THE LOGIC OF BELIEF CHANGE
AND NONADDITIVE PROBABILITY

I. INTRODUCTION AND PREVIEW

The present paper investigates the rationality conditions on belief change of a theory which has recently come to the forefront of philosophical logic and artificial intelligence — the Alchourrón-Gärdenfors-Makinson (AGM) theory of belief change.\(^1\) In contradistinction with the well-established Bayesian approach to belief revision, this one never explicitly refers to the individual’s decisions. Nor does it formalize the individual’s beliefs in measure-theoretic — let alone probabilistic — terms. The building blocks of the AGM theory are propositions. The major mathematical constraint is that these propositions are expressed in a language which in an appropriate sense includes the sentential calculus. Epistemic states, or states of belief, are captured by deductively closed sets of propositions. Epistemic attitudes — belief, disbelief, and indeterminacy — are then described by means of the membership relation. The epistemic input, that is the incoming information, is normally restricted to be propositional. Epistemic changes are axiomatized in terms of the following items: the input which bring them about, the initial epistemic state and the resulting epistemic state. There are three such operations: contraction, expansion and revision. In the principal case at least, contraction may be viewed as a move from belief or disbelief to indeterminacy, expansion as a move from indeterminacy to either belief or disbelief, and finally revision as a move from either determinate attitude to the opposite one. This is

\(^1\)The locus classicus of the AGM approach is Alchourrón, Gärdenfors and Makinson (1985), which should be complemented with the later results in Gärdenfors and Makinson (1988, referred to as GM 1988). Gärdenfors’s Knowledge in Flux (1988, referred to as G1988) methodically surveys both the formal theory and its philosophical applications. Makinson (1985) provides a general introduction.
all good common sense in an initially barren mathematical framework. The
somewhat of the approach makes it the more remarkable that it leads to
important results, one of which will occupy the center stage here.

In a significant development of the AGM framework, Gärdenfors (1988)
and Gärdenfors and Makinson (1988) introduced the novel concept of episte-
tic entrenchment. This concept is meant to capture the relative priority
of a proposition over another in the initial epistemic state. It then has a
bearing on what in the initial state is given up, and what is retained, when
the contraction operation takes place. More precisely, the enlarged AGM
approach axiomatizes epistemic entrenchment as a binary relation on pro-
positions subject to an ordering and further special constraints. It is then shown
that the epistemic entrenchment axioms can be recovered from those already
defined on contraction, and conversely. This result is described as a "grande
finale" in Gärdenfors (1988, p. 96). It is a significant achievement because
the axioms of epistemic entrenchment, on the one hand, and contraction, on
the other, have much to say for themselves, each in a seemingly different
sphere of epistemic intuition. Furthermore, epistemic entrenchment relations
are more concrete objects than contractions, the definition of which is natural
and plausible, but nonconstructive. Broadly speaking, the former relations
play a semantic role, in a way somewhat analogous to the so-called partial
meet contraction functions, which provided the constructive counterpart of
the latter operation in the 1985 version of the AGM theory.

The primary aim of this paper is to improve on the 1988 theorem. For
one, we shall complement it with a dual variant — one which connects suit-
ably modified axioms on epistemic entrenchment with the AGM axioms of the
revision operation. For another, we shall show that the contraction-revision
pair is linked through the corresponding epistemic entrenchment relations
with those two nonadditive measures which have been extensively studied
in the artificial intelligence literature under the labels of necessity and possi-
ability (Dubois and Prade, 1985). The Boolean duality of contraction-and revision-
induced epistemic entrenchment relations will be seen to be exactly reflected
in the elementary duality of necessity and possibility functions.2 The proper-
ties of these functions are surprising to those trained in the probabilistic

2The connection between contraction-induced epistemic entrenchment and necessity was noted in
Dubois and Prade (1991). We make it algebraically definite and state the dual connection of revision
with possibility.

Weltanschauung of Bayesianism and related doctrines. They lead to a curious
comparison between the AGM theory and its rivals.

In accordance with the constructive purpose of this paper, all of the
results of this paper are reached by semantic means. We redefine epistemic
states and inputs in the obvious way, i.e. as classes of models. Epistemic atti-
itudes, belief change operations and epistemic entrenchment are reformulated
accordingly, using the standard Boolean algebra structure of model sets. This
semantic shift leads to variant proofs of known results that are sometimes
more transparent than the original, syntactical proofs of the AGM theory.
However, the focus of this paper is on the above duality themes, and we shall
here state only the proofs of new facts. Sections 2 and 3 survey the main
axioms of the AGM theory along with our suggested, revision-oriented notion
of epistemic entrenchment. Section 4 clarifies the connection with nonad-
titive probability, and section 5 provides some philosophical perspective on this
connection.

II. AXIOMS AND BASIC IDENTITIES OF
BELIEF CHANGE OPERATIONS

For simplicity reasons we shall deal with the AGM theory as if it had been
stated in the propositional calculus strictly speaking (rather than in any logic
which includes it, such as modal propositional logics). Then, classes of models
are simply valuation sets, the properties of which can be taken for granted.
We also assume some familiarity with Tarski's (1935–36) calculus of systems.

Define \( VP = \{p_1, \ldots, p_n, \ldots\} \) to be a denumerable set of propositional
variables. (The role of this cardinality restriction will become clear in section
3.) \( L(VP) \) stands for the set of all propositions, \( \vdash \) for the inference relation,
and \( \mathcal{A} \) for \( L(VP)/\vdash \), i.e. the quotient of the proposition set by the logical
equivalence relation. Following AGM, the individual's epistemic states are
closed under \( \vdash \), that is they are systems in Tarski's sense.3 We denote by \( S \)
the set of all systems and use for them the letters \( S, S', \ldots, K, K', \ldots \). \( \mathcal{S} \)
stands for the set of axiomatizable systems, i.e. of those \( S^0 \in S \) such that

3Gärdenfors (1988) uses "belief sets" instead. The closure constraint is occasionally relaxed in AGM,
1985. It is clearly inconvenient for artificial intelligence applications. Fuhrmann (1991) and Hanson
(1991) analyze the problem of how to define belief change operations on "belief bases" that are not
deductively closed.
$S' = Cn(\{\varphi\})$ for some $\varphi \in L(VP)$. By assumption, the individual's initial state of belief $K$ may be any system whatsoever, including the tautology set $S^T$ of $L(VP)$ and the inconsistent system $S^\bot = L(VP)$, two limiting cases which are not deprived of epistemic relevance.\footnote{As David Makinson helped us to realize.} Note emphatically that $K$ does not have to be complete. We recall that $S$ has a lattice structure when the inf operation is defined as $S \cap S' = Cn(S \cup S')$. $S$ can be endowed with a stronger algebraic structure by introducing Tarski's notion of logical complementation: for any $S \in S$, define $\bar{S} = \bigcap_{\varphi \in S} S^\bot$. As is well-known, this strengthening is enough to turn $S^\bot$ into a full-fledged Boolean algebra (which is isomorphic with $\mathcal{A}$). But $S$ is not a Boolean algebra, except for the case of finite $VP$.

On the semantic side, $V = 2^{VP}$ stands for the set of all valuations and $S = \models$ for the tautological consequence relation. Epistemic states are now valuation sets $V, V', \ldots, W, W', \ldots \in 2^{V}$. For any $\varphi \in L(VP)$ and any $S \in S$, $V$ and $V^\varphi$ stand for the set of models of $\varphi$ and $S$, respectively. Denote by $2^V$ the set of those valuation sets which are models of some proposition $\varphi$ (of some axiomatizable system). The special cases above of noninformative (tautological), inconsistent and complete epistemic states are now captured by the following elements in $2^V$: $\emptyset, \varphi$ and the singletons. This, and other routine correspondences between syntax and semantics are summarized in the following two facts: $\sigma : S \rightarrow 2^V, S \rightarrow V^\varphi$ is a lattice monomorphism from $(S, \subseteq, S^T, S^\bot, \cup, \cap)$ into $(2^V, \subseteq, V, \emptyset, \cup, \cap)$, and the restriction of this function to $S^\bot$ is a Boolean algebra isomorphism into $2^\varphi$ (with set-theoretic complementation standing as a counterpart of logical complementation). In the reminder of this paper, we shall virtually forget about the original, syntactical expression of the AGM theory, using $\sigma$ as a mechanical translation device.

If $W \in 2^V$ is the individual's current epistemic state, his epistemic attitudes — acceptance, rejection and indeterminacy with respect to a proposition $\varphi$ — are clearly rendered as: $W \subseteq V^\varphi$, $W \subseteq V^{\neg \varphi}$ and $W \cap V^{\neg \varphi} \neq \emptyset$ and $W \cap V^\varphi \neq \emptyset$. To define the belief change operations, we introduce the following three functions $2^V \times 2^V \rightarrow 2^V$: $(W, V^\varphi) \rightarrow W^\varphi_\varphi$ ("the expansion of $W$ by $V^\varphi"), (W, V^{\neg \varphi}) \rightarrow W^{\neg \varphi}_\varphi$ ("the contraction of $W$ by $V^{\neg \varphi}"), and $(W, V^\varphi) \rightarrow W^\varphi_\neg \varphi$ ("the revision of $W$ by $V^\varphi"$). There is a definite loss of generality in the restricting of the domain to $2^V \times 2^V$ instead of $2^V \times 2^{V^\varphi}$.

To facilitate comparison, we shall give the semantic axioms of expansion, contraction, revision, and epistemic entrenchment the same labels as those given by AGM to their syntactical counterparts.\footnote{A generalization of this feature is suggested in Gärdenfors and Makinson (1991).}

**EXTRACTION AXIOMS:** $W^\varphi_\varphi = W \cap V^\varphi$

**CONTRACTION AXIOMS:**

\[
\begin{align*}
(K - 2) & \quad W \subseteq W^\varphi_\varphi \\
(K - 3) & \quad \text{If } W \cap V^{\neg \varphi} \neq \emptyset, \text{ then } W^\varphi_\neg \varphi = W \\
(K - 4) & \quad \text{If } V^\varphi \neq V, \text{ then } W^\varphi_\varphi \cap V^{\neg \varphi} \neq \emptyset \\
(K - 5) & \quad W^\varphi_\varphi \cap V^{\neg \varphi} \subseteq W \\
(K - 7) & \quad W^\varphi_\neg \varphi \subseteq W^\varphi_\varphi \cup W^{\neg \varphi}_\varphi \\
(K - 8) & \quad \text{If } W^{\neg \varphi}_\varphi \cap V^{\neg \varphi} \neq \emptyset, \text{ then } W^\varphi_\neg \varphi \subseteq W^{\neg \varphi}_\varphi \\
\end{align*}
\]

Let us call $(K - 2)$ to $(K - 5)$ the minimum AGM axioms for contraction. Among them, we single out $(K - 5)$ — the "recovery postulate" — for future discussion. In words, it says that the contraction of $W$ by $V^\varphi$ should contain no valuation of $V^{\neg \varphi}$ which is not already in $W$. This is surely not a definitional requirement. $(K - 7)$ and $(K - 8)$ may be referred to as the AGM coherence axioms for contraction. In the presence of the minimum axioms, they can be seen to be equivalent to:

\[
(K - V) \quad W^\varphi_\neg \varphi = W^\varphi_\varphi \text{ or } W^\varphi_\neg \varphi = W^\varphi_\neg \varphi \text{ or } W^{\neg \varphi}_\neg \varphi = W^{\neg \varphi}_\varphi \text{ or } W^{\neg \varphi}_\neg \varphi = W^{\neg \varphi}_\varphi
\]

**REVISION AXIOMS:**

\[
\begin{align*}
(K^*2) & \quad W^\varphi_\varphi \subseteq V^\varphi \\
(K^*3) & \quad W \cap V^{\neg \varphi} \subseteq W^\varphi_\varphi \\
(K^*4) & \quad \text{If } W \cap V^{\neg \varphi} \neq \emptyset, \text{ then } W^\varphi_\neg \varphi \subseteq W \cap V^{\neg \varphi} \\
(K^*5) & \quad W^\varphi_\neg \varphi = W^\varphi_\varphi \text{ only if } V^{\neg \varphi} = \emptyset \\
(K^*7) & \quad (W^\varphi_\neg \varphi)_\varphi \subseteq W^{\neg \varphi}_\varphi \\
(K^*8) & \quad \text{If } W^{\neg \varphi}_\varphi \cap V^{\neg \varphi} \neq \emptyset, \text{ then } W^{\neg \varphi}_\neg \varphi \subseteq (W^\varphi_\neg \varphi)_\varphi
\end{align*}
\]

\footnote{As David Makinson helped us to realize. Compare with Gärdenfors, 1988, ch. 3-4 and Makinson, 1988. AGM 1985 have a slightly different axiomatic approach to revision. The missing items in our list, $(K - 1), (K - 6), (K^*1)$, and $(K^*6)$ are taken care of in the very definition of $W^\varphi_\varphi$ and $W^\varphi_\neg \varphi$.}
The minimum AGM axioms for revision, i.e. (K*2) to (K*2), do not seem to include any such problematic restriction as the "recovery postulate". In the presence of them, the AGM coherence axioms for revision (K*7) and (K*8) can be shown (G1988, p. 57) to be equivalent to:

\((K^*V)\quad W_{\varphi\psi} = W_\psi'\) or \(W_{\varphi\psi} = W_\psi = W_\psi'\lor W_\psi'\).

The AGM theory connects revision and contraction with each other by means of the following facts (which are theorems rather than axioms in the current version):

(i) If \(W_\varphi^\varphi\) satisfies (K-2), (K-3), (K-4), the function defined as \(W_\varphi^\varphi = W_\varphi^\varphi \cup V^\varphi\) ("Levi's identity") satisfies the minimum AGM axioms for revision (theorem 3.2 in G1988).

(ii) If \(W_\varphi^\varphi\) satisfies the minimum AGM axioms for revision, the function defined as \(W_\varphi^\varphi = W_\varphi \cup W_\varphi^\varphi\) ("Harper's identity") satisfies the minimum AGM axioms for contraction (theorem 3.4 in G1988).

We pursue the implications of (i) and (ii) in the following observation:

**Observation 1.** The minimum revision axioms imply that \(W_\varphi^{\varphi\varphi} = W_\varphi^\varphi\). Axioms (K-2), (K-3), and (K-4) on contraction imply that:

\((K-5)\) holds if and only if \(W_\varphi^{\varphi\varphi} = W_\varphi^\varphi\).

**Proof.** Using Levi's and Harper's identities in succession, we see that \(W_{\varphi_{\varphi\varphi}} = (W_{\varphi_{\varphi\varphi}}) = (W_{\varphi_{\varphi\varphi}}) = (W_{\varphi_{\varphi\varphi}}) = (W_{\varphi_{\varphi\varphi}}) = W_\varphi\) from (K-2), (K-3) and (K-4) [note that K*5 is not needed]. Now, suppose that \(W_\varphi^{\varphi\varphi} = W_\varphi\) and (K-2), (K-3), (K-4) hold. Applying (i) and (ii) in succession, we conclude that (K-5) also holds. To see the converse statement, assume the minimum contraction axioms and use Harper's and Levi's identities in succession: \(W_{\varphi_{\varphi\varphi}} = W_{\varphi_{\varphi\varphi}}\) and (K-2), (K-3) and (K-4) (K-4 is not needed here).

This observation relates to earlier ones by Makinson (1987). It shows that the "recovery postulate" imposes much structure on the set of contraction functions and the set of revision functions (in particular, these two sets become bijectively related to each other).

**III. EPISTEMIC ENTRICHMENT RELATIONS**

There are two accounts of epistemic entrenchment (e.e.) in the current AGM theory (G1988, ch. 4 and GM 1988). (The point of the 1988 theorem is to show that they are equivalent in an appropriate sense.) The first account is an axiom set which does not involve contraction as a primitive term.

**AGM AXIOMS FOR E.E.:** Define an e.e. relation to be any binary relation \(\leq\) on \(2^V\) that satisfies the following properties:

\[(EE1)\quad\text{Transitivity}\quad\text{if } V' \subseteq V, \text{ then } V' \leq V'\]

\[(EE2)\quad\text{Monotonicity}\quad\forall V', V^\varphi, \text{ either } V^\varphi \leq V'^{\varphi\wedge\varphi} \text{ or } V^\varphi \leq V'^{\varphi\wedge\varphi}\]

\[(EE3)\quad\text{Conjunctiveness}\quad\forall V^\varphi, V'^\varphi, \text{ either } V^\varphi \leq V'^{\varphi\wedge\varphi} \text{ or } V^\varphi \leq V'^{\varphi\wedge\varphi}\]

\[(EE4)\quad\text{Minimality}\quad\text{when } W \neq \emptyset, W \cap V^\varphi \neq \emptyset \text{ iff } V^\varphi \leq V\]

\[(EE5)\quad\text{Maximality}\quad\text{if } V^\varphi \leq V^\varphi \text{ for all } V^\varphi, \text{ then } V^\varphi = V\]

The relation \(\leq\) implicitly depends on the initial epistemic state \(W\), a feature that has been subjected to criticism.\(^7\) The five axioms are meant to capture normatively desirable properties, granting the basic intuition: \(\psi\) is more entrenched than \(\varphi\) if the individual has more confidence in \(\psi\) than in \(\varphi\), ceteris paribus. (EE1) and (EE2) are easy to justify along this line, as is the requirement of connectedness.\(^8\) (EE4) means that the set of propositions with the lowest e.e. degree is exactly the set of those propositions which are not believed in the initial epistemic state. (EE5) says that only tautologies have maximal e.e. (EE3) is clearly the most problematic of the lot. Along with (EE2), it implies the rather strong condition:

\[\forall V, V^\varphi \text{ either } V^\varphi \leq V^\varphi \text{ or } V^\varphi \leq V^\varphi.\]

(We use the standard symbolism: \(\leq\) is a binary relation, \(\sim\) and \(\prec\) refer to its symmetric and asymmetric parts respectively.)

The second AGM account of e.e. consists in introducing a link between e.e. and contraction. The guiding idea is now: \(\psi\) is more entrenched than \(\varphi\) if the individual is more willing to give up \(\varphi\) than he is to give up \(\psi\), when

\(^7\)See Schlocho's (1991) elaboration of this point.

\(^8\)Connectedness of \(\leq\) follows from applying (EE3) and (N) below.
he is given the choice.9

DEFINITION (C ≤) OF E.E.: Given a function \((W, V^0) \mapsto W^*_\psi\) define \(\leq^*\) as the following binary relation on \(2^{V^0}\):

\[ V^\psi \leq^* V^0 \iff W^*_\psi \cap V^\neg\psi \neq \emptyset \text{ or } V^\psi = V = V^0. \]

Thus, in the AGM theory, e.e. has a direct connection with contraction and relates to revision only through the Levi identity. One may wonder how both the axioms and the above definition should be modified if e.e. had instead to connect with revision directly, and with contraction only through the Harper identity.

ALTERNATIVE AXIOMS FOR E.E.: Define an e.e. relation to be any binary relation \(\leq^*\) on \(2^{V^0}\) that satisfies:

\[ (EE1') \quad \text{Transitivity} \]
\[ (EE2) \quad \text{Monotonicity} \]
\[ (EE3) \quad \text{Disjunctiveness: } \forall V^0, V^1, \text{ either } V^0 \cup V^1 \leq^* \psi \vee V^0 \text{ or } V^0 \psi \leq^* \psi \quad \text{for all } V^0 \]
\[ (EE4) \quad \text{Alternative maximality: when } W \neq 0, W \psi \psi = \emptyset \text{ iff } V^\psi \psi \leq^* \psi \quad \text{for all } V^0 \]
\[ (EE5') \quad \text{Alternative minimality: if } V^\psi \psi \leq^* \psi \quad \text{for all } V^0, \text{ then } V^\psi = \emptyset. \]

DEFINITION (R ≤) OF E.E.: Given a function \((W, V^0) \mapsto W^*_\psi\), define \(\leq^*\) as the following binary relation on \(2^{V^0}\):

\[ V^\psi \leq^* V^0 \iff W^*_\psi \cap V^\neg\psi \neq \emptyset \text{ or } V^\psi = \emptyset = V^0. \]

To motivate these new concepts: \((EE4')\) means that the set of propositions whose negates are maximally entrenched is exactly the set of those propositions which are not believed in the initial epistemic state. One should expect this liberal criterion to allow for more beliefs than the corresponding \((EE4)\). \((EE5')\) says that only contradictions have minimal e.e. In the presence of \((EE2)\), \((EE5')\) implies that

\[ (P) \quad \forall V^0, V^\psi \text{ either } V^0 \psi \psi \leq^* \psi \quad \text{or } V^0 \psi \psi \leq^* \psi. \]

9We slightly distort the AGM way of presentation by stating their \((C \leq)\) condition as the definition of an e.e. relation.

On the face of it, this rather demanding condition is no more, no less difficult to justify than \((N)\) above. As far as \((R \leq)\) is concerned, the guiding idea is: \(\psi\) is more entrenched than \(\psi\) if the individual is more willing to give up \(\neg\psi\) than he is to give up \(\neg\psi\), when he is given the choice.

We record the simple fact: \((C \leq)\) and \((R \leq)\) define complete relations when \((K-1)\) and \((K-2)\) and \((K-5)\) hold of \(W^*_\psi\) and \(W^*\), respectively. Now, to clarify the algebraic connection between the two definitions of e.e. some more notation and terminology are needed. Given any functions \(W^*_\psi\), \(W^*_\psi\) the symbols \(\leq \quad \text{and} \quad \leq^* \quad \text{refer to well-defined} \quad \text{binary relations: the former results from applying} \quad (R \leq) \quad \text{to} \quad W^*_\psi\), \text{the latter from applying} \quad (C \leq) \quad \text{to} \quad W^*_\psi\). More generally, meaningful symbols of binary relations result from alternating \(\neg \quad \text{and} \quad \neg \quad \text{at any length whatsoever. Now, if} \quad B \text{is any Boolean algebra and} \quad x^* \quad \text{refers to the complementary of} \quad x, \text{we define two complete binary relations} \quad R, R' \subseteq B \times B \text{ to be dual to each other if}: \quad (a, b) \in R = (b^*, a^*) \in R'. \]

We are ready to state:

OBSERVATION 2. Suppose that \(W^*_\psi\), \(W^*_\psi\) are any functions \(2^{V^0} \times 2^{V^0} \rightarrow 2^{V^0}\). Then

\[ V^\psi \leq V^0 \Rightarrow V^\psi \leq^* V^0 \Rightarrow V^\psi \leq V^0 \Rightarrow V^\psi = V^0 \leq V^0. \]

Suppose that \(W^*_\psi\), \(W^*_\psi\) satisfy the minimum contraction and revision axioms respectively. Then, the converse implications also hold, and the relations \(\leq^* \quad \text{and} \quad \leq \quad \text{as well as} \quad \leq^* \quad \text{and} \quad \leq, \text{are dual to each other}. \text{There is a one-to-one relationship between the set} \quad R \quad \text{of those} \quad \leq\quad \text{which come from a} \quad W^*_\psi \quad \text{satisfying the minimum contraction axioms, and the set} \quad R^* \quad \text{of those} \quad \leq^* \quad \text{which come from a} \quad W^*_\psi \quad \text{satisfying the minimum revision axiom.} \]

Proof. Suppose that \(V^\psi \leq V^0\). Then either \(V^\psi = V^0 \Rightarrow V^\psi \leq^* V^0 \leq V^0 \Rightarrow V^\psi = V^0 \leq V^0\). Suppose that \(V^\psi \leq^* V^0\). Then either \(V^\psi = V^0 \Rightarrow V^\psi \leq^* V^0 \leq V^0 \Rightarrow V^\psi = V^0 \leq V^0\).

10This concept of dual complete relations is adapted from Halmo's definition of dual functions on B, (1974, p. 8); apply Halmo's definition to the characteristic function \(1_x\) and \(1_{\neg x}\), using the fact that \(R\) and \(R^*\) are complete. Note that the binary relation introduced by Grove (1988) and taken up by Gardenfors (1988, p.95) has sometimes been described as "dual" to \(\leq\). The Grove relation appears to be distinct from \(\leq^*\) as defined here, hence it must be dual in a different algebraic sense from Halmo's.
To show that there are cyclic $\leq^*$ it is enough to take $V^p, W$ and $W^\omega_\subseteq$ as above, and apply the duality property of $\leq^*$ and $\leq^{*\rightarrow}$ stated in observation 2.

**Observation 4:** If $\leq^*$ comes from a $W^\omega_\subseteq$ that satisfies the minimum contraction axioms, it satisfies (EE2) to (EE5). If $\leq^*$ comes from a $W^\omega_\subseteq$ that satisfies the minimum revision axioms, it satisfies (EE2)' to (EE5). 

**Sketch of the proof.** The proof of the $\leq^*$ part of observation 4 is a semantic variant of the syntactical proof in GM 1988, p. 94. The proof of the $\leq^*$ part is a simple exercise in duality, using observation 2: given $\leq^*$ in $\mathcal{R}^*$, there is a uniquely defined $\leq_\subseteq^*$ in $\mathcal{R}^\omega$ such that $\leq_\subseteq^* \subseteq \leq^*$; the properties of $\leq^*$ then follow from the corresponding properties of $\leq_\subseteq^*$ and the fact that $V^p \leq^* V^p \text{ if } V^p \leq V^p$.

**IV. EPISTEMIC ENTRENCHMENT AS NONADDITIVE PROBABILITY**

This section states the main observations: For one, if $\leq$ and $*$ satisfy the complete (rather than the minimum) set of AGM axioms, then $(C \leq)$ and $(R \leq)$ lead to e.e. relations that can be represented by numerical functions of the necessity and possibility types, respectively. For another, starting with a necessity and a possibility defined on the Boolean algebra of model sets, we can always construct belief change operations $\leq$ and $*$ that satisfy the complete AGM axiom set and are related to the initial necessity and possibility by means of $(C \leq)$ and $(R \leq)$, respectively. The $(C \leq)$ part of these statements is a simple corollary to the 1988 theorem, while the $(R \leq)$ part will be derived as before by using a duality argument.

First of all, we need some rudimentary facts and notations from the recently developed theories of nonadditive probability. Given a nonempty set $\Theta$ and a Boolean algebra of subsets $\mathcal{B}$ on it, a function $f : \mathcal{B} \to [0, 1]$ will be said to be a *nonadditive probability* if it satisfies:

1. $f(\emptyset) = 0$
2. $f(\Theta) = 1$

11*Capacity* is sometimes used instead, see Chateauneuf and Jaffray's (1989) survey of set functions satisfying (i), (iii) and (iii). The terminology of the field is not unified.
(iii) $\forall A, B \in \mathcal{B}, \ A \subseteq B \Rightarrow f(A) \leq f(B)$ (0-monotonicity).

The dual function $g$ which is defined by putting $g(A) = 1 - f(A^c)$ is again a nonadditive probability. Stronger monotonicity requirements than (iii) are often considered, as in Shafer's (1976) theory of "belief functions" and "upper probabilities". Even restricted in the way suggested by Shafer, the class of nonadditive probabilities remains wide. We shall here concentrate on two particular cases which are both very simple to handle and arguably relevant to the modelling of belief. In the terminology of Dubois and Prade (1985), a necessity is a function Nec : $\mathcal{B} \rightarrow [0, 1]$ that satisfies (i) and (ii) as well as:

$$(\text{NEC}) \quad \forall A, B \in \mathcal{B} \quad \text{Nec}(A \cap B) = \min(\text{Nec} A, \text{Nec} B),$$

and a possibility is a function Pos : $\mathcal{B} \rightarrow [0, 1]$ that satisfies (i) and (ii) as well as:

$$(\text{POS}) \quad \forall A, B \in \mathcal{B} \quad \text{Pos}(A \cup B) = \max(\text{Pos} A, \text{Pos} B).$$

Plainly, the (NEC) and (POS) clauses imply (iii). The dual of Nec, denoted by $P_{\text{Nec}}$, is a possibility, and the dual of Pos, denoted by $N_{\text{Pos}}$, is a necessity. The following properties can straightforwardly be derived in succession: for any $A \in \mathcal{B}$,

(iv) $\min(\text{Nec} A, \text{Nec} A^c) = 0$ and $\max(\text{Pos} A, \text{Pos} A^c) = 1$,

hence: $\text{Pos} A < 1 \Rightarrow N_{\text{Pos}} A = 0$ and $\text{Nec} A > 0 \Rightarrow P_{\text{Nec}} A = 1$

(v) $\text{Nec} A + \text{Nec} A^c \leq 1$ and $\text{Pos} A + \text{Pos} A^c \geq 1$,

hence: $\text{Nec} A \leq P_{\text{Nec}} A$ and $\text{Pos} A \geq N_{\text{Pos}} A$.

The properties in group (iv) are specific to the Nec and Pos functions, whereas those in group (v) are satisfied by a wider class of nonadditive probabilities, in effect Shafer's already mentioned "belief functions" (of which Nec is but a particular case) and "upper probabilities" (of which Pos is but a particular case). The generic properties (v) underlie the case for using nonadditive rather than Kolmogorov probabilities in the modelling of uncertain belief. If the agent is totally uncertain about whether $A$ or $A^c$ will take place, probability theory leaves us no other choice than putting $\text{Prob}(A) = \text{Prob}(A^c) = \frac{1}{2}$. This "principle of indifference" has led to celebrated paradoxes which are recalled for instance in Keynes's treatise (1921). The cause of the paradoxes is that the Prob function deals with complete uncertainty as if it were the certainty of equiprobability. This undesirable feature disappears from the nonadditive approach. Typical of the latter is the fact that the relevant information is contained in a pair of dual functions rather than in a single function. For instance, Nec $A$ conveys only limited information on Nec $A^c$, which can, however, be retrieved from the dual $P_{\text{Nec}} A$.

We can now state:

**Observation 5**: Suppose that $W_\omega$ satisfies the complete set of AGM axioms on contraction. Then, the induced epistemic entrenchment relation $\preceq$ is represented by a necessity function Nec, and the dual relation $\preceq^*$ is represented by the possibility function $P_{\text{Nec}}$ which is dual to Nec. Similarly, suppose that $W_\omega$ satisfies the complete set of AGM axioms on revision. Then, the induced e.e. relation $\preceq$ is represented by a possibility function Pos, and the dual relation $\preceq^*$ is represented by the necessity function $N_{\text{Pos}}$ which is dual to Pos.

**Sketch of the proof**: The main step in the derivation of Observation 5 is to show that in the presence of the minimum contraction axioms, $(K-7)$ and $(K-8)$ imply that $\preceq$ is transitive. This step is a major achievement of the 1988 theorem: see especially GM1988, theorem 4 (the earlier proof in G1988 is more roundabout). Our proof opera is a semantic variant that need not to be recast here. Once $(EE1)$ holds, we get a numerical representation for $\preceq$ at once; for $VP$, hence $L(VP)$, we have been taken to be denumerable. From observation 3, the ordering $\preceq$ satisfies clause (N), whence it readily follows that any numerical representation $f$ of $\preceq$ satisfies (NEC). It remains to be checked that $f$ can be chosen such that its range is $[0, 1]$ and (i) and (ii) hold. This is clearly the case in view of $(EE2)$ and of the fact that $\preceq$ is nontrivial. This last fact has to be derived from either $(EE4)$ or $(EE5)$.

Now, starting from a $\preceq$ which satisfies the complete AGM set of revision axioms, we know from the AGM-theory that $\preceq^*$ satisfies the complete AGM set of contraction axioms, and from observation 2 that $V^* \preceq^* V^v$.
iff $V^{-w} \leq^* V^{-w}$. This establishes the transitivity of $\leq^*$. Once $(EE1')$ holds, $\leq^*$ is represented by a numerical function $Pos$ which is readily seen to meet the definitional requirements (POS), (i) and (ii). Note the role of either $(EE4')$ or $(EE5')$ as a nontriviality condition.

We have not yet used the full force of $(EE4) - (EE5)$ and $(EE4') - (EE5')$. They impose the following necessary conditions on the representations $Nec$ and $Pos$ of $\leq^*$ and $\leq^*$:

(4) When $W \neq \emptyset, W \cap V^{-w} \neq \emptyset$ iff $Nec V^{-w} = 0$

(5) If $Nec V^{-w} = 1$, then $V^{-w} = V$

(4') When $W \neq \emptyset, W \cap V^{-w} \neq \emptyset$ iff $Pos V^{-w} = 1$

(5') If $Pos V^{-w} = 0$, then $V^{-w} = \emptyset$.

In words, (4) [(4')] says that if the initial belief set is consistent, a proposition $\phi$ belongs to it iff the necessity of $\phi$ is more than minimal (resp. the possibility of $\neg \phi$ is less than maximal). Using one of the specific properties (iv) of $Pos$, we have:

(4') $\Rightarrow$ (4'') : When $W \neq \emptyset, W \subseteq V^{-w} \Rightarrow Pos V^{-w} = 1$.

While (4) and (4') are vacuously satisfied in the case of initial inconsistent belief sets, (5) and (5') are always binding conditions. They impose on $Nec$ (Pos) the special feature that contingent propositions never enjoy maximal necessity (resp. minimal possibility). A moment's thought shows that this feature is rooted in the chosen definitions of $\leq^*$ and $\leq^*$, irrespective of the contraction and revision axioms.

Observation 5 admits of a kind of converse statement.

**Observation 6**: Suppose that $Nec$ is a necessity function on $2^V$ and is such that (4) and (5) hold. Then, one can construct a function $W^{-w} : 2^V \times 2^V \rightarrow 2^V$ which satisfies the complete AGM axiom set for contraction and is such that $Nec$ represents $\leq^*$. Suppose that $Pos$ is a possibility function on $2^V$ and is such that (4') and (5') hold. Then, one can construct a function $W^+ : 2^V \times 2^V \rightarrow 2^V$ which satisfies the complete AGM axiom set for revision and is such that $Pos$ represents $\leq^*$.

**Sketch of the proof**. The $\leq^*$ part is a corollary to theorem 4 in GM1988 as semantically repressed. The $\leq^*$ part derives from the following duality argument: starting with $Pos$, we construct $N^Pos$ and apply the first part to find $W^{-w}$ and $\leq^*$ such that $W^{-w}$ satisfies the complete AGM set of contraction axioms and $N^{Pos}$ represents $\leq^*$. Then, the AGM theory ensures that $\leq^*$ satisfies the complete AGM set of revision axioms and observation 2 implies that $Pos$ represents $\leq^*$.

Observations 5 and 6 complete the semantic exercise of this paper. Before discussing its significance, we need further to clarify the respective roles of the $Nec$ and $Pos$ representations. Contrary to its earlier version (which adds Harper's identity to the minimal revision axioms, see 1985, p. 519), the current AGM theory axiomatizes contraction and revision separately. Taken literally, the theory does not exclude the possibility that an individual contracts and revises his initial epistemic state $W$ in a completely disconnected way (so that the Levi and Harper associates $W^{-w}$ and $W^{+w}$ would describe purely notional operations). This disconnected behaviour leads to a $Nec$ and a $Pos$ representations which are related to each other only through (4) and (4') above, hence only through the condition that

$Nec V^{-w} = 0$ iff $Pos V^{-w} = 1$.

Now, the reading just made of the AGM theory is little more than a logical possibility. One should expect the individual to satisfy not only the contraction and revision axioms as they stand, but also one of the two basic identities. Of the two, Levi's has the best philosophical credentials. Whichever identity is added to the axioms, the individual's behaviour will be described by a pair of dual functions. The choice of $(Nec, P^{Pos})$ fits the preference for the Levi identity, whereas the choice of $(Pos, N^{Pos})$ fits the (less likely on theoretical grounds) preference for the Harper identity.

A comment may be added in connection with the dual role of the $Nec$ and $Pos$ representations. The bias of this paper is to connect them to contraction and revision, respectively, but it should be clear that even the pure theory of e.g. calls for a pair of dual concepts. The argument of the next paragraph is intended to convey the point; it makes no reference to the belief change operations.

It would seem as if the following made good commonsense:

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15 Use G1988, theorem 3.3.
16 See Levi's discussion of contraction and the corresponding identity as early as 1977, and his later assessments in 1983 and 1991.
(⋆) “If the individual has greater confidence in ψ than φ, then he has lesser confidence in ¬ψ and in ¬¬φ”.

Is it possible to include this principle in the notion of e.e.? Assume that (EE1) to (EE4) holds of ≤ and add:

\[ V^φ \leq V^ψ \iff V^{¬ψ} \leq V^{¬φ}. \]

(⋆⋆) Assume also that \( W \neq \emptyset \). Then, from the proof of observation 5 ≤ is represented by a Nec. But (⋆⋆) implies that (EE3’) also holds, and this in turn implies that Nec satisfies the functional equation of Pos – a degenerate case that can only be satisfied by Boolean valuations. Thus, either the individual entertains contradictory beliefs (\( W = \emptyset \)), or he gives the same maximal value to any member of his belief set. Either case trivializes the theory of e.e. The upshot of this argument is that (⋆⋆) cannot be the formal rendering of principle (⋆). By contrast, the use in this paper of two distinct e.e. relations ≤ and ≤’ arguably accounts for the latter principle.

V. DISCUSSION

The main objective of this paper was to provide the AGM logic of belief change with a semantic counterpart in terms of functional representations of epistemic entrenchment. Observations 5 and 6 have some remote analogy with a soundness and completeness theorems, respectively. (The same could be said, of course, of the two parts of the 1988 GM theorem on which these two observations improve marginally.) Although observation 6 is more constructive — offering as it does sufficient conditions to generate contractions and revisions —, we find observation 5 philosophically more useful. The necessary conditions that observation 5 states on e.e. puts the AGM theory to an indirect test.

1) One could object to the special shape of Nec and Pos imposed by the maximality and minimality requirements (5) and (5’). This might suggest to modify (C ≤) and (R ≤) slightly, because these requirements are built in the chosen definitions of e.e. relations.

2) More importantly, one could object to e.e. being representable by such highly specific nonadditive probabilities as are Nec and Pos. Among the properties listed in section 4, those in group (iv) lead to a puzzling consequence. Take an individual who believes with certainty that it is equally likely that the coin will fall heads (H) and that it will fall tail (T). If we are to model this individual’s belief by means of one of the functions under review, we must put Nec (H) = Nec (T) = 0 or Pos (H) = Pos (T) = 1. That is to say, the Nec and Pos functions deal with the certainty of equiprobability as if it were also something else: complete ignorance of the result H or T of the tossing of the coin (in the Nec case), and complete confidence in either result (in the Pos case). Nec and Pos raise an interpretative issue which is (informally) opposite to the classic problem of equiprobability, as briefly reported in section 4.

3) Writers of the Bayesian persuasion should object to the involvement of the AGM theory with any concept of nonadditive probability whatsoever. There is a problem here that cannot be evaded by referring to the otherwise interesting reformulation of the AGM belief change axioms in a probabilistic rather than propositional context.\(^\text{17}\) The present writer is inclined to sharpen the differences between Bayesianism and the AGM approach. There is no agreement among Bayesians on how to extend Bayes’ rule to the case in which the incoming information had zero a priori probability. By contrast, the AGM revision axioms apply universally, and their application to the case in which the epistemic input contradicts the initial belief set leads to intuitively plausible results. At this juncture, the Bayesian counterargument could be that the AGM approach is indeterminate in a sense different from, but perhaps no less worrying than the Bayesian one: the revision axioms do not define a unique procedure to reconcile the current information with initial beliefs.

It goes beyond the scope of this paper to pursue objections 2) and 3) in any detail. But note emphatically that even if they succeeded against the AGM theory as it stands, the final result would be ambiguous. One could decide against either the AGM axioms of belief change or against (C ≤) and (R ≤). The latter solution is much more attractive than the former.

\(^\text{17}\) Gardenfors (1988, ch. 5) devises a variant of the AGM theory in which belief change axioms apply to an initially given probability (instead of belief set). He then shows that the newly introduced notions of contraction and revision encompass the old ones (when belief sets are defined as sets of sentences with probability 1). This is a step towards Bayesianism, as it were. However, generalizations of the AGM framework may be sought in other directions as well. One wonders about the alternative variant in which belief change axioms would apply to an initially given nonadditive probability, typically a Nec or a Pos.\)
given the strong direct and indirect warrant of the contraction and revision axioms. Hence the research programme: how should one modify the definition of contraction-and revision-induced e.e. relations in order to generate other functional representations than Nec and Pos?

What has just been said suggests that we might have rendered the AGM theory an ambiguous service by stressing its connection with these special functions. However, the relation goes both ways, and we might wonder how observations 5 and 6 (or by this token, Dubois and Prade's already-mentioned result) should influence the "theory of possibilistic reasoning". Here is an example of the indirect support that this theory could draw from the exhibited connection with the AGM system.

A PUZZLE OF NECESSITY AND POSSIBILITY: Suppose that propositions A and B are such that Nec A > 0 and Nec B > 0. Then, Nec (A ∧ ¬B) ∨ (¬A ∧ B)) = 0. Suppose that A and B are such that Pos A < 1 and Pos B < 1. Then Pos (A ⊴ B) = 1.18

(Proof: Apply properties (NEC) and (iv) along with the fact that A ∧ B ∧ ((A ∧ ¬B) ∨ (¬A ∧ B)) is a contradiction. Then use duality.)

This observation is something of a puzzle to those trained in the (standard) probabilistic reasoning. To focus on the Nec case, why should the individual have zero degree of belief in the statement that A and B are incompatible as soon as he gives positive — however small — degrees of belief to A and B? The appearance of paradox disappears when one refers Nec to the AGM system associated with it by observation 6. To be specific, assume that the initial belief set is consistent and apply (4): then, the observation records the intuitively unproblematic fact that if the individual believes that A and B, he does not believe that A and B are mutually inconsistent. More generally, the AGM logic might provide some epistemic foundation to the use of the Nec and Pos functions in Artificial Intelligence.

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18Rot (1991) discusses a somewhat similar puzzle in the qualitative context of the e.e. relation.

LOGIC OF BELIEF CHANGE

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