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# Social Choice Theory in the Case of Von Neumann-Morgenstern Utilities

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**Abstract.** Part I of this paper offers a novel result in social choice theory by extending Harsanyi's well-known utilitarian theorem into a "multi-profile" context. Harsanyi was contented with showing that if the individuals' utilities $u_i$ are von Neumann-Morgenstern, and if the *given* utility $u$ of the social planner is VNM as well, then the Pareto indifference rule implies that $u$ is affine in terms of the $u_i$. We provide a related conclusion by considering $u$ as *functionally dependent* on the $u_i$, through a suitably restricted "social welfare functional" $(u_1,\dots,u_n)\mapsto u=f(u_1,\dots,u_n)$. We claim that this result is more in accordance with contemporary social choice theory than Harsanyi's "single-profile" theorem is. Besides, Harsanyi's initial proof of the latter was faulty. Part II of this paper offers an alternative argument which is intended to be both general and simple enough, contrary to the recent proofs published by Fishburn and others. It finally investigates the affine independence problem on the $u_i$ discussed by Fishburn as a corollary to Harsanyi's theorem.

## 1. Introduction

In 1955 John Harsanyi proved the following theorem: if the individuals' utility functions $u_i$ as well as the social utility function $u$ are von Neumann-Morgenstern (VNM), and if the Pareto indifference rule holds, then the social utility function is an affine transformation of individual utilities, i.e. $u=\sum a_i u_i+b$. This is clearly an important result on utilitarianism, indeed one of the fundamental ones along with the result in d'Aspremont and Gevers' classic paper (1977). However, it is by no means as assertive as the latter, or, by the same token, any standard result in the

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post-Arrowian social choice literature. As Sen has aptly pointed out (1986, p 1124), it deals with "single-profile exercises" only. That is, Harsanyi's theorem claims that a *given* VNM $u$ can be represented as an affine transformation of *given* VNM $(u_1,\ldots,u_n)$, provided that $u$ and $(u_1,\ldots,u_n)$ are connected by a Pareto-like condition. There is no implication, of course, that the coefficients $a_i$ and $b$ should remain the same when $(u_1,\ldots,u_n)$ is changed into $(u'_1,\ldots,u'_n)$. On the other hand, d'Aspremont and Gevers' theorem is typically a "multi-profile exercise". That is, they consider various axiomatic restrictions on a *social welfare functional* (SWF) $f:(u_1,\ldots,u_n)\mapsto f(u_1,\ldots,u_n)$, from which they derive utilitarian properties for $f$. That the latter concept bears no relationship to the approach taken by Harsanyi is worth emphasizing, since there is a vacillation in his own account. In 1977 (p 69), he claims that Bentham's utility sum rule, or rather its means utility variant, logically derives from his theorem. This cannot be the case, for utilitarian rules can only be captured within the mathematical framework of "multi-profile exercises". In particular, the anonymity requirement, which is implied by any utilitarian variant, cannot be defined in terms of a *single* $(n+1)$-tuple $(u_1,\ldots,u_n,u)$.

The primary aim of this paper is to build a "multi-profile exercise" upon Harsanyi's suggestive result, i.e. to extend it into a result on SWF's when the utility space is restricted to von Neumann-Morgenstern functions. This much is achieved by Proposition 1 below, which roughly says that any such SWF is affine provided it satisfies a suitable Pareto condition, called "extended neutrality" after a related condition in the literature. Once this is proved, it is routine to derive utilitarian shapes for $f$ from the usual anonymity requirement. Proposition 1 may be compared with d'Aspremont and Gevers' result, or rather a variant of it which is suited to our definition of extended neutrality. Proposition 1 is more precise than the variant, which only claims that $f(u_1,\ldots,u_n)(x)\geq f(u_1,\ldots,u_n)(y)$ iff $\sum u_i(x)\geq\sum u_i(y)$, but some might find the VNM requirement an exacting one. Indeed, the axioms mentioned in the variant of d'Aspremont and Gevers' theorem are *prima facie* more economical. But it should be stressed that Proposition 1 does *not* require any assumption on cardinal comparability. The normal cocardinality property of utilitarian rules turns out to be a *consequence* of our axioms, or rather only of the VNM restriction and the extended neutrality condition irrespective of the anonymity requirement.

As it happens, the "multi-profile exercise" casts some light on the "single-profile" one. Harsanyi's initial argument in his seminal 1955 paper lacked definiteness and we shall pinpoint a mistake in the 1977 version of his theorem. It was not until recently that social choice theorists grew dissatisfied with the mathematics of Harsanyi's theorem and began to publish either alleged counter-examples (Resnik 1983) or novel proofs (Domotor 1979; Fishburn 1984; Border 1985; Selinger 1986). Among those contributions, Domotor's and Fishburn's are at the highest level of generality. They do not make any irrelevant restriction on the utility space and their formalism relies on *mixture sets* and *mixture-preserving functions*, the most general way of capturing the intuitive notions of a lottery set and a VNM utility. This algebraic approach is also adhered to all throughout this paper. We were able to offer a novel simple proof of the "single-profile" result by adapting the basic argument of Proposition 1 to the case of given $(u_1,\ldots,u_n)$ and $u$. As it happens, little work is required to move from our "multi-profile" theorem to

Proposition 2, i.e. Harsanyi's theorem expressed in the formalism of mixture sets. The resulting proof is much shorter than Fishburn's and turns out to be related to some degree to Domotor's. The end of this paper investigates the independence problem on the $u_1, \ldots, u_n$, discussed in Fishburn's corollary to Harsanyi's theorem.

## 2. Basic Facts About Mixture Sets

In the formalism of this paper, a lottery set and a VNM utility are defined as a mixture set (*MS*) and a mixture-preserving (*MP*) function respectively. Recall that a *mixture set* $\mathscr{M}$ is any set together with an operation

$$[0,1] \times \mathscr{M} \times \mathscr{M} \rightarrow \mathscr{M}$$

$$(\lambda, x, y) \mapsto x\lambda y$$

as defined by the following set of axioms:

$$x1y = x \tag{A1}$$

$$x\lambda y = y(1-\lambda)x \tag{A2}$$

$$(x\lambda y)\mu y = x(\lambda\mu)y \tag{A3}$$

Intuitively, $(\lambda, x, y) \mapsto x\lambda y$ should be seen as the operation of "mixing" lotteries $x$ and $y$ according to the "weights" $\lambda$ and $1-\lambda$ respectively. As an example of a mixture set, take $\mathscr{M}$ = the set of all probability distributions on a given measurable set $E$. More generally, any convex subset $C$ of a linear space $F$ is a mixture set; there is a converse of a sort to this statement, as pointed out in footnote 2. Comparing linear spaces with *MS*'s makes it plain how weak an algebraic structure the latter have[1].

Despite their convenient generality, mixture sets are not widely resorted to among social choice theorists. A more common practice is to assume a set $E$ of basic, nonrandom alternatives ("pure prospects") and build the lottery set $L(E)$ from $E$ after making suitable assumptions on $E$. As an example of this procedure in the context of our problem, Harsanyi (1977, p 64) and Border (1985) assume that $E$ is measurable and define $L(E)$ as the set of probabilities on $E$. A trivial, rather uninteresting particular case of the latter assumption obtains when $E$ is taken to be a finite set, as in Selinger (1986). As an alternative specification, consider defining a set of "feasible endowments" $E \subset R^n$ and a set of binary lotteries on $E$, $L(E)$ = the convex hull of $E$. This is, of course, an attractive assumption when the issue under investigation is income or wealth distribution among $n$ individuals. However, it should be clear that such two-step procedures are unduly specific from the vantage point of a general social choice theory. First of all, they all neatly fall under the mixture set concept as particular cases. Second, and even more importantly, it may happen that there is no such thing as a "set of pure prospects" *prior to and distinct of* the lottery set from which the social planner is to choose.

[1] Mixture sets were first introduced by Herstein and Milnor (1953). For some details and examples see Fishburn (1982, Chap. 1).

A function $f: \mathcal{M} \to \mathbb{R}$ is said to be *mixture-preserving* ($MP$) if:

$$\forall (\lambda, x, y) \in [0,1] \times \mathcal{M} \times \mathcal{M} \ ,$$

$$f(x\lambda y) = \lambda f(x) + (1-\lambda) f(y) \ .$$

A function $\mathcal{M} \to \mathbb{R}^n$ which is $MP$ component by component will be called $MP$ as well. Note that the case just made in the last paragraph for the mixture set formalism does not carry with it the suggestion that mixture-preserving functions are *the* relevant utility concept. That is, mixture sets may also be used in the context of a theory of rational decision-making under risk which is *not* von Neumann-Morgenstern.

The set of all $MP$ functions on $\mathcal{M}$, denoted by $\mathcal{L}(\mathcal{M})$, is clearly a linear space[2]. As an example of $\mathcal{L}(\mathcal{M})$, take $\mathcal{M} = \mathbb{R}^n$ and consider the set $\mathcal{A}$ of all affine real functions on $\mathbb{R}^n$. Obviously, $\mathcal{A} \subset \mathcal{L}(\mathcal{M})$, but it is also clear that $\mathcal{L}(\mathcal{M}) \subset \mathcal{A}$, since a function which preserves convex combinations on $\mathbb{R}^n$ is both concave and convex on $\mathbb{R}^n$, that is affine on $\mathbb{R}^n$. This result easily extends to any linear space $\mathcal{M}$, a fact which we shall record here without proof:

*Remark 1.* If $\mathcal{M}$ is a linear space, $\mathcal{L}(\mathcal{M})$ is the set of all affine functions on $\mathcal{M}$.

We may also note the fact that, if $f: \mathcal{M} \to \mathbb{R}$ is $MP$, and $T: \mathbb{R} \to \mathbb{R}$ is affine, $T \circ f$ is $MP$ as well.

A further remark may be clarifying. Given $\mathcal{M}$, consider those $x, y \in \mathcal{M}$ that no element in $\mathcal{L}(\mathcal{M})$ "separates" from each other, i.e. those verifying $f(x) = f(y)$ for *every* mixture-preserving $f$ on $\mathcal{M}$. In the context of a paper centering upon the VNM theory of social choice, it seems to be reasonable to regard such $x$ and $y$ as one and the same element. This is formally possible since the suggested equivalence relation $x \sim y$ preserves the mixture set structure. Actually, slightly more than that is true:

*Remark 2.* Suppose $X$ is any set provided with any operation $[0,1] \times X \times X \to X$, $(\lambda, x, y) \mapsto \varphi(\lambda, x, y)$. Define a $\varphi$-preserving function as a $f: X \to \mathbb{R}$ such that

$$(\forall (\lambda, x, y) \in [0,1] \times X \times X) f(\varphi(\lambda, x, y)) = \lambda f(x) + (1-\lambda) f(y) \ .$$

Also define the following equivalence relation $\sim$:

$$(\forall (x, y) \in X^2) x \sim y \Leftrightarrow (\forall f \ \varphi\text{-preserving on } X) f(x) = f(y) \ .$$

Then, $\varphi$ induces an operation $\Phi$ on $X/\sim$ which satisfies the mixture set axioms.

If a mixture set $\mathcal{M}$ is such that its $\mathcal{L}(\mathcal{M})$ consists of constant functions only, $\mathcal{M}/\sim$ reduces to a singleton – a case which is hardly worth studying. The "multi-profile" theorem of Sect. 3 does not apply to such *degenerate MS*.

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[2] Using a duality argument, it may be seen that every mixture set $\mathcal{M}$ is essentially identical with a convex subset of a linear space. Define $\mathcal{L}(\mathcal{M})'$ the space of linear forms on $\mathcal{L}(\mathcal{M})$ and $i$ the function: $\mathcal{M} \to \mathcal{L}(\mathcal{M})'$, which maps $x$ into $E_x$, the evaluative function with respect to $x$ ($E_x$ is the linear form mapping every $u \in \mathcal{L}(\mathcal{M})$ into $u(x)$). Clearly, $i$ is injective and maps mixtures in $\mathcal{M}$ into convex combinations in $\mathcal{L}(\mathcal{M})'$. This fact suggests a technique for proving results on mixture sets. The resulting proofs may be simpler, though they should be less illuminating than the direct ones used in this paper. We are indebted in this footnote to a hint of L. Haddad.

## 3. The Multi-Profile Theorem

Before proving the multi-profile theorem, some social choice terminology needs to be introduced. Given $n$, the number of individuals in the society, and $\mathcal{M}$ a mixture set, $\mathcal{E}$ is a set (usually a vector space) of real functions on $\mathcal{M}$. A *social welfare function* is a function

$$f: \mathcal{E}^n \to \mathcal{E}$$
$$U = (u_1, \dots, u_n) \mapsto f(U) \ .$$

We shall here be concerned with SWF's verifying $\mathcal{E} = \mathcal{L}(\mathcal{M})$. Let us call *von Neumann-Morgenstern restriction* (*VNM*) this domain and range restriction on the permissible SWF's.

Define *independence of irrelevant alternatives* as:

$$(\forall U \in \mathcal{E}^n)(\forall U' \in \mathcal{E}^n)(\forall x \in \mathcal{M}) \quad U(x) = U'(x) \Rightarrow f(U)(x) = f(U')(x) \ . \tag{IR}$$

Note that this is stronger than d'Aspremont and Gevers' own independence axiom (1977, pp 201–202), which only implies ordinal invariance in case of partially identical $U$ and $U'$. The *strong Pareto principle* (*SP*) will be defined as the conjunction of:

$$(\forall U \in \mathcal{E}^n)(\forall (x, y) \in \mathcal{M}^2) \quad U(x) = U(y) \Rightarrow f(U)(x) = f(U)(y) \tag{SP_1}$$

and

$$(\forall U = (u_1, \dots, u_n) \in \mathcal{E}^n)(\forall (x, y) \in \mathcal{M}^2) \tag{SP_2}$$

$$u_i(x) \geq u_i(y),\ i = 1, \dots, n \text{ and } u_j(x) > u_j(y) \text{ for a } j \in \{1, \dots, n\} \Rightarrow f(U)(x) > f(U)(y) \ .$$

(*IR*) and ($SP_1$) together are clearly equivalent to the following property on utilities, which we shall call *extended neutrality*:

$$(\forall U \in \mathcal{E}^n)(\forall U' \in \mathcal{E}^n)(\forall (x, y) \in \mathcal{M}^2) \quad U(x) = U'(y) \Rightarrow f(U)(x) = f(U')(y) \ . \tag{XN}$$

We have borrowed our terminology from d'Aspremont and Gevers (1977, p 202), although their own condition is again weaker than ours in view of their weaker independence axiom. As defined here, extended neutrality is best seen as an extension of the standard Pareto indifference rule. It is *the* relevant welfare condition for the "multi-profile" version of Harsanyi's theorem. It has the following important consequence: for *a* given $a \in \bigcup_{U \in \mathcal{E}^n} \operatorname{Rge} U$, one may find a unique real number $F(a)$ associated with it by considering *any* $U$ and *any* $x$ such that $U(x) = a$, and putting $F(a) = f(U)(x)$. Thus, what (*XN*) says in effect is that $f(U): \mathcal{M} \to \mathbb{R}$ can be *factored out* as $\mathcal{M}^n \xrightarrow{U} \bigcup_{U \in \mathcal{E}^n} \operatorname{Rge} U \xrightarrow{F} \mathbb{R}$, $F$ being independent of

$U$ and unique with respect to $f$. In the case where $\mathscr{E}=\mathscr{L}(\mathscr{M})$, all the constant functions on $\mathscr{M}$ belong to $\mathscr{E}$, and clearly $\bigcup_{U\in\mathscr{E}^n} Rge\, U=\mathbb{R}^n$. Let us record these facts:

*Remark 3.* Suppose a SWF $f$ satisfies $(VNM)$ and $(XN)$. Then, there is a unique $F:\mathbb{R}^n\to\mathbb{R}$ such that

$$(\forall U\in[\mathscr{L}(\mathscr{M})]^n)(\forall x\in\mathscr{M})\, f(U)(x)=F(U(x)) .$$

Now, we are ready to prove the "multi-profile" theorem:

**Proposition 1.** *Given a non-degenerate mixture set $\mathscr{M}$, any* SWF *$f$ satisfying $(VNM)$ and $(XN)$ is affine on $\mathscr{M}$. That is, there are unique real numbers $a_1,\ldots,a_n,b$ such that:*

$$(\forall U=(u_1,\ldots,u_n)\in[\mathscr{L}(\mathscr{M})]^n)\, f(U)=\sum_{i=1}^{n} a_i u_i+b .$$

Since $\mathscr{M}$ is non-degenerate, there are $\varphi\in\mathscr{L}(\mathscr{M})$ and $(x,y)\in\mathscr{M}^2$, such that $\varphi(x)\neq\varphi(y)$. We know from Remark 3 that $f$ can uniquely be expressed in terms of the $u_i$. The proof consists in using the mixture-preserving property of $U$ and $f(U)$ to show that the auxiliary $F$ is affine on $\mathbb{R}^n$. From Remark 1 above, this conclusion would result from the fact that $F$ preserves convex combinations of vectors in $\mathbb{R}^n$, i.e. it will obtain if we prove:

$$\begin{aligned}&(\forall(a,b)\in\mathbb{R}^n\times\mathbb{R}^n)(\forall\lambda\in[0,1])\\ &F(\lambda a+(1-\lambda)b)=\lambda F(a)+(1-\lambda)F(b) .\end{aligned} \tag{P}$$

Take $a=(a_1,\ldots,a_n)$ and $b=(b_1,\ldots,b_n)$ in $R^n$, $a\neq b$. There is a $MP$ function $U=(u_1,\ldots,u_n)$ verifying $U(x)=a$ and $U(y)=b$. For instance, consider the affine transform of $\varphi$ given by:

$$u_i(\xi)=a_i\,\frac{\varphi(\xi)-\varphi(y)}{\varphi(x)-\varphi(y)}+b_i\,\frac{\varphi(\xi)-\varphi(x)}{\varphi(y)-\varphi(x)} , \qquad i=1,\ldots,n ,$$

which is a $MP$ function as well. Now, the left-hand member of $(P)$ can be written as

$$\begin{aligned}F(\lambda U(x)+(1-\lambda)\,U(y))&=F(U(x\lambda y))\\ &=f(U)(x\lambda y)=\lambda f(U)(x)+(1-\lambda)f(U)(y)\\ &=\lambda F(U(x))+(1-\lambda)F(U(y)) ,\end{aligned}$$

which is the right-hand member of $(P)$. □

Proposition 1 *by itself* implies certain invariance properties on $f$. To make this precise, define the *individual origins of utilities property* as:

$$\begin{aligned}&(\forall(u_1,\ldots,u_n)\in\mathscr{E}^n)(\forall(u'_1,\ldots,u'_n)\in\mathscr{E}^n)(\forall\alpha\in\mathbb{R}^{+*})\\ &(\forall(\beta_1,\ldots,\beta_n)\in\mathbb{R}^n)(\forall(x,y)\in\mathscr{M}^2)\\ &f(u_1,\ldots,u_n)(x)\geq f(u_1,\ldots,u_n)(y)\Rightarrow\\ &f(\alpha u_1+\beta_1,\ldots,\alpha u_n+\beta_n)(x)\geq f(\alpha u_1+\beta_1,\ldots,\alpha u_n+\beta_n)(y) .\end{aligned} \tag{IOU}$$

This is exactly d'Aspremont and Gevers' definition of *(IOU)* (1977, p 200). When $\mathscr{E}=\mathscr{L}(\mathscr{M})$, the *(IOU)* condition can be restated in a more precise way. As is well-known, if a *VNM* preference ordering $\mathscr{R}$ is represented by a mixture-preserving function $u$, any positive affine transform of $u$ is a *MP* representation of $\mathscr{R}$ as well, and conversely, any *MP* representation of $\mathscr{R}$ is a positive affine transform of $u$ (e.g. Fishburn 1982, p 14). Thus, when $\mathscr{E}=\mathscr{L}(\mathscr{M})$, *(IOU)* is equivalent to

$$\begin{aligned}&(\forall (u_1,\dots,u_n)\in[\mathscr{L}(\mathscr{M})]^n)(\forall \alpha\in\mathbb{R}^{+*})\\&(\forall(\beta_1,\dots,\beta_n)\in\mathbb{R}^n)(\exists\mu\in\mathbb{R}^{+*})(\exists\nu\in\mathbb{R})\\&f(\alpha u_1+\beta_1,\dots,\alpha u_n+\beta_n)=\mu f(u_1,\dots,u_n)+\nu\ .\end{aligned}\qquad (VNM\text{-}IOU)$$

Following the standard contemporary approach to utilitarianism, *(IOU)* has to be *assumed* as one among the a priori restrictions imposed on $f$. In the utilitarian theory sketched in this paper, *(IOU)*, i.e. *(VNM-IOU)*, is a *consequence* of the axioms. It is worth noting that it holds irrespective of the anonymity requirement defined below, as a straightforward corollary to Proposition 1:

**Corollary 1.1.** *Given a mixture set $\mathscr{M}$, any* SWF *$f$ satisfying (VNM.) and (XN) satisfies (IOU).*

It is by now an easy task to derive utilitarian rules from the hypotheses in Proposition 1 and the usual *anonymity* requirement:

$$\begin{aligned}&(\forall (u_1,\dots,u_n)\in\mathscr{E}^n)\ (\forall\sigma \text{ permutation of } \{1,\dots,n\})\\&f(u_{\sigma(1)},\dots,u_{\sigma(n)})=f(u_1,\dots,u_n)\ .\end{aligned}\qquad (A)$$

Any affine $f$ on $\mathscr{E}^n$ satisfying $(A)$ has its $a_i$ equal to each other. This follows from taking $u_1=1$, $u_i=0$, $1<i\leq n$, in $(A)$. Note also that positivity of the $a_i$'s in Proposition 1 obtains when $(SP_2)$, i.e. the second part of the strong Pareto condition, is super imposed to the initial set of hypotheses. To see this, recall from the proof of Proposition 1 that, for $i=1,\dots,n$, there are mixture-preserving $U_i$ such that, for some $z_i$, $x_i$ in $\mathscr{M}$, $U_i(z_i)=0$, $U_i(x_i)=e_i$, the $i$th vector in the canonical basis of $\mathbb{R}^n$; then, apply $(SP_2)$ to the $U_i$. Let us sum up the facts stated in this paragraph:

**Corollary 1.2.** *Given a mixture set $\mathscr{M}$ and a* SWF *$f$ satisfying (VNM), (IR), (SP) and (A), there are unique real numbers $a$ and $b$, with $a>0$, such that:*

$$(\forall U=(u_1,\dots,u_n)\in\mathscr{E}^n)\quad f(U)=a\sum_{i=1}^{n}u_i+b\ .$$

Corollary 1.2 may be compared with d'Aspremont and Gevers' classic characterisation of utilitarianism (1977, p 203). However, since their independence axiom is different from ours, such a comparison could not be relied on to show what is the added value of imposing the *(VNM)* restriction. Somewhat artificially, we shall rather compare our "multiprofile" results with the following variant of d'Aspremont and Gevers' characterisation:

**Proposition** (see d'Aspremont and Gevers 1977). *Given a mixture set $\mathscr{M}$, take $\mathscr{E}=\mathscr{F}(\mathscr{M})$, the set of all real functions on $\mathscr{M}$. Any $f:\mathscr{E}^n\to\mathscr{E}$ satisfying (IR), (SP), (A)* *and (IOU) is such that:*

$$(\forall U=(u_1,\ldots,u_n)\in\mathscr{E}^n)(\forall(x,y)\in\mathscr{M}^2)$$

$$f(U)(x)\geq f(U)(y)\Leftrightarrow\sum_{i=1}^{n}u_i(x)\geq\sum_{i=1}^{n}u_i(y)\ .$$

It has been pointed out that (*IR*) implied d'Aspremont and Gevers' own independence axiom. Thus, the above proposition is a straightforward logical consequence of their theorem. The logical connection between this variant and our utilitarian result is entirely clarified by Corollary 1.1. The latter in effect claims that the (*VNM*) restriction implies (*IOU*) in the presence of (*IR*) and (*SP*). Since the converse is obviously false, the hypotheses in the variant are strictly weaker than those of Proposition 1. It comes to no surprise that the result is strictly weaker as well, i.e. $f(u)$ may be increasing with $\sum_{i=1}^{n}u_i$ without being of the prescribed affine form. As long as we consider the planner's problem of ordering social utility *amounts*, there is no difference between this result and ours. The added value of strengthening the hypotheses with the (*VNM*) restriction becomes apparent only when we move to the problem of ordering *first differences* in social utility. Our affine result for $f(u)$ of course means that the latter ordering reflects the ordering of first differences in utility sum. This is not so under the weaker hypotheses of the proposition, which ensure that $f(U)(x)-f(U)(y)$ and $f(U)(x')-f(U)(y')$ are ordered in the same way as $\sum_{i=1}^{n}(u_i(x)-u_i(y))$ and $\sum_{i=1}^{n}(u_i(x')-u_i(y'))$ only in the particular case of quantities opposite in sign. Let us rehearse this simple point as a further corollary:

**Corollary 1.3.** *Given a mixture set* $\mathscr{M}$, *any* SWF *f satisfying* (*VNM*), (*IR*), (*SP*), *and* (*A*) *is such that:*

$$(\forall U=(u_1,\ldots,u_n)\in\mathscr{E}^n)(\forall(x,y)\in\mathscr{M}^2)(\forall(x',y')\in\mathscr{M}^2)$$

$$f(U)(x)-f(U)(y)\geq f(U)(x')-f(U)(y')$$

$$\Leftrightarrow\sum_{i=1}^{n}(u_i(x)-u_i(y))\geq\sum_{i=1}^{n}(u_i(x')-u_i(y'))\ .$$

One may be willing to go the opposite way, i.e. to derive results that are even more akin to Benthamism than Corollary 1.2. To get a social welfare function which is $\sum_{i=1}^{n}u_i$ or $\sum_{i=1}^{n}\frac{1}{n}u_i$, specific normalizations should be resorted to. Since $a>0$, it is always permissible for the social planner to change his VNM representation $f(U)$ into another one, which is identical with the desired sum or average utility rule. This procedure is very much in the spirit of Harsanyi's work, where Benthamite forms for $f(U)$ are derived for particular representations of the moral observer's ordering[3].

[3] See Harsanyi (1955, 1977). This procedure is in the vein of Fleming's early "single-profile" result (1952), where the assumed individual and social orderings are those of standard consumer theory and a utilitarian result $u=\sum u_i$ is derived for well-chosen utility representations.

## 4. The Single-Profile Theorem Reconsidered

Harsanyi's theorem calls for a separate proof from its "multi-profile" counterpart. Considering *all* elements in $[\mathscr{L}(\mathscr{M})]^n$ makes it possible to factor out any $f(U)$ as $\mathscr{M}\xrightarrow{U}\mathbb{R}^n\xrightarrow{F}\mathbb{R}$, an especially convenient decomposition. When a *single* element $U=(u_1,\ldots,u_n)$ is given, the Pareto indifference condition implies that $f(U)$ can be decomposed as $\mathscr{M}\xrightarrow{U}Rge\,U\xrightarrow{F_U}\mathbb{R}$. Since it is not the case in general that $Rge\,U=\mathbb{R}^n$, a different result from Remark 1 above is needed if an argument paralleling the proof of Proposition 1 is to succeed:

**Lemma.** *Suppose $K$ is a convex subset of $\mathbb{R}^n$ and $F: K\to\mathbb{R}$ is mixture-preserving. Then, there are real numbers $a_1,\ldots,a_n$, $b$ such that*

$$\forall(x_1,\ldots,x_n)\in K\ ,\qquad F(x_1,\ldots,x_n)=\sum_{i=1}^{n} a_i x_i+b\ .$$

Recall that any non-empty finite-dimensional convex set $K$ has a non-empty relative interior, i.e. $K$ has at least one interior point $X_0$ with respect to the affine subspace it spans (Berger 1978, vol 3, p 29). As a result, the translated convex $\tilde{K}=K-X_0$ has 0 as an interior point with respect to Vect $\tilde{K}$, the vector space spanned by $\tilde{K}$. Define $G$ on $\tilde{K}$:

$$(\forall X\in\tilde{K})\ G(X)=F(X+X_0)-F(X_0)\ .$$

$G$ is $MP$ and satisfies $G(0)=0$. Write $\tilde{G}(X)=\lambda G\left(\frac{X}{\lambda}\right)$, with $\lambda$ any positive number large enough for $\frac{X}{\lambda}\in K$. Since $G$ satisfies positive homogeneity, $\tilde{G}$ is a well-defined function which extends $G$ on Vect $\tilde{K}$. It is easy to check that $\tilde{G}$ is linear. Now, call $P$ the (linear) projection of $\mathbb{R}^n$ onto Vect $\tilde{K}$. $\tilde{G}P$ is a linear form which extends $G$ on $\mathbb{R}^n$. Write $\tilde{G}P(X)=\sum_{i=1}^{n}\alpha_i x_i$. Going back to $F$, one constant term is added, and the conclusion of the lemma holds. □

Now, let us state and prove Harsanyi's theorem:

**Proposition 2** (Harsanyi). *Suppose $\mathscr{M}$ is a mixture set; suppose $U=(u_1,\ldots,u_n)$, $u$ are $MP$ functions from $\mathscr{M}$ into $\mathbb{R}^n$, $\mathbb{R}$ respectively and satisfy the following ("Pareto indifference") condition:*

$$(\forall x\in\mathscr{M})(\forall y\in\mathscr{M})\,U(x)=U(y)\Rightarrow u(x)=u(y)\ . \qquad (P)$$

*Then there are real numbers $a_1,\ldots,a_n$, $b$ such that*

$$u=\sum_{i=1}^{n} a_i u_i+b\ .$$

It is clear that the $(P)$ condition is equivalent to the following one:

$$\exists F: U(\mathscr{M})\to\mathbb{R}\quad \text{s.t.}\quad u=F\circ U\ .$$

$U(\mathcal{M})$ is a convex subset of $\mathbb{R}^n$. For if $X_1=U(x_1)$, $X_2=U(x_2)$, the mixture-preserving property of $U$ implies that, for any $\lambda\in[0,1]$:

$$\lambda X_1+(1-\lambda)X_2=U(x_1\lambda x_2)\ .$$

Harsanyi's theorem follows at once from the lemma if we prove that $F$ is $MP$ on $K=U(\mathcal{M})$. But the latter is an easy consequence of the mixture-preserving property of $u$: for any $X_1, X_2\in U(\mathcal{M})$, $\lambda\in[0,1]$ ,

$$F(\lambda X_1+(-\lambda)X_2)=F(U(x_1\lambda x_2))=u(x_1\lambda x_2)$$
$$=\lambda u(x_1)+(1-\lambda)u(x_2)=\lambda F(X_1)+(1-\lambda)F(X_2)\ .\quad\square$$

The proof offered here is much shorter than Fishburn's lengthy inductive argument (1984). Since it uses mixture sets, it is more general than the technically elegant argument used by Border (1985). It bears some relationship to Domotor's (1979) earlier proof, which also relied on proving the linearity of the auxiliary function $F$. Some comments on Harsanyi's latest published proof (1977) may also be called for. First of all, it involves changing the zero points and measurement units of the $u_i$ and $u$, so that the affine formula holds for affine transforms of the original functions. This is slightly confusing, but unimportant after all since all the transformations used by Harsanyi turn out to be invertible, so that he could have easily reverted to the original functions. Second, Resnik (1983) has noted a curious feature of Harsanyi's proof: it appears to "assume" some $x\in\mathcal{M}$ with special properties. As it happens, the so-called "assumptions" can be rationalized *ex post* as simple facts on convexity such as were used in the proof of the lemma. But Harsanyi's wording is technically faulty, in particular his discussion of negative homogeneity which would hold good only if his initial $F$ were replaced by a suitable $G$, as defined above.

A seemingly curious feature of our proof is that it makes no use of the axioms of a mixture set, but only of the definition of mixture-preservation. This is an optical illusion, as can be seen from Remark 1 in Sect. 3. The very definition of mixture-preservation on $X$ makes $X/\sim$ a mixture set. Proving $u(x)=\sum_{i=1}^{n} a_i u_i(x)+b$ for any $x\in X$ is equivalent to proving it for any representative $x\in x\in X/\sim$, which involves using the mixture set structure of $X/\sim$ after all.

A particular case of Theorem 1 obtains when $\mathcal{M}$ is taken to be a vector space. From Remark 1 above, a $MP$ function is affine, i.e. the sum of a linear form and a constant term. If $f$ is a linear form on $\mathcal{M}$, denote its null space by $\operatorname{Ker} f$. Proposition 2 is then easily seen to follow from a well-known fact in linear algebra:

**Proposition.** *Suppose $\mathcal{M}$ is a vector space and $u_1,\dots,u_n$, $u$ are linear forms on $\mathcal{M}$ with*

$$\operatorname{Ker} u\supset\bigcap_{i=1}^{n}\operatorname{Ker} u_i\ .$$

*Then, there are real numbers $a_1,\dots,a_n$ such that*

$$u=\sum_{i=1}^{n} a_i u_i\ .$$

As stated, Harsanyi's theorem does not say anything on uniqueness and positivity of the $a_i$. Both issues will be addressed in the rest of this section. They are usefully discussed in Fishburn (1984, p 27). We shall briefly restate his results and investigate the rôle of his ($C$)-condition, as defined below. The main clarification here is provided by Corollary 2.1, which states that, when applied to a set of mixture-preserving functions, affine independence turns out to be not only implied by, but *equivalent to* the seemingly stronger ($C$)-property.

**Corollary 2.1.** *Let $\mathscr{L}(\mathscr{M})$ be the vector space of all real MP functions on a mixture set $\mathscr{M}$. Then, for any family $u_1,\ldots,u_n$ of $\mathscr{L}(\mathscr{M})$, affine independence is equivalent to the following independence condition:*

$$(\forall j \in \{1,\ldots,n\})(\exists (x_j, y_j) \in \mathscr{M}^2)$$
$$u_j(x_j) \neq u_j(y_j) \quad \textit{and} \quad u_i(x_j) = u_i(y_j) \ , \quad i \neq j \ . \tag{C}$$

Suppose $u_1,\ldots,u_n$ are affinely dependent. Then, there is a $j \leq n$, and there are $n$ scalars $\lambda_i$, $1 \leq i \leq n$, $i \neq j$ and $\mu$, such that:

$$(\forall x \in \mathscr{M}) u_j(x) = \sum_{i \neq j} \lambda_i u_i(x) + \mu \ .$$

Clearly, $j$ makes it impossible for ($C$) to hold.

Conversely, suppose that ($C$) does not hold. Then, there is a $j \leq n$ such that:

$$(\forall x \in \mathscr{M})(\forall y \in \mathscr{M}) \ u_i(x) = u_i(y), \ 1 \leq i \leq n, \ i \neq j \Rightarrow u_j(x) = u_j(y) \ .$$

This is the ($P$) condition of Proposition 2 with $u_j$ instead of $u$. From Proposition 2, it follows that $u_j = \sum_{i \neq j} \lambda_i u_i + \mu$ for some $\lambda_i$ and $\mu$, i.e. the family $(u_i,\ldots,u_n)$ is affinely dependent. □

If no restriction is posed on the $u_i$, it is of course most easy to find affinely independent functions that do not satisfy ($C$):

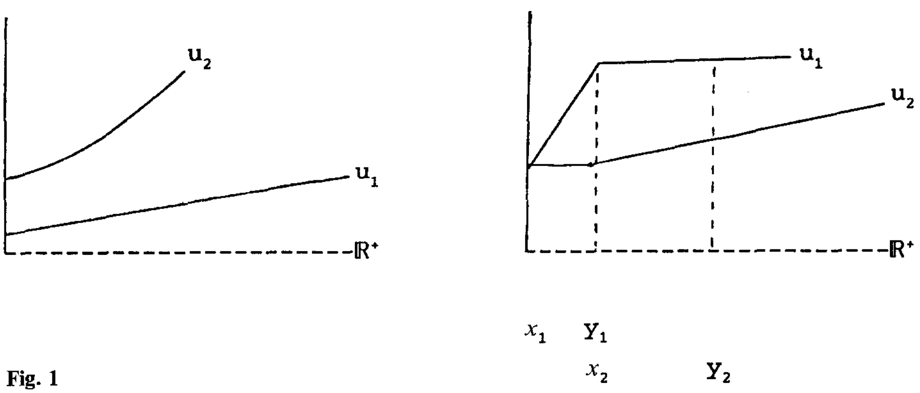


**Fig. 1**

(Contrary to the right-hand side, where both ($C$) and affine independence are met on $\mathscr{M} = \mathbb{R}^+$, the left-hand side exhibits two affinely independent $u_1$ and $u_2$ which cannot be $C$-independent since they are strictly monotonic).

Let us now restate Fishburn's results on uniqueness and positivity of the $a_i$:

**Corollary** (Fishburn). *Suppose the hypotheses of Proposition 2 hold. Then, there are unique* $a_1, \dots, a_n$ *and* $b$ *such that* $u = \sum_{i=1}^{n} a_i u_i + b$ *if and only if* $(C)$ *holds. Suppose the following ("strong Pareto") condition also holds:*

$$\begin{aligned}&(\forall x \in \mathcal{M})(\forall y \in \mathcal{M})\\ &U(x) \geq U(y) \quad \text{and} \quad u_i(x) > u_i(y) \quad \text{for some} \quad i \Rightarrow u(x) > u(y) \ .\end{aligned} \tag{SP}$$

*Then, the* $a_i$ *and* $b$ *such that* $u = \sum_{i=1}^{n} a_i u_i + b$ *satisfy* $a_i > 0$, $i = 1, \dots, n$, *if* $(C)$ *holds.*

See Fishburn 1984, p 27. □

In his early article (1955), Harsanyi appeared to believe that adding $(SP)$ to the hypotheses of his theorem would ensure positivity. What this conjecture exactly means is very unclear when the $u_i$ are affinely dependent, since, from Fishburn's corollary, there are many, indeed infinitely many, feasible $a_i$ in such a case. It is a straightforward matter to find examples of affinely dependent $u_1, \dots, u_n$, where *each* feasible set involves one non-positive element; so that the above conjecture is false, however it may be interpreted in the case where the $u_i$ are affinely dependent[4].

Leaving aside the $(SP)$ condition, Fishburn's corollary, supplemented with the equivalence result stated in Corollary 2.1, makes it very easy to construct cases where the hypotheses of Harsanyi's theorem fail to lead to a determinate expression of social utility $u$ in terms of the $u_i$: take any society where one of the $u_i$ is affinely dependent on some other $u_j$. How is this indeterminateness problem to be solved? The following purports to offer an answer:

**Corollary 2.2.** *Suppose the hypotheses of Proposition 2 hold. Then, if not all of the* $u_i$ *are constant functions, there is a maximal subset* $\{u_{i_1}, \dots, u_{i_k}\}$, $1 \leq k \leq n$, *for which there are unique* $a_{i_1}, \dots, a_{i_k}$, $b$ *verifying*

$$u = \sum_{l=1}^{k} a_{i_l} u_{i_l} + b \ .$$

*Furthermore, if* $(SP)$ *holds, the* $a_{i_1}, \dots, a_{i_k}$ *are positive.*

From the very definition of $(C)$, or more visibly from the definition of affine independence, it follows that either all of the $u_i$'s are constant functions or there is a non-empty subset $\{u_{i_1}, \dots, u_{i_k}\}$ satisfying $(C)$. Apply Fishburn's corollary to the latter. □

This result offers a partial solution to the indeterminateness problem raised above. If all of the $u_i$ are constant, the $(P)$ condition of Harsanyi's theorem implies that $u$ is constant as well – a hardly interesting case for social choice theory, which can be dispensed with as was done in the last section. (Still, one may note the difference: the "single-profile" theorem trivially holds true, whereas the "multi-profile" result would not apply). If not all of the $u_i$ are constant, we may express $u$ as

[4] Resnik's example (1983) is to the point here.

an affine transform of utilities taken from a maximal subset $\{a_{i_1}, \ldots, a_{i_k}\}$, putting $a_i = 0$ for the remaining $n-k$ coefficients. The resulting expression is not a canonical one; there are several maximal subsets for a given family $\{u_1, \ldots, u_n\}$ as soon as the latter is affinely dependent. Thus, arbitrariness in the decomposition of $u$ can only be alleviated. Some would possibly claim that it is immaterial which maximal subset is chosen, since any of them contains the same information as any other – roughly speaking the more basic characteristics out of which the rest of society is made. We feel that this argument is slippery. To make it precise, an axiom is needed such as the anonymity requirement of the last section, but it has been emphasized that the latter did not make sense in a "single-profile" exercise. Indeed, the main point of this discussion on independence and uniqueness is to show that a "single-profile" approach to *VNM* social choice is surrounded with unnecessary complexities and should leave ground for the "multi-profile" one.

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