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A note on affine aggregation

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Abstract

If a vector-valued function has convex range and one of its components is related to the others by a Pareto-like condition, that component must be affine w.r.t. the others; sign restrictions on the coefficients follow from suitably strengthening the unanimity condition. The theorem is applied to social choice and decision theories.

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1. Introduction

This paper proves a simple and general theorem on affine aggregation. Consider a vector-valued function $F = (f_0, f_1, \ldots, f_n)$ having convex range. Assuming that f_0 is related to the n remaining components by a unanimity condition, we investigate the functional relationship between f_0 and $f_1, \ldots, f_n$. When the f_i are utilities, the unanimity conditions are simply those of Pareto. Our theorem states that for most of the envisaged conditions, f_0 is affine in terms of the $f_1, \ldots, f_n$ and that sign restrictions on the coefficients result from a suitable choice of the unanimity condition.

By specifying the type of the f_i we generate relevant choice-theoretic applications. In particular, we derive variants of Harsanyi's (1955) Aggregation Theorem and give a general solution to the well-known problem of *signing* the coefficients in Harsanyi's affine social choice rule. We discuss further applications to Anscombe and Aumann's (1963) expected utility theory and to some recent results on probability aggregation (Mongin, 1993).

2. Definitions and basic facts

Consider any nonempty set X and $F = (f_0, f_1, \ldots, f_n): X \to \mathbb{R}^{n+1}$. We study the effect of imposing one of the following conditions: for any $x, y \in X$,

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$(P_0) f_i(x) = f_i(y), i = 1, \dots, n \Rightarrow f_0(x) = f_0(y)$.

$(P_1) f_i(x) \geq f_i(y), i = 1, \dots, n \Rightarrow f_0(x) \geq f_0(y)$.

$(P_2) f_i(x) > f_i(y), i = 1, \dots, n \Rightarrow f_0(x) > f_0(y)$.

$(P_3) f_i(x) \geq f_i(y), i = 1, \dots, n \& \exists j : f_j(x) > f_j(x) \Rightarrow f_0(x) > f_0(y)$.

$(P_4) f_i(x) \geq f_i(y), i = 1, \dots, n \& f_0(x) \leq f_0(y) \Rightarrow F(x) = F(y)$.

If the f_i are utility functions, then (P_0), (P_1), (P_2), and (P_3) become standard Pareto conditions, i.e. those of Pareto-indifference, Pareto-weak preference, Weak Pareto, and Strict Pareto, respectively. The conjunction of (P_0) and (P_3) is then the Strong Pareto condition, while the nonstandard condition (P_4) says that any conflict between the weak preferences of the social observer and those of society as a whole are resolved by general indifference. We also introduce the following condition of *minimum agreement* among the f_i:

$$\exists x^*, y^* \in X, \quad \forall i = 1, \dots, n, \ f_i(x^*) > f_i(y^*). \tag{C}$$

The following lemma clarifies the relations between the (P_s):

Lemma $(P_4) \Rightarrow (P_3) \Rightarrow (P_2)$; $(P_4) \Rightarrow (P_1) \Rightarrow (P_0)$; $(P_4) \Leftrightarrow (P_0) \& (P_3)$. *If* $F(X)$ *is convex*, $(C) \& (P_3) \Rightarrow (P_4)$.

Proof. Since the other relations are easy to check, we prove only the last statement. Assume that the antecedent of (P_4) holds, i.e.

$$f_i(x) \geq f_i(y), \quad i = 1, \dots, n \quad \text{and} \quad f_0(x) \leq f_0(y).$$

Then, (P_3) clearly implies that $f_i(x) = f_i(y)$, $i = 1, \dots, n$. Assume by way of contradiction that $f_0(x) < f_0(y)$ and define the vector

$$u = (1 - \varepsilon)(F(x) - F(y)) + \varepsilon(F(x^*) - F(y^*)),$$

where x^*, y^* are as in (C) and $\varepsilon \in]0,1[$. For ε small enough, $u_0 < 0$ and $u_i > 0$, $i = 1, \dots, n$. Since the set $\{F(x) - F(y) | x, y \in X\}$ is convex, there are $\xi, \xi' \in X$ such that $u = F(\xi) - F(\xi')$, contradicting (P_3). □

We need a few definitions and basic facts of convex analysis in finite-dimensional vector spaces. A subset D of $E = \mathbb{R}^{n+1}$ is said to be *polyhedral* if it is the set of solutions of a finite system of linear weak inequalities. Polyhedral sets are clearly convex and closed. Several of the arguments below crucially depend on assuming finite dimensionality. For $i = 0, 1, \dots, n$, let e_i denote the ith vector of the canonical basis. Given D, $D' \subset E$, let $V(D)$ denote the vector space spanned by D in E, and $D - D'$ the set $\{z - z' | z \in D, z' \in D'\}$.

The linear form ν is said to separate the sets K_1, $K_2 \subset E$ if

$$\inf_{k_1 \in K_1} \langle \nu, k_1 \rangle \geq \sup_{k_2 \in K_2} \langle \nu, k_2 \rangle, \text{ and } \exists k_1 \in K_1, k_2 \in K_2 : \langle \nu, k_1 \rangle > \langle \nu, k_2 \rangle ,$$

and to separate them *strictly* if the first inequality is strict. The standard separating hyperplane theorem states that there is a separating ν if K_1 and K_2 are nonempty, convex, and mutually disjoint. It is also known that there is a strictly separating ν if the previous conditions hold, and furthermore K_1 is closed and K_2 compact (Rockafellar, 1970, Corollary 11.4.2.). It is also the case that there is a strictly separating ν whenever $K_1, K_2 \subset E$ are polyhedral, nonempty, and mutually disjoint (Rockafellar, 1970, Corollary 19.3.3.).

3. Affine aggregation results

Proposition 1. Assume that $K = F(X)$ is convex. Then, (P_0) *holds if and only if there are real numbers $\lambda_1, \ldots, \lambda_n$, μ such that*

$$\forall x \in X, f_0(x) = \sum_{i=1}^{n} \lambda_i f_i(x) + \mu . \qquad (*)$$

(P_1) $[(P_4)]$ *holds if and only if there are non-negative* [*resp. strictly positive*] *numbers $\lambda_1, \ldots, \lambda_n$ and a real number μ satisfying* $(*)$.

Proof. The sufficiency part in each statement is obvious. To prove necessity in the case of (P_s), $s = 0$, 1, we shall reformulate these conditions appropriately. Notice first that (P_0) is equivalent to: $\forall x, y \in X$, $[f_i(x) = f_i(y), i = 1, \ldots, n \Rightarrow f_0(x) \geq f_0(y)]$. Define:

$$R_0 = \{z \in \mathbb{R}^{n+1} \mid z_0 < 0 \text{ and } z_i = 0, i = 1, \ldots, n\} ,$$
$$R_1 = \{z \in \mathbb{R}^{n+1} \mid z_0 < 0 \text{ and } z_i \geq 0, i = 1, \ldots, n\} ,$$

and let $\bar{R}_0$ and $\bar{R}_1$ denote the closures of R_0 and R_1, respectively. The following equivalence clearly holds for $s = 0, 1$:

$$(P_s) \Leftrightarrow R_s \cap K^- = \emptyset ,$$

where K^- denotes the set $K - K$. Using the fact that K^- is convex and symmetric with respect to 0, it can be checked that for $s = 0, 1$,

$$(P_s) \Leftrightarrow R_s \cap V(K^-) = \emptyset .$$

[To see that, assume that there is $z \in R_s \cap V(K^-)$ and write $z = \Sigma_{i=1}^m \alpha_i z_i$, where $Z_1, \ldots, z_m$ are elements of K^-. The symmetry property of K^- means that the α_i can be taken to be non-negative. The case $z = 0$ is trivial. If $z \neq 0$, the convexity property implies that $z' = (\Sigma \alpha_i)^{-1} \Sigma_{i=1}^m \alpha_i z_i$ is in K^-, a contradiction.] Since $\bar{R}_s - e_0 \subset R_s$,

$$(P_s) \Rightarrow (\bar{R}_s - e_0) \cap V(K^-) = \emptyset .$$

Thus, we have just reduced the problem to that of separating two sets that are polyhedral, nonempty, and mutually disjoint. Applying the latter strict separation theorem in Section 2, there is $\nu = (\nu_0, \nu_1, \ldots, \nu_n)$ such that for $s = 0, 1$,

$$\langle \nu, z - e_0 \rangle > \langle \nu, k \rangle \ , \quad \forall k \in V(K^-), \forall z \in \bar{R}_s \ .$$

The linear space property of $V(K^-)$ implies that

$$\forall k \in V(K^-), \langle \nu, k \rangle = 0 \ . \tag{1}$$

Applying the above inequality to $z = 0$, we conclude that $\nu_0 < 0$. Thus, Eq. (1) as restricted to K^- can be written as

$$\forall x, y \in X, f_0(x) - f_0(y) = \sum_{i=1}^{n} \nu_i(-\nu_0)^{-1}[f_i(x) - f_i(y)] \ . \tag{2}$$

Fixing $y \in X$ leads to coefficients $\lambda_1, \ldots, \lambda_n$, μ satisfying (*), which proves the (P_0) case. It remains to be shown that (*) holds for non-negative λ_i when (P_1) holds. For any $i = 1, \ldots, n$ and $\alpha > 0$, one has $\alpha e_i \in \bar{R}_1$, whence

$$\langle \nu, \alpha e_i \rangle > \langle \nu, e_0 \rangle \ ,$$

and the conclusion that $\nu_i \geq 0$ follows from dividing by α and letting $\alpha \rightarrow +\infty$.

To deal with (P_4) define:

$$R_4 = \{z \in \mathbb{R}^{n+1} \,|\, z_0 \leq 0 \text{ and } z_i \geq 0, i = 1, \ldots, n\} \backslash \{0\} \ ,$$

$$\Delta = \left\{ z \in R_4 \sum_{i=1}^{n} z_i - z_0 = 1 \right\} .$$

From the definition of R_4 we have

$$(P_4) \Leftrightarrow R_4 \cap V(K^-) = \emptyset \ ,$$

whence $(P_4) \Rightarrow \Delta \cap V(K^-) = \emptyset$. Since Δ is convex and compact, and $V(K^-)$ is convex and closed, the former strict separation theorem in Section 2 applies. There is $\nu = (\nu_0, \nu_1, \ldots, \nu_n)$ such that

$$\langle \nu, \delta \rangle > \langle \nu, k \rangle \ , \quad \forall \delta \in \Delta, \forall k \in V(K^-) \ .$$

As before, (1) holds, leading to

$$\langle \nu, \delta \rangle > 0 \ , \quad \forall \delta \in \Delta \ .$$

Applying this inequality to $\delta = -e_0, e_1, \ldots, e_n$ in turn, we conclude that $\nu_0 < 0$ and $\nu_i > 0$, $i = 1, \ldots, n$. Hence, (2) again follows from (1), and (*) holds with strictly positive $\lambda_i, i = 1, \ldots, n$. □

Proposition 2. Assume that $K = F(X)$ is convex. If (P_2)[(P_3)] holds, then there are non-negative numbers $\lambda_1, \ldots, \lambda_n$, not all of them zero [resp. strictly positive numbers $\lambda_1, \ldots, \lambda_n$], a non-negative number κ, and a number μ, such that

$$\forall x \in X, \kappa f_0(x) = \sum_{i=1}^{n} \lambda_i f_i(x) + \mu \,. \tag{$**$}$$

Assuming (P_2) *and* (C) [(P_3) *and* (C)], *there are non-negative* λ_i, *not all of them zero* [*resp. strictly positive* λ_i], *and there is* μ, *such that* ($*$) *in Proposition 1 holds.*

Proof. To prove the first part, define the following convex sets:

$$R_2 = \{z \in \mathbb{R}^{n+1} \mid z_0 \leq 0 \text{ and } z_i > 0, i = 1, \ldots, n\}$$

and $R_2' = \bar{R}_2 + \Sigma_{i=1}^n e_i$. The polyhedral sets R_2' and $V(K^-)$ can be separated strictly. Sign restrictions in ($**$) derive from considering the following vectors: for $\gamma > 0$, $-\gamma e_0 + \Sigma_{i=1}^n e_i$ and $\gamma e_j + \Sigma_{i=1}^n e_i$, for $j = 1, \ldots, n$, and using a limiting argument. A related argument takes care of the case R_3. [For instance, define:

$$\begin{aligned} R_3 &= \{z \in \mathbb{R}^{n+1} \mid z_0 \leq 0, z_i \geq 0, i = 1, \ldots, n \text{ and } \exists j \neq 0 \colon z_j > 0\} \\ &= \{z \in \mathbb{R}^{n+1} \mid z_0 \leq 0, z_i \geq 0, i = 1, \ldots, n\} \setminus \{(z_0, 0, \ldots, 0) \mid z_0 \leq 0\} \,, \end{aligned}$$

and

$$R_3' = R_3 \cap \{z \in \mathbb{R}^{n+1} \mid \Sigma_{i=1}^n z_i = 1\}.]$$

As to the second part, note that if (C) is added to (P_2), there is $(\xi_0, \xi_1, \ldots, \xi_n) \in K^-$ such that $\xi_i > 0$, for $i = 0, 1, \ldots, n$. Then, ($**$) implies that

$$\kappa \xi_0 = \lambda_1 \xi_1 + \cdots + \lambda_n \xi_n \,,$$

and hence that $\kappa < 0$. To deal with the case in which (C) is added to (P_3), apply Proposition 1 together with the Lemma. □

There is no hope of strengthening the first part of Proposition 2, as the following shows. Take $X = \mathbb{R}^2$, $f_0(x, y) = x$, $f_1(x, y) = y$, $f_2(x, y) = -y$. Clearly, the range of (f_0, f_1, f_2) is convex and (P_3), hence (P_2), trivially holds. This example illustrates the importance of the nontriviality condition (C).

Importantly, (P_2) and (P_3) behave differently in the presence of (P_0). To see that, take $X = \mathbb{R}$, $f_0(x) = x$, $f_1(x) = -x$, $f_2(x) = 0$. Here (P_2) and (P_0) hold but it is impossible to have non-negative coefficients. Obviously, this example violates (P_3). To assume (P_3) and (P_0) together is tantamount to assuming (P_4), which has been said to imply the existence of strictly positive coefficients.

4. Applications

Following Harsanyi's (1955) Aggregation Theorem, if the individuals and the social observer have von Neumann–Morgenstern (VNM) utility functions, $f_1, \ldots, f_n$ and f_0, respectively, and if f_0 satisfies the Pareto-indifference condition with respect to the f_i, then f_0 is affine in terms of the f_i. Under the VNM assumption, however formalized, the range of $F = (f_0, f_1, \ldots, f_n)$ is convex, so that Harsanyi's theorem in any axiomatic variant follows from the (P_0) part of Proposition 1. There has been much discussion on how (and even whether) unanimity conditions other than Pareto-indifference could be used to *sign* the coefficients of individual utilities in Harsanyi's affine conclusion. This was a relevant question to raise because the theorem was intended as a technical step towards deriving truly utilitarian rules. The first attempt to give a complete answer was probably Domotor (1979). Fishburn (1984) showed that Pareto-weak preference implied non-negative coefficients. He defines VNM utilities as mixture-preserving ('linear') functions on a mixture set, so that his proof is at the highest possible level of generality; it is, however, more complicated than the geometric argument used here to solve the (P_1) case. Weymark (1993, forthcoming) has recently analyzed the role of the Weak and Strong Pareto conditions with identical results to those reached here with respect to (P_2) and (P_4). His proofs use the Lemma of the Alternative and are thus implicitly related to the convex analysis pursued here. Unfortunately, Weymark's proofs are not at the highest possible level of generality. He defines VNM functions as expected-utility functionals on a lottery set which is constructed from a *finite* set of pure prospects. Besides simplicity, the present approach has the advantage of being completely general. Also, we analyze the respective contributions of Pareto-indifference and Strict Pareto within the Strong Pareto condition.

Another easy application is to the theory of subjective expected utility derived by Anscombe and Aumann (1963). Take X to be the set of acts, i.e. of functions from I to $\mathscr{L}$, where I is a finite state space and $\mathscr{L}$ a lottery set. Define $\mathscr{L}^*$ to be the set of simple probabilities on X. As is well known, Anscombe and Aumann assume that there are VNM functions v and v^* on $\mathscr{L}$ and $\mathscr{L}^*$, respectively, then connect them with each other through the Monotonicity and the Reversal of Order axioms in order to derive a (unique) subjective probability on I. Here is a quick variant of their proof. Using Reversal of Order, it is easy to define functions f_i, $i \in I$, that are mixture-preserving on $\mathscr{L}^*$ and satisfy

$$f_i(x) = v(x(i)) , \quad \forall i \in I, \forall x \in X .$$

Putting $f_0 = v^*$, the vector $F = (f_0, f_1, \ldots, f_n)$ is seen to have convex range. Monotonicity together with Reversal of Order imply that F satisfies (P_1) above. Proposition 1 then implies the existence of a subjective probability on I (the uniqueness of which is secured by an affine independence argument). More could be said on the use of Proposition 1 in the context of Anscombe and Aumann's theory when state-dependent utilities are allowed, as in Karni et al. (1983) and Drèze (1987).

A further application results from taking the f_i to be atomless probabilities as required by Savage's (1954) own version of subjective expected utility. By Lyapunov's (1940) theorem, the range of a vector-valued, atomless finite measure is convex, so that Propositions 1 and 2 can again be put to use. Mongin (1993) has recently explored the aggregative properties of atomless probabilities as a step in the analysis of the aggregative properties of Savagean orderings. Notice that the probabilistic application automatically satisfies (C), hence avoids the technical problems connected with condition (P_2) or (P_3) in the utility context.

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