On the Determination of Subjective Probability by Choices

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The paper explores the uniqueness properties of the subjective probabilities in two axiomatizations of state-dependent preferences. Karni, Schmeidler, and Vind's (KSV 1983) system depends on selecting an arbitrary auxiliary probability, and as such, does not guarantee the uniqueness of the derived subjective probability. However, an axiom system initially designed by Karni and Schmeidler (KS 1981) and further elaborated upon here does guarantee the desired uniqueness as well as a useful property of "stability" of the derived solution. When the preference relation displays state-independence, even the KS probabilities may not agree with those derived from the classic Anscombe-Aumann (AA 1963) theorem. However, we claim that, in this case, the KS rather than the AA probabilities are the appropriate representation of the agent's beliefs.
(*Subjective Probability; Subjective Expected Utility; State-Dependent Utility*)

1. Introduction

Bayesian decision theory (e.g., Savage 1954, Anscombe and Aumann 1963) implies that there is a probability measure on the state space and a utility function on the set of consequences such that the decision maker's preferences over acts (i.e., functions from states to consequences) have a subjective expected utility (SEU) representation, i.e., the preference relation can be represented as the expectation of the utility function with respect to the probability measure. Moreover, the probability-utility pair is unique in the sense that any alternative SEU representation of the decision maker's preferences must involve the same probability measure and a utility function which is a positive affine transformation of the original one.

It is well known that the uniqueness property of the probability component crucially depends on the utility function being state-independent. What is not always appreciated is that this condition entails two distinct requirements. First, the decision maker's *preferences* should satisfy state-independence. This means in particular that his attitude towards risk should be independent of the underlying states. Second, the *utility functions* chosen to represent his preferences in each state should be exactly the same functions. The former requirement concerns the preference relation and can be stated as an axiom. However, and most importantly, the latter requirement does not express a property of the preference relation, and does not belong to the axioms. Yet if the *two* requisites are not satisfied, the decision maker's subjective probability will not be uniquely determined. We will pursue this point at some length in the paper.

It has long been recognized that state-independence is an inappropriate specification of preferences in a number of circumstances involving economic decisions; for instance, the choice of health insurance or life insurance coverage, and the selection of nuclear waste dumping grounds.[1] The literature offers a num-

[1] Early statements include Savage (1954) and Drèze (1963).

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ber of different approaches to determining the decision maker's subjective probability when his preferences are state-dependent.[2] The present paper is concerned with the method of resolution that is associated with the work of Karni, Schmeidler, and Vind (1983, henceforth KSV). These authors modify Anscombe and Aumann's (1963, henceforth AA) framework of Bayesianism so as to permit observation of the decision maker's preferences among *hypothetical objects* which are lotteries (i.e., probability measures) over the set of state-outcome pairs. The interpretation of the prizes in such a lottery is "to be faced with outcome x when the state of nature is s," for instance "to be at home when the sun is shining" or "to be in the stadium when it is raining." There are two versions of this construction. In the specific contribution of Karni et al. (1983), the decision maker expresses his preferences only over a subset of the set of all state-outcome lotteries. This subset is determined by the property that the state-outcome lotteries in it have some (arbitrarily chosen) *fixed* marginal distribution on the state space.[3] There is, however, a variant of the construction, which was originally due to Karni and Schmeidler (1981, henceforth KS), and which will be further elaborated upon in this paper. Following the KS axiomatization the decision maker expresses preferences over the set of all state-outcome lotteries. So by contrast with the KSV setting, he can also express preferences between lotteries implying *different* marginal state probabilities. In either version the decision maker's (hypothetical) preferences on the set of hypothetical objects are connected with his (actual) preferences over acts by a consistency requirement. In either version the results are two SEU representations, one for the hypothetical, the other for the actual preference relation.

The uniqueness properties of the subjective probabilities obtained under the two versions are essentially different. This point was mentioned in passing by Karni et al. (1983) but has not been fully appreciated. In this paper we elaborate on the differences between the two constructions. First, we show by means of an example that, under the weak (KSV) version, subjective probabilities on the state space may depend on the (arbitrarily chosen) marginal probability on the states of the state-outcome lotteries. Then, we explain how this element of arbitrariness is eliminated in the strong (KS) version. Second, we introduce the notion of a *stable* solution for a procedure. Essentially, a subjective probability is a stable solution if it is selected whenever it is *itself* taken to be the arbitrary marginal of the procedure. The KS version always leads to stable solutions but the KSV version does not.

Mongin (1998b) mentioned the uniqueness problem in the KSV resolution when discussing a case of particular relevance to a multiagent context, i.e., the case of *event-dependent* preferences. If the conditional preference on some event (i.e., nonempty subset of states) is state-independent, then the ordinary Anscombe-Aumann (AA) theorem applies to this conditional preference, and using the convention discussed above, it is possible to derive a state-independent probability on the event under consideration. We show that neither the KSV nor the KS version guarantees that the conditional of the derived subjective probability will coincide with this state-independent probability. This discrepancy leads to a normative question—which of the two probabilities is the appropriate representation of the agent's beliefs conditional on the given event? We claim that the KS rather than the state-independent AA probabilities should be selected whenever both are jointly available. This claim leads us to a strong and seemingly paradoxical conclusion, which is perhaps the most important message of this paper. Since complete state-independence, as in AA, is a particular case of event-dependence, we conclude that *even under state-independence*, the KS rather than the conventional AA probability should be taken to be the decision maker's subjective probability. The KS approach provides direct information on the utility representations that by-passes the need for a conventional choice of these representations as in the Anscombe-Aumann approach. This claim extends

[2] For an overview see, e.g., Schervish (1990).

[3] According to KSV's suggested interpretation of this formalism, the decision maker states what his preferences among lotteries *would be* if the states hypothetically had a specified probability distribution rather than his own subjective probability distribution. We will discuss this counterfactual interpretation below.

the KS axiomatization much beyond its initially intended domain of application.

2. Definitions

The notation generally follows Karni et al. (1983). There is a finite set of distinct states $S = \{1, \ldots, s, \ldots, T\}$ with $T \geq 2$. For simplicity, we assume that the set of outcomes (or "prizes") X is also finite, with $|X| = M \geq 2$, and we take it to be the same in each state. The conclusions of this paper could be rephrased in the more general setting in which the availability (and not only the evaluation) of consequences varies with the state. As required by the AA framework of analysis, consequences are stochastic.[4] Define the set Y of consequences to be the set of all probability measures on X. If f is a function from S to Y (an "act" in Bayesian terminology), then, for all s, $f(s)$ is a probability measure; let us denote by $f(s, x)$ the value it gives to element x. Writers in SEU theory usually identify Y with the unit simplex of R^M, and then rewrite the set of acts $L = Y^S$ as a set of vectors in the following way: $L = \{f \in (R_+)^{MT} | \Sigma_X f(s, x) = 1, \forall s \in S\}$.

The AA framework takes the agent's alternatives of choice to be all simple probability measures on L, which we may denote by $(\lambda_1, f_1; \ldots; \lambda_k, f_k)$, where $f_1, \ldots, f_k \in L$, and $\lambda_i \geq 0$, $\Sigma \lambda_i = 1$. A further identification step is possible, but it raises a conceptual problem. We might identify the probability measure just introduced with the following, definitionally distinct mathematical object: $\lambda_1 f_1 + \cdots + \lambda_k f_k \in L$. This identification of the set of alternatives with L is formally equivalent to assuming AA's "Reversal of Order" condition, and Drèze (1987, Chapter 2) has argued against this condition because it excludes the possibility that the agent could influence the realization of states of nature (even marginally). We are following here the standard practice of assuming "Reversal of Order," while taking note of Drèze's critical point.

The agent's weak preference relation on L (i.e., his actual preference relation) will be denoted by $\geq$, with $>$ and $\approx$ standing for the induced strict preference and indifference relations. As usual, the agent's conditional preference relation $\geq_s$ on L is defined by the following: for all $s \in S$ and all $f, g \in L$, $f \geq_s g$ iff $f^* \geq g^*$ for all $f^*, g^* \in L$, such that $f^*(s) = f(s)$, $g^*(s) = g(s)$, and $f^* = g^*$ outside s (i.e., $f^*(t) = g^*(t)$, $\forall t \neq s$). A state s is said to be null if $f \geq_s g$ for all $f, g \in L$; otherwise, it is nonnull. To simplify the exposition, we will assume throughout that all states are nonnull. Formally:

Axiom (A0). $\forall s \in S$, $\exists f_s, g_s \in L$ *s.t.* $f_s >_s g_s$.

Now, we explain what the KSV construction specifically adds to the AA framework. Fix a probability p' on S which has full support (i.e., $p'(s) > 0$ for all s); p' is called an *auxiliary* (or *hypothetical*) *probability*. We define a set of *hypothetical objects of choice* as follows:

$$L_{p'} = \left\{ f' \in \Delta(S \times X) \,\middle|\, \sum_X f'(s, x) = p'(s), \ \forall s \in S \right\},$$

where, as usual, $\Delta(\cdot)$ denotes the set of all probabilities on a given set. In words, hypothetical objects are those state-outcome lotteries which have fixed marginal probability p' on S. They can also be written as vectors in R^{MT}:

$$L_{p'} = \left\{ f' \in (R_+)^{MT} \,\middle|\, \sum_X f'(s, x) = p'(s), \ \forall s \in S \right\},$$

a formula that must be compared with the already given formula for acts, i.e.,

$$L = \left\{ f \in (R_+)^{MT} \,\middle|\, \sum_X f(s, x) = 1, \ \forall s \in S \right\}.$$

The two formulas capture the difference between a lottery-valued function on states like f, and a state-outcome lottery, like f'. Each hypothetical object f' can be paired in a one-to-one way with a suitable act f by the mapping

$$H_{p'}: L_{p'} \to L, \quad H_{p'}(f') = f, \quad f(s, x) = f'(s, x)/p'(s).$$

In words, $H_{p'}$ "strips out" the state probabilities $p'(s)$ from the hypothetical object of choice f', thus turning

[4] Technically, what is needed is only that consequences belong to a mixture set. For an algebraic investigation of the mixture set assumption, see Mongin (1998a).

it into an act. Finally, define $\geq^{p'}$ to be the agent's preference relation on $L_{p'}$. For each s, $(\geq^{p'})_s$ denotes this preference relation conditional on s.

The following example (adapted from Karni 1985) illustrates the construction. There are two states, s_1 = "it is raining" and s_2 = "the sun is shining," and two final outcomes both available in each state, x_1 = "watch the football game at home (on television)" and x_2 = "watch the football game in the stadium." Given these descriptions of states and outcomes, the act f "to stay home" is defined as:

$$f(s_1, x_1) = 1, \quad f(s_1, x_2) = 0,$$
$$f(s_2, x_1) = 1 \quad \text{and} \quad f(s_2, x_2) = 0,$$

and the act g "to go to the stadium" as:

$$g(s_1, x_1) = 0, \quad g(s_1, x_2) = 1,$$
$$g(s_2, x_1) = 0 \quad \text{and} \quad g(s_2, x_2) = 1.$$

The AA framework assumes that the decision maker is also able to contemplate acts leading to randomized consequences, e.g., the following act h:

$$h(s_1, x_1) = 1/2\,, \quad h(s_1, x_2) = 1/2\,,$$
$$h(s_2, x_1) = 1/2\,, \quad h(s_2, x_2) = 1/2\,.$$

which corresponds to flipping a coin to decide between home and the stadium, regardless of the state. The KSV approach assumes that the decision maker can make even further preference comparisons. Supposedly, he is able to contemplate state-outcome lotteries with fixed marginal probabilities on states. What values these marginal probabilities take is decided once and for all by the observer. For example, suppose that the observer decides to fix the probability of rain to be $p'(s_1) = \frac{1}{3}$. Then, the decision maker can compare the state-outcome lotteries f' and g' defined as follows:

$$f'(s_1, x_1) = 1/3\,, \quad f'(s_1, x_2) = 0,$$
$$f'(s_2, x_1) = 2/3\,, \quad f'(s_2, x_2) = 0, \quad \text{and}$$
$$g'(s_1, x_1) = 0, \quad g'(s_1, x_2) = 1/3\,,$$
$$g'(s_2, x_1) = 0, \quad g'(s_2, x_2) = 2/3\,,$$

since they satisfy the marginal probability condition that $p'(s_1) = 1/3$.

KSV interpret the elements f', g' as being *acts* of a special sort, that is, "acts contingent upon the hypothetical probability" p' (1983, p. 1024). Thus, following KSV, to express preferences between f' and g' means expressing preferences between staying home and going to the stadium *if the probability of rain hypothetically were* 1/3. We prefer to follow Karni's (1985) exposition in terms of extended lotteries, that is, lotteries on the Cartesian product $S \times X$. In this alternative interpretation, to express preferences between f' and g' means expressing preferences between a lottery ticket offering "rain and home" with probability 1/3 and "sun and home" with probability 2/3, and another lottery ticket offering "rain and stadium" with probability 1/3 and "sun and stadium" with probability 2/3. The latter interpretation has the advantage of not making explicit reference to counterfactual conditionals relative to state probabilities. Of course, a definite value for the hypothetical probability distribution is mathematically contained in the datum of any state-outcome lottery. But by presenting the objects of choice in this way rather than the other, the Bayesian observer avoids mentioning the problematic notion of a probability distribution on states which is unrelated to the agent's own subjective probability. We see this as a definite advantage of the state-outcome interpretation. We pursue the semantic point here at some length in §4.2, where we also discuss a methodological objection raised by Drèze (1987) against using hypothetical preference data.

We also need to introduce the set of all state-outcome lotteries, or rather (for convenience) the following slightly less inclusive set:

$$L' = \{f' \in \Delta(S \times X) \mid \forall\, s \in S,\ \exists\, x \in X: f'(s, x) > 0\}.$$

That is, we exclude from consideration those state-outcome lotteries which give zero probability to some s. We write L' as the following subset of R^{MT}:

$$L' = \left\{ f' \in (R_+)^{MT} \,\middle|\, \sum_S \sum_X f'(s, x) = 1 \quad \text{and} \quad \sum_X f'(s, x) > 0,\ \forall\, s \in S \right\}.$$

The symbol $\succeq'$ will denote the agent's preference relation on L'.

We also want to introduce a notion of conditional preference $\succeq'_s$. It is not clear how to compare, conditionally on s, two state-outcome lotteries f' and g' having different marginal probabilities on s. So we define $\succeq'_s$ as follows. For all $s \in S$ and all $f', g' \in L$ having the same marginal on s, $f' \succeq'_s g'$ iff $f'^* \succeq g'^*$ for all $f'^*, g'^* \in L'$, such that $f'^*(s) = f'(s)$, $g'^*(s) = g'(s)$, and $f'^* = g'^*$ outside s (i.e., $f'^*(t, x) = g'^*(t, x)$ $\forall x$, $\forall t \neq s$).[5]

For any fixed p' with $p'(s) > 0$ for all s, $L_{p'} \subset L'$, and it is possible to extend $H_{p'}$ to a mapping H from L' to L by putting $H(f') = f$, and $f(s, x) = f'(s, x)/\Sigma_{y \in X} f'(s, y)$. In words, the counterpart in L of the state-outcome lottery f' is that act f which associates with s the distribution conditional on s that is implied by f'. Obviously, H on L' loses the one-to-one property that $H_{p'}$ enjoyed on $L_{p'}$.

If p'' is another full support probability on S, we may want to compare $L_{p'}$ with

$$L_{p''} = \left\{ f'' \in (R_+)^{MT} \,\middle|\, \sum_X f''(s, x) = p''(s),\ \forall s \in S \right\}.$$

There is a mapping $H_{p''}$ analogous to $H_{p'}$. The two mappings can easily be related to each other by introducing the (obviously bijective) G: $L_{p'} \to L_{p''}$ defined as:

$$G(f') = f'' \quad \text{iff} \quad H_{p'}(f') = H_{p''}(f''),$$

or equivalently,

$$G(f') = f'', \quad f''(s, x) = f'(s, x)p''(s)/p'(s).$$

In words, we associate f' and f'' with each other whenever they are associated with the same act f after being "stripped out" of their respective probabilities p' and p''. In the sequel we will generally omit mentioning the mappings H, $H_{p'}$, or $H_{p''}$. We will simply say that "f' is the associate of f in L (respectively $L_{p'}$, $L_{p''}$)."

By themselves, the inclusions $L_{p'} \subset L'$ and $L_{p''} \subset L'$ do not imply that $\succeq^{p'}$ and $\succeq^{p''}$ are restrictions of $\succeq'$. The latter preference relation has been introduced here in order to facilitate comparison between preference relations $\succeq^{p'}$ and $\succeq^{p''}$, for particular choices of p' and p''. In order to make these comparisons we will need the following connecting axiom:

Axiom (A0'). *For any full support probability p' on S, $\succeq^{p'}$ is the restriction of $\succeq'$ to $L_{p'}$.*

[5] In §4 we will discuss preferences that are conditional not on states, but on *events*.

3. Uniqueness and Stability Properties of the KSV and KS Probabilities

3.1. Additive Separable Representations and the AA State-Independence Axiom

The sets L, L', and (for any fixed p') $L_{p'}$ are convex sets. So we can impose the familiar von Neumann-Morgenstern (VNM) axioms on the preference relations that have been defined on each of these sets. (Any version of the axioms might do; see, e.g., Fishburn 1970.)

Axiom (A1). *$\succeq$ on L satisfies the VNM axioms.*

The same VNM requirement will be imposed on $\succeq'$ and, for any fixed p', on $\succeq^{p'}$. We denote by (A1)' and (A1)$_{p'}$ the same postulate as (A1) with $\succeq'$ and $\succeq^{p'}$ instead of $\succeq$, respectively.

From a variant of the VNM theorem for Cartesian products, which can be found in Fishburn (1970, p. 176), we know that $\succeq$ can be represented quasi-uniquely by an additively separable utility function. Formally, there exist mixture-preserving (i.e., affine) functions $w(1,.), \ldots, w(T,.)$ on the set of consequences Y such that:

$$\forall f, g \in L, f \succeq g \quad \text{iff} \quad \sum_S w(s, f(s)) - w(s, g(s)) \geq 0. \quad (1)$$

Hence there exist real-valued functions $w(1,.), \ldots, w(T,.)$ on the set of prizes X such that:

$$\forall f, g \in L, f \succeq g \quad \text{iff} \quad \sum_S \sum_X w(s, x)(f(s, x) - g(s, x)) \geq 0. \quad (2)$$

Any other functions $w^*(1,.), \ldots, w^*(T,.)$ satisfying either (1) or (2) must also satisfy the condition that $(w^*(s,.))_{s\in S} = \alpha(w(s,.))_{s\in S} + \beta$ for some $\alpha > 0$ and some $\beta \in R^T$.

Each function $w(s,.)$ may be decomposed as: $w(s,.) = q(s)\ u(s,.)$, where u is a state-dependent utility function on X and $q(s) > 0$, so that (after being normalized) the vector q defines a probability distribution on S. However, this probability q would be conceptually meaningless since there are infinitely many such decompositions. In particular, for any choice of $a(s) > 0$, $s = 1, \ldots, T$, the alternative decomposition: $w(s,.) = r(s)v(s,.)$, with $r(s) = q(s)a(s)/\Sigma_{t\in S}\ q(t)a(t)$ and $v(s,.) = u(s,.)/a(s)$, will also preserve the additively separable representation (2). In words, the uniqueness property stated after (2) is not strong enough to determine an implicit subjective probability. This is the classic problem of state-dependent utility theory that was alluded to in the introduction.

In their seminal work, Anscombe and Aumann (1963) were able to go beyond representation (2) by assuming in effect the following *State-Independence* axiom:[6]

AXIOM (SI). *For any two nonnull states s, t, and all constant $f, g \in L$, $f \geq_s g$ if and only if $f \geq_t g$.*

This axiom involves the important mathematical consequence that each $w(s,.)$ can now be decomposed as: $w(s,.) = c(s)\ v(\cdot)$, where v is a state-independent function on X and $c(s) > 0$ for all s, so that the vector c can be used to define a probability p. Anscombe and Aumann claim that this p *is* the individual's subjective probability. However, the same argument as before applies, and there exist many other decompositions than Anscombe and Aumann's. Nothing in the added axiom itself indicates that a state-independent decomposition should be selected rather than any of the infinitely many state-dependent decompositions considered in the previous paragraph. It is just a *convention* to rewrite (2) by plugging $v(\cdot)$ into every term of the sum, instead of plugging one distinct $u(s,.)$ into each term. No doubt, because it is uniform across states, the convention is both convenient and natural. But again, it is not justified axiomatically.

The above remarks (in the style of Schervish et al. 1990 and Karni 1996) show in what limited sense Anscombe and Aumann's state-independence assumption can be said to resolve the problem of defining a meaningful subjective probability. We will pursue this critique in §4.

3.2. The KSV Construction

Here we assume that the decision maker can make many more preference comparisons than in AA's framework. Beside comparing acts in terms of $\geq$, he is now able to compare state-outcome lotteries in $L_{p'}$ in terms of $\geq^{p'}$ for any given (full support) p'. We will sometimes consider p' as being fixed and sometimes consider two different choices of p' at a time. Axiom $(A1)_{p'}$ is assumed to hold for all possible choices of p'.

For any such p', the usual VNM theorem applies, so that there exists a mixture-preserving function $u^{p'}$ on $L_{p'}$ with the property that:

$$\forall f', g' \in L_{p'}, f' \geq^{p'} g' \quad \text{iff} \quad u^{p'}(f') \geq u^{p'}(g'). \quad (3)$$

Hence there are utility functions $u^{p'}(1,.), \ldots, u^{p'}(T,.)$ on X such that:

$$\forall f', g' \in L_{p'}, f' \geq^{p'} g' \quad \text{iff}$$
$$\sum_{S} \sum_{X} u^{p'}(s, x)(f'(s, x) - g'(s, x)) \geq 0. \quad (4)$$

Any other u^* satisfying (3) or (4) must be such that $u^* = \gamma u^{p'} + \delta$ for some $\gamma > 0$ and some $\delta \in R^T$.

No axiomatic restriction has yet connected the preference relations $\geq$ and $\geq^{p'}$ with each other. This is where Karni et al. (1983, p. 1025) consistency axiom comes into play. The form of consistency that they impose between $\geq^{p'}$ and $\geq$ amounts to requiring that *conditional* preference judgments are preserved from one set of preferences to another. The agent's conditional preference among two state-outcome lotteries f' and g' should faithfully reflect his preference between the associated actual lotteries $f = H_{p'}(f')$ and $g = H_{p'}(g')$. Formally,

AXIOM $(A2)_{p'}$. *$\forall s \in S$, $\forall f', g' \in L_{p'}$, $f'(\geq^{p'})_s g'$ iff $f \geq_s g$.*

This axiom is called "weak consistency" in Karni

[6] Literally, their axiom is *Monotonicity in Prizes* and has a different but equivalent expression. We need not go into these details here.

(1985, p. 19). Its simple formulation here depends on the assumption (A0) that there are no null states. We now restate the KSV main result for this case. For a detailed analysis of the null states the reader is referred to the original paper.

PROPOSITION 1 (KSV THEOREM WITHOUT NULL STATES). *Suppose that the relation $\geq$ on L satisfies (A0) and (A1), the relation $\geq^{p'}$ on $L_{p'}$ satisfies $(A1)_{p'}$, and they jointly satisfy $(A2)_{p'}$. Then, there exist utility functions $u(1,.), \ldots, u(T,.)$ on X and a full support subjective probability p on S such that: for all $f, g \in L$,*

$$f \geq g \quad iff \quad \sum_S \sum_X p(s)u(s, x)(f(s, x)) - g(s, x)) \geq 0 \qquad (*)$$

and

$$f' \geq^{p'} g' \quad iff \quad \sum_S \sum_X u(s, x)(f'(s, x) - g'(s, x)) \geq 0, \qquad (**)$$

where f′ and g′ are the state-outcome lotteries associated with f and g, respectively. Any alternative representations $u^*(1,.), ,\ldots, u^*(T,.), p^*$ satisfying the two conditions (*) and (**) must be such that: $(u^*(s,.))_{s\in S} = a(u(s,.))_{s\in S} + b$ for some $a > 0$ and $b \in R^T$, and $p^* = p$.

PROOF. See Karni et al. (1983).

The simple idea underlying the KSV procedure can be stated as follows. The *actual* subjective probability p (as opposed to the *hypothetical* one p') is obtained from the equation:

$$p(s) = c(s) / \sum_{t\in S} c(t),$$

where $c(s)$ is the uniquely determined positive number given by:

$$w(s, x) = c(s)u^{p'}(s, x) + d(s).$$

To show why for each s, there exists such a number $c(s)$, compare (2) and (4), and make use of the consistency Axiom $(A2)_{p'}$. Then, $w(s,.)$ and $u^{p'}(s,.)$ are seen to be equivalent representations of one and the same VNM preference.

In the equation $w(s, x) = c(s)\, u^{p'}(s, x) + d(s)$, the left-hand side does not depend on p', while the right-hand side does. This suggests that p is not invariant with respect to the chosen parameter p'. We now establish this point formally.

Take another full support probability $p'' \neq p'$, and suppose that we have derived a subjective probability q from (A0), (A1), $(A1)_{p''}$, and $(A2)_{p''}$. Hence, for each s,

$$q(s) = k(s) / \sum_{t\in S} k(t),$$

where $k(s)$ is a positive number given by:

$$w(s,.) = k(s)u^{p''}(s,.) + l(s).$$

In view of (A0) a necessary and sufficient condition to conclude that $p = q$ is that:

$$c(s)/k(s) = c(t)/k(t) \quad \text{for all} \quad s, t \in S. \qquad (+)$$

We may renormalize the $u^{p'}(s,.)$ and $u^{p''}(s,.)$ so that $d(s) = l(s) = 0$ for all s. The equations connecting the renormalized representations with each other are:

$$\forall s \in S,\ \forall x \in X,\ u^{p'}(s, x) = (k(s)/c(s))u^{p''}(s, x). \qquad (++)$$

Using (+), we conclude that it is necessary and sufficient for $p = q$ that (++) holds with multiplicative coefficients $k(s)/c(s)$ that are independent of the particular s. Let us record this condition formally:

LEMMA. *Given the KSV axioms when there are no null states, and two distinct hypothetical probabilities p' and p'', the derived subjective probabilities p and q are equal if and only if there are $\lambda > 0$ and $\mu \in R^T$ such that:*

$$((u^{p''}(s,.))_{s\in S} = \lambda(u^{p'}(s,.))_{s\in S} + \mu.$$

Using this lemma, we can now formally state the major drawback of the KSV axiomatization.

PROPOSITION 2. *The KSV system does not imply that for $p'' \neq p'$, the derived p and q are equal.*

In order to go from the lemma to this observation, it is enough to show that the KSV axioms can be satisfied with $u^{p'}(s,.))_{s\in S}$ and $((u^{p''}(s,.))_{s\in S}$ being related to each other by positive affine transformations (PAT) involving *state-dependent* multiplicative coefficients. This is done in the example below.

A further word of motivation might be in order before we state this key example. Let us say that the preference $\geq^{p'}$ and the actual preference $\geq$ *agree with* *each other* if for all $f, g \in L$, $f' \geq^{p'} g'$ if and only $f \geq g$, where, as usual, f' and g' are the associates of f, g in $L_{p'}$. (The notion of agreement is taken up in §3.4.) Now, suppose that for two distinct p' and p'', $\geq$ and $\geq^{p'}$ agree with each other, *and similarly*, $\geq$ and $\geq^{p''}$ agree with each other. Loosely speaking (since agents are generally not aware of their subjective probabilities), it is as if the agent would say when presented with p': "That's my own probability distribution," and again, when presented with p'': "That's my own probability distribution." Such a situation would strongly suggest that the subjective probability distributions on S that are obtained on the basis of p' and p'' are *distinct*. The example is a particularization of this scenario.

EXAMPLE 1. Take $S = \{1, 2\}$, any finite X, and define:

$$u^{p'}(1, x) = x, \quad u^{p'}(2, x) = x^{1/2},$$

$$u^{p''}(1, x) = 3x/2, \quad u^{p''}(2, x) = 3x^{1/2}/4.$$

Set $p = (1/2, 1/2)$ and $q = (1/3, 2/3)$. Define $\geq$ as follows: $\forall f, g \in L, f \geq g$ iff $1/2 \sum_X x(f(1, x) - g(1, x)) + 1/2 \sum_X x^{1/2}(f(2, x) - g(2, x)) \geq 0$, (equivalently: iff $1/3 \sum_X 3x/2(f(1, x) - g(1, x)) + 2/3 \sum_X 3x^{1/2}/4(f(2, x) - g(2, x)) \geq 0$). Take $p' = p$ and $p'' = q$, and define $\geq^{p'}$ and $\geq^{p''}$ as follows: $\forall f', g' \in L_{p'}$, $f' \geq^{p'} g'$ iff $\sum_X x(f'(1, x) - g'(1, x)) + \sum_X x^{1/2}(f'(2, x) - g'(2, x)) \geq 0$, and $\forall f'', g'' \in L_{p''}, f'' \geq^{p''} g''$ iff $\sum_X 3x/2(f''(1, x) - g''(1, x)) + \sum_X 3x^{1/2}/4(f''(2, x) - g''(2, x)) \geq 0$, where f', g', f'', g'' are defined in terms of the chosen p' and p''. Clearly, the relations $\geq$, $\geq^{p'}$ and $\geq^{p''}$ satisfy all the KSV axioms, yet the derived actual probabilities p and q are distinct.

If the elements of X have the interpretation of wealth levels, Example 1 also illustrates the following important point. The KSV axioms certainly ensure that the *risk-attitudes* measured by $u^{p'}$ and $u^{p''}$ are the same in each state (take for instance the Arrow-Pratt index of absolute risk-aversion). This much is guaranteed by the conjunction of $(A1)_{p'}$ and $(A1)_{p''}$. However, more than equality of risk attitudes would be needed for the conclusion that $p = q$, as is shown by the lemma.

It is worth noting that the preference comparisons allowed in this section can be relative to an arbitrary state-dependent lottery f' in L'. However, f' can be compared only with elements f'' in the *same* $L_{p'}$. This restriction is essential to the KSV approach. It is removed in the more demanding construction that we are to consider now.

3.3. The KS Construction

The construction we investigate here was first made in an unpublished paper by Karni and Schmeidler (1981) and discussed in print by Karni (1985, Chapter 1).[7] This variant is based on the auxiliary preference relation $\geq'$ on L'. On top of the VNM assumption: $(A1)'$. The relation $\geq'$ satisfies the following axiom of consistency with the preference relation $\geq$.

AXIOM (A2). *$\forall s \in S$, $\forall f', g' \in L'$ s.t. f' and g' have same marginal on s, $f' \geq'_s g'$ iff $f \geq_s g$, where $f = H(f')$ and $g = H(g')$.*

This is labelled "strong consistency" in Karni (1985, p. 16). Here is the KS theorem for the case when (A0) holds:

PROPOSITION 3 (KS THEOREM WITHOUT NULL STATES). *Suppose that the relation $\geq$ on L satisfies (A0) and (A1), the relation $\geq'$ on L' satisfies $(A1)'$, and both jointly satisfy (A2). There exist utility functions $u(1,.), \ldots, u(T,.)$ on X and a full support subjective probability p on S such that for all $f, g \in L$, and all state-outcome lotteries f', g' in L':*

$$f \geq g \quad iff \quad \sum_S \sum_X p(s)u(s, x)(f(s, x)) - g(s, x)) \geq 0 \tag{\#}$$

and

$$f' \geq' g' \quad iff \quad \sum_S \sum_X u(s, x)(f'(s, x) - g'(s, x)) \geq 0. \tag{\#\#}$$

Any alternative representations $u^*(1,.), \ldots, u^*(T,.)$, p^* satisfying this property must be such that $(u^*(s,.))_{s \in S} = a(u(s,.))_{s \in S} + b$ for some $a > 0$ and $b \in R^T$, and $p^* = p$.

PROOF. See Karni (1985, pp. 24–26). □

The statement is essentially different from that of

[7] Notice also that Schervish et al. (1990, p. 846) choose to follow the KS rather than the KSV procedure. See also their collective choice application of the procedure in Schervish et al. (1991).

Proposition 1. It does not refer to any auxiliary probability p', and as a result is not open to the nonuniqueness objection unfolded in §3.2. The added value of moving to the KS variant is further stressed in the following restatement, which we introduce for the sake of comparison with Proposition 2:

PROPOSITION 3 (VARIANT). *Suppose that the relation $\geq$ on L satisfies (A0) and (A1), the relation $\geq'$ on L' satisfies (A1)', and both jointly satisfy (A2). Suppose also that (A0') holds. Then, there exist utility functions $u(1,.), \ldots, u(T,.)$ on X and a full support subjective probability p on S such that for all (full-support) probabilities p' on S the following holds: for all $f, g \in L$,*

$$f \geq g \quad \text{iff} \quad \sum_{S} \sum_{X} p(s)u(s, x)(f(s, x)) - g(s, x)) \geq 0 \tag{*}$$

and

$$f' \geq^{p'} g' \quad \text{iff} \quad \sum_{S} \sum_{X} u(s, x)(f'(s, x) - g'(s, x)) \geq 0, \tag{**}$$

where f' and g' are the state-outcome lotteries associated with f and g, respectively. Any alternative representations $u^*(1,.), \ldots, u^*(T,.)$, p^* satisfying the previous property must be such that $(u^*(s,.))_{s\in S} = a(u(s,.))_{s\in S} + b$ for some $a > 0$ and $b \in R^T$, and $p^* = p$.

We leave it for the reader to check that the conclusion in Proposition 3 implies the conclusion in the variant, once axiom (A0')—which connects the preference relation $\geq'$ with the relations $\geq^{p'}$—is added to the assumptions. The variant spells out the sense in which the KS system implies that different hypothetical probabilities lead to one and the same subjective probability distribution.

Why does the KS construction succeed in solving the problem on which the KSV construction collapses? Assuming (A0'), the KS assumptions (i.e., (A1), (A1)' and (A2)) become logically comparable with, and indeed stronger than, the KSV assumptions (i.e., (A1), and for all p', $(A1)_{p'}$ and $(A2)_{p'}$). Then, a direct proof of the variant from the lemma becomes available. The KS axioms imply that $u^{p'}$ and $u^{p''}$ in the Lemma coincide with each other up to a common state-independent PAT.

Axiom (A1)' is stronger than (for all p', $(A1)_{p'}$) since it makes it possible to compare state-outcome lotteries with different marginals. As to (A2), it can be seen to be equivalent to (for all p', $(A2)_{p'}$). So it is (A1)', not (A2) which is responsible for the improved uniqueness conclusion stated in the variant. As suggested by the last paragraph, the crucial element is to use one and the same utility function to represent *both* the rankings $\geq^{p'}$ and $\geq^{p''}$. The role of (A2) is only to ensure that for any two p' and p'', the consistency conditions $(A2)_{p'}$ and $(A2)_{p''}$ hold at the same time. By comparing the KS and KSV constructions one should not stress the difference between two kinds of consistency, but the distinctive ways of allowing for hypothetical preference comparisons.

To further clarify the role of (A1)' note the following. In the statement of both Proposition 3 and the variant, it is possible to replace (A2) with $(A2)_{p'}$ for one arbitrarily chosen p'.[8] In other words, the added strength of (A1)' with respect to (for all p', $(A1)_{p'}$) makes it possible to weaken the other assumption, i.e., for all p', $(A2)_{p'}$.

3.4. Stability

The comparison between the two procedures can be approached from a slightly different angle. Given either the KSV or KS procedure, let us say that a solution p for the actual subjective probability (as defined by either Proposition 1 or the variant of Proposition 3) is *stable* if the following holds: p is obtained as a solution when it is taken to be the auxiliary probability, i.e., whenever $p = p'$. Even more obviously than uniqueness, stability seems to embody a requisite of internal consistency. If the observer has derived p by following a certain procedure, then he will regard p as the agent's "true" probability. Hence, he should expect to derive p again when the decision maker is faced with lotteries made conditional on this very probability. Were this expectation not fulfilled, one would have all reason to declare the procedure followed by the observer to be nonsensical. Logically, stability is a *weaker* notion than uniqueness: If p does not depend on the auxiliary p',

[8] This follows from inspecting the proof of Proposition 3 in Karni (1985, p. 24–26).

it follows that p is stable, but the converse need not hold.

To formally define stability in the KSV case, we assume (A0), (A1), and for all p', both $(A1)_{p'}$ and $(A2)_{p'}$. To define it in the KS case we need (A0), (A0′), (A1), (A1)′, and (A2). Now, given the appropriate set of axioms, we say that a solution p in either Proposition 1 or the variant of Proposition 3 is *stable* if the two preference relations $\geq^p$ and $\geq$ agree with each other (in the sense formally introduced before Example 1). The KS procedure automatically satisfies stability; this follows at once from the uniqueness property stated in the variant of Proposition 3. The fact that the KSV construction admits of multiple stable solutions is demonstrated by Example 1. In the next example we establish that the KSV construction can also lead to nonstable solutions.

EXAMPLE 2. Take $S = \{1, 2\}$, any finite X, and put $p = (1/2, 1/2)$ and $q = (1/3, 2/3)$, $p' = (9/10, 1/10)$. We will construct preference relations $\geq$, $\geq^{p'}$ and $\geq^p$ such that p is a KSV solution for the auxiliary probability p', but q is the KSV solution when p is taken to be the auxiliary probability. Hence p will not be a stable solution. Define $\geq$ as follows:

$$\forall f, g \in L, f \geq g \quad \text{iff}$$
$$1/2 \sum_X x(f(1, x) - g(1, x)) + 1/2 \sum_X x^{1/2}(f(2, x) - g(2, x)) \geq 0, \qquad (*)$$

or equivalently:

$$\forall f, g \in L, f \geq g \quad \text{iff}$$
$$1/3 \sum_X 3x/2(f(1, x) - g(1, x)) + 2/3 \sum_X 3x^{1/2}/4(f(2, x) - g(2, x)) \geq 0. \qquad (*')$$

Define $\geq^{p'}$ and $\geq^p$ as follows:

$$\forall f', g' \in L_{p'}, f' \geq^{p'} g' \quad \text{iff}$$
$$9/10 \sum_X x(f(1, x) - g(1, x)) + 1/10 \sum_X x^{1/2}(f(2, x) - g(2, x)) \geq 0; \qquad (**)$$

and finally:

$$\forall f'', g'' \in L_p, f'' \geq^p g'' \quad \text{iff}$$
$$\sum_X x(f(1, x) - g(1, x)) + \sum_X x^{1/2}/2(f(2, x) - g(2, x)) \geq 0. \qquad (**')$$

Thus, $u^{p'}(1, x) = x$, $u^{p'}(2, x) = x^{1/2}$, and $u^p(1, x) = x$, $u^p(2, x) = x^{1/2}/2$. The relations $\geq$, $\geq^{p'}$ and $\geq^p$ satisfy all the KSV axioms, while the properties (*)–(**) and (*′)–(**′) imply that both p and q are solutions.

Example 1 had uncovered the nonuniqueness of the KSV solution. Example 2 uncovers another, in some sense even more severe drawback. It might have been conceivable that the multiple KSV solutions were all stable. This is not the case.

4. Two Kinds of Subjective Probabilities

4.1. Event-dependent Preferences

Intermediate between state-dependent preferences and state-independent preferences is the case of *event-dependent* preferences. For instance, an agent may have one attitude towards risk if healthy and another if sick, while given his state of health, his risk attitude does not depend on whether or not the national team will win the Mondial 1998. To model this and similar situations the state space can be partitioned in such a way that the decision maker's preferences are state-independent (i.e., satisfy Anscombe and Aumann's state-independence axiom) *within* each event in the chosen partition, but are not state-independent for states belonging to distinct such events.[9] Admittedly, in a single-agent context it is often possible to deal with this case by suitably redefining the state space. In the example just considered we can analyze the problem of choosing an optimal level of health insurance in terms of the simplified set of states {Healthy, Sick} and a pure state-dependent model. However, a more thorough understanding of event-dependent preferences

[9] Event-dependent preferences have been discussed in particular by Karni (1992), Karni and Schmeidler (1993) and, for the KSV context, Mongin (1998b).

is crucial to the multi-agent context of decision. The agents' different sets of risk attitudes across states often imply partitions that differ from one to the other, for instance when one agent's injury affects his risk attitude, but not that of the others.

Formally, we consider a preference relation $\succeq$ on L satisfying "Reversal of Order," (A0) and (A1), and partition S into maximal cells $S_1, \ldots, S_N$ such that for each cell, the axiom (SI) of 3.1 holds, i.e., such that for all $i = 1, \ldots, N$, all $s, t \in S_i$, and all constant $f, g \in L$, $f \geq_s g$ if and only if $f \geq_t g$. This construction encompasses the state-independent case (when the partition is trivial), the pure state-dependent case (when the S_i are the singletons $\{s\}$), as well as any intermediary case between these two. (If there were null states, they would be lumped together into a separate cell, but (A0) applies here as elsewhere in the paper.)

Now, for each nonempty cell S_i, we introduce the preference conditional on S_i, to be denoted by $\geq_i$. (It is formally defined in the same way as the preference conditional on states, i.e., by fixing values for the states outside S_i and redefining acts accordingly.) From (A1), $\geq_i$ satisfies the VNM axioms. Moreover, $\geq_i$ is nontrivial because of (A0) and by construction, it satisfies the axiom (SI) across S. (To apply this axiom, notice that states outside S_i count as null states for $\geq_i$.) Hence, we can apply the Anscombe-Aumann theorem, and conclude that $\geq_i$ has a state-independent Subjective Expected Utility representation with respect to some probability π_i on S. Necessarily, $\pi_i(s) = 0$ for $s \notin S_i$. Also, assuming the relevant axioms, we apply either the KSV or the KS construction to obtain a (full-support) subjective probability p on S. Denote by $p_i(s)$ the conditional probability $p(s)/p(S_i)$, for $i = 1, \ldots, N$.

Is it necessarily the case that $p_i(\cdot) = \pi_i(\cdot)$? The answer is negative under the KSV version, and more importantly, also under the KS version. By assuming (A0′) we make the latter logically stronger than the former, so that the following example will apply to both.

EXAMPLE 3. Let $S = \{1, 2, 3\}$. Define $\geq$ as follows:

$$\forall f, g \in L, f \geq g \quad \text{iff}$$

$$1/3 \sum_X x(f(1, x) - g(1, x))$$
$$+ 1/6 \sum_X 2x(f(2, x) - g(2, x)))$$
$$+ 1/2 \sum_X x^{1/2}(f(3, x) - g(3, x))) \geq 0 \qquad (\#)$$

so that (A0) and (A1) hold. Define $\geq'$ on L' as:

$$\forall f', g' \in L', f' \geq' g' \quad \text{iff}$$
$$\sum_X x(f'(1, x) - g'(1, x))$$
$$+ \sum_X 2x(f'(2, x) - g'(2, x)))$$
$$+ \sum_X x^{1/2}(f(3, x) - g(3, x))) \geq 0. \qquad (\#\#)$$

That is, (A1)′ holds with a v representation given by: $v(1, x) = x$, $v(2, x) = 2x$, $v(3, x) = x^{1/2}$. Also, (A2) holds. In view of (#) the above formalism of event-dependent preferences applies with $S_0 = \varnothing$, $S_1 = \{1, 2\}$, $S_2 = \{3\}$, and $\geq_1$ is the preference relation on L_1 given by: $f \geq_1 g$ iff $1/2 \sum_X x(f(1, x) - g(1, x)) + 1/2 \sum_X x(f(2, x) - g(2, x))) \geq 0$, so that $\pi_1(1) = \pi_1(2) = 1/2$. Comparing (#) and (##), we see that the probability $p = (1/3, 1/6, 1/2)$ is the KS solution, and that $p_1 \neq \pi_1$.

In this example, the conditional derived probability p_1 does not coincide with the state-dependent probability π_1 on S_1 because p_1 is determined by utility functions $u(1,.)$ and $u(2,.)$ that do not satisfy the convention of state-independent representations; indeed, they are *distinct* utility functions. Under the KS system the ratios $u(1,.)/u(2,.)$ are uniquely determined and can have any numerical value. There is no reason to expect this ratio to be 1, and it is not permissible to renormalize the functions in order to make them equal to each other.

Given this discrepancy between two kinds of probabilities, which one should the observer select to represent the decision maker's beliefs? We now turn to this issue.

4.2. KS versus AA Probabilities

In any situation where both the AA and KS axiomatizations of subjective probability can be applied, so that two kinds of probabilities can be made available, one should select the KS probabilities rather than the state-independent AA proba-

bilities. This is a very strong claim to make, and before embarking upon a defence, we would like single out one of its implications which definitely runs counter to the received wisdom in decision theory: *When complete state-independence prevails, the KS axiomatization of subjective probability should replace the standard AA axiomatization of subjective probability.*

This consequence follows from the claim because a situation to which Anscombe and Aumann's state-independence axiom applies can also be analyzed by *adding* the relevant KS axioms to the AA axioms. Complete state-independence is but a particular case of *event*-dependence, with the events reduced to singletons.

We first defend our claim by arguing that the KS approach is more informative than the AA one. The former makes it possible to determine u, and thus p, entirely from the preference axioms. As we emphasized at the outset of this paper, the latter approach does not determine u and p completely from its stated axioms. A further, purely conventional step is needed in order to select the state-independent representations among all those compatible with the axioms. Example 3 can be used to highlight the difference in motivation between the two approaches. AA would have us select identical probabilities for states 1 and 2 after making an arbitrary normalization of the utility functions in states 1 and 2. By contrast, the KS axioms provide a *reason* for the conclusion that the probability of state 2 is one half the probability of state 1. This reason lies with the agent's preference judgments, as constrained by the stated axioms.

Note carefully that we are not claiming that the KS axiomatization should replace Anscombe and Aumann's in all and every circumstances. When the aim is to investigate the properties of the decision maker's utility, and in particular his risk attitude, rather than his subjective probability, there is a convenience reason for selecting the state-independent normalization. For this sort of applications (and there are many of them in information economics) the added precision of the KS approach would clearly be irrelevant. As we understand it, the superiority of the KS approach is limited to the axiomatization of *subjective probability*. As an axiomatization of *subjective expected utility*, the AA system is still promised to a long life.

Notice also that the claim defended here for the KS system cannot be consistently made for the KSV one. Because the latter fails to determine the agent's subjective probability independently of the auxiliary probability p', it involves an element of arbitrariness comparable with the arbitrary normalization of AA theory. Actually, the choice of p' *is* an arbitrary normalization: u is normalized so as to be consistent with p' when the two are combined into in an expected utility representation of the auxiliary preference. In case of event-dependent preferences, the KSV probabilities fare worse than the AA probabilities: The arbitrary normalization implied by the latter has a simplicity advantage which the former evidently lack.

An arbitrary convention is better than a bad reason, and it would indeed be a worse evil if the added preference axioms were absurd or indefensible. So we are led, as a more crucial step in the argument, to defend the two axioms that are specific to the KS approach, i.e., (A1)′ and (A2). Our aim here is limited to showing that Axiom (A1)′, which was said to be the crucial one of the two, *delivers significant preference information*. We will not discuss the VNM axioms themselves, but only the particular kind of objects they are applied to in Axiom (A1′).

Axiom (A1′) refers to a possible experiment of verbal elicitation of preferences and can be discussed in terms of the stylized form this experiment would take. As we already suggested, there are two interpretations for hypothetical objects, and the questionnaire to be put to the subject will differ according to which of the two is selected. In terms of *hypothetical acts*, the questionnaire will roughly go as follows:

(i) "Which one would you prefer of the act of staying home or the act of going to the stadium if the probabilities of rain and sunshine were 1/3 and 2/3 ?"

And in terms of the *state-outcome lottery* interpretation:

(ii) "Which one would you prefer of the lottery stipulating that, with probability 1/3, you will watch the game at home and it is raining, and with probability 2/3, you will watch the game at home and the sun is shining, or the lottery stipulating that. . . ."

We reiterate the point that despite its more roundabout formulation, (ii) is a clearer statement than (i). Not only does it eschew the counterfactual "if," but it can be made concrete by resorting to the existing technology of experimental choice between VNM lotteries. Once reinterpreted in terms of state-outcome lotteries, the KS stylized experiment scheme and today's routine experimental work have much in common. They differ only because of the *stakes* in the lotteries to be presented to the subjects. Those used to respond to questions about VNM lotteries with monetary stakes will perhaps not find unanswerable a question about VNM lotteries involving just a more roundabout arrangement of stakes. The verbal elicitation of preferences in the KS way is much more complex, but does not seem to be of a different nature from, say, the verbal elicitation of risk attitudes in elementary decision theory.

There remains a classic objection in our way. Drèze has objected against the idea of a KSV (or for that matter, KS) stylized experiment that "verbal answers to these questions do not lend themselves to verification through material behaviour" (1987, p. 69). This objection is directed against the hypothetical choice approach as a whole. It connects with the tradition of postwar economics which—after Machlup (1963) and Friedman (1953) in particular—has constantly been dismissive of questionnaire information. This formidable attack must be faced squarely.

4.3. Is Verbal Evidence Relevant?

Let us first clear the ground from a possible ambiguity. The "verbal" versus "material behaviour" distinction relates to two very distinctive considerations. The first one is the observer's way of becoming informed of the agent's behaviour. Accordingly, the agent's behaviour will be said to be verbal-1 if it consists of answers to questions put by the observer, and material if otherwise. All experimental behaviour is verbal in this sense. If in an experiment, a subject points at the lottery he prefers instead of saying "I prefer this lottery," this should count as verbal behaviour since it is a way of answering a question from the observer. On the other side of the line, the agent's purchases and sales on a market, or his contracts with other agents, should count as material behaviour. That these economic actions often involve verbal utterances on the agent's part plays no role here. What underlies the distinction is, to repeat, the observer's mode of inquiry.

The second consideration is the significance—as construed by the observer—of the agent's behaviour for the agent himself. Taking into account the economists' longstanding convictions as to individual motivations, let us say that the subject's behavior is verbal-2 if it entails no significant financial consequences for him, and it is material if otherwise. This way of drawing the line is very different from the previous one. Take any experiment in decision theory: It is verbal-1, and might or might not be verbal-2, depending on what incentive scheme the observer has included in his experiment. Conceivably, it could be argued that the two senses of "verbal" coincide in actual practice although they do not in principle. But this is not a reasonable argument to make. We can take for granted that at least *some* of the incentive schemes devised by experimentators succeed in motivating the subjects. Hence, the two distinctions do not coincide in practice. This is not to say that the second demarcation line is easy to draw in concrete cases. It is shakier than the first because it involves a strong element of interpretation on the observer's part. Experimentators often disagree on the respective merits of practical reward schemes.

The old-style rebuttal of questionnaires, e.g., in Machlup (1963) and Friedman (1953) made no exceptions at all. We may interpret the defenders of this strong line as being insufficiently sophisticated. They did not pay sufficient attention to the two different ways of drawing the line between "verbal" and "material" behavior. They did not contemplate the logical possibility, which we have claimed has been made real by experimental economics, that the economist can study verbal-1 behaviour which is more than verbal-2. We cannot interpret Drèze as simply reiterating the Machlup-Friedman objection. As we read him, he is above all making a point against the specific use of questionnaire evidence in the KSV (or KS) procedure. This further critical point can easily be stated at the desired level of sophistication: Drèze is in effect complaining that a KS stylized experiment would be not only verbal-1, but also verbal-2. The underlying argument here is that contrary to the ordinary lotteries of experimental economics, state-outcome lotteries cannot be "played out" since they involve assigning arbitrary probabilities to states of nature. Then, there is no fine reward structure to motivate the subject in carefully answering that part of the KS questionnaire which specifically deals with hypothetical objects. Subjects who agree to take part in the experiment can be offered a fee, but no rewards contingent on their particular answers.

To fully appreciate this objection, let us pursue it a little beyond the purview of the present discussion. If the observer has doubts about the subject's answers about hypothetical objects, and just relies on a KS framework, he will not be able to separate this subject's probabilities and utilities from each other. Without embarking on a review of alternative frameworks of state-dependent utility, we may mention that Drèze's (1987) raises difficulties of its own. His theory connects state-dependence with "moral hazard," and attempts to infer (nonunique) subjective probabilities only from preferences over actual lotteries, but by assuming that the agent can influence the states of the world. It has been objected to Drèze (e.g., Karni 1992) that his "moral hazard" assumption is not met in many applications in which economists would like to attribute subjective probabilities to agents. Hence, to separate probabilities from utilities might turn out to be as difficult in Drèze as it is in Karni and Schmeidler (1981) although for different reasons. There is still another road, which is to give up the project of separating probabilities and utilities, and just employ any utility representation of the agent's preferences between actual acts. Less radical resolutions—leading to at least partial separation—have also been explored recently (e.g., Nau 1995).

Returning now to the main argument, the crucial problem we have singled out for the KS construction is that the subject's answers about hypothetical objects are not properly motivated. But does this imply that one should discard them? We think not. Our argument will take an *ad hominem* form: If this inference were drawn by the observer, he should consistently become suspicious of the subject's answers about *actual* acts. Doubts will spread from the questionable preference data to those which are apparently safe. This seems unavoidable because expressed preferences over actual acts are, on further reflection, not well motivated either. Depending on the set of states and the set of consequences, the number of required answers about actual acts can be very large indeed. Any comparison made between two actual acts can in principle be turned into a choice between real bets. But when there are many states and many consequences, there will be so many acts to consider that this possibility will mean little for actual practice. Besides, some of the acts to be evaluated will be far-fetched. Technically, this is because of the "Cartesian product" domain assumptions underlying most of SEU axiomatizations. Those who are truly concerned with the motivational force of actual payments should view the more standard parts of subjective expected utility with at least some dose of scepticism. If motivation is the crucial issue that these critics say, they should consistently conclude that there is a difference in degree, not in nature, between choices made between AA acts (choices over L in our framework), and choices made between hypothetical objects (choices over L').[10]

It will be instructive to locate the major figure of post-war Bayesianism, i.e., Savage (1954), in the methodological debate just sketched. Drèze (1987, p. 69) draws attention to the following passage: "Attempts to define the relative probability of a pair of events in terms of the answers people give to direct interrogation has justifiably met with antipathy from most statistical theorists . . . . If the state of mind in question is not capable of manifesting itself in some sort of extraverbal behavior, it is extraneous to our main interest . . ." (1972, pp. 27–28). But as the first sentence makes clear, Savage is worried by questions like: "Do you regard event A as more likely than event B?". His polemical target here is a specific position in subjective probability theory which, like Savage's, can be traced back to de Finetti's seminal paper (1937) and was taken up later by writers like Suppes or de Groot.[11]

[10] The present argument is somewhat related to an ad hominem objection that has often been raised against revealed preference theorists, i.e., that their empiricism is faked and they conjure up unpalatable possibilities; e.g., Sen (1973).

[11] It is quite clearly Suppes's (e.g., 1994). It also underlies de Groot's (1970) analysis of qualitative probability relations.

Briefly put, it says that qualitative probability relations are well-understood by subjects and that they constitute the relevant starting point, both behaviorally and axiomatically, for the analysis of subjective probability. Savage's passage says nothing against gathering data about preferences between hypothetical objects.

Here is another passage confirming that Savage's position is by no means as sanguine as what has sometimes been suggested:

> There is a mode of interrogation between what I called the behavioral and the direct. One can, namely, ask the person, not how he feels but what he would do in such and such situation. In so far as the theory of decision under development is regarded as an empirical one, the intermediate mode is a compromise between economy and rigor. But in the theory's more normative interpretation as a set of criteria of consistency for us to apply to our decisions, the intermediate mode is just the right one. (1972, p. 28)

It would appear as if the sort of questions implied by the KS system agreed with the notion of "an intermediary mode" in these lines. We reconstruct Savage's position in terms of the following abstract statement: a subject's answers, *provided they are expressed as choices among alternatives*, constitute admissible evidence regarding either the empirical or the normative value of the decision theory at hand. The conception of scientific evidence underlying this statement is itself "intermediate" rather than extreme. It appears to be compatible with various philosophical views of how preferences relate to choices. Most important for our argument, Savage is drawing the line between acceptable evidence not within two classes of choice statements (as Drèze and many economists do), but between choice statements and direct epistemic statements, such as "my probability of s is greater than my probability of t" or—a fortiori—"my probability of s is $1/3$."

5. Summary

This paper has argued that the KSV approach to subjective probability should be replaced by the lesser-known KS alternative. The latter is not as economical as the former, because it requires the decision maker to express preferences over a wider set of hypothetical objects. However, despite this informational shortcoming, the comparison between the uniqueness and stability properties of the two tilts the balance in favour of the KS approach. We have also argued—a novel claim in the Bayesian literature, it seems—that the techniques motivated by state-dependent preferences should also be applied to state-independent preferences. The KS axiomatization of subjective probability, if not of subjective expected utility, should be preferred to Anscombe and Aumann's. In defending this claim we have been led to touch on the crucial methodological issue of what kind of evidence is relevant to Bayesian decision theory.[12]

[12] Part of the research for this paper was done when the authors were visiting the Faculty of Management, Tel Aviv University (May 1997). The second author's visit was made possible by the European Network FMRX-CT96-0005. The authors would like to thank an anonymous referee and editor Robert Nau for their most careful reviews of the initial version. They have also benefitted from comments made by Jacques Drèze, Isaac Levi, David Schmeidler, and Karl Vind.

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