from the long established social choice theory, and part of its agenda consists in revisiting the classic problems of that theory at a higher level of formal abstraction. The aim is not simply to reorganize existing results, but also to add new items to the stock. On the face of it, judgment aggregation theory has been more successful in achieving the former than the latter. Leaving aside voting, on which there have been very definite advances, and concentrating on preference aggregation *per se*, what the theory most typically contributes is to recover classic results, such as Arrow's impossibility theorem and related variants. (See the sample reviewed in Mongin, 2010, with particular reference to Dietrich and List, 2007, and Dokow and Holzman, 2010b.) Without by any means diminishing the value of unifying this existing body of knowledge, one may wonder whether the theory will be able to extend it significantly.

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It may be that judgment aggregation theory has not yet reached the proper technical stage. The bulk of the current work employs a *binary* notion of judgment, as in classical logic (the judgment either validates or does not validate the proposition), and Dokow and Holzman's (2010c) use of a *nonbinary* notion, as in some non-classical logics (the judgment can evaluate the proposition in more than two ways) remains exceptional. The nonbinary case is significantly more complex than the binary one (compare with Dokow and Holzman, 2010a and b, on the latter). But there are parts in social choice theory that judgment aggregation theorists cannot subsume unless they move to the more complex case. A prominent example is the aggregation of *classifications* - a typically nonbinary concept - as against the aggregation of *preferences* - a binary concept by definition. It comes as no surprise that Dokow and Holzman (2010c) apply their main aggregative theorem to *assignments*, which are a particular case of classifications.

In this note, we apply the same theorem to explore another facet of the aggregation of assignments. While Dokow and Holzman take the individual and collective assignments to be one-to-one (each item enters a different category), we take them to be onto (each category contains at least one item). The ensuing result can be added to the list of those in social choice theory which judgement aggregation theory makes it possible to discover, and not simply to recover.

A problem in the aggregative theory of classification

Our problem is best introduced by means of examples. Where military conscription applies, draft boards allocate conscripts into various defence services, and they are expected to fill each category with some conscripts. Some committees in charge of evaluating scientific projects begin by dispatching them to inside or outside referees, while making sure that each referee takes some share of the work. Trusts and charities must legally decide who, among the members, will be the Chairperson, the Secretary, the Treasurer, or just an ordinary member. Some countries divide their population according to ethnic or religious criteria, with none of the relevant categories being left empty. In all of these cases, the individuals who are called upon to make the classification (the citizens or the lawmakers in the last example) may have divergent views, and an aggregation problem arises.

The task involved in all these examples is to *classify* the objects of interest. The categories could obey a hierarchy, in which case a *ranking* task would also be involved, but the examples make perfect sense without this assumption, and the aggregation problem should be addressed also without imposing it. What social choice theory has to say on this score appears to be scant; however, see Fishburn and Rubinstein, 1986, and some recent ensuing literature, among which Chambers and Miller, 2011, and Dimitrov, Marchant and Mishra, 2012. Our examples share two specific features, (i) the individuals agree among themselves and with the collective on what categories are to be used, and (ii) neither the individuals nor the collective leave any category unfilled. Feature (i) may be taken as specifying *assignments* within the generic family of classifications. Feature (ii) is the particular restriction on assignments that we plan to investigate. Our aggregative problem is fully described by these two features.

An interesting complication occurs when the set of ranking individuals and the set of objects to be ranked overlap, which happens in some though not all of the above examples. The two sets are equal in the last, which has been explored in some detail by Kasher and Rubinstein (1997) and subsequent writers, like Miller(2008). This work illuminates our problem by the same token, but it is limited by the assumption that there are only two categories. (Kasher and Rubinstein motivated their work by the question "What is a J?", meaning that the Js and non-Js exactly partition the society.) List (2007) has usefully noted the connection between the Kasher-Rubinstein literature and judgment aggregation theory. However, in its basic form, the theory shares the same binariness as this literature, and it can elegantly recover some of its results, but not extend them to any number of categories, as would be desirable. Our theorem permits such an extension, and this is a further way of motivating it.

The formal set up

The set up is Dokow and Holzman's (2010c) in the same notation.

$J = \{1, ..., m\}$ is the set of *issues*;

$P = \{1, ..., p\}$ is the set of *positions*;

$N = \{1, ..., n\}$ is the set of *individuals*.

Cardinality restrictions will occur in the theorems below.

An *evaluation* is a vector $(x_1, ..., x_m) \in P^m$ and $X \subseteq P^m$ is the subset of feasible evaluations. For all $j \in J$, X_j is the j-projection of X, i.e., the set of feasible evaluations relative to j. Unlike Dokow and Holzman, we assume that $X_j = P$ for all $j \in X$ because this proves sufficient for our application. Observe that one may still have $X \subsetneq \Pi_{j \in J} X_j = P^m$. A domain X is *nonbinary* if $p \geq 3$.

An *aggregator* is a mapping $F : X^n \to X$, $((x_1^1, ..., x_m^1), ..., (x_1^n, ..., x_m^n)) \longmapsto (x_1, ..., x_m)$.

Think of profiles $((x_1^1, ..., x_m^1), ..., (x_1^n, ..., x_m^n))$ as being $n \times m$ matrices $\mathbf{x}$. Denote by $\mathbf{x}^i$ and $\mathbf{x}_j$ the i-th line and j-th column respectively.

F satisfies *Independence* (I) if for all $(\mathbf{x}^1, ..., \mathbf{x}^n), (\mathbf{y}^1, ..., \mathbf{y}^n) \in X^n$, and for all $j \in J$,

$$x_j^i = y_j^i \text{ for all } i \in N \ \Rightarrow x_j = y_j \ .$$

F satisfies *Unanimity* (U) if for all $(\mathbf{x}^1, ..., \mathbf{x}^n) \in X^n$, for all $j \in J$, for all $u \in P$,

$$x_j^i = u \text{ for all } i \in N \ \Rightarrow x_j = u.$$

F satisfies *Supportiveness* (S) if for all $(\mathbf{x}^1, ..., \mathbf{x}^n) \in X^n$, and for all $j \in J$, for all $u \in P$,

$$x_j = u \ \Rightarrow \ \exists i : x_j^i = u.$$

F satisfies *Dictatorship* (D) if there is $i \in N$ such that for all $(\mathbf{x}^1, ..., \mathbf{x}^n) \in X^n$ and for all $j \in J$,

$$x_j^i = x_j.$$

X is an *impossibility domain* if every F satisfying I and S also satisfies D.

Theorem 1 *(Dokow and Holzman, 2010) Take $p \geq 3$, $m \geq 2$, $n \geq 1$. If X is multiply constrained and totally blocked, X is an impossibility domain.*

The following auxiliary notions are needed to explain the two conditions ("multiply constrained" and "totally blocked").

A *subbox* in P^m is a subset of the form $B = \Pi_{j\in J}\ B_j$, with $B_j \subseteq X_j = P$ for all $j \in J$, and as particular case, a *2-subbox* has $|B_j| = 2$ for all $j \in J$. Only the evaluations $(x_1, ..., x_m) \in B \cap X$ are feasible.

For all $(x_1, ..., x_m) = (x_j)_{j\in J} \in P^m$, we may fix $K \subseteq J$ and consider the K*-evaluation* defined by the subvector $(x_j)_{j\in K} \in P^k$.

Given a subbox $B = \Pi_{j\in J}\ B_j$, a K-evaluation is *within* B if it is in the K-projection of B, i.e., if $(x_j)_{j\in K} \in \Pi_{j\in K}\ B_j$.

A K-evaluation $(x_j)_{j\in K}$ that is within B is *feasible within* B if there is $(x_j)_{j\in J} \in B \cap X$ having $(x_j)_{j\in K}$ as a subvector, it is *infeasible within* B otherwise, and it is *minimally infeasible within* B (briefly: a B*-MIPE*) if it is infeasible within B and all its subvectors are feasible within B.

A binary relation, called *relative conditional entailment* (RCE), will be defined on the set

$$G = \{(u, u'; j) \in P \times P \times J : u \neq u'\}.$$

Let us say that $(u, u'; k)$ RCE $(v, v'; l)$ if $k \neq l$ and there are a 2-subbox $B = \Pi_{j\in J}\ B_j$ with $B_k = \{u, u'\}$ and $B_l = \{v, v'\}$ and a B-MIPE $(x_j)_{j\in K}$ such that $k, l \in K$ and $x_k = u$ and $x_l = v'$.

(Here is the interpretation: taking position u rather than u' on issue k entails taking position v rather than v' on issue l, given the positions taken on the other issues in the K-evaluation, and given what positions on the issues are made available by the 2-subbox B.)

Now to define the two conditions in the theorem.

X is **multiply constrained** if there is a subbox $B \subseteq P^m$ and a B-MIPE $(x_j)_{j\in K}$ such that $|K| \geq 3$.

X is **totally blocked** if, for all $(u, u'; k)$, $(v, v'; l) \in G$ with $k \neq l$, $(u, u'; k)$ $\overline{RCE}$ $(v, v'; l)$, where $\overline{RCE}$ is the transitive closure of the RCE relation (i.e., (v, v', l) can be reached from (u, u', k) by a chain of triples each of which is related by RCE to the next).

The Social Choice Theorem

Our problem can be fitted in the judgment aggregation framework, and as far as nonbinary classifications are concerned, in the framework of last section. We are interested in evaluations that fill *every available position with some issue.* Call the relevant X an *ontoness* domain. Formally,

$$X = \{(x_1, ..., x_m) \in P^m : \forall u \in P, \exists j \in J,\ x_j = u\}.$$

We will investigate aggregators $F : X^n \to X$ when the ontoness domain is *nonbinary*, i.e., when $p \geq 3$.

Theorem 2 *Take X to be an ontoess domain with $n \geq 1$ and either $m = p \geq 4$ or $m > p \geq 3$. If $F : X^n \to X$ satisfies I and U, it also satisfies D.*

The proof is based on the previous theorem and the following lemmas.

Lemma 3 *For $m \geq p \geq 3$, U is equivalent to S.*

Proof. Suppose that for some profile $(\mathbf{x}^1, ..., \mathbf{x}^n) \in X^n$ and some issue $j^* \in J$, $x_j = u$ but $x_j^i \neq u$ for all $i \in N$.

Fix a subset J^* of $m - p + 1$ issues not containing j^* and construct a profile $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^n)$ as follows. For all $i \in N$, (i) $y_{j_*}^i = x_{j_*}^i$; (ii) $y_j^i = u$ for all $j \in J^*$ (iii) the $p - 2$ issues $j \neq j^*$, $j \notin J^*$ are distributed in $\mathbf{y}^i$ so as to cover the $p - 2$ positions $v \neq u, y_{j_*}^i$. Hence $(\mathbf{y}^1, ..., \mathbf{y}^n) \in X^n$. By U, $y_j = u$ for all $j \in J^*$, and by I, $y_{j_*} = u$. There are $p - 1$ positions to be filled with only $p - 2$ issues, hence $\mathbf{y} \notin X$, a contradiction. ■

Lemma 4 *For $m = p \geq 4$ or $m > p \geq 3$, X is multiply constrained.*

Proof. (i) Case $m = p \geq 4$. Take K to be a subset of 4 distinct positions, say w.l.g. positions 1, 2, 3 and 4, and take the subbox $B = \Pi_{j \in J} B_j$ s.t. $B_1 = \{1, 4\}$, $B_2 = \{2, 4\}$, $B_3 = \{3, 4\}$, $B_4 = \{1, 2, 3\}$, and $B_j = P$ if $5 \leq j \leq m$. Take $K = \{1, 2, 3\}$ and define $(x_j)_{j \in K} = (1, 2, 3)$. This K-evaluation is within B, and there infeasible, since it cannot be completed so as to cover p positions (any completion will lack position 4). It is also minimally infeasible within B since taking out any one component makes a full completion available (4 will be recovered). Thus, $(x_j)_{j \in K}$ is a B-MIPE, and since $|K| = 3$, X is multiply constrained.

(ii) Case $m > p \geq 3$. Take K to be a subset of $m - p + 2$ distinct issues and the K-evaluation assigning position 1 to each of them, i.e., $(x_j)_{j \in K} = (1, ..., 1)$. By construction, $|K| \geq 3$. This K-evaluation is infeasible within the maximal subbox $B = P^m$ because the $p - 2$ issues that remain can cover at most the same number of distinct positions, so any completion can have at best $p - 1$ of them. Taking any one component out makes a full completion available, so $(x_j)_{j \in K}$ is a B-MIPE, as requested. ■

Lemma 5 *For $m \geq p \geq 3$, X is totally blocked.*

Proof. We need to prove that $(u,u';k)\ \overline{RCE}\ (v,v';l)$ under the definitional constraints $k \neq l$, $u \neq u'$, $v \neq v'$. In view of the definition of RCE and the fact that $X_j = P$ for all $j \in X$, the proof will not depend on the choice of k and l. Thus, we can take $k = 1$, $j = 2$ without loss of generality. We first prove a particular case of the statement, i.e.,

$$(*)\quad (u,u';1)RCE(v,u;2)$$

We can take $u = 1$, $u' = 2$, $v = 3$ without loss of generality. The proof has two cases.

(i) For $m = p \geq 3$, define $B_1 = \{1,2\}$, $B_2 = \{1,3\}$, $B_3 = \{2,3\}$, and $B_j = \{1,j\}$, $4 \leq j \leq m = p$. Take $K = \{1,2\}$ and $(x_j)_{j\in K} = (1,1)$. Thus, $B = \Pi_{j\in J}B_j$ is a 2-subbox with $B_k = \{u,u'\}$ and $B_l = \{v,u\}$; $x_k = u$ and $x_l = v'$; $(x_j)_{j\in K}$ is infeasible within B since it cannot be extended to cover more than $p-1$ distinct positions (either 2 or 3 is excluded); $(x_j)_{j\in K}$ is also minimally infeasible within B since deleting either component permits finding extensions that cover the p positions (both 2 or 3 can now be obtained). Hence $(x_j)_{j\in K}$ is a B-MIPE as required.

(ii) For $m > p \geq 3$, define

- if $1 \leq j \leq m-p+2$, $B_j = \{1,2\}$ if j is odd and $B_j = \{1,3\}$ is j is even;
- if $j = m-p+3$, $B_j = \{2,3\}$,
- if $m-p+4 \leq j \leq m$, $B_j = \{1, j+p-m\}$, i.e., $B_{m-p+4} = \{1,4\}$, ..., $B_m = \{1,p\}$.

For example if $m = 4$ and $p = 3$, $B_1 = \{1,2\}$, $B_2 = \{1,3\}$, $B_3 = \{1,2\}$, $B_4 = \{2,3\}$.

Take $K = \{1,...,m-p+2\}$ and $(x_j)_{j\in K} = (1,1,..,1)$. The 2-subox $B = \Pi_{j\in J}B_j$ and the K-evaluation $(x_j)_{j\in K}$ meet the conditions on the indexes. An argument similar to that of case (ii) shows that $(x_j)_{j\in K}$ is minimally infeasible within B, hence is a B-MIPE, as required.

We prove another particular case of the statement, i.e.,

$$(**)\quad (u,u';1)RCE(u',u;2).$$

For case (i), it is enough to redefine B_2 as $\{1,2\}$. For case (ii), take

- if $1 \leq j \leq m-p+2$, $B_j = \{1,2\}$;
- if $m-p+3 \leq j \leq m$, $B_j = \{1,j+p-m\}$, i.e., $B_{m-p+3} = \{1,3\}$, ..., $B_m = \{1,p\}$.

Either argument now carries through.

Repeated application of $(*)$ and $(**)$ deliver more particular cases of the statement. For instance, $(u,u';1)\overline{RCE}(u',v';2)$ follows from

$$(u,u';1)RCE(u',u;3)RCE(u',v';2)$$

and $(u,u';1)\overline{RCE}(u,u';2)$ from

$$(u,u';1)RCE(u',u;3)RCE(u,u';2).$$

The cases $(u,u';1)\overline{RCE}(u,v';2)$ and $(u,u';1)\overline{RCE}(v,u';2)$ similarly follow. In a final easy step, we show that $(u,u';1)\overline{RCE}(v,v';2)$ for distinct u,u',v,v', thus completing the proof that X is totally blocked. ■

More comments

Dokow and Holzman's (2010c, Example A and Corollary 1) explore a different case of aggregating assignments. They consider the problem of distributing several people in P among several jobs in J under the natural restrictions that a person can only do one job and that the people (or candidates) are at least as many as are the jobs. (Equally well, they could have considered the problem in which jobs are assigned to candidates, any job is assigned to at most one candidate, and there are at least as many jobs as there are candidates.) Formally, they have an *injectivity* domain:

$$X = \{(x_1,...,x_m) \in P^m : \forall j,j' \in J,\ x_j \neq x_{j'}\}$$

with $p \geq m$, so that their application and ours complement each other rigorously. Using their main theorem, Dokow and Holzman conclude that X is an impossibility domain for $p \geq m \geq 4$, a corollary that overlaps with Theorem 2 for all $p = m \geq 4$. This leaves undecided the case $p = m = 3$. However, by another theorem in their paper, they show that X is an impossibility domain also in this case (see 2010c, Example 3).

One may wonder whether these new social choice results could not be proved directly by the tools of social choice theory. This is indeed the case for Theorem 2, which the authors can also obtain from an ultrafilter construction (available on request). This approach might strike social choice theorists as more congenial, but were it used exclusively, it would miss the interesting connections that judgment aggregation theory brings out, for instance the complementarity with the injectivity domain.

A word may be added in connection with the motivating examples of this note. In our view, the conflict between U, I and ontoness is neither simply predictable from this list, nor easy to resolve after it is recognized. If the aim was to aggregate preferences, with the axioms U and I being redefined accordingly, the first suggestion would be to relax ontoness as regards the *range* of the aggregator (see Maniquet and Mongin, forthcoming, for a voting context in which this appears to be the way out). Another, more drastic departure would be to allow for categories that differ among the individuals, or differ between the individuals and the collective, and this would entail considering classifications more generally than assignments, as in Fishburn and Rubinstein (1986) and followers. However, our examples were meant to show that shared categories and ontoness are often part of the institutional setting of the aggregation problem. When this is the case, there is a thorny choice to be made between U and I, two *prima facie* defensible conditions.

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