Chapter 38
Judgment Aggregation

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Abstract Judgment aggregation theory generalizes social choice theory by having the aggregation rule bear on judgments of all kinds instead of barely judgments of preference. The theory derives from Kornhauser and Sager's doctrinal paradox and Pettit's discursive dilemma, which List and Pettit turned into an impossibility theorem – the first of a long list to come. After mentioning this formative stage, the paper restates what is now regarded as the "canonical theorem" of judgment aggregation theory (in three versions due to Nehring and Puppe, Dokow and Holzman, and Dietrich and Mongin, respectively). The last part of paper discusses how judgment aggregation theory connects with social choice theory and can contribute to it; it singles out two representative applications, one to Arrow's impossibility theorem and the other to the group identification problem.

38.1 A New Brand of Aggregation Theory

It is a commonplace idea that collegial institutions generally make better decisions than those in which a single individual is in charge. This optimistic view can be traced back to Enlighment theorists, such as Rousseau and Condorcet, and it permeates today's western judiciary organization, which is heir to this philosophical tradition. The more important a legal case, the more likely it is to be entrusted to a collegial court; appeal courts are typically collegial, and at the top of the legal organization, constitutional courts always are. However, the following, by now classic example from legal theory challenges the Enlightment view.

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A plaintiff has brought a civil suit against a defendant, alleging a breach of contract between them. The court is composed of three judges A, B and C, who will determine whether or not the defendant must pay damages to the plaintiff (d or $\neg d$). The case brings up two issues, i.e., whether the contract was valid or not (v or $\neg v$), and whether the defendant was or was not in breach of it (b and $\neg b$). Contract law stipulates that the defendant must pay damages if, and only if, the contract was valid and he was in breach of it. Suppose that the judges have the following views of the two issues, and accordingly of the case:

A	v	$\neg b$	$\neg d$
B	$\neg v$	b	$\neg d$
C	v	b	d

In order to rule on the case, the court can either directly collect the judges' recommendations on it, or collect the judges' views of the issues and then solve the case by applying contract law to these data. If the court uses majority voting, the former, *case-based* method delivers $\neg d$, whereas the latter, *issue-based* method returns first v and b, and then d. This elegant example is due to legal theorists Kornhauser and Sager [21]. They describe as a *doctrinal paradox* any similar occurrence in which the two methods give conflicting answers. What makes the discrepancy paradoxical is that each method is commendable on some ground, i.e., the former respects the judges' final views, while the latter provides the court with a rationale, so one would wish them always to be compatible. The legal literature has not come up with a clear-cut solution (see Nash [32]). This persisting difficulty casts doubt on the belief that collegial courts would be wiser than individual ones. Clearly, with a single judge, the two methods coincide unproblematically.

An entire body of work, now referred to as *judgment aggregation theory,* has grown out of Kornhauser and Sager's doctrinal paradox. As an intermediary step, their problem was rephrased by political philosopher Pettit [39], who wanted to make it both more widely applicable and more analytically tractable. What he calls the *discursive dilemma* is, first of all, the generalized version of the doctrinal paradox in which a group, whatever it is, can base its decision on either the *conclusion-based* or the *premiss-based* method, whatever the substance of conclusions and premisses may be. What holds of the court equally holds of a political assembly, an expert committee, and many other deliberating groups; as one of the promoters of the concept of deliberative democracy, Pettit would speculatively add political society as a whole. Second, and more importantly for our purposes, the discursive dilemma shifts the stress away from the conflict of methods to *the logical contradiction within the total set of propositions that the group accepts*. In the previous example, with $d \longleftrightarrow v \wedge b$ representing contract law, the contradictory set is

$$\{v, b, d \longleftrightarrow v \wedge b, \neg d\}.$$

Trivial as this shift seems, it has far-reaching consequences, because all propositions are now being treated alike; indeed, the very distinction between premisses and conclusions vanishes. This may be a questionable simplification to make in the legal context, but if one is concerned with developing a general theory, the move has clear analytical advantages. It may be tricky to classify the propositions into two groups, and it is definitely simpler to pay attention to whole sets of accepted propositions – briefly *judgment sets* – and inquire when and why the collective ones turn out to be inconsistent, given that the individual ones are taken to be consistent. This is already the problem of judgment aggregation.

In a further step, List and Pettit [24] introduce an aggregation mapping F, which takes profiles of individual judgment sets $(A_1, \ldots, A_n)$ to collective judgment sets A, and subject F to axiomatic conditions which they demonstrate are logically incompatible. Both the proposed formalism and impossibility conclusion are in the vein of social choice theory, but they are directed at the discursive dilemma, which the latter theory cannot explain in terms of its usual preference apparatus. At this stage, the new theory exists in full, having defined its object of study – the F mapping, or *collective judgment function* – as well as its method of analysis - it consists in axiomatizing F and investigating subsets of axioms to decide which result in an impossibility and which, to the contrary, support well-behaved rules (such as majority voting).

List and Pettit's impossibility theorem was shortly succeeded, and actually superseded, by others of growing sophistication, due to Pauly and van Hees [38], Dietrich [3], Dietrich and List [6], Mongin [29], Nehring and Puppe [35, 36], [10–12], and Dietrich and Mongin [9]. This lengthy, but still incomplete list, should be complemented by two papers that contributed differently to the progress of the field. Elaborating on earlier work in social choice theory by Wilson [45] and Rubinstein and Fishburn [42], and in a formalism that still belongs to that theory, Nehring and Puppe [33] inquired about which *agendas* of propositions turn the axiomatic conditions into a logical impossibility. Agendas are the rough analogue of preference domains in social choice theory. This concept raised to prominence in mature judgment aggregation theory, and Nehring and Puppe's characterization of impossibility agendas was eventually generalized by Dokow and Holzman [11], whose formulation has become the received one. On a different score, Dietrich [4] showed that the whole formalism of the theory could be deployed without making reference to any specific logical calculus. Only a few elementary properties of the formal language and the logic need assuming for the theorems to carry through. The so-called *general logic* states these requisites (see Dietrich and Mongin [9], for an up-to-date version). The first papers relied on propositional calculi, which turns out to be unnecessary. This major generalization underlies the theory as it is presented here, as well as in the more extensive overviews by Mongin and Dietrich [31] or Mongin [30]. (These two papers actually use the tag "logical aggregation theory" instead of the standard one "judgment aggregation theory" to emphasize the particular angle they adopt.)

The next section “A Logical Framework for Judgment Aggregation Theory” provides a syntactical, framework for the F function, using the general logic as a background. It states the axiomatic conditions on F that have attracted most attention, i.e., systematicity, independence, monotonicity and unanimity preservation. The issue of agendas arises in the ensuing Sect. 38.3, which presents an impossibility theorem in three variant forms, due to Nehring and Puppe, Dokow and Holzman, and Dietrich and Mongin, respectively. This is the central achievement of the theory by common consent – hence the label “canonical theorem” adopted here – but many other results are well deserving attention. For them, the reader is referred to the two reviews just mentioned, or at a more introductory level, those of List and Puppe [25] and Grossi and Pigozzi [17]. The final Sect. 38.4 sketches a comparison with social choice theory and discusses how judgment aggregation theory relates and contributes to the latter.

Several topics are omitted here. One is *probability aggregation*, which gave rise to a specialized literature already long ago (see Genest and Zidekh’s [15] survey of the main results). Both commonsense and traditional philosophy classify judgments into certain and uncertain ones, so probability aggregation theoretically belongs to the topic of this chapter. However, we will comply here with the current practice of taking judgments in the restricted sense of judgments passed under conditions of certainty. Another, no doubt more questionable omission concerns those logics which the general logic excludes despite its flexibility; prominent among which are the *multi-valued* logics investigated by Pauly and van Hees [38], van Hees [43], and Duddy and Piggins [14], and the *non-monotonic* logics investigated by Wen [44]. Finally, we have omitted the topic of *belief merging*, or *fusion*, which emerged in theoretical computer science independently of judgment aggregation theory, but is now often associated with it. Although they represent the information stored in databases rather then by human agents, the computer scientists’ “belief sets” or “knowledge bases” are analogous to judgment sets, and the problem of “merging” or “fusing” these items is analogous to the problem of defining a collective judgment function. Pigozzi [40] was one of the first to make this connection, and the reader can consult one of her up-to-date surveys (e.g., Pigozzi [41]). The computer scientists’ solutions are particular cases of *distance-based judgment aggregation*, i.e., they depend on defining what it means for a judgment set to be closer to one judgment set than another. Miller and Osherson [28] thoroughly explore the abstract properties of distance metrics, while Lang et al. [22] provide a classification.

38.2 A Logical Framework for Judgment Aggregation Theory

By definition, a *language* $\mathcal{L}$ for judgment aggregation theory is any set of formulas $\varphi, \psi, \chi, \ldots$ that is constructed from a set of logical symbols $\mathcal{S}$ containing $\neg$, the Boolean negation symbol, and that is closed for this symbol (i.e., if $\varphi \in \mathcal{L}$, then $\neg\varphi \in \mathcal{L}$). In case $\mathcal{S}$ contains other elements, such as symbols for the remaining Boolean connectives or modal operators, they satisfy the appropriate closure properties. A *logic* for judgment aggregation theory is any set of axioms and rules that regulates the inference relation $\vdash$ on $\mathcal{L}$ and associated technical notions – logical truth and contradiction, consistent and inconsistent sets – while satisfying the general logic. Informally, the main requisites are that $\vdash$ be monotonic and compact, and that any consistent set of formulas can be extended to a complete consistent set. ($S \subset \mathcal{L}$ is *complete* if, for all $\varphi \in \mathcal{L}$, either $\varphi \in S$ or $\neg\varphi \in S$.) Monotonicity means that inductive logics are excluded from consideration, and compactness (which is needed only in specific proofs) that some deductive logics are. The last requisite is the standard Lindenbaum extendability property.

Among the many calculi that enter this framework, propositional examples stand out. They need not be classical, i.e., $\mathcal{S}$ may contain modal operators, like those of deontic, epistemic and conditional logics, each of them leading to a potentially relevant application. Each of these extensions should be double-checked, because some fail compactness. Although this may not be so obvious, first-order calculi are also permitted. When it comes to them, $\mathcal{L}$ is the set of *closed* formulas – those without free variables – and the only question is whether $\vdash$ on $\mathcal{L}$ complies with the general logic.

In $\mathcal{L}$, a subset X is fixed to represent the propositions that are in question for the group; this is the *agenda*, one of the novel concepts of the theory and one of its main focuses of attention. In all generality, X needs only to be non-empty, with at least one contingent formula, and to be closed for negation. The discursive dilemma reconstruction of the court example leads to the agenda

$$\overline{X} = \{v, b, d, d \leftrightarrow v \wedge b, \neg v, \neg b, \neg d, \neg(d \leftrightarrow v \wedge b)\}.$$

The theory represents judgments in terms of subsets $B \subset X$, which are initially unrestricted. These *judgment sets* – another notion specific to the theory – will be denoted by $A_i, A'_i, \ldots$ when they belong to the individuals $i = 1, \ldots n$, and by $A, A', \ldots$ when they belong to the group as such. A formula φ from one of these sets represents a proposition, in the ordinary sense of a semantic object endowed with a truth value. If φ is used also to represent a judgment, in the sense of a cognitive operation, this is in virtue of the natural interpretive rule:

(R) i judges that φ iff $\varphi \in A_i$, and the group judges that φ iff $\varphi \in A$.

Standard logical properties may be applied to judgment sets. For simplicity, we only consider two cases represented by two sets of judgments sets:

- the unrestricted set 2^X;
- the set D of *consistent* and *complete* judgment sets (consistency is defined by the logic and completeness is as above, but relative to X).

Thus far, the theory has been able to relax completeness, but not consistency (see, e.g., Dietrich and List [8]).

The last specific concept is the *collective judgment function* F, which associates a collective judgment set to each profile of judgment sets for the n individuals:

$$A = F(A_1, \ldots, A_n).$$

The domain and range of F can be defined variously, but we restrict attention to $F : D^n \to 2^X$, our baseline case, and $F : D^n \to D$, our target case, in which the collective sets obey the same stringent logical constraints as the individual ones. The present framework captures the simple voting rule of the court example, as well as less familiar examples. Formally, define *formula-wise majority voting* as the collective judgment function $F_{maj} : D^n \to 2^X$ such that, for every profile $(A_1, \ldots, A_n) \in D^n$,

$$F_{maj}(A_1, \ldots, A_n) = \{\varphi \in X : |\{i : \varphi \in A_i\}| \geq q\},$$

$$\text{with } q = \frac{n+1}{2} \text{ if } n \text{ is odd and } q = \frac{n}{2} + 1 \text{ if } n \text{ is even.}$$

Here, the range is not D because there can be unbroken ties, and so incomplete collective judgment sets, when n is even. More strikingly, for many agendas, the range is not D even when n is odd, because there are inconsistent collective judgment sets, as the court example neatly shows. By varying the value of q between 1 and n in the definition, one gets specific quota rules F^q_{maj}. One would expect inconsistency to occur with low q, and incompleteness with large q. Nehring and Puppe [33, 35] and Dietrich and List [7] investigate the F^q_{maj} in detail.

Having defined and exemplified F functions, we introduce some axiomatic properties they may satisfy.

Systematicity. For all formulas $\varphi, \psi \in X$ and all profiles $(A_1, \ldots, A_n)$, $(A'_1, \ldots, A'_n)$, if $\varphi \in A_i \Leftrightarrow \psi \in A'_i$ for every $i = 1, \ldots, n$, then

$$\varphi \in F(A_1, \ldots, A_n) \Leftrightarrow \psi \in F(A'_1, \ldots, A'_n).$$

Independence. For every formula $\varphi \in X$ and all profiles $(A_1, \ldots, A_n)$, $(A'_1, \ldots, A'_n)$, if $\varphi \in A_i \Leftrightarrow \varphi \in A'_i$ for every $i = 1, \ldots, n$, then

$$\varphi \in F(A_1, \ldots, A_n) \Leftrightarrow \varphi \in F(A'_1, \ldots, A'_n).$$

Monotonicity. For every formula $\varphi \in X$ and all profiles $(A_1, \ldots, A_n)$, $(A'_1, \ldots, A'_n)$, if $\varphi \in A_i \Rightarrow \varphi \in A'_i$ for every $i = 1, \ldots, n$, with $\varphi \notin A_j$ and $\varphi \in A'_j$ for at least one j, then

$$\varphi \in F(A_1, \ldots, A_n) \Rightarrow \varphi \in F(A'_1, \ldots, A'_n).$$

Unanimity preservation. For every formula $\varphi \in X$ and every profile $(A_1, \ldots, A_n)$, if $\varphi \in A_i$ for every $i = 1, \ldots, n$, then $\varphi \in F(A_1, \ldots, A_n)$.

By definition, F is a *dictatorship* if there is a j such that, for every profile $(A_1, \ldots, A_n)$,

$$F(A_1, \ldots, A_n) = A_j.$$

Given the unrestricted domain, there can only be one such j, to be called the *dictator*. The last property is

Non-dictatorship. F is not a dictatorship

It is routine to check that F_{maj} satisfies all the list. Systematicity means that the group, when faced with a profile of individual judgment sets, gives the same answer concerning a formula as it would give concerning a *possibly different* formula, when faced with a *possibly different* profile, supposing that the individual judgments concerning the first formula in the first profile are the same as those concerning the second formula in the second profile. Independence amounts to restricting this requirement to $\varphi = \psi$. Thus, it eliminates one claim made by Systematicity – i.e., that the identity of the formula does not matter – while preserving another – i.e., that the collective judgment of φ depends only on individual judgments of φ. That is, by Independence, the collective set A is defined *formula-wise* from the individual sets $A_1, \ldots, A_n$. By contrast, for any concept of distance envisaged in the distance-based literature (e.g., [22, 28]), if F is defined by minimizing the total distance of A to $A_1, \ldots, A_n$, F violates Independence. The collective judgment sets in this class of solutions are constructed from the individual sets *taken as a whole* and not formula-wise.

Systematicity was the condition List and Pettit's [24] impossibility theorem, but henceforth, the focus of attention shifted to Independence. The former has little to say for itself except that many voting rules satisfy it, but the latter can be defended as a *non-manipulability* condition. If someone is in charge of defining the agenda X, Independence will prevent this agent to upset the collective judgment on a formula by adding or withdrawing other formulas in X; this argument appears in Dietrich [3]. However, Independence does not block all and every form of manipulability, as Cariani, Pauly and Snyder [2] illustrate; they show that a suitable choice of the language $\mathcal{L}$ can influence the collective judgment.

Some writers take Monotonicity to be a natural addition to Independence. This condition requires that, when a collective result favours a subgroup's judgment, the same holds if more individuals join the subgroup. It can be defended in terms of democratic responsiveness, though perhaps not so obviously as the last two conditions, i.e., Unanimity preservation and Non-dictatorship.

The problem that has gradually raised to the fore is to characterize – in the sense of necessary and sufficient conditions – the agendas X such that no $F : D^n \to D$ satisfies Non-dictatorship, Independence, and Unanimity preservation. There is a variation of this problem with Monotonicity as a further axiomatic condition. The next section provides the answers.

38.3 The Canonical Theorem in Three Forms

The promised answers depend on further technical notions. First, a set of formulas $\mathcal{S} \subset \mathcal{L}$ is called *minimally inconsistent* if it is inconsistent and all its proper subsets are consistent. With a classical propositional calculus, this is the case for

$$\{v, b, d \leftrightarrow v \wedge b, \neg d\},$$

but not for

$$\{\neg v, \neg b, d \leftrightarrow v \wedge b, d\}.$$

Second, for $\varphi, \psi \in X$, it is said that φ *conditionally entails* ψ – denoted by $\varphi \vdash^* \psi$ – if $\varphi \neq \neg\psi$ and there is some minimally inconsistent $Y \subset X$ with $\varphi, \neg\psi \in Y$. This is trivially equivalent to requiring that $\{\varphi\} \cup Y' \vdash \psi$ holds for some minimal auxiliary set of premisses Y' that is contradictory neither with φ, nor with $\neg\psi$.

Now, an agenda X is said to be *path-connected* (another common expression is *totally blocked*) if, for every pair of formulas $\varphi, \psi \in X$, there are formulas $\varphi_1, \ldots, \varphi_k \in X$ such that

$$\varphi = \varphi_1 \vdash^* \varphi_2 \vdash^* \ldots \vdash^* \varphi_k = \psi.$$

Loosely speaking, agendas with this property have many, possibly roundabout logical connections. Finite agendas can be represented by directed graphs: the formulas φ, ψ are the nodes and there is an arrow pointing from φ to ψ for each conditional entailment $\varphi \vdash^* \psi$. The court agenda $\overline{X}$ is path-connected, as the picture below of conditional entailments illustrates (it does not represent all existing conditional entailments, but sufficiently many for the reader to check the claim) (Fig. 38.1).

(Here and in the next figures, an arrow pointing from one formula to another means that the former conditionally entails the latter, and the small print formulas near the head of the arrow are a choice of auxiliary premisses; $d \leftrightarrow v \wedge b$ is abridged as q.)

Now, we are in a position to state a version of the canonical theorem (see Dokow and Holzman [11], and Nehring and Puppe [36]; it originates in Nehring and Puppe [33]). From now on, we assume that $n \geq 2$.

Theorem (first form) *If X is path-connected, then no $F : D^n \to D$ satisfies Non-dictatorship, Unanimity preservation, Monotonicity and Independence. The agenda condition is also necessary for this conclusion.*

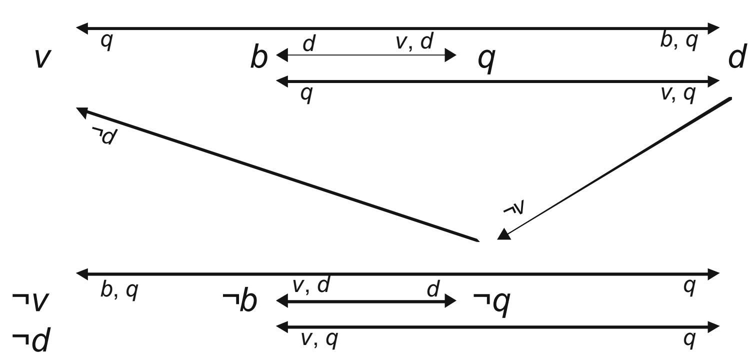

Fig. 38.1 The court agenda in the discursive dilemma version

To illustrate the sufficiency part, let us take $\overline{X}$ and F_{maj}, assuming that n is odd, so that F_{maj} has range D if and only if $F_{maj}(A_1, \ldots, A_n)$ is consistent for all profiles $(A_1, \ldots, A_n)$. The court example in the discursive dilemma version exhibits a profile contradicting consistency, and this shows that D is not the range of F_{maj}. The theorem leads to the same conclusion by a more general reasoning: since F_{maj} satisfies the four axioms and $\overline{X}$ is path-connected, D cannot be the range. By a converse to this entailment, when the agenda is *not* path-connected, there is no collective inconsistency even if the axiomatic conditions hold. This important addition is the necessity part of the theorem, which we do not illustrate here.

As it turns out, Monotonicity can be dropped from the list of axioms if the agenda is required to satisfy a further condition. Let us say that X is *even-number negatable* if there is a minimally inconsistent set of formulas $Y \subseteq X$ and there are distinct $\varphi, \psi \in Y$ such that $Y_{\neg\{\varphi,\psi\}}$ is consistent, where the set $Y_{\neg\{\varphi,\psi\}}$is obtained from Y by replacing φ, ψ by $\neg\varphi, \neg\psi$ and keeping the other formulas unchanged. This seems to be an unpalatable condition, but it is not demanding, as $\overline{X}$ illustrates: take

$$Y = \{v, b, d, \neg(d \leftrightarrow v \wedge b)\} \text{ and } \varphi = v, \psi = b,$$

and there are alternative choices of Y. The next result was proved by Dokow and Holzman [11] as well as, for the sufficiency part, by Dietrich and List [6].

Theorem (second form) *If X is path-connected and even-number negatable, then no $F : D^n \to D$ satisfies Non-dictatorship, Unanimity preservation, and Independence. If $n \geq 3$, the agenda conditions are also necessary for this conclusion.*

A further step of generalization is available. Unlike the work reviewed so far, it is motivated *not by the discursive dilemma, but by the doctrinal paradox*, and it is specially devised to clarify the premiss-based method, which is often proposed as a solution to this paradox (see Pettit [39], and some of the legal theorists reviewed by Nash [32]). Formally, we define the *set of premisses* to be a subset $P \subseteq X$, requiring only that it be non-empty and closed for negation, and reconsider the framework to account for the difference between P and its complement $X \setminus P$. Adapting the axioms, we define

Independence on premisses: same statement as for Independence, but holding only for every $p \in P$.

Non-dictatorship on premisses: there is no $j \in \{1, \ldots, n\}$ such that $F(A_1, \ldots, A_n) \cap P = A_j \cap P$ for every $(A_1, \ldots, A_n) \in D^n$.

Now revising the agenda conditions, we say that X is *path-connected in P* if, for every pair $p, p' \in P$, there are $p_1, \ldots, p_k \in P$ such that

$$p = p_1 \vdash^* p_2 \vdash^* \ldots \vdash^* p_k = p'.$$

Note that formulas in $X \setminus P$ may enter this condition via the definition of conditional entailment $\vdash^*$. We also say that X is *even-number negatable in P* if there are $Y \subseteq$

X and $\varphi, \psi \in Y$ as in the above definition for being even-number negatable, except that "$\varphi, \psi \in Y \cap P$" replaces "$\varphi, \psi \in Y$" (i.e., the negatable pair consists of premisses). The two conditions can be illustrated by court agendas in the doctrinal paradox style.

If we stick to the agenda $\overline{X}$, the subset

$$\overline{P} = \{v, b, d \leftrightarrow v \wedge b, \neg v, \neg b, \neg(d \leftrightarrow v \wedge b)\}$$

best captures the judges' sense of what premisses are. However, the following construal may be more to the point. Suppose that judges do not vote on the law, but rather take it for granted and apply it – a realistic case from legal theory (see [21]). We model this, first by reducing the agenda to

$$\overline{\overline{X}} = \{v, b, d, \neg v, \neg b, \neg d\},$$

and second by including the formula $d \leftrightarrow v \wedge b$ into the inference relation, now defined by

$$S \vdash_{d \leftrightarrow v \wedge b} \psi \text{ if and only if } S \cup \{\varphi, d \leftrightarrow v \wedge b\} \vdash \psi.$$

In this alternative model, the set of premisses reads as

$$\overline{\overline{P}} = \{v, b, \neg v, \neg b\}.$$

Technically, the two construals are wide apart: $\overline{X}$ is both path-connected and even-number negatable in $\overline{P}$, whereas $\overline{\overline{X}}$ is even-number negatable but not path-connected in $\overline{\overline{P}}$, thus failing the more important agenda condition. The next two pictures – the first for $\overline{P}$ and the second for $\overline{\overline{P}}$ – illustrate the stark contrast (Fig. 38.2).

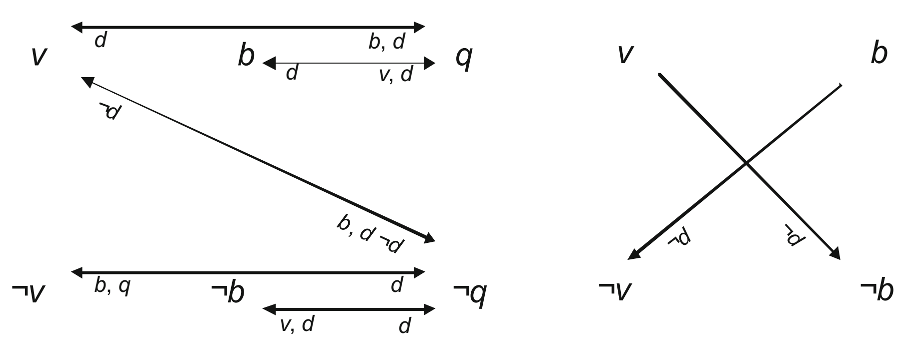

Fig. 38.2 Two sets of premisses $\overline{P}$ (left) and $\overline{\overline{P}}$ (right) for the court agenda in the doctrinal paradox version

(The first picture represents sufficiently many conditional entailments in $\overline{P}$ for the conclusion that $\overline{X}$ is path-connected in $\overline{P}$, and the second represents all conditional entailments in $\overline{\overline{P}}$, which are too few for $\overline{\overline{X}}$ to be path-connected in $\overline{\overline{P}}$.)

Having illustrated the new definitions, we state the result by Dietrich and Mongin [9] that puts them to use.

Theorem (third form) *If X is path-connected and even-number negatable in P, there is no $F : D^n \to D$ that satisfies Non-dictatorship on premisses, Independence on premisses and Unanimity preservation. If $n \geq 3$, the agenda conditions are also necessary for this conclusion.*

Note carefully that Unanimity preservation retains its initial form, unlike the other two conditions. If it were also restricted to premisses, one would check that no impossibility follows. Thus, the statement is best interpreted as an impossibility theorem for the premiss-based method, granting the normatively defensible constraint that unanimity should be preserved on all formulas. Anyone who accepts this addition – in effect, a whiff of the conclusion-based method – is committed to the unpleasant result that the premiss-based method is, like its rival, fraught with difficulties. As with the previous forms of the canonical theorem, solutions can be sought on the agenda's side by relaxing the even-number negatibility or – more relevantly – the path-connectedness condition. The $\overline{\overline{X}}$, $\overline{\overline{P}}$ reconstruction of the doctrinal paradox illustrates this way out; observe that F_{maj} is well-behaved in this case.

Legal interpretations aside, the third form of the theorem is more assertive than the second one. This is seen by considering $P = X$, a permitted limiting case. Having explored the canonical theorem in full generality, we move to the comparative topic of this paper.

38.4 A Comparison with Social Choice Theory

Judgment aggregation theory has clearly been inspired by social choice theory, and two legitimate questions are, how it formally relates, and what it eventually adds, to its predecessor. The F mapping resembles the *collective preference function* G, which takes profiles of individual preference relations to preference relations for the group. (Incidentally, the official terminology for G, i.e., the "social welfare function", is misleading since it jumbles up the concepts of preference and welfare.) The normative properties posited on judgment sets are evocative of those, like transitivity and completeness, which one encounters with preference relations, and the axiomatic conditions on F are most clearly related to those usually put on G. Systematicity corresponds to neutrality, Independence to independence of irrelevant alternatives, Monotonicity to positive responsiveness, and Unanimity preservation to the Pareto principle, not to mention the similar requisite of Non-dictatorship.

Conceptually, a major difference lies in the objects of the two aggregative processes. A *judgment*, as the acceptance or rejection of a proposition, is more general than a *preference* between two things. According to a plausible account, an agent, whether individual or collective, prefers x to y if and only if it judges that x is preferable to y, i.e., accepts the proposition that x is preferable to y. This clarifies the claim that one concept is more general than the other, but how does this claim translate into the respective formalisms?

We answer this question by following Dietrich and List's [6] footsteps. They derive a version of Arrow's [1] impossibility theorem in which the individuals and the group express *strict* preferences on the set of alternatives Z, and these preferences are assumed to be not only transitive, but also *complete*. Although these assumptions are restrictive from the viewpoint of social choice theory, the logical derivation elegantly shows how judgment aggregation theory can be linked to that theory. The first step is to turn the G mappings defined on the domain of preferences into particular cases of F. To do so, one takes a first-order language $\mathcal{L}$ whose elementary formulas xPy express "x is strictly preferable to y", for all $x, y \in Z$, and defines a logic for $\mathcal{L}$ by enriching the inference relation $\vdash$ of first-order logics with the axioms expressing the asymmetry, transitivity and completeness of P. The conditions for general logic hold. Now, if one takes X to be the set of elementary formulas of $\mathcal{L}$ and defines the set of judgment sets D from this agenda, it is possible to associate with each given G an $F : D^n \rightarrow D$ having the same informal content. The next step is to make good the results of judgment aggregation theory. Dietrich and List show that X satisfies the agenda conditions of the canonical theorem (second form). To finish the proof that Arrow's axiomatic conditions on G are incompatible, it is enough to check that they translate into those put on F in the theorem, so that the sufficiency part of the theorem applies.

Although social choice theory is primarily concerned with aggregating preferences, or related individual characteristics such as utility functions, it also extends in other directions, as illustrated by the work on *group identification*. Kasher and Rubinstein [19] consider a finite population N, each member of which is requested to partition N into two categories, conventionally labelled *J* and not-*J*. The question is to associate a collective partition with this process, and Kasher and Rubinstein answer it along social-choice-theoretic lines, i.e., by introducing a mapping H from profiles of individual partitions to collective partitions and submitting H to axiomatic conditions. Among other results, they show that if H determines the collective classification of i as J or not-*J* only from the individual classifications of i as J or not-*J*, and if H respects unanimous individual classifications of i as J or not-*J*, then H is a dictatorship. As List [23] suggests, this impossibility theorem can easily be derived from judgment aggregation theory by taking $\mathcal{L}$ to be a propositional language whose elementary formulas express "*i* is a *J*", for all $i \in N$. Then a reasoning paralleling that made for the Arrovian case leads to the dictatorship conclusion. The canonical theorem (second form) is again put to use, and just as in the earlier case, an important step is to check that the agenda conditions for this theorem hold. What turns out to be crucial in this respect is the assumption made by Kasher and Rubinstein that both individual and collective partitions are non-trivial (i.e., each partition classifies at least one individual as J and at least one individual as non-J).

There are other derivations of social choice results, most of them based on the canonical theorem or variants of it. For instance, Dokow and Holzman [12] recover a theorem by Gibbard [16] on quasi-transitive social preferences and oligarchies, and Herzberg and Eckert [18] explain how the infinite population variants of Arrow's theorem, as in Kirman and Sondermann [20], relate to infinite population extensions of the canonical theorem. Moreover, some of the derived social choice results are novel. Thus, Dokow and Holzman [12] obtain unnoticed variants of Gibbard's theorem. More strikingly, Dokow and Holzman's [13] analysis of collective judgment aggregation functions in the non-binary case delivers entirely new results concerning *assignment problems* (such as the problem of assigning a given number of jobs to a given number of candidates), and these problems arguably belong to social choice theory, although taken broadly.

Against this reassuring evidence, two reservations are in order. For one thing, the derivations from judgment aggregation theorems are often complex, which may discourage social choice theorists to use them despite the powerful generality of these theorems. A basic example is Arrow's theorem, which the canonical theorem permits recovering only in the version singled out by Dietrich and List [6]. To obtain the theorem in full, i.e., with weak preferences instead of strict ones, one way, due to Dokow and Holzman [12], is to derive first Gibbard's oligarchy theorem and then reinforce the assumptions, and another way, due to Dietrich [5], requires one to move to a richer judgment aggregation framework in which "relevance" constraints are put on the formulas of the agenda X. Either way is subtle, but perhaps disappointingly roundabout for social choice theorists. Another, less standard example concerns the generalization of Kasher and Rubinstein's [19] impossibility theorem to more than two categories. This turns out to be a non-trivial problem, and it can be solved using Dokow and Holzman's [13] apparatus of non-binary evaluations (see Maniquet and Mongin [26]). However, this resolution may seem to be exceedingly complex, given that a direct proof can be offered using standard tools in social choice theory (compare with Maniquet and Mongin [27]).

For another thing, the canonical theorem is an impossibility theorem, and so are other results we did not review here, like the theorems on oligarchy that generalize the canonical theorem. Admittedly, all these results fully *characterize* the agenda conditions for impossibility, so that they should not be interpreted only negatively; any failure of the necessary conditions corresponds to a possibility. However, there is no way to infer the precise form of the possibility in question, so these results should clearly be complemented with others, which will directly axiomatize judgment aggregation rules that are neither dictatorial nor oligarchical. The theory has actively followed this direct approach for voting rules, especially majority voting and its refinements [7, 34, 35, 37], but it should be applied more systematically elsewhere. The literature on belief merging may provide heuristic keys, although not every procedure for fusing databases delivers a plausible way of aggregating human judgments, and help resolve the legal and political issues that are at the core of judgment aggregation theory.

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