

Asymmetric Information: Walrasian Equilibria, and Rational Expectations Equilibria

1 Basic Setup

- Two periods: 0 and 1.
- One riskless asset with interest rate r .
- One risky asset which
 - pays a normally distributed dividend d in period 1:

$$d \sim N(\bar{d}, \sigma^2);$$

- has price p at $t = 0$;
- has supply $S > 0$.

- N agents:
 - consume only in period 1;
 - have exponential utility with CARA coefficient α ;
 - out of the N agents, N_I are informed, and N_U are uninformed: $N = N_I + N_U$.
 - * The informed agents observe a signal about d :

$$s = d + \varepsilon, \quad \varepsilon \sim N(0, \sigma_\varepsilon^2),$$

such that ε and d are independent.

- * The uninformed observe no signal.

- The previous model is called the **asymmetric information** model
 - Asymmetric information models analyze how prices *incorporate* information.
- An alternative model, called the **differential information** model, is one where each agent observes a different signal:

$$s^i = d + \varepsilon^i, \quad \varepsilon^i \sim N(0, \sigma_\varepsilon^2),$$

such that ε^i and d are independent.

- Differential information model analyze how prices *aggregate* information.

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2 Walrasian Equilibrium

- We first determine the Walrasian equilibrium (WE) of this economy. We will show that, in the presence of asymmetric information, the WE is not a satisfactory equilibrium concept.
- To determine the WE, we need to determine the agent demands.
- Consider an **uninformed agent** with wealth W_0 , who buys x shares of the risky asset. His wealth in period 1 will be

$$W = (W_0 - xp)(1 + r) + xd.$$

To calculate the expected utility of wealth, use the following formula: if $X \sim N(\mu, \sigma^2)$,

$$\mathbf{E}(\exp(X)) = \exp(\mu + \frac{1}{2}\sigma^2).$$

The expected utility of the uninformed is:

$$-\mathbf{E}(\exp(-\alpha W)) = -\exp\left(-\alpha((W_0 - xp)(1 + r) + x\bar{d}) + \frac{1}{2}\alpha^2 x^2 \sigma^2\right).$$

So the uninformed solves:

$$\max_x \left((W_0 - xp)(1 + r) + x\bar{d} - \frac{1}{2}\alpha\sigma^2 x^2 \right).$$

The solution to this problem is the demand of the uninformed (given price p):

$$x_U(p) = \frac{\bar{d} - (1 + r)p}{\alpha\sigma^2}.$$

The demand is independent of W_0 because of exponential utility.

- Consider an **informed agent** with wealth W_0 , who buys x shares of the risky asset. We have the same formulas as above, except that the agent optimizes *conditional* on his signal s . The demand of the informed is then:

$$x_I(s, p) = \frac{\mathbf{E}(d|s) - (1 + r)p}{\alpha \text{var}(d|s)}.$$

I will show below that

$$\mathbf{E}(d|s) = \bar{d} + \beta_s(s - \bar{d}),$$

where

$$\beta_s = \frac{\text{cov}(s, d)}{\text{var}(s)} = \frac{\text{cov}(d + \varepsilon, d)}{\text{var}(d + \varepsilon)} = \frac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2}.$$

I will also show that

$$\text{var}(d|s) = \mathbf{E}(\text{var}(d|s)) = \text{var}(d) - \beta_s^2 \text{var}(s) = \frac{\sigma^2 \sigma_\varepsilon^2}{\sigma^2 + \sigma_\varepsilon^2}.$$

Denote

$$\sigma_{d|s}^2 = \frac{\sigma^2 \sigma_\varepsilon^2}{\sigma^2 + \sigma_\varepsilon^2}.$$

- I will prove a more general result. Consider $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ jointly normal. Then we have the following results conditional on x (where $\text{var}(x)$ is the covariance $n \times n$ matrix):

$$\begin{aligned} \mathbf{E}(y|x) &= E(y) + (x - \mathbf{E}(x))^\top \beta_y, \quad \text{with} \quad \beta_y = \text{var}(x)^{-1} \text{cov}(x, y), \\ \text{var}(y|x) &= \text{var}(y) - \beta_y^\top \text{var}(x) \beta_y = \text{var}(y) - \text{cov}(x, y)^\top \text{var}(x)^{-1} \text{cov}(x, y). \end{aligned}$$

The way to prove this is to consider an orthogonal decomposition

$$y - \mathbf{E}(y) = (x - \mathbf{E}(x))^\top \beta_y + z,$$

where z and x are independent (as normal variables, it is enough if they are uncorrelated). Multiply the above equation on the left by $x - \mathbf{E}(x)$, and take expectations on both sides:

$$\text{cov}(x, y) = \text{var}(x) \beta_y.$$

But this is equivalent to the desired formula for β_y . Now, use again the above decomposition, and take variances on both sides. Since z and x are uncorrelated, we get:

$$\text{var}(y) = \beta_y^\top \text{var}(x) \beta_y + \text{var}(z).$$

But in this case, $\text{var}(z) = \text{var}(y|x)$, so we are done. To prove this last equation, start with $\text{var}(y|x) = \mathbf{E}\left((y - \mathbf{E}(y|x))^2 \mid x\right) = \mathbf{E}(z^2|x) = \mathbf{E}(z^2) = \text{var}(z)$, with the next to last equation coming from the independence of z and x . In particular, I also showed that $\text{var}(y|x)$ is constant, i.e., does not depend on x .

- To determine equilibrium price, consider the market clearing equation:

$$N_I x_I(s, p) + N_U x_U(p) = S.$$

Combining the various equations, the equilibrium price is:

$$p = \frac{\bar{d}}{1+r} + \frac{N_I \beta_s (s - \bar{d})}{(1+r)(N_I + N_U \frac{\sigma_{d|s}^2}{\sigma^2})} - \frac{S \alpha \sigma_{d|s}^2}{(1+r)(N_I + N_U \frac{\sigma_{d|s}^2}{\sigma^2})}.$$

The price is the sum of three terms:

- the discounted value of the expected dividend;
- the price reaction to the signal of the informed;
- the risk premium.

- The problem with the WE concept is the following: The price fully reveals the signal of the informed. However, the uninformed do not take this into account when forming their demand.

3 Rational Expectations Equilibrium

- The REE captures the notion that agents learn from prices.
- To define the REE in the most general setting, start with a setup as before, with N agents. Suppose agent i has expected utility U^i , initial wealth W_0^i , and receives signal s^i .

- **DEFINITION:** An REE is determined by a price function $p(s^1, \dots, s^N)$ and a vector of demands $(x^1(s^1, p), \dots, x^N(s^N, p))$, such that:

- (Optimization) For agent i , $x = x^i(s^i, p)$ solves the problem

$$\max_x \mathbf{E} \left(U^i((W_0^i - xp)(1 + r) + xd) \mid s^i, p \right);$$

- (Market clearing)

$$\sum_{i=1}^N x^i(s^i, p) = S.$$

- The differences between REE and WE are the following:
 - The WE involves a price, while the REE involves a price *function*, mapping agents' private information to a price.
 - In the WE agents maximize conditional on their private information, while in the REE they maximize conditional on their private information *and* the price. An agent can use the price to extract information on the signals of the other agents, since the price is a function of the signals.
 - Therefore, in the REE the price plays a dual role: (i) it affects agents' budget constraint, and (ii) it affects the inferences agents make about the signals of the other agents. In the WE the price plays only the first role.
- In the **asymmetric information** setup, there are N_I informed agents who get a signal s , and N_U uninformed agents. An REE is given by:
 - A price function $p(s)$ mapping the signal of the informed to the price.
 - A demand function $x_I(s, p)$ for each informed agent.
 - A demand function $x_U(p)$ for each uninformed agent.

- **THEOREM:** The solution to this problem is given by (assume $N_I > 0$):

$$\begin{aligned} p(s) &= \frac{\bar{d}}{1+r} + \frac{\beta_s(s - \bar{d})}{1+r} - \frac{S\alpha\sigma_{d|s}^2}{(1+r)N}, \\ x_I(s, p) &= \frac{\bar{d} + \beta_s(s - \bar{d}) - (1+r)p}{\alpha\sigma_{d|s}^2}, \\ x_U(p) &= \frac{S}{N}. \end{aligned}$$

- **PROOF:** One can do a constructive proof, by assuming linear price $p(s)$ and demands $x_I(s, p)$ and $x_U(p)$. Alternatively, let us prove the above formulas directly. Show first that the demands are optimal. The optimization problem of the informed is the same as for the WE, since the price does not convey any additional information relative to the signal. Then

the optimal demand $x_I(s, p)$ is the same as for the WE. The optimization problem of the uninformed is different, since in a REE they condition on price:

$$\max_x -\mathbf{E} \exp\left(-\alpha((W_0^i - xp)(1+r) + xd) \mid p\right).$$

To compute this, we use the results from the previous section. We need to determine the conditional distribution of d given p . The conditional mean of d given p is:

$$\mathbf{E}(d|p) = \bar{d} + \beta_p(p - \bar{p}),$$

where (using the given formula for p)

$$\begin{aligned} \bar{p} &= \frac{\bar{d}}{1+r} - \frac{S\alpha\sigma_{d|s}^2}{(1+r)N}, \quad \text{and} \\ \beta_p &= \frac{\text{cov}(p, d)}{\text{var}(p)} = \frac{\frac{\beta_s}{1+r}\text{cov}(s, d)}{\left(\frac{\beta_s}{1+r}\right)^2 \text{var}(s)} = \frac{1+r}{\beta_s} \frac{\text{cov}(s, d)}{\text{var}(s)} = \frac{1+r}{\beta_s} \beta_s = 1+r. \end{aligned}$$

The conditional variance of d given p is:

$$\sigma_{d|p}^2 = \text{var}(d) - \beta_p^2 \text{var}(p) = \text{var}(d) - (1+r)^2 \left(\frac{\beta_s}{1+r}\right)^2 \text{var}(s) = \text{var}(d) - \beta_s^2 \text{var}(s) = \sigma_{d|s}^2.$$

Therefore the optimal demand of the uninformed is:

$$x_U(p) = \frac{\mathbf{E}(d|p) - (1+r)p}{\alpha\sigma_{d|p}^2} = \frac{\bar{d} - (1+r)\bar{p}}{\alpha\sigma_{d|s}^2} = \frac{S}{N}.$$

Next we prove market clearing. Plug the price formula into the demand of the informed:

$$x_I(s, p) = \frac{\bar{d} + \beta_s(s - \bar{d}) - (1+r)\left(\frac{\bar{d}}{1+r} + \frac{\beta_s(s - \bar{d})}{1+r} - \frac{S\alpha\sigma_{d|s}^2}{(1+r)N}\right)}{\alpha\sigma_{d|s}^2} = \frac{S}{N}.$$

Therefore

$$N_I x_I(s, p) + N_U x_U(p) = N_I \frac{S}{N} + N_U \frac{S}{N} = S.$$

- The price function in the REE fully reveals the signal of the informed, and is thus fully informative. Moreover, the price function is the same as the WE price in a fictitious economy where all agents are informed.

– Indeed, by setting $N_I = N$ and $N_U = 0$ in the price equation for the WE, we get exactly the price equation for the REE.

The allocation of the REE is also the same as the WE allocation in the fictitious economy, since both the informed and the uninformed agents get S/N .

- The intuition for these results is that the uninformed demand inelastically S/N , which is their allocation of the asset supply according to the optimal risk-sharing rule (the uninformed thus submit a “market order”). Given this demand of the uninformed, the informed can only get S/N . The price is the same as the WE price in the fictitious economy, since the price is set by the informed who in both cases get the same allocation.
- In the REE, the informed thus set the price, and the uninformed free-ride on the price discovery process. The intuition why the uninformed submit price-inelastic demands is that the price is fully revealing. If the price is low, this increases demand, holding information constant. However, if the price is low, this also reveals negative information, and decreases demand. The two effects cancel, and demand is price-inelastic.
- The REE describes well a phenomenon that we observe in the real-world: indexing. Indeed, the uninformed demand inelastically their allocation of the asset supply according to the optimal risk-sharing rule. We can thus interpret the uninformed as “indexers,” following a passive investment strategy of buying their share of the market portfolio.
- Since the informed and the uninformed get the same allocation, they make also the same profits. This creates a paradox: the **Grossman–Stiglitz paradox**: If information acquisition is costly, why would any agent acquire information? One resolution of the Grossman–Stiglitz paradox is to introduce noise.

4 Noisy Rational Expectations Equilibrium

- Same model as for the REE, except that supply of the risky asset is random: $S + u$, where the “noise” $u \sim N(0, \sigma_u^2)$. The random supply can be interpreted literally as a random number of shares issued by the firm, but can also be interpreted as a random number of shares that some traders outside the model dump into the market. We refer to such traders as “noise traders.”
- There are two reasons for introducing noise:
 - It can provide a resolution of the Grossman–Stiglitz paradox. In the presence of noise, the price will not be fully informative, and the uninformed will make smaller profits than the informed.
 - It can make the model more realistic. In the absence of noise, the price is fully informative because the uninformed know all the parameters of the model except the signal of the informed. Therefore, it is realistic to assume that agents do not know other market parameters. The simplest way to introduce parameter uncertainty is to assume a random asset supply.
- When the supply is random, REE generalizes to noisy REE. To define the noisy REE, we consider the general setting where there are N agents, and where the i 'th agent has expected utility U^i , wealth W_0^i , and receives signal s^i .
- **DEFINITION:** A noisy REE is determined by a price function $p(s^1, \dots, s^N, u)$ and a vector of demands $(x^1(s^1, p), \dots, x^N(s^N, p))$, such that:

– (Optimization) For agent i , $x = x^i(s^i, p)$ solves the problem

$$\max_x \mathbf{E} \left(U^i((W_0^i - xp)(1+r) + xd) \mid s^i, p \right);$$

– (Market clearing)

$$\sum_{i=1}^N x^i(s^i, p) = S + u.$$

- The price function in the noisy REE depends on agents' signals and on the noise. In our asymmetric information model, a noisy REE is given by (i) a price function $p(s, u)$, (ii) a demand $x_I(s, p)$ for each informed agent, and (iii) a demand $x_U(p)$ for each uninformed agent.
- **THEOREM:** There exists a noisy REE in which the price is given by

$$p(s, u) = A + B((s - \bar{d}) - Cu).$$

for three constants A , B and C .

- **PROOF:** First, determine agents' demands. As in the case of REE, the optimization problem of the informed is the same as for the WE, since the price does not convey any additional information relative to the signal. Then the optimal demand $x_I(s, p)$ is the same as for the WE. The optimization problem of the uninformed is different:

$$\max_x -\mathbf{E} \exp \left(-\alpha((W_0^i - xp)(1+r) + xd) \mid p \right).$$

To compute this, we need to determine the conditional distribution of d given p . The conditional mean of d given p is:

$$\mathbf{E}(d|p) = \bar{d} + \beta_p(p - A),$$

where

$$\beta_p = \frac{\text{cov}(p, d)}{\text{var}(p)} = \frac{B \text{cov}(s, d)}{B^2(\text{var}(s) + C^2 \text{var}(u))} = \frac{\sigma^2}{B(\sigma^2 + \sigma_\varepsilon^2 + C^2 \sigma_u^2)}.$$

The conditional variance of d given p is:

$$\sigma_{d|p}^2 = \text{var}(d) - \beta_p^2 \text{var}(p) = \sigma^2 - \frac{\sigma^4}{\sigma^2 + \sigma_\varepsilon^2 + C^2 \sigma_u^2} = \frac{\sigma^2(\sigma_\varepsilon^2 + C^2 \sigma_u^2)}{\sigma^2 + \sigma_\varepsilon^2 + C^2 \sigma_u^2}.$$

The optimal demand of the uninformed is:

$$x_U(p) = \frac{\mathbf{E}(d|p) - (1+r)p}{\alpha \sigma_{d|p}^2} = \frac{\bar{d} + \beta_p(p - A) - (1+r)p}{\alpha \sigma_{d|p}^2}.$$

Next we need to show that there exist values A , B and C so that the market clears, i.e.

$$N_I x_I(s, p) + N_U x_U(p) = S + u.$$

Using the formulas above, this is the same as:

$$N_I \frac{\bar{d} + \beta_s(s - \bar{d}) - (1+r)p}{\alpha\sigma_{d|s}^2} + N_U \frac{\bar{d} + \beta_p(p - A) - (1+r)p}{\alpha\sigma_{d|p}^2} = S + u.$$

Plug the price equation $p = A + B((s - \bar{d}) - Cu)$ into the above equation, to get a formula linear in $s - \bar{d}$ and u . Identifying the constant term and the coefficients, we get three equations. The constant terms yield:

$$N_I \frac{\bar{d} - (1+r)A}{\alpha\sigma_{d|s}^2} + N_U \frac{\bar{d} - (1+r)A}{\alpha\sigma_{d|p}^2} = S.$$

Solve for A :

$$A = \frac{\bar{d}}{1+r} - \frac{S\alpha\sigma_{d|s}^2}{(1+r)\left(N_I + N_U \frac{\sigma_{d|s}^2}{\sigma_{d|p}^2}\right)}.$$

From the equality of the coefficients of $s - \bar{d}$, we get:

$$N_I \frac{\beta_s - (1+r)B}{\alpha\sigma_{d|s}^2} + N_U \frac{\beta_p B - (1+r)B}{\alpha\sigma_{d|p}^2} = 0.$$

This is equivalent to solving the equation:

$$B = \frac{N_I\beta_s + N_U \frac{\sigma_{d|s}^2}{\sigma_{d|p}^2} \beta_p B}{(1+r)\left(N_I + N_U \frac{\sigma_{d|s}^2}{\sigma_{d|p}^2}\right)}.$$

From the equality of the coefficients of u , we get:

$$N_I \frac{(1+r)BC}{\alpha\sigma_{d|s}^2} + N_U \frac{-\beta_p BC + (1+r)BC}{\alpha\sigma_{d|p}^2} = 1.$$

Multiplying the equation for $s - \bar{d}$ by C and adding it to the equation for u , we get

$$N_I \frac{\beta_s C}{\alpha\sigma_{d|s}^2} = 1.$$

This implies

$$C = \frac{\alpha\sigma_\varepsilon^2}{N_I}.$$

We have now determined A , B and C , so we have showed the existence of a linear equilibrium.

- The constant C is an important parameter of the equilibrium. It measures the relative effect of information and noise shocks on the price: A unit noise shock has the same effect on the price as an information shock of size C . The formula shows that C decreases in the number of informed agents N_I , and increases in the CARA coefficient α , and the signal noise σ_ε^2 .

- The intuition for these results is that the relative effect of information and noise shocks on the price is determined by the informed agents, who can distinguish between the two shocks.
 - Noise shocks will have a small effect ($C \approx 0$) if there are many informed agents, if informed agents are not very risk-averse, or if they get precise signals.
 - This is because in these cases informed agents trade aggressively on their information.
- Define the **price informativeness** by the inverse of the conditional variance of d given p :

$$\tau = \frac{1}{\sigma_{d|p}^2} = \frac{\sigma^2 + \sigma_\varepsilon^2 + C^2\sigma_u^2}{\sigma^2(\sigma_\varepsilon^2 + C^2\sigma_u^2)}.$$

The price informativeness decreases in the supply noise σ_u^2 and in C . It thus increases in the number of informed agents N_I , and decreases in the CARA coefficient α and signal noise σ_ε^2 .

- The price is fully informative, i.e., $\sigma_{d|p}^2 = \sigma_{d|s}^2$ only in the limit where there is an infinite number of informed agents, informed agents are risk-neutral, or get perfect signals.
- Define the **price sensitivity of the uninformed** by:

$$\frac{dx_U(p)}{dp} = \frac{\beta_p - (1+r)}{\alpha\sigma_{d|p}^2}.$$

Using the formulas for β_p and B , one can show that the price sensitivity has the same sign as

$$\frac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2 + C^2\sigma_u^2} - \frac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2} < 0.$$

The uninformed therefore submit price-elastic demands, which is in contrast to the case where there is no noise. This is because of the two canceling effects:

- A high price decreases demand, holding information constant.
- A high price reveals positive information, hence increases demand.

With noise, the second effect is weaker, because the price is not fully informative.

- This implies that, since demand is price-elastic, the uninformed sometimes trade “against” the information of the informed. They do so because they confuse information with noise: Suppose the informed get a negative signal, but that the realization of u is low, i.e., the price p is higher than it should be in the absence of noise. Then the uninformed interpret the higher price as coming (at least partially) from positive information – so they will demand more of the asset. So the uninformed would trade in the opposite direction to the informed.
- Define the **price responsiveness to information** as $dp/ds = B$. One can show:

$$\frac{N_I\beta_s}{(1+r)\left(N_I + N_U\frac{\sigma_{d|s}^2}{\sigma^2}\right)} < B < \frac{\beta_s}{1+r}.$$

The price responsiveness to information in the noisy REE is thus greater than in the WE, but smaller than in the (non-noisy) REE. This makes intuitive sense:

- In the WE the uninformed agents do not learn from price, and thus trade “against” the information of the informed. Therefore, price responds less to information than in the noisy REE or the REE.
- In the noisy REE the uninformed agents learn from price, but sometimes trade against the informed because they confuse their information with noise. Therefore, price responds less to information than in the REE.
- In the noisy REE the uninformed sometimes trade against the informed. Therefore, their profits should be lower than those of the informed. This is shown in the following proposition. Assume that all agents have the same initial wealth W_0 , and denote by W_I and W_U the final wealth of the informed and the uninformed, respectively.

- **PROPOSITION:** The ratio of the utilities of informed to uninformed is:

$$\frac{-\mathbf{E} \exp(-\alpha W_I)}{-\mathbf{E} \exp(-\alpha W_U)} = \frac{\sigma_{d|s}}{\sigma_{d|p}}.$$

- **PROOF:** The expected utility of an informed agent, conditional on s and p , is:

$$-\exp\left(-\alpha\left((W_0 - xp)(1 + r) + x\mathbf{E}(d|s) - \frac{1}{2}\alpha\sigma_{d|s}^2x^2\right)\right).$$

This is maximized at $x = x_I(s, p)$, so the expected maximum expected utility is:

$$-\mathbf{E} \exp(-\alpha W_I) = -\exp(-\alpha(1 + r)W_0) \cdot \mathbf{E}_{s,p} \exp\left(-\frac{(\mathbf{E}(d|s) - (1 + r)p)^2}{2\sigma_{d|s}^2}\right),$$

where the unconditional expectation is taken over s and p . Similarly, the unconditional maximum expected utility of an uninformed is:

$$-\mathbf{E} \exp(-\alpha W_U) = -\exp(-\alpha(1 + r)W_0) \cdot \mathbf{E}_p \exp\left(-\frac{(\mathbf{E}(d|p) - (1 + r)p)^2}{2\sigma_{d|p}^2}\right),$$

where the unconditional expectation is taken over p . Instead of calculating these expectations, I will use a shortcut, and prove instead that:

$$\mathbf{E}_s \exp\left(-\frac{(\mathbf{E}(d|s) - (1 + r)p)^2}{2\sigma_{d|s}^2}\right) = \exp\left(-\frac{(\mathbf{E}(d|p) - (1 + r)p)^2}{2\sigma_{d|p}^2}\right) \cdot \frac{\sigma_{d|s}}{\sigma_{d|p}},$$

where the first expectation integrates out s , but remains conditional on p . So we need to calculate

$$\mathbf{E}\left(\exp\left(-\frac{z^2}{2}\right) \mid p\right),$$

where

$$z = \frac{\mathbf{E}(d|s) - (1 + r)p}{\sigma_{d|s}}.$$

For this, we use the formula: if $y \sim N(\mu, \sigma^2)$,

$$\mathbf{E} \exp\left(-\frac{y^2}{2}\right) = \frac{1}{\sqrt{1 + \sigma^2}} \exp\left(-\frac{\mu^2}{2(1 + \sigma^2)}\right).$$

Conditional on p , the variable z is normal with mean

$$\mu_{z|p} = \frac{\mathbf{E}(d|p) - (1 + r)p}{\sigma_{d|s}},$$

and variance

$$\sigma_{z|p}^2 = \frac{\text{var}(\mathbf{E}(d|s) | p)}{\sigma_{d|s}^2} = \frac{\text{var}(d|p) - \mathbf{E}(\text{var}(d|s) | p)}{\sigma_{d|s}^2} = \frac{\sigma_{d|p}^2 - \sigma_{d|s}^2}{\sigma_{d|s}^2},$$

where the second inequality comes from the EVE formula. For the third equality, we also used the fact that $V(d|s)$ is independent of p (because of normality). Therefore

$$\mathbf{E}\left(\exp\left(-\frac{z^2}{2}\right) | p\right) = \frac{\sigma_{d|s}}{\sigma_{d|p}} \cdot \exp\left(-\frac{(\mathbf{E}(d|p) - (1 + r)p)^2}{2\sigma_{d|p}^2}\right),$$

and we are done.

- Notice that $\sigma_{z|p}^2 > 0$ implies $\sigma_{d|p} > \sigma_{d|s}$. This can be proved directly: recall that

$$\sigma_{d|s}^2 = \frac{\sigma^2 \sigma_\varepsilon^2}{\sigma^2 + \sigma_\varepsilon^2} \quad \text{and} \quad \sigma_{d|p}^2 = \frac{\sigma^2 (\sigma_\varepsilon^2 + C^2 \sigma_u^2)}{\sigma^2 + \sigma_\varepsilon^2 + C^2 \sigma_u^2}.$$

- This means that the ratio of the expected utility of the informed to the expected utility of the uninformed is always smaller than 1. Since expected utilities are negative, this means that the informed have higher expected utility than the uninformed. Introducing noise thus provides a resolution to the Grossman–Stiglitz paradox.
- The ratio of expected utilities gets closer to 1 as $\sigma_{d|p}$ decreases, i.e., as the price becomes more informative. The ratio is equal to 1 when the price is fully informative.
- Suppose that information acquisition is endogenous. Any agent can become informed by paying a cost c . Then in equilibrium we have

$$\frac{-\mathbf{E} \exp(-\alpha(W_I - c))}{-\mathbf{E} \exp(-\alpha W_U)} = 1 \quad \implies \quad \frac{\sigma_{d|s}}{\sigma_{d|p}} = \exp(-\alpha c).$$

From this, using the formulas for $\sigma_{d|p}$ and $\sigma_{d|s}$, we determine C , and since

$$C = \alpha \sigma_\varepsilon^2 / N_I,$$

this determines the number of agents N_I who become informed in equilibrium.

- Notice that the equilibrium price informativeness is:

$$\tau = \frac{1}{\sigma_{d|p}^2} = \exp(-2\alpha c) \frac{1}{\sigma_{d|s}^2},$$

which is independent of the noise:

- When the noise σ_u^2 increases, price informativeness decreases. However, this induces more agents to become informed, and in equilibrium price informativeness is the same.

5 Differential Information REE

- The model is the same as the asymmetric information REE, except for the information structure. There are no more informed and uninformed, but all agents have some piece of information. Agent i observes a signal $s^i = d + \varepsilon^i$, where $\varepsilon^i \sim N(0, \sigma_\varepsilon^2)$ is independent of d , and independent of the other ε 's.
- **THEOREM:** There exists a REE in which the price is given by the linear formula

$$p = A + B \frac{\sum_{i=1}^N (s^i - \bar{d})}{N},$$

for two constants A and B .

- **PROOF:** First, determine agents' demands. Agent i solves the optimization problem:

$$\max_x -\mathbf{E} \exp\left(-\alpha((W_0^i - xp)(1+r) + xd) \mid s^i, p\right).$$

To calculate this, we need to know the conditional distribution of d given s^i and p . But all variables here are normal, so we only need to know the conditional mean and variance of d given s^i and p . The conditional variance of d given s^i and p is a constant. And for the conditional mean the sum of the signals is a sufficient statistic, i.e., it is enough to know p . This implies that the conditional distribution of d given s^i and p is the same as the conditional distribution of d given p . The conditional mean of d is:

$$\mathbf{E}(d|p) = \bar{d} + \beta_p(p - A),$$

where

$$\beta_p = \frac{\text{cov}(p, d)}{\text{var}(p)} = \frac{B \text{cov}\left(\frac{\sum_{i=1}^N s^i}{N}, d\right)}{B^2 \text{var}\left(\frac{\sum_{i=1}^N s^i}{N}\right)} = \frac{\sigma^2}{B(\sigma^2 + \frac{\sigma_\varepsilon^2}{N})}.$$

The conditional variance of d is

$$\sigma_{d|p}^2 = \text{var}(d) - \beta_p^2 \text{var}(p) = \frac{\sigma^2 \frac{\sigma_\varepsilon^2}{N}}{\sigma^2 + \frac{\sigma_\varepsilon^2}{N}}.$$

The optimal demand of agent i is:

$$x^i(p) = \frac{\mathbf{E}(d|p) - (1+r)p}{\alpha \sigma_{d|p}^2} = \frac{\bar{d} + \beta_p(p - A) - (1+r)p}{\alpha \sigma_{d|p}^2}.$$

We next show that there exist values of A and B so that markets clear, i.e., so that

$$\sum_{i=1}^N x^i(p) = N \frac{\bar{d} + \beta_p(p - A) - (1+r)p}{\alpha \sigma_{d|p}^2} = S.$$

This is a linear formula in the average (excess) signal $\sum_{i=1}^N (s^i - \bar{d})/N$, so we require that the coefficients be equal. Looking at the constant terms, we get:

$$N \frac{\bar{d} - (1+r)A}{\alpha \sigma_{d|p}^2} = S,$$

which is true if

$$A = \frac{\bar{d}}{1+r} - \frac{S\alpha\sigma_{d|p}^2}{(1+r)N}.$$

Looking at the coefficients of $\sum_{i=1}^N (s^i - \bar{d})/N$, we get:

$$N \frac{\beta_p B - (1+r)B}{\alpha\sigma_{d|p}^2} = 0,$$

which is true if $\beta_p = 1+r$, which is equivalent to:

$$B = \frac{\sigma^2}{(1+r)(\sigma^2 + \frac{\sigma_\varepsilon^2}{N})}.$$

- The price function in the differential information REE is a function of the average signal. Since the average signal is a sufficient statistic for all the signals, the price perfectly aggregates the information of all the agents.
- Since the price perfectly aggregates all the information, agents ignore their own signal when forming their demand. This creates another paradox: the **Grossman paradox**. If agents do not use their information when forming their demand, then how does the price aggregate the information?
- To understand the Grossman paradox more formally, consider the demand of agent i . Using the formulas for A and B , this can be shown to simplify to S/N . Agents thus demand inelastically their allocation of the asset supply under the optimal risk sharing rule. The intuition why demand demand is price-inelastic is that price movements are only due to information. Therefore, agents do not want to trade against the market.
- The demand S/N is independent of agents' information and the price. Therefore, if all agents submit this demand, any price can be a market-clearing price. There is thus no reason for the market-clearing price to be the REE price. One resolution of the Grossman paradox is, as before, to introduce noise. In the presence of noise, the price is not fully informative. Therefore, agents use their signals and submit price-elastic demands.
- With noise, one gets a noisy differential information REE. The equilibrium price can be shown to be a function of the average signal and the noise:

$$p = A + B \left(\frac{\sum_{i=1}^N (s^i - \bar{d})}{N} - Cu \right).$$

- However, notice that there is still a problem with these equilibria. This is what Hellwig calls the schizophrenia of the agents: agent i understands that price incorporates his signal s^i , but does not understand that modifying his demand can change the price. This is weird! Another way of saying this: when the agent writes his maximization problem, he does not make price $p = p(x)$, but treats p as a constant.
- All the models we discussed so far are called competitive models, because agents are not strategic. The problem is solved by non-competitive models, such as in Kyle (1989).

6 The No-Trade Theorem

- Consider a one-period model, with a risky asset:
 - At $t = 0$ price is p ;
 - At $t = 1$ the liquidation value is \tilde{v} (random).
- There are N informed traders. Agent i :
 - gets a signal s^i about the true value \tilde{v} ;
 - observes price p ;
 - trades quantity t^i , and has profit $\pi^i = t^i(\tilde{v} - p)$;
 - maximizes expected utility of profit $u^i(\pi^i)$, given signal s^i and price p :

$$\max_{t_i} \mathbf{E}(u^i(\pi^i) \mid s^i, p).$$

- **Assumption:** There exists common prior on signals, i.e., the expectation $\mathbf{E}(\cdot \mid s^i, p)$ is computed in the same way by all agents.
- **THEOREM:** (Milgrom and Stokey, 1982) In the absence of noise or insurance motives to trade, differently informed agents do not trade with each other.
- **PROOF:** By assumption, agents in this model only trade to maximize profit. Denote by t_*^i the optimal amount traded by agent i in equilibrium, and by $\pi_*^i = t_*^i(\tilde{v} - p)$ the corresponding profit. Rescale utilities so that $u^i(0) = 0$. No trade ($t_i = 0$) is in the choice set, so

$$\mathbf{E}(u^i(t_*^i(\tilde{v} - p)) \mid s^i, p) \geq 0.$$

The utility function u^i is concave, so by Jensen's inequality $u^i(\mathbf{E}(\pi_*^i \mid s^i, p)) \geq \mathbf{E}(u^i(\pi_*^i) \mid s^i, p)$. Therefore, $u^i(\mathbf{E}(\pi_*^i \mid s^i, p)) \geq 0$, and since u^i is monotone non-decreasing, we also have $\mathbf{E}(\pi_*^i \mid s^i, p) \geq 0$, i.e.,

$$\mathbf{E}(t_*^i(\tilde{v} - p) \mid s^i, p) \geq 0.$$

By iterated expectations, we can integrate out s^i :

$$\mathbf{E}(t_*^i(\tilde{v} - p) \mid p) \geq 0.$$

Sum over i , to get

$$\mathbf{E}\left(\sum_{i=1}^N t_*^i(\tilde{v} - p) \mid p\right) \geq 0.$$

But, because of market clearing, the sum of all trades has to equal zero (this is where the “no noise” assumption comes in – otherwise, the sum would equal the noise u as in the noisy REE):

$$\sum_{i=1}^N t_*^i = 0,$$

so all the inequalities above must be equalities. In particular, for all i :

$$\mathbf{E}(t_*^i(\tilde{v} - p) \mid s^i, p) = 0.$$

i.e., in equilibrium there is no gain from trade. Therefore agents do not trade.