DEF 24, 59 – 69 (2001)

Decisions in Economics and Finance

Notes and Comments

A note on mixture sets in decision theory

Philippe Mongin

Laboratoire d'économétrie, Centre National de la Recherche Scientifique
et Ecole Polytechnique
e-mail: mongin@u-cergy.fr

Received: 5 April 2000 / Accepted: 3 September 2000

Mathematics Subject Classification (2000): 52A01

Journal of Economic Literature Classification: D80

1. Introduction

The algebra of mixture sets provides a mathematical framework in which the basic questions of von Neumann–Morgenstern (VNM) utility theory can be addressed. At a high level of abstraction, mixture sets (MS) and mixture-preserving (MP) functions formalize the decision-theoretic notions of "lottery sets" and "linear" utility functions, respectively. These tools were first introduced into decision theory by Herstein and Milnor (1953) in an attempt to clarify the algebra of expected utility of von Neumann and Morgenstern (1944), p. 26. They are still part of the modern treatment of the subject. The major example is Fishburn's authoritative text *The Foundations of Expected Utility* (1982), in which he states a mixture-set version of the VNM representation theorem before moving to less abstract versions of this result.

Despite this and other references, the properties of MS have hardly been studied. Herstein and Milnor (1953), p. 292, contented themselves with mentioning that "a convex set in a real vector space is easily seen to be a mixture

The author wishes to thank Peter Fishburn, Peter Hammond, Edi Karni, Jean-François Laslier, John Weymark and an anonymous referee for commenting on this note. He is grateful to Peter Wakker for drawing his attention to relevant references. He would like to thank CORE for hospitality and the European network FMRX-CT96-0055 for support when he was preparing the paper.

set". In the main, later writers in decision theory have shown little interest in any further algebraic investigation. Are convex sets the only examples of MS, and, if not, how should Herstein and Milnor's axioms be strengthened in order to characterize convex sets? This note is mostly concerned with these two questions. Section 2 makes it clear by means of counterexamples that MS have a much weaker structure than convex sets. Section 3 states the main result, namely, an isomorphism theorem which amounts to a MS axiomatization of convexity. The main advantage of this result is that it delivers the familiar modelling of "lotteries" as elements of a convex set, starting with primitive operations which are in some sense more natural than the vector space operations. As we discovered, our theorem is a variation on an earlier one proved by the mathematician Stone (1949); this and related references are discussed in Section 3. Finally, Section 4 elaborates on the applications of the algebraic approach that is presented here.

2. Mixture sets: definitions and examples

As Herstein and Milnor (1953) define it, a *mixture set* (MS) is any non-empty set $\mathcal{M}$ which is endowed with an external operation:

$$\begin{aligned} [0,1] \times \mathcal{M} \times \mathcal{M} &\rightarrow \mathcal{M} \\ (\lambda, x, y) &\rightarrow x\lambda y \end{aligned}$$

such that:

(A1) $x1y = x$;
(A2) $x\lambda y = y(1-\lambda)x$;
(A3) $(x\lambda y)\mu y = x(\lambda\mu)y$.

These axioms have the following consequences (see Luce and Suppes (1965) or Fishburn (1982), pp. 15–16):

$$x0y = y, \; x\lambda x = x$$
$$(x\lambda y)\mu(x\nu y) = x(\lambda\mu + (1-\mu)\nu)y.$$

Thus, (A1) may be replaced with

(A1$^+$) $x\lambda x = x$

to provide an equivalent axiomatization.

A function u from $\mathcal{M}$ to a real vector space is said to be *mixture-preserving* (MP) if

$$\forall (x,y) \in \mathcal{M} \times \mathcal{M}, \forall \lambda \in [0,1], u(x\lambda y) = \lambda u(x) + (1-\lambda)u(y).$$

Denote by $\mathcal{L}(\mathcal{M})$ the set of all real-valued MP functions on $\mathcal{M}$. Clearly, $\mathcal{L}(\mathcal{M})$ is a vector space and all constant functions are in it.

Convex subsets are the obvious examples of MS, with the λ-operation interpreted as

$$(\lambda, x, y) \mapsto \lambda x + (1-\lambda) y.$$

MP functions on convex subsets are well-known to be affine.

Fact 1. If $\mathcal{M}$ is a convex subset of a vector space $\mathcal{V}$, $u \in \mathcal{L}(\mathcal{M})$ if and only if u is *affine* on $\mathcal{M}$ (i.e., if and only if there is a linear form φ and a constant function k on $\mathcal{V}$ such that, for all $x \in \mathcal{M}$, $u(x) = \varphi(x) + k$). (A proof of this fact can be found in Coulhon and Mongin (1989), p. 183.)

In the wake of Herstein and Milnor (1953), p. 292, decision theorists – such as Fishburn (1982) – have regularly observed that convex subsets of a real vector space – e.g., convex subsets of measures – are MS, but they have abstained from investigating the reverse inclusion. As the examples below illustrate, the reverse inclusion does not hold.[1]

Fact 2. Not every mixture set is isomorphic to a convex subset of a vector space.

Example 1. Take $\mathcal{M} = \{x, y, z\}, x \neq y, x \neq z, y \neq z$, and all mixtures equal to x except for:

$$y\lambda y = y, z\lambda z = z, \quad \forall \lambda \in [0, 1],$$
$$x0y = y1x = z0y = y1z = y \quad \text{and} \quad x0z = z1x = y0z = z1y = z.$$

Example 2. Take $\mathcal{M}$ to be the triangle in $\mathbb{R}^2$ which has vertices $x = (0, 1)$, $y = (0, 0)$ and $z = (1, 0)$. It is the case that any $\xi \in \mathcal{M}$ can be written as $\xi = \alpha x + (1-\alpha)[\beta y + (1-\beta)z]$ for some $\alpha, \beta \in [0, 1]$. We define the λ-operation on $\mathcal{M}$ as follows.

For $\xi_1, \xi_2 \in \mathcal{M}$ with $\xi_1 = \alpha_1 x + (1-\alpha_1)[\beta_1 y + (1-\beta_1)z]$ and $\xi_2 = \alpha_2 x + (1-\alpha_2)[\beta_2 y + (1-\beta_2)z]$, the mixture $\xi_1 \lambda \xi_2$ is:

- z if $\xi_1 = \xi_2 = z$ and $\lambda \in [0, 1]$, or if $\xi_1 = z$ and $\lambda = 1$, or if $\xi_2 = z$ and $\lambda = 0$,
- $[\lambda\alpha_1 + (1-\lambda)\alpha_2]x + (1 - [\lambda\alpha_1 + (1-\lambda)\alpha_2])y$ otherwise.

In words, leaving aside trivial decompositions of z, we project the two elements of a mixture onto the vertical axis, and then compute the convex combination of the projections.

In these two examples, Axioms (A1), (A2) and (A3) hold, while two basic properties of convex sets fail, namely,

(A0) $\qquad$ for any $\lambda \in]0, 1[$, for any $x \in \mathcal{M}$, $y \mapsto x\lambda y$ is injective,

[1] In the recent literature, Wakker's book (1989), pp. 136–137, is exceptional in pointing out this fact.

and

(A0′) for any x, y in $\mathcal{M}$ such that $x \neq y$, $\lambda \mapsto x\lambda y$ is injective.

Example 3. Take $\mathcal{M} = \{x, y, z\}, x \neq y, x \neq z, y \neq z$, and define the λ-operation as follows:

$$x\lambda y = y\lambda x = y, \quad y\lambda z = z\lambda y = z, \quad z\lambda x = x\lambda z = x, \quad \forall\lambda \in]0, 1[,$$

and

$$x\lambda x = x, \quad y\lambda y = y, \quad z\lambda z = z, \quad \forall\lambda \in [0, 1],$$

and

$$x0y = y1x = z0y = y1z = y, \quad y0z = z1y = x0z = z1x = z,$$
$$z0x = x1z = y0x = x1y = x.$$

Thus, except for trivial mixtures involving the coefficients 0 and 1, each element prevails over another, i.e., y prevails over x, z over y, and x over z.

Example 3 illustrates a failure not only of (A0) and (A0′), but also of two associativity properties which are satisfied by convex sets, namely,

$$(x\lambda y)\mu z = x(\lambda\mu)\left(y\frac{\mu(1-\lambda)}{1-\lambda\mu}z\right) \quad \forall\lambda, \mu \in [0, 1], \quad \lambda\mu \neq 1, \qquad \text{(A4)}$$

and

$$x\lambda(y\mu z) = \left(x\frac{\lambda}{\lambda+\mu-\lambda\mu}y\right)(\lambda+\mu-\lambda\mu)z$$
$$\forall\lambda, \mu \in [0, 1], \quad \lambda \neq 0 \text{ or } \mu \neq 0. \qquad \text{(A4′)}$$

We note incidentally that Example 3 is the most general of its kind. It would seem possible to generalize it slightly by introducing the following threshold behavior: for some $\lambda_0 \in]0, 1[$,

$$x\lambda y = y \text{ if } 0 \leq \lambda < \lambda_0,$$

and

$$x\lambda y = x \text{ if } \lambda_0 \leq \lambda \leq 1,$$

and similarly for the other pairs of elements. Intuitively, y would prevail over x if and only if the proportion of x is sufficiently small. However, threshold behavior is incompatible with the mixture set axioms. (Take λ such that $\lambda_0 < \lambda < 1$. Then, $x\lambda y = x$, and from (A3) and (A1$^+$), $x\lambda^n y = x$ for all $n \geq 1$. But clearly, for some n large enough, $x\lambda^n y = y$.)

The main objective of this note is to axiomatize convex sets by strengthening Herstein and Milnor's mixture set axioms suitably. In the next section we show that the properties which were found to be missing in Examples 1, 2 and 3 are exactly those needed to recover the familiar notion of convexity.

3. An axiomatization of convex sets

We first note that there are redundancies in the missing conditions. It is readily seen that (A4) and (A4′) are equivalent. Also:

Fact 3. If (A1), (A2) and (A3) hold, then (A0) implies (A0′).

Proof. Take $x, y \in \mathcal{M}$ and $\lambda, \mu \in [0, 1]$ such that $x \neq y$ and $x\lambda y = x\mu y$. To prove that $\lambda = \mu$, assume first that $0 < \lambda < \mu$. Then, $\lambda = \mu\nu$ for some $\nu \in]0, 1[$, and $x\lambda y = x(\mu\nu)y = (x\nu y)\mu y$ from (A3), so that $(x\nu y)\mu y = x\mu y$. Using (A2) and (A1$^+$), we reach the contradiction that $x = y$. The case $\lambda = 0$, $\mu > 0$ leads to a similar contradiction. □

Now we proceed to show that (A0), (A1), (A2), (A3) and (A4) taken together characterize convex sets. An equivalent and more economical axiomatization is (A0), (A1$^+$), (A2) and (A4). (This follows from the fact that when (A1$^+$) is assumed instead of (A1), (A3) is a particular case of (A4).) The proof of the theorem will consist in showing that if, and only if, a mixture set satisfies (A0) and (A4), is it isomorphic to a convex subset of a vector space.

Given a mixture set $\mathcal{M}$, we introduce the following equivalence relation on $\mathcal{M}$:

$$x \sim y \ \text{ iff } \ \forall u \in \mathcal{L}(\mathcal{M}), u(x) = u(y).$$

This equivalence relation is compatible with the mixture operation on $\mathcal{M}$.

Hence, there is an induced mixture operation on classes: $\dot{x} \ \lambda \ \dot{y} = \dot{\widehat{x\lambda y}}$ for which $\mathcal{M}/\sim$ is itself a MS. Clearly, $\mathcal{M}/\sim$ satisfies (A0).

We now define a *non-degenerate* $\mathcal{M}$ to be a MS such that all classes of $\mathcal{M}/\sim$ are singletons, i.e., $x \sim y \Rightarrow x = y$. Examples 1, 2 and 3 illustrate the opposite case of degenerate MS.

The notion of a non-degenerate MS will serve here as a link between the axiom set and the convex representation. We investigate it by means of the following construction. Let $\mathcal{L}(\mathcal{M})'$ be the dual space of $\mathcal{L}(\mathcal{M})$, i.e., the space of all linear forms on $\mathcal{L}(\mathcal{M})$. For any $x \in \mathcal{M}$, define the *evaluation function at* x, $E_x \in \mathcal{L}(\mathcal{M})'$, by

$$\forall u \in \mathcal{L}(\mathcal{M}), \quad E_x(u) = u(x).$$

Then, consider the function $E : \mathcal{M} \to \mathcal{L}(\mathcal{M})'$ which is defined by $E(x) = E_x$. Clearly, E is MP on $\mathcal{M}$. It is injective if $\mathcal{M}$ is non-degenerate. The image (i.e., set of values) of E in $\mathcal{L}(\mathcal{M})'$ is convex. Accordingly, we have just proved.

Fact 4. If $\mathcal{M}$ is non-degenerate, $\mathcal{M}$ is isomorphic to a convex subset of the linear space $\mathcal{L}(\mathcal{M})'$.

In words: points in a non-degenerate MS can be identified with corresponding evaluation functions, so that $\mathcal{M}$ is isomorphically embedded in a vector space. Now, if we can show that MS satisfying (A0) and (A4) are non-degenerate, we will obtain the desired implication from our axiom set to the convex representation. The following terminology will be needed in the proof below. If $\mathcal{M}$ is a MS and $N \subset \mathcal{M}$, we define N to be a *mixture subset* if N is a MS for the restriction of the λ-operation to N. Clearly, $N \subset \mathcal{M}$ is a mixture subset if and only if

$$\forall x, y \in N \quad \forall \lambda \in [0, 1], \quad x\lambda y \in N.$$

Proposition. *If $\mathcal{M}$ is a MS, the following statements are equivalent:*

(i) *$\mathcal{M}$ satisfies* (A0) *and* (A4)*;*
(ii) *$\mathcal{M}$ is non-degenerate;*
(iii) *$\mathcal{M}$ is isomorphic to a convex subset of some linear space.*

Proof. Trivially, (iii) ⇒ (i), and we have just seen that (ii) ⇒ (iii). It remains to prove that (i) ⇒ (ii). This implication holds when $\mathcal{M}$ is a singleton; so we assume that $\#\mathcal{M} \geq 2$. Take $\bar{x} \neq \bar{y}$; we want to construct $u \in \mathcal{L}(\mathcal{M})$ with $u(\bar{x}) \neq u(\bar{y})$. Take any pair of distinct values for $u(\bar{x})$ and $u(\bar{y})$. Define the subset

$$\mathcal{M}_{\bar{x},\bar{y}} = \{z \in \mathcal{M} | \exists \alpha \in [0, 1] : z = \bar{x}\alpha\bar{y}\}.$$

From one of the properties listed after the MS axioms,

$$\forall z, z' \in \mathcal{M}_{\bar{x},\bar{y}}, \ \forall \lambda \in [0, 1], \quad z\lambda z' = \bar{x}[\alpha\lambda + (1-\lambda)\beta]\bar{y},$$

so that $\mathcal{M}_{\bar{x},\bar{y}}$ is a mixture subset. Now, for any $z = \bar{x}\alpha\bar{y}$, define

$$u(z) = \alpha u(\bar{x}) + (1-\alpha)u(\bar{y}).$$

(A0′) implies that this is a well-defined function. It is easily seen to be MP on $\mathcal{M}_{\bar{x},\bar{y}}$.

Suppose now that u has been extended from $\{\bar{x}, \bar{y}\}$ to a mixture subset N in such a way that u is MP on N. Suppose that there is $x \in \mathcal{M} \setminus N$. We want to show that

$$N_x = \{z \in \mathcal{M} | \exists \alpha \in [0, 1], \exists y \in N : z = x\alpha y\}$$

is a mixture subset and that u can be extended to a MP function on N_x. Take any two elements $z = x\alpha y$ and $z' = x\beta y'$ in N_x. Consider the mixture $z\lambda z'$. If $\lambda, \alpha, \beta \notin \{0, 1\}$, we can apply (A4) and (A4′) in succession and conclude that

$$z\lambda z' = (x\alpha y)\lambda(x\beta y') = x[\beta(1-\lambda) + \alpha\lambda]\left(y \frac{\lambda(1-\alpha)}{1-\alpha\lambda-\beta(1-\lambda)} y'\right).$$

Hence, $z\lambda z' \in N_x$. This also holds in the case where $\lambda, \alpha, \beta \in \{0, 1\}$. We conclude that N_x is a mixture subset. Now, fix a value for $u(x)$, and for any $z = x\alpha y$, define

$$u(z) = \alpha u(x) + (1 - \alpha)u(y).$$

This is a well-defined function. To see that, we note first that, if $z = x\alpha y = x\beta y'$, (A0) implies that either $y = y'$ and $\alpha = \beta$, or $y \neq y'$ and $\alpha \neq \beta$. Consider the latter case and assume that $\alpha < \beta$. Then, (A2), (A3) and (A0) imply that

$$y = x\left(\frac{\beta - \alpha}{1 - \alpha}\right)y',$$

so that the value $u(z)$ is the same whichever of the two decompositions is chosen. It is easy to check that u is MP on N_x.

Now, consider all mixture subsets S such that $\bar{x}, \bar{y} \in S$ and u has been extended to a MP function u_s on S. Partially order the set P of pairs (S, u_s) in the following way:

$$(S, u_s) \leq (S', u_{s'}) \quad \text{iff } S \subseteq S' \text{ and } u_{s'} \text{ extends } u_s.$$

Take any *totally* ordered chain of P, $(S_i, u_{s_i})_{i \in I}$. It has an upper bound $(\cup_{i\in I} S_i, \bar{u})$, with $\bar{u}$ defined by: $\bar{u}(x) = u_{s_i}(x)$ if $x \in S_i$. (The point here is that $\cup_{i\in I} S_i$ is a mixture subset, and that $\bar{u}$ is well-defined and MP on $\cup_{i\in I} S_i$.) Thus, we can apply Zorn's Lemma and conclude that P has a maximal element for $\leq$. Denote it by $(\tilde{S}, \tilde{u})$. Suppose $\mathcal{M} \neq \tilde{S}$. Then we could take $x \notin \tilde{S}$ and apply the second part of the proof to construct (T, v) such that $(\tilde{S}, \tilde{u}) \leq (T, v)$ and $T \neq \tilde{S}$, a contradiction. Hence $\mathcal{M} = \tilde{S}$, and $\tilde{u}$ extends u to the whole domain. □

The salient features of this proof are, on the one hand, the transfinite induction argument which makes it parallel to the proof of the Hahn–Banach theorem, and, on the other hand, the use of the added axioms (A0) and (A4) in order to achieve the inductive step.

After deriving the above proposition, we found two early variants of it in Hausner (1954) and Stone (1949), respectively. In essence, Hausner (1954), Theorem 3.2, states that if a set $\mathcal{M}$ is endowed with a λ-operation satisfying (A0) to (A4), it can be isomorphically embedded into a convex subset of a vector space. Hausner's embedding device is not the same as ours, and his statement does not make any reference to the intermediary characterization of convex sets as non-degenerate mixture sets. Stone (1949), Theorem 2, also provides an axiomatic characterization of convex sets without assuming any vector space structure, but his primitive operation is different from the λ-operation investigated here. For any nonempty set $\mathcal{M}$, he defines an external operation $(x, y; \alpha, \beta), x, y \in \mathcal{M}, \alpha, \beta \geq 0, \alpha + \beta > 0$, which can be interpreted as the operation of assigning weights α and β to the elements

x and y, respectively.[2] Stone's axioms on his weighting operation can be related to axioms imposed on the λ-operation. Although this is only implicit in his work, Stone may be credited with first showing that a set $\mathcal{M}$ endowed with a λ-operation which satisfies (A0) to (A4) can be represented by a convex set.[3]

We might follow Hausner (1954), Theorem 3.4, by complementing the above proposition with a suitable *uniqueness* result. Suppose that a mixture set $\mathcal{M}$ is shown to be isomorphic to two convex subsets C and C' of linear spaces L and L', respectively; i.e., there are two MP bijections $\varphi : \mathcal{M} \to C \subset L$ and $\varphi' : \mathcal{M} \to C' \subset L'$. Evidently, C and C' are themselves isomorphic in the sense that there is a MP bijection $f = \varphi' \circ \varphi^{-1}$ from C onto C'. In this trivial sense, the convex representation of a MS satisfying (A0) and (A4) can be said to be unique. However, a stronger (and perhaps not completely obvious) uniqueness property also holds. We recall that if X is a subset of some linear space, the *affine span of* X is the set of all affine combinations of elements of X, i.e., of all $\sum_{i=1}^{n} \lambda_i x_i$, for $n \geq 1, x_1, \cdots, x_n \in X$, and $\sum_{i=1}^{n} \lambda_i = 1$.

Fact 5. Assume that L, L' are linear spaces, and that $\varphi : \mathcal{M} \to C \subset L$ and $\varphi' : \mathcal{M} \to C' \subset L'$ are MP bijections. Then, $f : \varphi' \circ \varphi^{-1}$ can be extended uniquely to an affine bijection from $\mathrm{Aff}(C)$ onto $\mathrm{Aff}(C')$, where $\mathrm{Aff}(C)$ [$\mathrm{Aff}(C')$] is the affine span of C in L [of C' in L', respectively].

The proof relies on a standard extension device which is used, for instance, in Coulhon and Mongin's (1989), p. 183, proof of Fact 1, and so we do not give it here. A significant consequence of Fact 5 is that it makes it possible to define the *dimension* of a MS satisfying axioms (A0) to (A4). For such a mixture set $\mathcal{M}$, we define its dimension to be the affine dimension of any convex set C to which $\mathcal{M}$ is isomorphic. This statement makes sense since, from Fact 5, any other convex representation C' of $\mathcal{M}$ must have the same affine dimension as that of C.[4] When (A0) and (A4) do not hold, it is not clear to us how the dimension of a MS can be defined.

4. Applications

Elaborating on Stone's and Hausner's results, Gudder (1977) and Gudder and Schroeck (1980) have argued that the λ-operation provides a satisfactory idealization of various natural and psychological phenomena involving the

---

[2] The weights may belong to any ordered field.

[3] Gudder (1977) credits Stone with this finding.

[4] Recall that the *affine dimension* of a (non-empty) subset X of a linear space L is the linear dimension of the translated set $X - x_0$, where x_0 is any element of X. See, e.g., Kelly (1979). Importantly, the dimension concept which is preserved by the isomorphism f in Fact 5 is the *affine* dimension, not the *linear* dimension.

intuitive notion of mixing. For instance, in chemistry, the effect of blending two substances depends on both the particular substances and the surrounding circumstances, so that each of the axioms (A0) to (A4) may or may not hold, depending on the case in hand. A curious fact reported by Gudder and Schroeck (1980), p. 985, is that even the apparently unexceptionable (A1$^+$) may sometimes be violated. A gasoline with octane number 100 can be mixed with a different gasoline with identical octane number to get a gasoline with octane number 105. This and similar counterexamples mean that the mixture axioms have empirical content, at least as far as chemistry is concerned.

Gudder (1977) makes a similar claim for an altogether different field, i.e., color vision. By surveying some salient results of the theory of visual perception, he interestingly shows that, again in this area, the mixture axioms can be endowed with empirical content. Relevant in this connection is, for instance, the familiar fact that intensity (or brightness) influences the perceptual effect of mixing one color with another, so that, at very low intensity levels, a person with normal vision tends to react like a monochromat. This, and other facts of the matter, can be described in the language of mixture sets. Krantz (1975) also discusses applications of MS (which he calls “Grassman structures”) to color perception.

In decision theory, *lotteries* are the virtually unique application of the MS algebra. One way of connecting the latter with the empirical concept of a lottery is as follows. (Since the construction is elementary, we do not give the full details.) Start with a non-empty set X_0, to be thought of as the set of final results; we take it to be finite for simplicity. Given X_{n-1}, $n \geq 1$, define X_n to be the set of all elements $(\lambda, x; (1-\lambda), y)$, where $\lambda \in [0, 1]$ and $x, y \in X_{n-1}$. Intuitively, X_n is the set of n-level lotteries based on X_0. Elements $(1, x; 0, y) \in X_n$ are identified with the element $x \in X_{n-1}$. This implies that $X_{n-1} \subset X_n$, which makes it possible to define a λ-operation on $X = \cup_{n\in\mathbb{N}} X_n$: for all $x, y \in X$, we define $x\lambda y$ to be $(\lambda, x; (1-\lambda), y) \in X_{n+1}$ where n is the level of the higher-level of the two elements x or y. By construction, this λ-operation satisfies (A1). We make it satisfy (A2) and (A3) by imposing the corresponding restrictions at each level n.

An interesting feature of this construction is that it involves only a limited form of the principle of reduction of compound lotteries (for a statement, see Luce and Raiffa (1957), p. 26). Not all compound lotteries can be reduced to simpler ones. For instance, $(\frac{1}{4}, 1\$; \frac{3}{4}, (\frac{1}{3}, 1\$; \frac{2}{3}, 2\$))$ can be reduced to $(\frac{1}{2}, 1\$; \frac{1}{2}, 2\$)$, but

$$\left(\frac{1}{4}, 3\$; \frac{3}{4}, \left(\frac{1}{3}, 1\$; \frac{2}{3}, 2\$\right)\right) \text{ and } \left(\frac{1}{4}, 1\$; \frac{3}{4}, \left(\frac{1}{3}, 3\$; \frac{2}{3}, 2\$\right)\right)$$

are treated as distinct elements, and the reduced form of these two lotteries, i.e.,

$$\left(\frac{1}{4}, 1\$; \frac{1}{2}, 2\$; \frac{1}{4}, 3\$\right)$$

is not even defined in the present framework. The MS version of the VNM theorem (Herstein and Milnor (1953); see also Fishburn (1982), p. 14) applies here, which means that *the full force of the compound lottery principle is not needed for the VNM theorem to hold*. To the best of our knowledge, this fact had never been brought to light by decision theorists.

Notice that the previous construction is drastically simplified if we impose (A4) on top of the other axioms. Then, lotteries of three or more final results can be defined meaningfully in terms of the initial, exclusively two-element concept of lottery. To add (A4) amounts to requiring the full force of the compound lottery principle.

It is a matter for empirical examination whether or not the weakening of the reduction principle that is suggested by the MS axioms has any decision-theoretic relevance. However, scepticism might be prompted by the fact that the previous construction allows for generous reduction of *some* high-level lotteries (i.e., when the same result occurs at each successive level), whereas level 1 lotteries having three distinct results cannot even be taken into account. The *prima facie* conclusion is that there is no significant gain of content in proving the VNM theorem within such a weak axiom system as that of Herstein and Milnor.

This author's recommendation is that VNM theory should be developed either in the context of the stronger axiom system (A0) to (A4) or by assuming directly that the decision-maker's alternative set is a convex set (typically but not necessarily, a convex set of measures). Either of these frameworks has its merits. The fact that MS satisfying (A0) to (A4) have been shown to be essentially identical with convex sets does not mean that it is better to start with sets than with axioms. Indeed, we follow Gudder (1977), p. 230, in thinking that the axiomatic approach might be justified even if there exists a convenient representation. One argument for this claim is that the linear operations which underlie the convex representation of MS are introduced *ex post* and somewhat artificially; only the mixture operation is primitive. Another relevant argument is that to formalize the latter makes it possible to perceive curious analogies between such diverse fields as chemistry, color vision and decision theory.

References

Coulhon, T., Mongin, P. (1989): Social choice theory in the case of von Neumann–Morgenstern utilities. *Social Choice and Welfare* **6**, 175–187

Fishburn, P.C. (1982): The foundations of expected utility. D. Reidel, Dordrecht

Gudder, S.P. (1977): Convexity and mixtures. *SIAM Review* **19**, 221–240

Gudder, S.P., Schroeck, F. (1980): Generalized convexity. *SIAM Journal on Mathematical Analysis* **11**, 984–1001

Hausner, M. (1954): Multidimensional utilities. In: Thrall, R.M. et al. (eds.), *Decision Processes*. Wiley, New York, pp. 167–180

Herstein, I.N., Milnor, J. (1953): An axiomatic approach to measurable utility. *Econometrica* **21**, 291–297

Kelly, P.J., Weiss, M.L. (1979): Geometry and Convexity. Wiley, New York

Krantz, D.H. (1975): Color measurement and color theory. I. Representation theorem for Grassmann structures. *Journal of Mathematical Psychology* **12**, 283–303

Luce, R.D., Raiffa, H. (1957): Games and decisions: introduction and critical survey. Wiley, New York

Luce, R.D., Suppes, P. (1965): Preference, utility and subjective probability. In: Luce, R.D. et al. (eds.), *Handbook of Mathematical Psychology*, Volume III, Chapters 15–21, Wiley, New York, pp. 250–410

Stone, M.H. (1949): Postulates for the barycentric calculus. *Annali di Matematica Pura ed Applicata* **29**, 25–30

von Neumann, J., Morgenstern, O. (1944): Theory of games and economic behavior. Princeton University Press, Princeton, NJ

Wakker, P. (1989): Additive representations of preferences. Kluwer, Dordrecht