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STRONG COMPLETENESS THEOREMS FOR WEAK LOGICS OF COMMON BELIEF

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ABSTRACT. We show that several logics of common belief and common knowledge are not only complete, but also strongly complete, hence compact. These logics involve a weakened monotonicity axiom, and no other restriction on individual belief. The semantics is of the ordinary fixed-point type.

KEY WORDS: common belief, common knowledge, logical omniscience, strong completeness

1. INTRODUCTION

Common belief and common knowledge constitute one of the areas in which the foundational concerns of game theorists and mathematical economists can be best met by importing the logicians' particular methods of analysis. True, the informal definition of common knowledge ("everybody knows that everybody knows that...") and the corresponding one for common belief ("everybody believes that everybody believes that...") are sufficient for several practical purposes. In games of finite size, depending on the particular equilibrium concept, the standard assumption that "the rules of the game are common knowledge" may be replaced by the less formidable one that the rules of the game are shared belief (rather than shared knowledge), that these shared beliefs are themselves shared beliefs, and so on up to some fixed level k which depends on the size of the game at hand. Here, some use of epistemic logic may help to handle the finite sequence of shared beliefs, but the logic of common belief is irrelevant. However, for infinite games generally and for particular equilibrium concepts even in finite games, the common belief or common knowledge assumption becomes genuinely needed, and its logic definitely helps to clarify the various asymptotic and fixed-point properties that are part and parcel of this assumption.

The formal analysis of common knowledge, which was lacking in Lewis's pioneering work, *Convention* [16], later came from two quarters, namely, the foundational work in game theory and mathematical economics that ensued from Aumann's paper [1], and the no less foundational work pursued by logicians and computer scientists in relation to AI and Distributed Systems (a good deal of which is covered by Fagin, Halpern, Moses and Vardi in [8]). At least initially, there was a stark contrast between the methods adopted by the two groups of scholars. Aumann's paper and his first followers' work are exclusively set-theoretical. The general idea is to fix a "universal" set of states of nature and express facts of nature and epistemic facts alike in terms of this set, i.e., by identifying them with subsets of it. In this framework, an individual's beliefs are conveniently summarized by a mapping that transforms subsets into other subsets, i.e., factual events into epistemic events, and epistemic events into higher-order epistemic events. Epistemic regularities can then be stated in terms of elementary properties imposed on the belief mappings. Over the years, the method of belief mappings has become part of the bread-and-butter of game-theory, and it has also been used in the general equilibrium context to clarify the assumptions underlying the "no trade" theorems.[1]

From the logicians' point of view, belief mappings and other related set-theoretic objects are just semantics, and the economists' and game theorists' analysis lacks the crucial component of a syntax. Desirable or plausible properties of individual belief and knowledge should also be formulated as axioms or inference rules in a formal language. Given the peculiar finiteness constraints normally embodied in the syntax, it will provide a perspective that is complementary to that offered by the semantics. Sometimes more intuition can be gained from turning from the semantics to the syntax. A classic example is the axiomatic decomposition of the game theorists' "partitional model of information", which is arguably clearer than the corresponding list of set-theoretic properties. More important, however, is the fact that the syntax allows for a kind of investigation that cannot be carried on the semantic side. The standard method of epistemic logic is to describe each belief held by an individual with a modal formula. Thus, the problem of investigating the individual's inferences reduces to that of investigating a particular subset of the system's inferences.

Once defined in their own right, the syntax and semantics of belief and knowledge should ideally be related by means of a soundness and completeness theorem. Modal logicians (e.g., Hughes and Cresswell [13]) usually distinguish between completeness, i.e.:

($*$) every formula that is a semantic truth can be proven formally, and strong completeness, i.e.:

($**$) every formula that can be obtained as a semantic consequence from some set of assumptions can also be proven formally from the same set of assumptions.

The present paper is concerned with a specific, actually very weak, logic of common belief that we demonstrated in [19] to be sound and complete according to definition ($*$). We will now establish that the stronger form ($**$) can be obtained, a much more satisfactory result from the point of view of finding a "well-behaved" axiomatization of common belief. As far as we can judge, to prove a *strong* completeness theorem is an innovation in the subarea of common belief logic. We emphasize this fact and make relevant comparisons at the end of the paper.

To return to the game-theoretic motivation of this work, there is a supplementary reason to move from a purely semantic framework of epistemic analysis to a proper logic, and it has to do with the time-honoured project of introducing bounded rationality into game theory. One among the (many) facets of bounded rationality is well captured by what is commonly called *the logical omniscience problem*. We just emphasized that epistemic logic makes it possible to describe the individual's own inferences in terms of subsets of the system's inferences. How small should the subsets be? How small can they become without wrecking the project of developing a well-behaved formalism? These syntactical questions provide a useful perpective on logical omniscience, hence on bounded rationality. They are important to the present writers, whose constant objective has been to make sense of common belief under very weak assumptions put on individual belief. Unexpectedly, the syntactical theme of logical omniscience connects with the logical topic discussed above, i.e., the two forms of completeness. As we will explain, strong completeness can be obtained only for those common belief systems where logical omniscience assumptions are somehow kept to a minimum. The more popular systems *à la* Kripke, as in [9] and [18], make much more demanding epistemic assumptions, and these systems are demonstrably only complete, not strongly complete. The intriguing connection between logical omniscience and strong completeness is the topic of the present paper.

2. Common Belief as a Fixed-Point of a Set-Theoretic Mapping

The semantics of common belief or common knowledge is mainly of two types, i.e., the iterative and the fixed-point one. Here, we will exclusively rely on fixed-points constructions, referring to other papers for comparisons with the other type.[2] Nearly unexceptionally, the fixed-point constructions of common belief are based on the simple observation (usually attributed to Tarski) that a monotonic set-function has a greatest fixed point. We state a variant of this well-known fact in the first proposition below.

Given a nonempty set W, we denote the power set of W by $\mathfrak{P}(W)$ and define a mapping $f : \mathfrak{P}(W) \to \mathfrak{P}(W)$ to be *monotonic* (respectively: *quasi-monotonic, reflective-monotonic*) if for all $X, Y \subseteq W$,

$$X \subseteq Y \Longrightarrow f(X) \subseteq f(Y)$$

(respectively: $X \subseteq f(X) \cap Y \Longrightarrow f(X) \subseteq f(Y)$, and: $X \subseteq f(X) \Longrightarrow f(X) \subseteq f(f(X))$). Clearly, any monotonic mapping is quasi-monotonic, and any quasi-monotonic mapping is reflective-monotonic. A subset $F \subseteq W$ will be said to be a *fixed-point* of f whenever $f(F) = F$. A *greatest* fixed-point is a fixed-point that includes any other. Observe that these definitions do not prevent $\emptyset$ from being a fixed-point (or even a greatest fixed point) of f.

PROPOSITION 1. *If f is a quasi-monotonic mapping, then f has a greatest fixed-point, which is*

$$F = \bigcup\{X \subseteq W \mid X \subseteq f(X)\}.$$

Proof. If X is a fixed-point of f, i.e., if $f(X) = X$, then $X \subseteq F$ from the definition of F. If $F = \emptyset$, F is a greatest fixed-point by the following argument. Because $\emptyset = F \subseteq f(F)$ and f is quasi-monotonic, hence reflective-monotonic, $f(F) \subseteq f(f(F))$, so that $f(F)$ is one of the sets X in the definition of F. This implies that $f(F) = \emptyset$. From the observation in the first sentence, there is no other fixed point than $F = \emptyset$.

We may now assume that there is $w \in F$. There is X such that $w \in X$, and $X \subseteq f(X) \cap F$. Since f is quasi-monotonic, $f(X) \subseteq f(F)$, so that $w \in f(F)$, and thus $F \subseteq f(F)$. To prove the converse inclusion, we use again the fact that $F \subseteq f(F) \Longrightarrow f(F) \subseteq f(f(F))$, so that $f(F) \subseteq F$ from the definition of F. □

To define common belief semantically, we introduce the key notion of a *belief closed event*. In this paper, an "event" means any subset of the set W, which is to be thought of as a set of possible worlds (or states of the world in the game theorist's terminology). We denote by $\mathfrak{P}(W)$ the power set of W. Intuitively, an event is belief closed iff everybody believes it to take place whenever it effectively takes place. Symbolically, if $P \subseteq W$ is an event, $b_a(P) \subseteq W$ denotes the event that agent $a \in A$ believes P, and

$$b_E(P) = \bigcap_{a \in A} b_a(P)$$

denotes the event that every agent believes P, we say that P is belief closed if

$$P \subseteq b_E(P).$$

Lismont and Mongin in [19] and [20] propose that common belief be semantically defined as follows:

(LM) P is *common belief* at $w \in W$ if there exists X which is belief closed, believed at w, and included in P.

Denoting by $F_P \subseteq W$ the event that "P is common belief", the definition translates into the following:

$$F_P = \bigcup \{b_E(X) \mid X \subseteq b_E(X) \cap P\}.$$

The next proposition restates this definition in a perhaps less interpretable, but convenient way.

PROPOSITION 2. *Define* $F'_P = \bigcup \{X' \mid X' \subseteq b_E(X' \cap P)\}$. *If* $b_E : \mathfrak{P}(W) \to \mathfrak{P}(W)$ *is quasi-monotonic, then*

$$F_P = F'_P.$$

Proof. Suppose there is $w \in F_P$. There is X such that $w \in b_E(X)$ and $X \subseteq b_E(X) \cap P = b_E(X) \cap (b_E(X) \cap P)$. Because b_E is quasi-monotonic, $b_E(X) \subseteq b_E(b_E(X) \cap P)$, so that $b_E(X) \subseteq F'_P$ and $w \in F'_P$. This proves one of the two inclusions. For the other, take $w' \in X'$, with $X' \subseteq b_E(X' \cap P)$. The event $X' \cap P$ satisfies the desired property that $X' \cap P \subseteq b_E(X' \cap P) \cap P$, so that $w' \in F_P$. □

In view of Proposition 2, Proposition 1 entails that common belief of P, in the (LM) sense, is the greatest fixed point of the following mapping indexed by P:

$$f_P : \mathfrak{P}(W) \to \mathfrak{P}(W),\ X \to f_P(X) = b_E(X \cap P).$$

Related notions of common belief or common knowledge have appeared in the game-theoretic literature. In [22], Mertens and Zamir construct a particular set of states of the world W from sequences of (probabilistic) mutual beliefs of any order. They are concerned with common *knowledge* specifically. Abstracting from probabilistic features, their definition states that P is common knowledge at $w \in W_0$, where W_0 is a finite subset of W, if P includes the *smallest* belief closed event X such that $w \in X$. We will restate this definition in relation to the whole set W instead of restricting attention to finite subsets, and accordingly drop reference to the smallest belief closed event, which may not exist. Hence:

(MZ) P is common knowledge at $w \in W$ iff P includes a belief closed event X such that $w \in X$.

Here is the resulting (MZ) definition for the event that "P is common knowledge":

$$G_P = \bigcup \{X \subseteq W \mid X \subseteq b_E(X) \cap P\}.$$

This is again the greatest fixed point of a mapping indexed by P, i.e.:

$$g_P : \mathfrak{P}(W) \to \mathfrak{P}(W),\ X \to g_P(X) = b_E(X) \cap P.$$

In [23] Monderer and Samet propose still another definition of common belief (rather than common knowledge). Abstracting again from any probabilistic features, their variant reads as:

(MS) P is common belief at $w \in W$ iff there is a belief closed event X such that $w \in X$, and P is believed at every w at which X takes place.

Accordingly, the event that "P is common belief" is now:

$$H_P = \bigcup \{X \subseteq W \mid X \subseteq b_E(X) \cap b_E(P)\}$$

which makes this definition a greatest fixed-point once again. The relevant mapping is:

$$h_P : \mathfrak{P}(W) \to \mathfrak{P}(W),\ X \to h_P(X) = b_E(X) \cap b_E(P).$$

Comparing the three definitions, we observe that in (MZ), the event P actually *occurs* at every state where it is commonly believed, whereas the other definitions just require P (in (MS)) or even a smaller event than P (in (LM)) to be *believed* at every state where it is commonly believed. Given the standard construal of knowledge as true belief, which we take here for granted despite its obvious shortcomings, it becomes clear that (MZ) are concerned with the (objective) notion of common knowledge, and (LM) and (MS) with the (subjective) notion of common belief.

Except in [19], the three formalizations above were introduced for individual belief operators b_a that were at least monotonic, a property inherited by the shared belief operator $b_E \equiv \bigcap_a b_a$. From the epistemic point of view, monotonicity captures a form of *logical omniscience*: for any two events X and Y, if X implies Y, the agents' reasoning automatically replicates this implication. In some Artificial Intelligence applications, monotonicity, as well as other forms of logical omniscience, may be unproblematic – or so it is argued by computer scientists. However, in relation to human reasoners, logical omniscience appears to be a questionable assumption even for normative purposes. The standard modelling of the agents' reasoning both in epistemic logic and game theory is thus open to a significant objection.

In epistemic logic, the prevailing semantic model is that of a *Kripke structure*. It is equivalent to assuming monotonicity, plus the following, hardly less exacting requirement on the agents' part: first, they should believe any truth that can proved in the system, a property that logicians call "necessitation"; second, if the agents believe in two events, they automatically believe in their conjunction, a property labelled "conjunctiveness". Symbolically:

- $b_a(W) = W$,
- for all $X, Y \subseteq W$, $b_a(X) \cap b_a(Y) = b_a(X \cap Y)$,

and the same hold of b_E as a matter of consequence. Game theory is in an even worse predicament than standard epistemic logic. Its celebrated model of *information partitions* turns out to be equivalent to the three properties that underlie a Kripke structure, plus further assumptions, mostly imposed on the agents' introspective abilities. The introspective axioms have been famously criticized.[3]

The argument just sketched against logical omniscience has led the present writers to investigate common belief in an epistemic context that is *not* monotonic. Although more radical departures from logical omniscience have been proposed elsewhere in epistemic logic (notably in Rantala's work [24], [25] on "non-normal worlds"), [19] appears to be the only attempt made to analyze common belief under a truly weak assumption imposed on individual belief. The only requirement on b_a (and b_E) is precisely that which we singled out at the outset, i.e., quasi-monotonicity. The next section takes up the project of this earlier paper by further clarifying the logic of quasi-monotonic constructions.

Before we proceed, we record some relevant properties of the formal concepts of common belief or common knowledge. We note in passing that when b_E is quasi-monotonic, so are f_P, g_P and h_P for any $P \subseteq W$. The next proposition shows how the three concepts F_P, G_P, H_P relate to each other:

PROPOSITION 3. *If $b_E : \mathfrak{P}(W) \to \mathfrak{P}(W)$ is quasi-monotonic, $P \in \mathfrak{P}(W)$, then $G_P = F_P \cap P$, and $F_P \subseteq H_P$. Further, there are W, $P \in \mathfrak{P}(W)$ and a quasi-monotonic mapping b_E such that $G_P \subset F_P \subset H_P$.*

Proof. (1) Take first $w \in F_P \cap P$. There is $X \subseteq f_P(X) = b_E(X \cap P)$ such that $w \in X$. Thus $w \in X \cap P \subseteq b_E(X \cap P) \cap P = g_P(X \cap P)$, whence $w \in G_P$. Now, take $w \in G_P$. There is $X \subseteq b_E(X) \cap P$ such that $w \in X$. Since $w \in b_E(X)$, $w \in F_P$ (Proposition 2) so that $w \in F_P \cap P$.

(2) Let $X \subseteq b_E(X \cap P)$. Since $X \cap P \subseteq b_E(X \cap P) \cap X$ and $X \cap P \subseteq b_E(X \cap P) \cap P$, quasi-monotonicity implies that $b_E(X \cap P) \subseteq b_E(X) \cap b_E(P)$, which leads to the conclusion.

(3) Let:

- $W = \{0, 1, 2\}$;
- $b_E(\emptyset) = b_E(\{0\}) = b_E(\{2\}) = \{0\}$;
- $b_E(\{1\}) = b_E(\{0, 1\}) = b_E(\{0, 2\}) = b_E(\{1, 2\}) = \{0, 2\}$;
- $b_E(\{0, 1, 2\}) = \{0, 1, 2\}$.

Then $G_{\{1\}} = \emptyset \subset F_{\{1\}} = \{0\} \subset H_{\{1\}} = \{0, 2\}$. □

The second group of facts concerns the monotonicity or otherwise of the common belief concepts when F_P, G_P and H_P are regarded as mappings $\mathfrak{P}(W) \to \mathfrak{P}(W)$.

PROPOSITION 4. *For all $P, P' \in \mathfrak{P}(W)$,*

(1) *If b_E is quasi-monotonic, F_P is monotonic, i.e.,*

$$P \subseteq P' \Longrightarrow F_P \subseteq F_{P'}.$$

(2) *If b_E is quasi-monotonic, G_P is monotonic.*

(3) *If b_E is quasi-monotonic, H_P is quasi-monotonic, i.e.,*

$$P \subseteq H_P \cap P' \Longrightarrow H_P \subseteq H_{P'}.$$

Further, there are W and a quasi-monotonic mapping b_E such that H_P is not monotonic, i.e., such that for some $P \subseteq P'$, $H_P \nsubseteq H_{P'}$.

(4) *If b_E is monotonic, H_P is monotonic, i.e.,*

$$P \subseteq P' \Longrightarrow H_P \subseteq H_{P'}.$$

Proof. (1) Let $P \subset P'$ and $X \subseteq b_E(X \cap P)$. Then $X \cap P \subseteq b_E(X \cap P) \cap X \cap P'$. Quasi-monotonicity implies that $b_E(X \cap P) \subseteq b_E(X \cap P')$, which leads to the conclusion.

(2) This follows from the definitions of g_P and $g_{P'}$. (The assumption of quasi-monotonicity is needed only to ensure the existence of the greatest fixed-point.)

(3) Let $P \subseteq H_P \cap P'$ and $w \in P$. Since $w \in H_P$, there is $X \subseteq b_E(X) \cap b_E(P)$ with $w \in X$, so that $w \in b_E(P)$. Hence $P \subseteq b_E(P) \cap P'$.
Quasi-monotonicity entails that $b_E(P) \subseteq b_E(P')$, which leads to the conclusion.

Now, let:

- $W = \{0, 1, 2\}$;
- $b_E(\emptyset) = b_E(\{1, 2\}) = \emptyset$;
- $b_E(\{1\}) = b_E(\{0, 2\}) = \{0, 2\}$;
- $b_E(\{0\}) = b_E(\{2\}) = \{1\}$;
- $b_E(\{0, 1\}) = \{2\}$;
- $b_E(\{0, 1, 2\}) = \{0, 1, 2\}$.

Then $H_{\{1\}} = \{0, 2\}$ and $H_{\{1,2\}} = \emptyset$.

(4) Monotonicity implies that $b_E(P) \subseteq b_E(P')$, which again leads to the conclusion. □

Proposition 3 states that the closely related (LM) and (MS) concepts are nonetheless distinct, an observation already made by Heifetz [10]. There are more sets having the common belief property under the (LM) definition than there are under the (MS) definition. Proposition 4 points towards a conceptually relevant difference between the two concepts. That *common* belief is monotonic has been taken for granted nearly unexceptionally, and indeed appears to be an intuitively desirable property. Now, given what has been said about logical omniscience, it is worthwhile to obtain this property even when *shared* belief is not itself monotonic, but just quasi-monotonic. The (LM) concept achieves this result, but the (MS) concept does not.

When conjunctiveness is assumed on top of monotonicity, the (LM) and (MS) concepts collapse into each other. In particular, the Kripke and *a fortiori* the partitional modelling will confound them. Even so, the two concepts remain distinct from the (MZ) one. Introducing formally the "knowledge property":

- $b_a(X) \subseteq X$,

we see that:

$$\begin{aligned} b_E \textit{ monotonic and } b_E \textit{ satisfies the knowledge property} &\implies F_P \subseteq G_P \\ b_E \textit{ satisfies the knowledge property} &\implies H_P \subseteq G_P. \end{aligned}$$

The inclusions can be shown to be strict in general.

3. Three Axiom Systems for Common Belief

Our syntax makes it possible to state three axiomatic systems, corresponding to the (LM), (MZ) and (MS) notions, respectively. The axioms and rules will be provided for the three systems at one go. The formal language is exactly as usual in epistemic modal logic. It is built upon the following ingredients: a set of propositional variables $p \in PV$, which are meant to represent the basic nonepistemic facts; the standard symbols of propositional connectives, i.e., $\neg$ (to represent "not"), $\vee$ ("or"), $\wedge$ ("and"), $\rightarrow$ ("implies"), $\longleftrightarrow$ ("is equivalent to"); and finally several symbols of unary operators acting on formulae, with an intended epistemic interpretation. These syntactic operators are: B_a ("a believes that…"or "a knows that…", depending on the axiomatic context), where a ranges on the (finite) set of agents A; E ("everybody believes that…" or "everybody knows that…"); and last but not least, C ("it is common belief that…" or "it is common knowledge that…"). Actually, we will employ three variant symbols for C, i.e., C_f, C_g, C_h, whose intended interpretation is common belief, or common knowledge, in each of the (LM), (MZ) and (MS) versions respectively. The *formulae* of the language are obtained by concatenating the primitive symbols in the natural way; we skip the details. The important fact is that a formula always involves a finite number of symbols. This excludes infinite conjunctions or disjunctions, hence a direct axiomatic definition of common belief in terms of an infinite conjunction of shared belief formulas. Within the more standard framework of epistemic logic, the semantics of common belief may be either of the iterative or fixed-point type, but the syntax is of necessity of the fixed-point type.[4]

The three systems will be denoted by $QMCB_f$, $QMCB_g$, $QMCB_h$, with *QMCB* serving as a generic label. Here is the part of the axiomatization which is common to each of the three. It has to do with individual and shared belief:

(PL) *Any axiomatization of propositional logic*

$$(\mathrm{RE}_a)\ \frac{\phi \longleftrightarrow \psi}{B_a\phi \longleftrightarrow B_a\psi}, \quad a \in A$$

$$(\mathrm{Def.E})\ E\phi \longleftrightarrow \bigwedge_{a\in A} B_a\phi$$

$$(\mathrm{RQM}_E)\ \frac{\phi \rightarrow E\phi \wedge \psi}{E\phi \rightarrow E\psi}$$

As usual, the horizontal bar symbol is used to denote an inference rule, to be carefully distinguished from an axiom or a theorem. The axiomatization of propositional logic will typically contain the familiar rule of *Modus*

Ponens as well as suitable propositional axioms. We concentrate on the epistemic component of the systems. The rule (RE$_a$), or *Rule of Equivalence*, says in words that if $\phi \longleftrightarrow \psi$ holds as a theorem of the system, it is possible to infer $B_a\phi \longleftrightarrow B_a\psi$ as another theorem of the system. Its epistemic import is that each agent a has sufficient reasoning ability in order to reproduce the logical equivalences that can be proved in the system. (RE$_a$) claims the property only for logical equivalences, not for logical implications. Herein lies the important difference between the present (called "minimal" in Chellas's text [6]) and the much stronger monotonic logics such as the Kripke one. There is an ingredient of monotonicity in our systems, however. The rule (RQM$_E$) or *Rule of Quasi-Monotonicity*, says that if $\phi \rightarrow E\phi \wedge \psi$ holds as a theorem of the system, so does $\phi \rightarrow E\psi$. To paraphrase it, if (it is a theorem that) ψ and everybody's belief in φ hold whenever φ does, then (it is a theorem that) everybody's belief in ψ holds whenever everybody's belief in φ holds. Hence a subclass of the logical implications provable in the system can be reproduced by the individuals' reasoning. The authorized implications involve what amounts to a syntactical rendering of belief closure. *Roughly speaking, belief closed sentences support monotonic reasoning on the agents' part*. Notice that our restriction is stated directly in terms of the "everybody believes" operator E, not the individual operators B_a. As to the axiom (Def.E) it defines E in terms of the B_a in the obvious way.

Now to the specific part of each system. For $QMCB_f$:

$$(\mathrm{PF}_f)\ C_f\varphi \rightarrow E(C_f\varphi \wedge \varphi)$$

$$(\mathrm{RI}_f)\ \frac{\chi \rightarrow E(\chi \wedge \varphi)}{\chi \rightarrow C_f\varphi}$$

For $QMCB_g$:

$$(\mathrm{PF}_g)\ C_g\varphi \rightarrow EC_g\varphi \wedge \varphi$$

$$(\mathrm{RI}_g)\ \frac{\chi \rightarrow E\chi \wedge \varphi}{\chi \rightarrow C_g\varphi}$$

For $QMCB_h$:

$$(\mathrm{PF}_h)\ C_h\varphi \rightarrow EC_h\varphi \wedge E\varphi$$

$$(\mathrm{RI}_h)\ \frac{\chi \rightarrow E\chi \wedge E\varphi}{\chi \rightarrow C_h\varphi}$$

We define the $\vdash$ inference relation of a given system in the following way, which is standard in modal logic generally. First, for any formula φ, we say that $\vdash \varphi$ holds if φ can be obtained as the last element in a finite sequence of formulae of the following specified form: either they are instances of axioms of the given system, or they are derived from previous formulae in the sequence by applying one of the inference rules of the system. Second, for any set Σ of formulae, and for any formula φ, $\Sigma \vdash \varphi$ holds if there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\vdash \wedge\Sigma_0 \to \varphi$ can be obtained, where $\wedge\Sigma_0$ is the conjunction of all formulae in Σ_0, taken in any order (this does not matter since (PL) is part of the systems). Notice once again the finiteness restrictions which are typical of the syntax.

When $\vdash \varphi$ holds, we say in words that φ is a *theorem of the system*, and when $\Sigma \vdash \varphi$ holds, that formula φ can be *proved in the system* from the set of *assumptions* Σ.

It is easy to list epistemically relevant theorems for the systems just introduced. We single out those expressing the fact that common belief implies shared belief of any order. For $QMCB_f$, we prove that $\vdash C_f\varphi \to E\varphi$ in the following way:

$\vdash C_f\varphi \to E(C_f\varphi \wedge \varphi)$ from (PF_f)
$\vdash C_f\varphi \wedge \varphi \to E(C_f\varphi \wedge \varphi) \wedge \varphi$
$\vdash E(C_f\varphi \wedge \varphi) \to E\varphi$ from (RQM_E).

And we inductively prove that for all k, $\vdash C\varphi \to E^k\varphi$ from the following:

$\vdash C_f\varphi \wedge \varphi \to E(C_f\varphi \wedge \varphi) \wedge E^{k-1}\varphi$
$\vdash E(C_f\varphi \wedge \varphi) \to E^k\varphi$ from (RQM_E)

Proofs are similar in the cases of $QMCB_g$ and $QMCB_h$.

The semantics of our systems hinges on set-theoretic mappings with an epistemic interpretation, as in the previous section. Given a nonempty set W, to be thought of as a set of states of the world, there will be individual belief mappings b_a, $a \in A$, to serve as a counterpart for the syntactical operators B_a, as well as derived mappings to serve as counterparts for the syntactical E and $C = C_f, C_g, C_h$. The previous section has paved the way for the formal definitions to come. First, we define a *quasi-monotonic belief structure* (or *model*) to be a tuple

$$m = \langle W, (b_a)_{a\in A}, v\rangle,$$

where W is a nonempty set, the b_a are mappings $\mathfrak{P}(W) \to \mathfrak{P}(W)$ such that the mapping b_E defined by $b_E \equiv \bigcap_{a\in A} b_a$ is quasi-monotonic, and v (called a *valuation* function) is a mapping $PV \to \{0, 1\}$. In this definition, the individual belief mappings b_a are restricted only indirectly, i.e., through the quasi-monotonicity constraint put on b_E. Readers who would rather assume that each b_a is quasi-monotonic and that b_E inherits this property are welcome to do so. The epistemic motivation for assuming quasi-monotonicity, rather than monotonicity, was explained earlier in connection with the semantic mappings.

Second, using the items in m, truth conditions for formulae φ are defined inductively in a standard way. We say that φ is *true in structure m at state w*, to be denoted by $(m, w) \models \varphi$, if:

- $v(p) = 1$ whenever $\varphi = p \in PV$,
- $(m, w) \models \psi$ and $(m, w) \models \psi$ whenever $\varphi = \psi \wedge \chi$, and the appropriate clauses for the remaining connectives $\neg$, $\vee$, $\rightarrow$, and $\longleftrightarrow$,
- $w \in b_a(\|\psi\|_W)$ whenever $\varphi = B_a\psi$,
- $w \in F_{\|\psi\|_W}$ whenever $w = C_f\psi$ (or: $w \in G_{\|\psi\|_W}$ whenever $w = C_g\psi$, or: $w \in H_{\|\psi\|_W}$ whenever $w = C_h\psi$),

where the abbreviation $\|\psi\|_W$ denotes the set $\{w' \in W \mid (m, w') \models \psi\}$.

Third, we say that φ is *true in structure m*, to be denoted by $m \models \varphi$, if $(m, w) \models \varphi$ holds for all $w \in W$. Let us denote by $\mathfrak{QM}$ the generic notion of a class of quasi-monotonic structures. It will mean either $\mathfrak{QM}_f$, or $\mathfrak{QM}_g$, or $\mathfrak{QM}_h$, since each semantic clause for common belief generates a class of quasi-monotonic belief structures. We say that φ is *true in* $\mathfrak{QM}$, a property denoted by

$$\mathfrak{QM} \models \varphi \text{ (or when the context is clear: } \models \varphi)$$

if $m \models \varphi$ holds for all $m \in \mathfrak{QM}$. We say that is φ a *semantic consequence* of Σ, to be denoted by

$$\Sigma \models \varphi,$$

if for all $m = \langle W, (b_a)_{a \in A}, v\rangle \in \mathfrak{QM}$ and all $w \in W$, $(m, w) \models \varphi$ whenever $(m, w) \vDash \Sigma$ (i.e., whenever $(m, w) \models \psi$ for all $\psi \in \Sigma$). This completes the definitions of the semantics.

4. Soundness and Completeness Theorems

The previous sections have provided separate definitions of the syntax and semantics of belief (knowledge) and common belief (common knowledge). In this section we will reconcile the two viewpoints in terms of a two-way theorem. In one direction, the *soundness* part, the theorem states that every formula that can be proved in the system from some set of assumptions can also be obtained as a semantic consequence from that set. Formally, for all formulae φ, and all sets of formulae Σ,

$$\Sigma \vdash \varphi \Longrightarrow \Sigma \models \varphi.$$

The other direction, or *strong completeness* part, states that every formula that can be obtained as a semantic consequence from some set of formulae can also be proved from this set, i.e., for all formulae φ, and all sets of formulae Σ,

$$\Sigma \models \varphi \Longrightarrow \Sigma \vdash \varphi.$$

In actual fact, there will be three theorems, since each syntactical system $QMCB_f$, $QMCB_g$,and $QMCB_h$ is mapped into a distinct class of semantic structures, $\mathfrak{QM}_f$, $\mathfrak{QM}_g$, and $\mathfrak{QM}_h$, respectively. We have grouped the results because the proofs can be carried simultaneously.

THEOREM 1. *$QMCB_f$, $QMCB_g$,and $QMCB_h$ are sound and strongly complete axiom systems for the classes $\mathfrak{QM}_f$, $\mathfrak{QM}_g$, and $\mathfrak{QM}_h$, respectively.*

Proof (For the soundness part). In order to establish soundness, it is sufficient to check that the axioms are true, and the rules validly apply, in $\mathfrak{QM}$. A proof of soundness for $QMCB_f$ and $\mathfrak{QM}_f$ can be found in Lismont and Mongin ([19]). As a variant, we give here the proof for $QMCB_h$ and $\mathfrak{QM}_h$. The proof for the remaining case follows a similar pattern.

The proof is an immediate application of the chosen definition of common belief. As to (FP_h), we have to show that $\|C_h\varphi\|_W \subseteq \|EC_h\varphi \wedge E\varphi\|_W$ for all $W \in \mathfrak{QM}_h$. We fix W in the ensuing argument, so we may drop the subscript in $\|\cdot\|_W$. Take $w \in \|C_h\varphi\| = H_{\|\varphi\|}$. There is $X \subseteq b_E(X) \cap b_E(\|\varphi)\|) = b_E(X) \cap \|E\varphi\|$ such that (i): $w \in X$. Then $X \subseteq b_E(X) \cap H_{\|\varphi\|}$. Quasi-monotonicity entails that (ii): $X \subseteq b_E(X) \subseteq b_E(H_{\|\varphi\|}) = \|EC_h\varphi\|$. The conclusion follows from (i) and (ii).

Concerning (RI_h), we have to check that:

if for all $W \in \mathfrak{QM}_h$, $\|\chi\|_W \subseteq \|E\chi \wedge E\varphi\|_W$,
then for all $W \in \mathfrak{QM}_h$, $\|\chi\|_W \subseteq \|C_h\varphi\|_W$.

If the assumption holds, then for any given W, $\|\chi\|_W \subseteq \|E\chi \wedge E\varphi\|_W = b_E(\|\chi\|_W) \cap b_E(\|\varphi\|_W) = h_{\|\varphi\|_W}(\|\chi\|_W)$. Then, for this W, $\|\chi\|_W \subseteq H_{\|\varphi\|_W} = \|C_h\varphi\|_W$. The argument holds regardless of the chosen W, which means that the conclusion holds. □

Proof (For the strong completeness part). Following a well-known proof technique, we construct a so-called canonical model for each of the three systems. A canonical model is but a quasi-monotonic belief structure $m = \langle W, (b_a)_{a\in A}, v\rangle$, where each of the symbols is filled in a particular way using as data the axioms and rules of the given system. The construction of m will be done stepwise.

First, we take W to be the set of maximal consistent sets Γ of formulae of the given system, a set henceforth denoted by MC. We will use the standard notation $|\varphi|$ to refer to $\{\Gamma \in MC \mid \varphi \in \Gamma\}$, i.e., the set of maximal consistent sets that contain the formula φ.[5]

Second, the individual belief mappings $b_a : \mathfrak{P}(MC) \to \mathfrak{P}(MC)$ are defined as follows:

$$\begin{aligned} b_a(X) &= |B_a\varphi| \quad \text{if } X = |\varphi|, \\ b_a(X) &= \bigcup\{|E\varphi| \mid |\varphi| \subseteq |E\varphi| \cap X\} \quad \text{otherwise.} \end{aligned}$$

The first part of this definition is coherent because of (RE_a). To see that, suppose that $|\varphi| = |\varphi'|$, then from a property of maximal consistent sets of formulas, we have that $\vdash \varphi \longleftrightarrow \varphi'$, and from ($\mathrm{RE}_a$), $\vdash B_a\varphi \longleftrightarrow B_a\varphi'$, or $|B_a\varphi| = |B_a\varphi'|$, so that $b_a(X)$ is the same whether φ or φ' is chosen to define it. Repeatedly in this proof, we will have to rely on a similar argument based on (RE_a), but we will not spell it out anymore.

Third, we define $b_E : \mathfrak{P}(MC) \to \mathfrak{P}(MC)$ by putting $b_E(X) = \bigcap_a b_a(X)$ as required by the definition of $m \in \mathfrak{QM}$. It remains to check that this definition also satisfies the main restriction, i.e., that b_E is quasi-monotonic. From (Def.E), it is easily seen that:

$$\begin{aligned} b_E(X) &= |E\varphi| \quad \text{if } X = |\varphi|, \\ b_E(X) &= \bigcup\{|E\varphi| \mid |\varphi| \subseteq |E\varphi| \cap X\} \quad \text{otherwise.} \end{aligned}$$

We show that these set-theoretic equations ensure quasi-monotonicity by using a lemma on quasi-monotonic mappings (the proof is given afterwards):

LEMMA 1. *For any nonempty set W, suppose that $\mathfrak{B}(W) \subseteq \mathfrak{P}(W)$, and $f : \mathfrak{B}(W) \to \mathfrak{B}(W)$ is quasi-monotonic. Then, the following mapping, which extends f to the whole of $\mathfrak{P}(W)$:*

$$\begin{aligned} f^*(P) &= f(P) \quad \textit{if } P \in \mathfrak{B}(W), \\ f^*(P) &= \bigcup\{f(P') \mid P' \in \mathfrak{B}(W) \textit{ and } P' \subseteq f(P') \cap P\} \quad \textit{otherwise,} \end{aligned}$$

is quasi-monotonic.

In view of Lemma 1, we just have to check that b_E is quasi-monotonic on the subset

$$\mathfrak{B}(W) = \{|\varphi| \mid \varphi \text{ is a formula}\}.$$

That is to say, we have to check that for all φ and ψ, if $|\varphi| \subseteq b_E(|\varphi|) \cap |\psi|$, then $b_E(|\varphi|) \subseteq b_E(|\psi|)$. This follows from ($\mathrm{RQM}_E$) and relevant properties of maximal consistent sets. The proof that the chosen canonical model m belongs to $\mathfrak{QM}$ is now complete.

We move on to proving that for all φ:

$$|\varphi| = \|\varphi\|.$$

(We are now dropping the subscript in $\|\cdot\|_{MC}$ because there is no ambiguity.) The proof goes by induction on the complexity of φ. The cases involving propositional connectives, like $\varphi = \varphi_1 \wedge \varphi_2$, are immediate. If $\varphi = B_a\psi$, the desired equation follows from the definition of b_a and the inductive hypothesis:

$$\|B_a\psi\| = b_a(\|\psi\|) = b_a(|\psi|) = |B_a\psi|,$$

and similarly for $\varphi = E\psi$. The case $\varphi = C\psi$, with $C = C_f, C_g, C_h$ is the object of the remainder of the proof, except for the very last paragraph.

Our aim is to reach the equation $\|C\psi\| = |C\psi|$. We make two preparatory steps:

(i) $|C\psi| = \bigcup\{|\chi| \mid |\chi| \subseteq f(|\chi|)\}$
(ii) $\bigcup\{|\chi| \mid |\chi| \subseteq f(|\chi|)\} = \bigcup\{X \subseteq MC \mid X \subseteq f(X)\}$,

where f serves as a generic symbol for three distinct mappings $\mathfrak{P}(MC) \to \mathfrak{P}(MC)$:

- $f_{|\psi|} = b_E(X \cap |\psi|)$
- $g_{|\psi|} = b_E(X) \cap |\psi|$
- $h_{|\psi|} = b_E(X) \cap b_E(|\psi|)$.

(These mappings were introduced in the first section to analyze the three concepts of common belief.)

We prove that (i) holds in the case of $QMCB_f$. The inclusion from left to right is based on (FP$_f$). Suppose that $\Gamma \in |C\psi|$, or $C\psi \in \Gamma$. Then, $E(C\psi \wedge \psi) \in \Gamma$. Putting $\chi = C\psi$, we see that $\Gamma \in |\chi|$ with $|\chi| \subseteq |E(\chi \wedge \psi)| = b_E(|\chi| \cap \psi) = f_{|\psi|}(|\chi|)$, as required. The inclusion from right to left is based on (RI$_f$). Take χ such that $\Gamma \in |\chi| \subseteq |E(\chi \wedge \psi)|$. Then, $\vdash \chi \to E(\chi \wedge \psi)$, and $\vdash \chi \to C\psi$ follows, which is $|\chi| \subseteq |C\psi|$. So $\Gamma \in |C\psi|$. The argument for the other systems is similar: one direction depends on the appropriate version of (FP), the other on the appropriate version of (RI).

As to (ii), it will follow from another lemma on quasi-monotonic mappings (also proved below):

LEMMA 2. *For any nonempty set* W, *assume that* $\mathfrak{B}(W) \subseteq \mathfrak{P}(W)$, f_0 *and* f *are two mappings* $\mathfrak{B}(W) \to \mathfrak{B}(W)$, f *is quasi-monotonic, and for all* $X \in \mathfrak{B}(W)$,

$$X \subseteq f_0(X) \Longrightarrow X \subseteq f(X).$$

Take any mapping f^* *which coincides with* f *on* $\mathfrak{B}(W)$, *and otherwise satisfies either of these two clauses:*

(a) $\quad f^*(X) = f(X')$ *for some* $X' \in \mathfrak{B}(W)$ *s.t.* $X' \subseteq X$;

(b) $\quad f^*(X) = \bigcup\{f(X') \mid X' \in \mathfrak{B}(W), X' \subseteq f(X') \cap X\}$.

Then:

$$\bigcup\{X \in \mathfrak{P}(W) \mid X \subseteq f^*(X)\} = \bigcup\{X \in \mathfrak{B}(W) \mid X \subseteq f(X)\}.$$

We obtain (ii) by putting $W = MC$, $\mathfrak{B}(W) = \mathfrak{B}(MC)$, and showing that the assumptions of the lemma are met in the case of each system *QMCB*. Here is a detailed argument for $QMCB_f$. We have to check that $f_{|\psi|}$ can be viewed as the mapping f^* of the lemma, with:

$$\begin{aligned} f(|\chi|) &= f_{|\psi|}(|\chi|) = b_E(|\chi| \cap |\psi|), \\ f_0(|\chi|) &= b_E(|\chi|) \cap |\psi|. \end{aligned}$$

Clearly, the definitions for f_0 and f satisfy the condition in the lemma that for all $X \in \mathfrak{B}(MC)$,

$$X \subseteq f_0(X) \Longrightarrow X \subseteq f(X).$$

Now, to see that $f_{|\psi|}$ is as requested, suppose that $X \notin \mathfrak{B}(W)$. If $X \cap |\psi| = |\chi|$, $f_{|\psi|}(X) = b_E(|\chi|) = f(|\chi|)$, and clause (a) of the lemma is satisfied. Otherwise, from the definition of b_E:

$$f_{|\chi|}(X) = b_E(X \cap |\psi|) = \bigcup\{|E\chi| \mid |\chi| \subseteq |E\chi| \cap X \cap |\psi|\}.$$

Each of the χ in the union is such that $|\chi| = |\chi| \cap |\psi|$. Hence,

$$\begin{aligned} f_{|\chi|}(X) &= \bigcup\{b_E(|\chi| \cap |\psi|) \mid |\chi| \subseteq b_E(|\chi|) \cap |\psi| \cap X\} \\ &= \bigcup\{f(|\chi|) \mid |\chi| \subseteq f_0(|\chi|) \cap X\} \end{aligned}$$

which means that clause (b) is satisfied.

The other systems are handled similarly by means of Lemma 2. The mapping for $QMCB_g$, i.e., $g_{|\psi|}(X)$, is of the prescribed form f^* with the following definitions for f and f_0:

$$\begin{aligned} f(|\chi|) &= g_{|\psi|}(|\chi|) = b_E(|\chi|) \cap |\psi|); \\ f_0(|\chi|) &= b_E(|\chi|). \end{aligned}$$

So is the mapping for $QMCB_g$, i.e., $h_{|\psi|}(X)$, with:

$$\begin{aligned} f(|\chi|) &= h_{|\psi|}(|\chi|) = b_E(|\chi|) \cap b_E|\psi|), \\ f_0(|\chi|) &= b_E(|\chi|). \end{aligned}$$

This completes the proof of step (ii).

Once the two preparatory steps (i) and (ii) are secured, the proof that $\|C\psi\| = |C\psi|$ derives at once:

$$\begin{aligned} \|C\psi\| &= \bigcup\{X \subseteq MC \mid X \subseteq b_E(X \cap \|\psi\|)\} \\ &= \bigcup\{X \subseteq MC \mid X \subseteq b_E(X \cap |\psi|)\} \\ &\qquad \text{from the inductive hypothesis} \\ &= \bigcup\{X \subseteq MC \mid X \subseteq f_{|\psi|}(X)\} \\ &= \bigcup\{|\chi| \mid |\chi| \subseteq f_{|\psi|}(|\chi|)\} \quad \text{from (ii)} \\ &= |C_f\psi| \quad \text{from (i).} \end{aligned}$$

The end of the proof follows a familiar pattern in modal epistemic logic. Suppose that it is not the case that $\Sigma \vdash \varphi$. Then, $\Sigma \cup \{\neg\varphi\}$ is consistent, hence $\Sigma \cup \{\neg\varphi\} \subseteq \Gamma$ for some maximal consistent set Γ. In the canonical model m, the equation $|\varphi'| = \|\varphi'\|$ holds for all formulae φ'; hence $(m, \Gamma) \models \varphi'$ for all $\varphi' \in \Sigma \cup \{\neg\varphi\}$. We conclude that it is not the case that $\Sigma \models \varphi$. □

Proof of Lemma 1. We must show that for all $X, Y \subseteq W$,

$$X \subseteq f^*(X) \cap Y \Longrightarrow f^*(X) \subseteq f^*(Y).$$

The cases where $X, Y \in \mathfrak{B}(W)$ and $X, Y \notin \mathfrak{B}(W)$ directly follow from the assumptions. If $X \in B(W)$, $Y \notin B(W)$, the antecedent becomes $X \subseteq f(X) \cap Y$, whence $f(X) = f^*(X) \subseteq f^*(Y)$. For the case where $X \notin B(W)$ and $Y \in B(W)$, take $w \in f^*(X)$, i.e., $w \in f(X')$ with $X' \subseteq f(X') \cap X \subseteq f(X') \cap Y$. The quasi-monotonicity of f entails that $f(X') \subseteq f(Y) = f^*(Y)$, so we have shown that $w \in f^*(Y)$. □

Proof of Lemma 2. Put

$$\begin{aligned} F^* &= \bigcup\{X \in \mathfrak{P}(W)) \mid X \subseteq f^*(X)\} \quad \text{and} \\ F &= \bigcup\{X \in \mathfrak{B}(W) \mid X \subseteq f(X)\}. \end{aligned}$$

The inclusion $F \subseteq F^*$ is trivial. For the converse inclusion, take $w \in F^*$. There is $X \in \mathfrak{P}(W)$ such that $X \subseteq f^*(X)$. Then:

- If $X \in B(W)$, then $f^*(X) = f(X)$, and the conclusion that $w \in F$ follows immediately.
- If $f^*(X) = f(X')$ for some $X' \in \mathfrak{B}(W)$ such that $X' \subseteq X$, then the reflective-monotonic property of f entails that $f(X') \subseteq F$. As $w \in X \subseteq f^*(X) = f(X')$, the conclusion follows again.
- If $f^*(X) = \bigcup\{f(X') \mid X' \in \mathfrak{B}(W), X' \subseteq f_0(X') \cap X\}$, there is $X' \in \mathfrak{B}(W)$ such that $X' \subseteq f_0(X')$ and $w \in f(X')$. By assumption on f and f_0, $X' \subseteq f(X')$. So $f(X') \subseteq f(f(X'))$, $f(X') \subseteq F$, and the conclusion follows again. □

COROLLARY 1. *The rule*

$$\frac{\varphi \to \psi}{C\varphi \to C\psi}$$

can be derived in QMCB$_f$ and QMCB$_g$ but not QMCB$_h$. The rule

$$\frac{\varphi \to C\varphi \wedge \psi}{C\varphi \to C\psi}$$

can be derived in QMCB$_h$. The rule

$$\frac{\varphi}{C\varphi}$$

cannot be derived in any system.

Proof. The fact that the stated rules can be derived follows immediately from soundness and completeness.

To prove that the rule

$$\frac{\varphi \to \psi}{C\varphi \to C\psi}$$

cannot be derived in *QMCB$_h$*, take $p, p' \in PV$ and m a model where:

- W and b_E are the same as in point (3) of Proposition 4;
- the valuation is such that $\|p\|_m = \{1\}$ and $\|p'\|_m = \{1, 2\}$.

As $p \to p \vee p'$ is a theorem of *QMCB$_h$*, if the rule were true, $Cp \to C(p \vee p')$ should also be a theorem. By soundness it should be true in any world of any structure of the appropriate class. But this is not the case in m, as the following shows: $\|Cp\|_m = \{0, 2\}$ since $H_{\{1\}} = \{0, 2\}$, but $\|C(p \vee p')\| = \emptyset$ since $\|p \vee p'\|_m = \{1, 2\}$ and $H_{\{1,2\}} = \emptyset$.

To prove that the rule

$$\frac{\varphi}{C\varphi}$$

cannot be derived it is sufficient to exhibit a model in which F_W (resp. G_W, H_W) is not W. Let m satisfy $W = \{0\}$ and $b_E(\varnothing) = b_E(W) = \varnothing$. Then $F_W = G_W = H_W = \emptyset \neq W$. □

5. Technical Comments and Comparisons

The theorem of this paper contributes two things. For one, it extends the earlier theorem of Lismont and Mongin in [19] for quasi-monotonic logics to two notions of common belief that the authors did not consider at the time, i.e., those of Mertens and Zamir in [22] and Monderer and Samet in [23], respectively. It is interesting to find out that these alternative concepts are amenable to a satisfactory logical treatment under the very weak epistemic assumption of quasi-monotonicity.[6] This result agrees with the authors' position that common belief is not as demanding a notion as it first seems – or at least as it first seemed from the classic work, like Aumann's [1], that introduced it. Briefly put, it makes good sense to speak of common belief taking place between human agents who are very imperfect logicians, as in the present framework. This observation will perhaps not be wasted to game theorists since it heuristically leads to the following practical recommendation: *weaken the agents' cognitive abilities, but retain the standard assumption that the rules of the game are common belief among the players*.

Second, and more importantly from the logical viewpoint, this paper has proved what appears to be the first *strong* completeness theorem of the common belief and common knowledge logics. Using the notation of the previous section, the previously available theorems have stated that:

$$\models \phi \iff \ \vdash \phi,$$

which corresponds to the particular case of our result $\Sigma = \emptyset$. As modal logicians are well aware, the difference is irrelevant as far as soundness goes, but there is a major difference as far as completeness goes.

We briefly point out where in the proof we departed from the argument usually made. The previous axiomatizations of common belief used a filtration technique of proof which can only deliver completeness *simpliciter*. Essentially, the language is relativized to the particular formula φ for which the implication $\models \phi \implies \vdash \phi$ should be proved. This makes it possible to consider only a finite number of maximal consistent sets of sentences, a crucial step to take the reasoning to its successful end. Here, the device of filtration was made unnecessary by the use of Lemmas 1 and 2.

Strongly complete systems enjoy the desirable property of *compactness*, i.e.:

if in a set of formulae Σ, each finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable, then Σ is satisfiable.

"Satisfiable" is defined here as "true in some world w of some structure". The Kripke logics of common belief are *not* compact. The standard argument to establish this negative result involves taking

$$\Sigma = \{\neg C\varphi\} \cup \{E\varphi, E^2\varphi, \ldots, E^k\varphi, \ldots\},$$

where $E^k\varphi$ means $E \ldots E\varphi$ with k repetitions of E. Each finite subset $\Sigma_0 \subset \Sigma$ is such that $\Sigma_0 \subseteq \{\neg C\varphi, E\varphi, \ldots, E^k\varphi\}$ for some k. Using the semantic definitions for Kripke logics, Σ_0 can be shown to be satisfiable, whereas Σ cannot be so. The counterexample is powerful. It does not trade merely on the semantic definition of common belief in Kripke structures, i.e.:

(K) $\quad (m, w) \models C\varphi \quad \textit{iff} \quad (m, w) \models E^k\varphi \quad \textit{for all integers } k$

but on the proven fact that alternative semantic definitions in terms of fixed-points automatically collapse to clause (K). However, the counterexample does *not* apply to weaker common belief systems than the Kripke one. It has again been proven that for weaker logics such as the monotonic one, the equivalence between (K) and the fixed-point definitions of common belief does not hold anymore.[7] In other words, the counterexample does not work for these logics, given the fixed-point definition of common belief.

Of course, the failure of the standard counterexample in the non-Kripkean case does not amount to a proof. It just delivers a heuristic hint that weaker systems *may* be compact after all. The present article has exhibited a family of such systems that *are* compact, i.e., the quasi-monotonic systems $QMCB_f$, $QMCB_g$ and $QMCB_h$. We do not yet know whether the good news holds good of the monotonic axiomatizations. It seems as if the technique of proof used in this paper cannot be transferred straightforwardly to the case where (RQM_E) is replaced by the standard monotonicity rule:

$$\frac{\varphi \to \psi}{E\varphi \to E\psi}.$$

Systems of this sort raise an interesting open question for modal epistemic logic.

NOTES

[1] Some early examples of these applications are Bacharach's [3], Milgrom and Roberts's [21] and Monderer and Samet's [23].

[2] The distinction between the iterative and the fixed-point approaches was introduced by Barwise in [4]. It is further explored along Barwise's lines by Lismont in [17] and, from a different point of view, by Heifetz in [12].

[3] Game theorists and mathematical economists are growing increasingly aware of the rich potential of *nonpartitional* models, which in essence preserve the properties of a Kripke structure, but avoid the introspective assumptions. See the discussions by Dekel and Gul in [7] and Battigalli and Bonanno in [5].

[4] The approach of standard epistemic logic should be contrasted with that of infinitary logic, which allows for disjunctions or conjunctions of countably many formulae. Kaneko and Nagashima ([15]) and Heifetz ([11]) axiomatize the iterative notion of common knowledge within infinitary modal logics. Kaneko ([14]) and Heifetz ([12]) compare the common knowledge concept defined in this way with the finitary concept of standard epistemic logic.

[5] The existence and standard properties of maximal consistent sets of formulae will be taken for granted here; see modal logic texts such as Chellas's ([6]) or Hughes and Cresswell's ([13]).

[6] Compare with Heifetz's work in [10]. His paper contains a *monotonic* axiomatization of common belief in the MS sense, and a comparison with the *monotonic* axiomatization in the LM sense.

[7] The equivalences and non-equivalences of this paragraph were discussed by Halpern and Moses in the Appendix to [9], and by Lismont and Mongin in [19]. They are further clarified by Heifetz in [12].

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