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Research Note

A non-minimal but very weak axiomatization of common belief

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Abstract

The paper introduces a modal logic system of individual and common belief which is shown to be sound and complete with respect to a version of Neighbourhood semantics. This axiomatization of common belief is the weakest of all those currently available: it dispenses with even the Monotonicity rule of individual belief. It is non-minimal in that it does not use just the Equivalence rule but the conjunction of the latter with the specially devised rule of C-Restricted Monotonicity.

1. General

Informally, a proposition is *common belief* (CB) if every individual in the group believes it, believes that every individual in the group believes it, and so on *ad infinitum*. Given the usual definition of knowledge as true belief, the more standard notion of *common knowledge* (CK) follows: a proposition is CK if it is true, every individual in the group knows it, etc. The logic of these concepts has been thoroughly investigated in the context of modal propositional calculi and Kripkean variants of the possible world semantics: see [6] and [10, Sections 2 and 3] for up-to-date surveys.

A problem of axiomatizations of CB *á la* Kripke is that they involve heavy and questionable epistemic assumptions. From the definition of a K-System [2, Chapter 4] individual belief is required to reproduce any logical inference

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(Monotonicity rule), include any logical truth (Necessitation axiom), and preserve conjunctions (Conjunctiveness axiom). There certainly are computer science applications for which K, and even stronger systems such as $S4$ or $S5$, can be assumed as good approximations. For instance, Halpern and Moses [5] show how $S5$-based axiomatizations of CK apply to distributed systems analysis. In epistemic logic and artificial intelligence, however, the Monotonicity rule and Necessitation axiom have been repeatedly criticized on the grounds that they imply *logical omniscience* on the individuals' part [3,7,8,12–14]. For example, Fagin and Halpern [3] state the following reasons why it is unrealistic to assume that individual belief can reproduce any logical inference and include any logical truth: (i) lack of awareness; (ii) lack of computational resources; (iii) ignorance of the relevant mathematical rules; (iv) inconsistency between different "frames of mind".

These criticisms, in particular (iv), can arguably be extended to the Conjunctiveness axiom. As is well known, the latter raises epistemic problems of its own—it is incompatible with probabilistic belief except for the case of events having probability 1. In view of the development and applications of probability logics in AI (see [1] for a survey), this incompatibility might strike one as a further unpleasant feature of the K-system. A related reason for dispensing with Conjunctiveness has to do with the recent work in game theory on the quantified notion of *p-common belief* [11].

The above epistemic considerations motivate the attempt to axiomatize CB in terms of weaker systems of individual belief, and more powerful semantics than Kripke's. Several "nonstandard" frameworks are currently available in epistemic logic to tackle the logical omniscience problem [4, Chapter 5]; presumably, one of them will provide the path to the proper axiomatization of CB. The present paper uses the framework of Neighbourhood semantics, which stands prominently among the current alternatives because of its mathematical generality and its flexibility in epistemic applications. The notion of a *Neighbourhood* (or *Montague-Scott*) structure is known strictly to include that of a Kripke structure, while still delivering epistemically interpretable soundness and completeness theorems for modal logic systems [2, Chapters 7–9]. This convenient feature might explain why some of the recent work in AI inspired by the logical omniscience problem involves Neighbourhood structures or variants of them ([13,14]; see also the "frames of mind" construction in [3, Section 6]). In connection with the point made above on Conjunctiveness, it is worth stressing that this approach can be related to the probability calculus if needed. Further motivation can be found in [10, Section 4]. The latter paper also contains an informal review of axiomatizations of CB reached by the present writers in the Neighbourhood semantics framework.

These axiomatizations share the following feature: they dispense with Necessitation and Conjunctiveness, which is a step in the right direction, but retain Monotonicity, which is worrying. For there is general agreement that this rule is at the core of the logical omniscience problem. Fortunately, it can be weakened, as the novel result of this paper demonstrates. As in earlier axiomatizations, the CB operator is required to satisfy a Fixed-Point axiom and an Induction rule. Crucially, the individual belief axiom block replaces Monotonicity with the conjunction of the following two rules:

(i) the *Equivalence* rule, which requires the individual to reproduce logical inferences when the premises are logically equivalent to the conclusion and are believed by that individual;

(ii) the specially devised rule of *C-Restricted Monotonicity*, which requires him to reproduce logical inferences when the premises logically imply the conclusion and are common belief.

Taken individually, (i) and (ii) weaken the Monotonicity rule by restricting its application to a particular case of logical implication and a particular case of belief in the premises, respectively. Taken together, they are strictly weaker than Monotonicity, as will be checked. It would have been more satisfactory to use just (i), which is well known to be the *minimal* axiomatization of individual belief within Neighbourhood structures. There are, however, difficulties, both technical and conceptual, to make this further step. The present conjunction of (i) and (ii) is tailor-made to meet the *prima facie* conflicting requirements posed on the logic: When added to the CB axiom block, the individual belief axiom block should

(1) deliver a sound and complete axiomatization with respect to the Neighbourhood semantics of individual and common belief,

(2) be strong enough to imply the intuitively desirable properties of CB,

(3) make progress with the logical omniscience problem, hence be weak enough *not* to imply Monotonicity.

Section 2 explains the syntactical definitions as well as results supporting claim (2). Section 3 explains the semantics, the CB part of which is expressed in terms of *belief closure*, and states the determination (i.e., soundness and completeness) theorem required by claim (1). Section 4 contains a proof and states the simple independence corollary warranting claim (3). Modifications of the proof deliver the axiomatization results that are mentioned in [10, Section 4] and were derived in [9] under the unnecessarily strong rule of Monotonicity. Since the present paper derives the most powerful among the determination theorems relative to CB in Neighbourhood structures, it should be clear that it supersedes [9] as the relevant technical source for the review article [10].

2. Syntactical definitions and results

The vocabulary of our systems consists of a set PV of propositional variables (of any cardinality), the usual propositional connectives, and unary operators with intended epistemic applications. There is a *finite* set A of "agents" each of whom is endowed with a belief operator B_a, the CB operator C and (just for convenience) the shared belief operator E. Let Φ denote the set of well-formed formulas constructed from these components.

The first system, to be denoted by MC_A^-, consists of any axiomatization of the propositional calculus, and the following modal rules and axiom schemata:

Individual Belief Axiom Block.

(RE_a) $\dfrac{\varphi \leftrightarrow \psi}{B_a\varphi \leftrightarrow B_a\psi}$ for any $a \in A$;

(Def.E) $E\varphi \leftrightarrow \bigwedge_{a\in A} B_a\varphi$;

($C\text{-}RM_a$) $\dfrac{\varphi \rightarrow \psi}{C\varphi \rightarrow B_a\psi}$ for any $a \in A$;

Common Belief Axiom Block.

(FP) $C\varphi \rightarrow E(C\varphi \wedge \varphi)$;

(RI) $\dfrac{\varphi \rightarrow E\varphi}{E\varphi \rightarrow C\varphi}$;

(RM_C) $\dfrac{\varphi \rightarrow \psi}{C\varphi \rightarrow C\psi}$.

The original feature in block (i) is the *C-Restricted Monotonicity* rule ($C\text{-}RM_a$) that the previous section informally contrasted with the more standard *Monotonicity* rule:

$$(RM_a) \quad \frac{\varphi \rightarrow \psi}{B_a\varphi \rightarrow B_a\psi}.$$

The second system, to be denoted by MC_A, involves the following alternative block (i′) of individual belief: (RM_a) for any $a \in A$, plus the unproblematic (Def.E). Clearly, (i′) implies (i) in the presence of (FP). That this implication is strict will be clarified below.

Note the easy fact that (i) as well as (i′) are satisfied in models in which agents do not believe anything at all. This possibility would disappear if the already discussed *Necessitation* axiom were added to the individual belief axiom block:

(N_a) B_aT for any $a \in A$ (where T stands for any theorem).

Similarly, (i) and (i′) are satisfied in models in which agents hold beliefs without holding the conjunction of them, i.e., violate *Conjunctiveness*:

(C_a) $B_a\varphi \wedge B_a\psi \rightarrow B_a(\varphi \wedge \psi)$.

We recall in passing the formal definition of a *K-system for agent a*: (RM_a) + (N_a) + (C_a).

The (ii) block is common to MC_A^- and MC_A. A *Fixed-Point* axiom, (FP), says in effect that CB of any statement implies shared belief of that statement *and* of the statement that there is CB. The property obtained by changing the end of the last sentence into: "shared belief of that statement *and* shared belief of the statement that there is CB" is easily derived in MC_A. It also holds in MC_A^-, as shown in Proposition 1 (i) and (v) below.

(RI) says in effect that if a statement φ is inherently *public* belief—i.e., it is a theorem that φ cannot happen without everybody's believing it—then φ is inherently CB. The intuitive connection between public and common belief was first pointed out in the economics literature—see [10] for details. This paper also explains why rules like (RI) have been called *Induction* rules.

Finally, (RM_C) is the *Monotonicity of CB* rule. Its technical role will become clear from the syntactical results below. Readers aware of the axiomatization of CB in [5,6] could check that (RM_C) is implied by the particular version of the Induction rule adhered to in these papers, granting their assumption of a K-system for any $a \in A$.

Any rule or axiom schema in this paper is taken to hold for every a or for none. Hence the self-explanatory notations: (RE_A), (C-RM_A), etc. The formal inference relations $\vdash_{MC_A^-}$ and $\vdash_{MC_A}$ are defined in the standard way. We shall drop the subscript where there is no ambiguity.

Proposition 1.

(i) *Granting* (RM_C) *and* (Def.E), (C-RM_A) *is equivalent to the following axiom*:

$$C\varphi \rightarrow E\varphi.$$

Granting (FP), (RI) *and* (Def.E), (C-RM_A) *is equivalent to the following rule*:

$$\frac{\varphi \rightarrow E\varphi \wedge \psi}{E\varphi \rightarrow E\psi}.$$

The following theorems hold:

(ii) *For any* $k \geqslant 1$, $\vdash_{MC_A^-} C\varphi \rightarrow E^k\varphi$.

(iii) *For any* $k \geqslant 1$, $\vdash_{MC_A^-} C\varphi \rightarrow C^k\varphi$.

(iv) $\vdash_{MC_A^-} C\varphi \leftrightarrow E(C\varphi \wedge \varphi)$.

(v) $\vdash_{MC_A^-} C\varphi \rightarrow EC\varphi$.

(vi) $\vdash_{MC_A^-} EC\varphi \rightarrow CE\varphi$.

(vii) $\vdash_{MC_A^-} C\varphi \rightarrow CE\varphi$.

(viii) *(C_A)* $\vdash_{MC_A^-} CE\varphi \rightarrow EC\varphi$.

(ix) *(C_A)* $\vdash_{MC_A^-} C\varphi \leftrightarrow EC\varphi \wedge E\varphi$.

Proof. (Def.E) and (C-RM_A) clearly imply the axiom in (i); the proof of the converse depends on (Def.E) and (RM_C). To derive the rule in (i) from (Def.E) and (C-RM_A), use (RI); for the converse, use (FP).

For (ii): Assume that $\vdash C\varphi \rightarrow E^k\varphi$ has been proved up to some $k \geqslant 1$. Then, $\vdash C\varphi \wedge \varphi \rightarrow E^k\varphi$ holds, as well as (from (C-RM_A) and (Def.E))

$\vdash C(C\varphi \wedge \varphi) \to E^{k+1}\varphi$. Applying (FP) and (RI) in succession leads to $\vdash E(C\varphi \wedge \varphi) \to C(C\varphi \wedge \varphi)$, whence (from applying (FP) again) $\vdash C\varphi \to C(C\varphi \wedge \varphi)$. We conclude that $\vdash C\varphi \to E^{k+1}\varphi$.

To prove (iii) note that (RM_C) implies that $\vdash C(C\varphi \wedge \varphi) \to C^2\varphi$; hence $\vdash C\varphi \to C^2\varphi$, and from an inductive argument $\vdash C\varphi \to C^k\varphi$ for any $k \geqslant 2$.

(iv): From (FP), (RI) and (RM_C).

(v): From (iii), (C-RM_A) and (Def.E).

(vi): From (v), (RI) and (RM_C) as applied to (i).

(vii): From (v) and (vi).

(viii): From (i), (v), (C_A), (RI), (RM_C), (C-RM_A) and (Def.E). (ix): From (i), (v), (iv), (C_A) and (Def.E). $\square$

The properties of C-Restricted Monotonicity in (i) provide some further intuition on this rule. As indicated by the first restatement, (C-RM_A) can be defended on definitional grounds once the monotonicity of CB is taken for granted. The second restatement says in effect that monotonicity applies when the premisse φ is inherently public belief. Proposition 1 also lists intuitively desirable properties of C, two of which deserve special emphasis: (ii) captures the standard iterate notion of CB, while (iii) is the C-analogue of Positive Introspection, a property of CB which is taken for granted in a number of applications. The fact that (ii) to (ix) hold in MC_A is an immediate corollary to Proposition 1.

3. Semantic definitions. The soundness and completeness theorem

Our concept of structures is a specially devised variant of the standard concept of Neighbourhood structures [2, Chapter 7]. Define a *C-Restricted Monotonic Structure* to be any $(|A| + 2)$-tuple:

$$m = \langle W, (N_a)_{a\in A}, v\rangle,$$

where:

- W is a nonempty set (referred to as the set of possible worlds);
- for all $a \in A$, N_a is a mapping $W \to 2^{2^W}$ (i.e. the power set of the power set of W) such that for all $w \in W$, all $P, P' \subseteq W$,

$$\begin{aligned}&\text{if } P \in \textstyle\bigcap_{a\in A} N_a(w) \text{ and } P \subseteq \{w' \in W \mid P \in \bigcap_{a\in A} N_a(w')\},\\ &\text{then } P \subseteq P' \Rightarrow P' \in \textstyle\bigcap_{a\in A} N_a(w).\end{aligned} \qquad (*)$$

  (This condition, to be discussed below, will be called *C-Restricted Monotonic Closure.*)
- v is a mapping $W \times PV \to \{0,1\}$ (referred to as a valuation).

For convenience we introduce the following notation: for any $w \in W$,

$$N_E(w) = \bigcap_{a\in A} N_a(w).$$

Let us denote by $\mathcal{M}^{M^-}$ the class of C-Restricted Monotonic structures. Replacing condition $(*)$ in the above definition with the simpler condition of *Monotonic Closure*:

$$\text{if } P \in N_a(w), \text{ then } P \subseteq P' \Rightarrow P' \in N_a(w), \tag{$**$}$$

one would get the class of *Monotonic structures*, to be denoted by $\mathcal{M}^M$. Clearly, $(**) \Rightarrow (*)$, i.e., $\mathcal{M}^M \subseteq \mathcal{M}^{M^-}$. The inclusion is strict as the following easy example shows. Take $|A| = 2$, $W = \{w_1, w_2\}$ and v defined on PV in any way; $N_{a_1}(w_1) = N_{a_1}(w_2) = \{\phi, W\}$, $N_{a_2}(w_1) = N_{a_2}(w_2) = \{W\}$. This model is in $\mathcal{M}^{M^-}$ but not in $\mathcal{M}^M$.

The validation relation: $\langle m, w\rangle \models \varphi$ is defined for any $m \in \mathcal{M}^{M^-}$ and any $\varphi \in \Phi$ through a standard inductive process. We just state the modal clauses of the inductive definition:

- if $\varphi = B_a\psi$, $\langle m, w\rangle \models \varphi$ iff $[|\psi|]^m \in N_a(w)$, where $[|\psi|]^m$ denotes the truth set of ψ in m (i.e. $[|\psi|]^m \equiv \{w' \in W \mid \langle m, w'\rangle \models \psi\}$);
- if $\varphi = E\psi$, $\langle m, w\rangle \models \varphi$ iff $[|\psi|]^m \in N_E(w)$;
- if $\varphi = C\psi$, $\langle m, w\rangle \models \varphi$ iff there is $P \subseteq W$ s.t. $P \subseteq [|\psi|]^m$, $P \in N_E(w)$ and $P \subseteq \{w' \in W \mid P \in N_E(w')\}$.

The intuitive interpretation of the N_a functions is the usual one in an epistemic context. Given w, $N_a(w)$ is a system of belief for a at that world—i.e., the set of subsets that are a's objects of belief at w. The familiar validation clause for $B_a\psi$ connects this semantic account of individual belief with the syntactical one in the case when subsets are *propositions*. We proceed to the unfamiliar part of the semantics. For any $Q \subseteq W$, define Q to be *belief closed* (b.c.) if:

$$\forall\, w \in Q, Q \in N_E(w), \text{ i.e., } Q \subseteq \{w' \in W \mid Q \in N_E(w')\}.$$

Intuitively, Q is b.c. if it is an object of shared belief at every world at which it occurs. This is a semantic rendering of an event which is "public by nature". (Notice, however, that the belief closure property is relative to the given model.) The validation clause for $C\psi$ says that CB of ψ prevails at w iff there is a subset P (which may not be a proposition) that (i) implies the proposition corresponding to ψ in the model, (ii) is everybody's object of belief at w, and (iii) is b.c. This is one among several plausible *fixed-point* semantic definitions of CB. It embodies the basic intuition (also underlying (RI)) that events which are "public by nature" are also "CB by nature". The belief closure definition of CB *implies* the semantic account of CB in terms of a countable iteration, as in [5,6]: this follows from Proposition 1(ii) and the determination theorem below. (The converse implication, however, does not hold in $\mathcal{M}^{M^-}$; for details and further comparison of the iterate versus fixed-point semantics of CB, see [10, Section 4].)

Condition $(**)$ in our definition of structures is the familiar semantic notion of Monotonicity. The not so demanding $(*)$ imposes Monotonic Closure on those P only which satisfy conditions (ii) and (iii) of the last paragraph. The intended meaning is to allow for *just* that amount of monotonicity, hence of logical omniscience, which is required by the semantics of CB.

For any structure m and any relevant class of structures, $\mathcal{M}, m \models \varphi$ and $\mathcal{M} \models \varphi$ are defined in the usual way.

Theorem 2. *MC_A^- is a sound and complete axiomatization of the class of C-Restricted Monotonic structures, i.e. for any $\varphi \in \Phi$,*

$$\vdash_{MC_A^-} \varphi \quad \textit{iff} \quad \mathcal{M}^{M^-} \models \varphi.$$

4. Proof of the determination theorem

As usual, the soundness part is easy. The following observation will be useful:

Lemma 3. *For any $m \in \mathcal{M}^{M^-}$, if $P \subseteq [|\varphi|]^m$ and P is b.c., then $P \subseteq [|C\varphi|]^m$.*

Proof of the soundness part.

(C-RM_A): Assume that for all $m \in \mathcal{M}^{M^-}$, $[|\varphi|]^m \subseteq [|\psi|]^m$. Take any $m' \in \mathcal{M}^{M^-}$ and any $w' \in W'$. We assume that $\langle m', w'\rangle \models C\varphi$ and wish to prove that $\langle m', w'\rangle \models E\psi$. From the semantic definitions there is $P \subseteq [|\varphi|]^{m'}$, hence $P \subseteq [|\psi|]^{m'}$, s.t. $P \in N_E(w')$ and P is b.c. The property of C-Restricted Monotonicity implies that any superset of P is in $N_E(w')$; in particular $[|\psi|]^{m'}$ is. Hence the conclusion.

(FP): Take any $m \in \mathcal{M}^{M^-}$, any $w \in W$, and assume that $\langle m, w\rangle \models C\varphi$. There is P as in the relevant semantic clause. From Lemma 3, $P \subseteq [|C\varphi|]^m \cap [|\varphi|]^m = [|C\varphi \wedge \varphi|]^m$. P satisfies the condition for applying C-Restricted Monotonicity. Hence $[|C\varphi \wedge \varphi|]^m \in N_E(w)$, i.e., $\langle m, w\rangle \models E(C\varphi \wedge \varphi)$.

(RI): Assume that for all $m \in \mathcal{M}^{M^-}$, $[|\varphi|]^m \subseteq [|E\varphi|]^m$. Take any $m' \in \mathcal{M}^{M^-}$ and any $w' \in W'$. We assume that $\langle m', w'\rangle \models E\varphi$ and wish to prove that $\langle m', w'\rangle \models C\varphi$. Let $P = [|\varphi|]^{m'}$. From the assumptions P is b.c. and $P \in N_E(w')$; it trivially satisfies $P \subseteq [|\varphi|]^{m'}$. Hence the conclusion.

(RM_C), (RE_A): left to the reader. □

The proof of the completeness part is roundabout. We shall leave the more familiar parts for the reader. The general strategy is to prove the implication:

$$\mathcal{M}^{M^-} \models \psi \Rightarrow \vdash_{MC_A^-} \psi$$

for any *given* well-formed formula ψ, adapting to that formula the usual construction of a canonical model [2]. Completeness proofs of the "each formula" kind seem difficult to avoid whenever the system includes an axiomatization of CB in the style of the (ii) block; see also [6]. As a result of this technique, some information that could have been derived from a "universal" completeness proof—e.g. compactness—will be lost.

The first step is to define a *sublanguage* $\Phi[\psi]$ relative to ψ. Given any $\varphi \in \Phi$ the *depth* of φ, to be denoted by $dp(\varphi)$, is defined inductively:

- if $\varphi \in PV$, then $dp(\varphi) = 0$;
- $dp(\neg\varphi) = dp(\varphi)$;
- $dp(\varphi_1 * \varphi_2) = \max(dp(\varphi_1), dp(\varphi_2))$ for any two-place connective $*$;
- $dp(B_a\varphi) = dp(E\varphi) = dp(C\varphi) = dp(\varphi) + 1$.

We now define $\Phi[\psi]$ to be the subset of those formulas in Φ which are constructed from the propositional variables occurring in ψ and have depth at most $dp(\psi) + 1$. The subset $\Phi[\psi]^-$ is defined in the same way except for the following variant: "depth at most $dp(\psi)$".

$\mathcal{C}[MC_A^-]$ will denote the class of maximal consistent sets of Φ-formulas relative to the system MC_A^- (see [2] for the definition and properties of maximal consistent sets of formulas). We introduce the following equivalence relation on $\mathcal{C}[MC_A^-]$:

$$\forall\, \Gamma, \Delta \in \mathcal{C}[MC_A^-], \Gamma \overset{\psi}{\equiv} \Delta \text{ iff } \Gamma \cap \Phi[\psi] = \Delta \cap \Phi[\psi].$$

For any equivalence class $[\Gamma]^\psi$, let Γ^ψ stand for its intersection. Note carefully that Γ^ψ may contain formulas of any depth: it is the set of those formulas which are *deducible* from $\Gamma \cap \Phi[\psi]$. Clearly, $\Gamma \overset{\psi}{\equiv} \Delta$ iff $\Gamma^\psi = \Delta^\psi$. The general idea of the sequel is to construct a ψ-specific canonical model having the set of $\overset{\psi}{\equiv}$-equivalence classes, to be denoted by I^ψ, as its set of possible worlds.

Lemmas 4 and 5 record two technically useful properties of subsets of I^ψ. The proof of Lemma 4 hinges on the fact that I^ψ is finite; that of Lemma 5 on the precise use of the above definitions.

Lemma 4. *For all $P \subseteq I^\psi$ there is $\varphi \in \Phi[\psi]$ such that:*

$$P = \{[\Gamma]^\psi \in I^\psi \mid \varphi \in \Gamma^\psi\}.$$

For any $\varphi \in \Phi$ we shall denote sets such as P by $[\varphi]^\psi$.

Lemma 5. *For all $\varphi_1 \in \Phi[\psi]$ and $\varphi_2 \in \Phi$, if $[\varphi_1]^\psi \subseteq [\varphi_2]^\psi$, then $\vdash \varphi_1 \to \varphi_2$.*

Now, we define a *ψ-canonical* model to be a $(|A| + 2)$-tuple $m_\psi = \langle W, (N_a)_{a\in A}, v\rangle$ such that:

- $W = I^\psi$,
- for all $a \in A$ and $[\Gamma]^\psi \in I^\psi$,

$$N_a([\Gamma]^\psi) = \{P \subseteq I^\psi \mid \exists\, \varphi \in \Phi[\psi] \text{ s.t. } P = [\varphi]^\psi \text{ and } B_a\varphi \in \Gamma^\psi\};$$

- for all $\varphi \in PV$ and $[\Gamma]^\psi \in I^\psi, v([\Gamma]^\psi, \varphi) = 1$ iff $\varphi \in \Gamma^\psi$.

Lemma 6 clarifies this construction with a view of showing that ψ-canonical models are in $\mathcal{M}^{M^-}$.

Lemma 6. *For all $a \in A$ and $\Gamma \in \mathcal{C}[MC_A^-]$,*

(i) $\forall\, \varphi \in \Phi[\psi]^-, B_a\varphi \in \Gamma \Rightarrow [\varphi]^\psi \in N_a([\Gamma]^\psi)$;

(ii) $\forall\, \varphi \in \Phi[\psi], [\varphi]^{\psi} \in N_a([\Gamma]^{\psi}) \Rightarrow B_a\varphi \in \Gamma^{\psi}$;
(iii) $N_a([\Gamma]^{\psi})$ *satisfies C-Restricted Monotonic Closure.*

Proof.
(i) immediate.

(ii) Assume that $[\varphi]^{\psi} \in N_a([\Gamma]^{\psi})$. From the definition there is $\varphi_0 \in \Phi[\psi]$ s.t. $B_a\varphi_0 \in \Gamma^{\psi}$ and $[\varphi_0]^{\psi} = [\varphi]^{\psi}$. Lemma 5 implies that $\vdash \varphi \leftrightarrow \varphi_0$. Hence from ($\mathrm{RE}_a$), $B_a\varphi \in \Gamma^{\psi}$.

(iii) Take any P, P' s.t. $P \subseteq P' \subseteq I^{\psi}$ and: (I) $P \in N_a([\Gamma]^{\psi})$ for all $a \in A$; (II) $P \subseteq \{[\Gamma']^{\psi} \mid P \in \bigcap_{a\in A} N_a([\Gamma']^{\psi})\}$. We wish to prove that: (III) $P' \in N_a([\Gamma]^{\psi})$ for all $a \in A$. From Lemma 4 $P = [\varphi]^{\psi}$ for some $\varphi \in \Phi[\psi]$. Now, condition (I) and part (ii) imply that for all a, $B_a\varphi \in \Gamma^{\psi}$, hence that $E\varphi \in \Gamma^{\psi}$, using (Def.E). Condition (II) can be restated as: $[\varphi]^{\psi} \subseteq \{[\Gamma']^{\psi} \mid [\varphi]^{\psi} \in \bigcap_{a\in A} N_a([\Gamma']^{\psi}\}$ or (using part (ii) and (Def.E)): $[\varphi]^{\psi} \subseteq \{[\Gamma']^{\psi} \mid E\varphi \in \Gamma'^{\psi}\} = [E\varphi]^{\psi}$. Applying Lemma 5 once again, $\vdash \varphi \rightarrow E\varphi$ holds and (RI) implies that $\vdash E\varphi \rightarrow C\varphi$, whence $C\varphi \in \Gamma^{\psi}$. Now, Lemmas 4 and 5 imply that $P' \subseteq I^{\psi}$ can be expressed as $[\varphi']^{\psi}$ for some $\varphi' \in \Phi[\psi]$, and $\vdash \varphi \rightarrow \varphi'$ holds. ($\mathrm{C\text{-}RM}_A$) delivers the conclusion that $E\varphi' \in \Gamma^{\psi}$, which (using (Def.E) and part (i)) implies that (III) holds. □

The following lemma says in effect that ψ-canonical models are indeed *canonical* in the usual sense of modal logic.

Lemma 7. *Take a ψ-canonical model m_{ψ}. Then, for all $\Gamma \in \mathcal{C}[MC_A^-]$ and $\psi' \in \Phi[\psi]^-$,*

$$\langle m_{\psi}, [\Gamma]^{\psi}\rangle \models \psi' \quad \textit{iff} \quad \psi' \in \Gamma^{\psi}. \tag{$*$}$$

Proof. As usual with the proof of canonical lemmas, by induction on the propositional complexity of ψ'. The definition of v in m_{ψ} deals with the case of a zero-complexity ψ'. The inductive hypothesis can be stated as:

$$[|\varphi|]^{m_{\psi}} = [\varphi]^{\psi}.$$

The only cases of interest are $\psi' = B_a\varphi$ and $\psi' = C\psi$.

Case 1: $\psi' = B_a\varphi$. Equivalence $(*)$ then reads as:

$$[|\varphi|]^{m_{\psi}} \in N_a([\Gamma]^{\psi}) \Leftrightarrow B_a\varphi \in \Gamma^{\psi},$$

which follows from the inductive hypothesis together with Lemma 6(i) and (ii).

Case 2: $\psi' = C\varphi$. From left to right in $(*)$: Using the definition of semantic validation, there is $P \subseteq I^{\psi}$ s.t. conditions (I) and (II) in the proof of Lemma 6(iii) hold and $P \subseteq [|\varphi|]^{\psi} = [\varphi]^{\psi}$. Lemma 4 implies that $P = [\varphi_0]^{\psi}$ for some $\varphi_0 \in \Phi[\psi]$, and Lemma 5 that $\vdash \varphi_0 \rightarrow \varphi$. In the presence of (I), Lemma 6(ii) together with (Def.E) implies that $E\varphi_0 \in \Gamma^{\psi}$. Now, repeating an argument already made in the proof of Lemma 6(iii) we conclude from (II) that $\vdash E\varphi_0 \to C\varphi_0$. It then follows that $C\varphi_0 \in \Gamma^\psi$. Applying ($\mathrm{RM}_C$) leads to the desired conclusion that $C\varphi \in \Gamma^\psi$.

From right to left in ($*$): Assume that $C\varphi \in \Gamma^\psi$. (FP) implies that $E(C\varphi \wedge \varphi) \in \Gamma^\psi$. Clearly, $C\varphi \wedge \varphi \in \Phi[\psi]^-$, so that we may apply Lemma 6(i) and conclude that $[C\varphi \wedge \varphi]^\psi \in N_a([\Gamma]^\psi)$ for all $a \in A$. Putting $P = [C\varphi \wedge \varphi]^\psi$ we have thus shown that (I) holds of P. Using (FP), (Def.E) and Lemma 6(i) again, it can be checked that (II) also holds of P. Thus, we have found $P \subseteq I^\psi$ satisfying (I), (II) as well as—obviously—the condition that $P \subseteq [\varphi]^\psi = [|\varphi|]^\psi$. Hence $\langle m_\psi, [\Gamma]^\psi \rangle \models C\varphi$. □

End of the completeness proof. Use Lemma 7 and the already-mentioned consequence of Lemma 6(iii) that $m_\psi \in \mathcal{M}^{M^-}$, as in the standard argument for completeness [2, Chapter 2]. □

Corollary 8. *MC_A^- is decidable.*

Proof. From the standard argument spelled out in [2, pp. 62–64], and the fact that the ψ-canonical model constructed above is finite. □

Corollary 9. *MC_A^- is strictly weaker than MC_A.*

Proof. Assume that $MC_A = MC_A^-$. Then, from the soundness theorem, rule (RM_A) should be valid in $\mathcal{M}^{M^-}$. But this is not the case in view of the model used in Section 3 to prove that $\mathcal{M}^M \subsetneq \mathcal{M}^{M^-}$. □

Proposition 10. *MC_A is a sound and complete axiomatization of the class of Monotonic Structures, i.e., for any $\varphi \in \Phi$,*

$$\vdash_{MC_A} \varphi \quad \textit{iff} \quad \mathcal{M}^M \models \varphi.$$

Proof (Sketch).

The reader will readily check that (RM_A) is valid in $\mathcal{M}^M$. As far as soundness is concerned, no further direct verification is needed. For it has been shown that the remaining rules and axioms are valid in $\mathcal{M}^{M^-}$, hence in $\mathcal{M}^M$.

To prove completeness, the reader might repeat the construction in this section after replacing $\mathcal{C}[MC_A^-]$ with the class of maximal consistent sets relative to MC_A. Using whenever necessary the fact that MC_A is stronger that MC_A^-, Lemmas 4, 5, and 6(i)–(ii) carry through. Simplifying the argument used to prove Lemma 6(iii), it can be seen that $N_a([\Gamma]^\psi)$ satisfies Monotonic Closure. The rest of the proof carries through, using again the above syntactical fact. □

Proposition 10 is mentioned in the authors' survey of the logic of CB [10, Section 4, Theorem 8]. In [9] it received a direct proof, which is superseded by the present one. Given that Theorem 2 above provides the most powerful, both mathematically and conceptually, among the current axiomatizations of CB in Neighbourhood structures, it is best to regard Proposition 10 as a derivative result. A related comment applies to an alternative monotonic axiomatization of CB mentioned as [10, Section 4, Theorem 7]. This result trades on a semantic account of CB in terms of uncountable iterations. It can be derived from Proposition 10, hence also recovered within the present framework, once the connection between iterate and fixed-point validation clauses in Monotonic Neighbourhood structures is clarified (see [10, Section 4, Proposition 9]).

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References

[1] F. Bacchus, *Representing and Reasoning With Probabilistic Knowledge* (MIT Press, Cambridge, MA, 1990).
[2] B.F. Chellas, *Modal Logic* (Cambridge University Press, Cambridge, England, 1980).
[3] R. Fagin and J.Y. Halpern, Belief, awareness and limited reasoning, *Artif. Intell.* **34** (1988) 39—76; Preliminary version in: *Proceedings IJCAI-85*, Los Angeles, CA (1985) 491—501.
[4] R. Fagin, J.Y. Halpern, Y.O. Moses and M.Y. Vardi, *Reasoning about Knowledge* (MIT Press, Cambridge, MA, to appear).
[5] J.Y. Halpern and Y.O. Moses, Knowledge and common knowledge in a distributed environment, *J. ACM* **37** (3) (1990) 549—587; Preliminary version in: *Proceedings Third ACM Conference on Principles of Distributed Computing*, New York (1984) 50—61.
[6] J.Y. Halpern and Y.O. Moses, A guide to completeness and complexity for modal logics of knowledge and belief, *Artif. Intell.* **54** (3) (1992) 319-379; Preliminary version in: *Proceedings IJCAI-85*, Los Angeles, CA (1985) 480—490.
[7] J. Hintikka, Impossible possible worlds vindicated, *J. Philos. Logic* **4** (1975) 475—484.
[8] H.J. Levesque, A logic of implicit and explicit belief, in: *Proceedings AAAI-84*, Austin, TX (1984) 198—202.
[9] L. Lismont and P. Mongin, Belief closure: a semantics of common knowledge for modal propositional logic, CORE Discussion Paper 9339, Université Catholique de Louvain (1993).
[10] L. Lismont and P. Mongin, On the logic of common belief and common knowledge, *Theory Decis.* **37** (2) (1994).
[11] D. Monderer and D. Samet, Approximating common knowledge with common beliefs, *Games Econ. Behav.* **1** (1989) 170-190.
[12] R. Stalnaker, The problem of logical omniscience, I, *Synthese* **89** (1991) 425—440.
[13] M.Y. Vardi, On epistemic logic and logical omniscience, in: J.Y. Halpern, ed., *Proceedings First Conference on Theoretical Aspects of Reasoning about Knowledge* (Morgan Kauffmann, San Mateo, CA, 1986) 293—305.
[14] M.Y. Vardi, On the complexity of epistemic reasoning, in: *Proceedings Fourth Annual Symposium on Logic in Computer Science* (1989) 243—252.