ON THE DERIVATION OF THE BLACK–SCHOLES FORMULA

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ABSTRACT. Methods of proving the Black–Scholes formula for the price of an European call option fall into two categories: the bond replication method (the original one by Black and Scholes), and the call replication method (originated by Merton). These two methods are not equivalent. While the call replication argument is simple and requires only continuity of the call price C, the bond replication method puts more restrictions on C in order for the argument to work. Moreover, we show that the typical bond replication method fails if the call option delta is equal to one. That implies that at each point either C satisfies the Black–Scholes PDE, or it satisfies the linear PDE given by delta equal to one. We then show that these two PDEs cannot coexist if C is assumed continuously differentiable, which proves that the Black–Scholes PDE holds everywhere.

1. INTRODUCTION

Ever since the publication of the paper by Black and Scholes (1973), the literature on options and derivatives has been expanding at an exponential rate. There are by now many derivations of the famous Black–Scholes equation (1) for the price of a European call option. However, all these derivations essentially fall into two categories: the call replication method and the bond replication method.

The call replication method has been given extensive and rigorous treatment in the literature. The method originates with Merton, in his 1977 paper on contingent claims and the Modigliani–Miller theorem (reproduced in Merton (1992)). A clear presentation of the argument can be found in Duffie (2001). Another version of the call replication method is the martingale approach, pioneered by Harrison and Kreps (1979) (for a more recent description of this approach, see Karatzas and Shreve (1998)).

On the other hand, the bond replication method is much less clearly understood, despite the fact that it was the original method adopted by Black and Scholes (1973), and by Merton, in his 1973 paper on the theory of rational option pricing (reproduced in Merton (1992)). In fact, as far as we know, all the proofs that use the bond replication method run into the same problem, which we shall explain below. In the present paper we show how to get around this problem, and thus make the bond replication method rigorous.

What is then the difference between the two methods?[1] Suppose we are in the Black–Scholes environment, with a stock S, a bond B, and an European call option C (details of this environment will be given later). Then the call replication method proceeds by attempting to "replicate" C, i.e. by forming a portfolio $\Theta = \alpha S + \beta B$ with the stock and the bond, which at maturity has the same payoff as C. If we can find a replicating Θ which is self-financing, then a simple arbitrage argument shows that the price of C at any given time should equal that of Θ. Applying Itô's formula, one finds that the self-financing condition forces Θ to satisfy the Black–Scholes equation. Hence, since the replicating condition determines Θ at the time of maturity, Θ is uniquely determined as the solution to the Black–Scholes equation with

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[1]There is a widely held belief in the mathematical finance community that the two methods are essentially equivalent. To understand why we do not share this belief, see that next to last paragraph of this introduction.

specified terminal data. One of the many virtues of the call price replication method is that it works with minimal assumptions imposed on the call price process $C(t)$.

By contrast, the bond replication method requires us to make rather rigid assumptions about C. In particular, we have to assume that $C(t) = C\big(S(t), t\big)$ where $C(s,t)$ is a reasonably bounded, differentiable function of s and t. In addition, this method requires one to check that the portfolio involved in the arbitrage argument is riskless as well as self-financing. More precisely, the method proceeds as follows. One looks for a portfolio Π formed with the stock and the call, which replicates the bond in the sense that it riskless and self-financing, in which case it is said to be "hedging". To fix ideas, let $\Pi = aS - bC$. Assuming that we can find such a hedging portfolio Π, the self-financing condition implies $d\Pi = a\, dS - b\, dC$, which, together with the riskless condition, leads to $a = bC_s$. So we obtain $\Pi = b(SC_s - C)$. A simple arbitrage argument shows that Π must earn interest at the riskless rate r, hence $d\Pi = r\Pi\, dt$. Putting these together, one concludes, via Itô's formula, that C must satisfy the Black–Scholes PDE.

The difficulty we have with Black and Scholes (1973), as well as the rest of the literature on the bond replication method, is that, as far as we can tell, no one has bothered to check that there exists a non-vanishing choice of b for which $\Pi = b(SC_s - C)$ is self-financing. Of course, if such a b fails to exist, the whole strategy breaks down. There are in the literature proofs that such a b exists when C satisfies the Black–Scholes equation, but these cannot be used when what one is trying to show is that C is such a solution.

Resolution of the problem just raised is the the main goal of the present note. Namely, by carefully examining the requirement that $\Pi = b(SC_s - C)$ be self-financing, we provide a rigorous derivation of the Black–Scholes formula along the lines which Black and Scholes suggested originally. Our analysis has two important ingredients. The first of these is the localization of the arguments outlined above. That is, we show that a non-trivial, riskless, self-financing Π can be constructed so long as $SC_s - C$ stays away from 0; and this leads to the conclusion that, in order to avoid an arbitrage opportunity, C must satisfy the Black–Scholes equation wherever $sC_s - C \neq 0$. This is the free boundary value problem alluded to in the abstract. The second ingredient in our argument is the proof that this free boundary value problem is trivial in the sense that there is no boundary in the case under consideration. That is, we show that for the type of terminal boundary data which arise here, any smooth function which satisfies the Black–Scholes equation in the region where $sC_s - C \neq 0$ must satisfy the Black–Scholes equation everywhere. Thus, *a posteriori*, we find that $sC_s - C$ is strictly positive everywhere[2]. Had we assumed from the outset that C satisfied $SC_s - C > 0$ everywhere, we could have removed most of our difficulties. However, because we have no sound economic grounds for making such an assumption, we were motivated to confront these difficulties rather than assume them away.

Summarizing, what we show is that, with sufficient diligence, Black and Scholes' original bond replication method can be made to work. However, the argument required is significantly more difficult and more rigid than the one required by the call replication method. The reason we decided to carry it through, aside from mathematical curiosity, was to fill what we found to be a disturbing a gap in the literature that has existed ever since the publication of [1]. Since many mathematical finance textbooks employ the bond replication method (see for example the influential textbooks of Hull (1997), Ingersoll (1987), or Wilmott et al (1995)), we feel that our paper provides a useful reference in this area.

One may still wonder if the call replication and bond replication methods are not actually equivalent, perhaps after using some clever transformation. We argue that they are not. To see why, recall briefly how the two methods work. In the call replication, one constructs a portfolio $\Theta = \alpha S + \beta B$ that replicates the payoff of C at maturity. In the bond replication,

[2]It is interesting that Black and Scholes also made this observation in [1]. In their notation, it translates into the statement that $\delta > 1$, which means that the call is always more volatile than the stock.

one constructs a portfolio $\Pi = aS - bC$ that is self-financing and riskless. Both proofs proceed by showing that if C does not satisfy Black–Scholes, then one would be able to construct an arbitrage using Θ or Π, respectively. Now one may hope that there is some correspondence between the pairs of coefficients (α, β) and (a, b). However, even a summary inspection shows that this is not possible: while α and β depend only on the value of C at maturity, a and b depend on C at each intermediate time. Since we do not know yet that C satisfies Black–Scholes (so in principle it can be anything), such a correspondence cannot exist.

The structure of this note is as follows: Section 2 contains definitions as well as a proposition which shows that in the absence of arbitrage the value of a self-financing, riskless portfolio must grow at the interest rate. Section 3 contains the statement of the main theorem and an outline of the proof. Details of the proof are left for Section 4. Finally, in Section 5 we show how to generalize our results to the cases when the drift is stochastic and volatility may depend on the stock price.

2. Hedging Portfolios and Arbitrage

We start by recalling the usual setup of continuous-time finance, as described for example in Karatzas and Shreve (1998). Namely, we consider a market in which the uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, P)$ with a non-decreasing filtration $\{\mathcal{F}_t : \in [0, \infty)\}$ of sub-σ-algebras, and a standard $\mathbb{R}^d$-valued Brownian motion (i.e., Wiener process) $\big(W(t), \mathcal{F}_t, P\big)$. In this context, an adapted process $t \mapsto F(t)$ is a process such that $F(t)$ is $\mathcal{F}_t$-measurable for each $t \geq 0$. An Itô processes is an adapted process $t \mapsto X(t)$ for which there exist adapted processes $u : \Omega \times [0, T] \to \mathbb{R}$ and $v : \Omega \times [0, T] \to \mathbb{R}^d$, with $\int_0^T |u(t)| dt < \infty$ and $\int_0^T |v(t)|^2 dt < \infty$ almost surely, so that $dX(t) = u(t)\, dt + v(t)\, dW(t)$, in the sense of Itô calculus.

For $i = 0, \dots, n$, let $S_i(t)$ be the price process of asset i. In this paper $S_0(t) = B(t) = \exp\big(\int_0^t r(\tau)\, d\tau\big)$ is the price of a bond, which makes sure that money always earns interest at rate $r(t)$. Consider a portfolio Π for which $\theta_i(t)$ represents the holdings of S_i at time t, i.e. the number of units of i. Assume that $S_i(t)$ is an Itô process defined on $\Omega \times [0, T]$, while $\theta_i(t)$ is an Itô process on $\Omega \times [0, T)$ ($\theta_i(t)$ may be undefined at $t = T$, since no trading takes place time T). Then the value of the portfolio at time t is

$$\Pi(t) = \theta(t) \cdot S(t) = \sum_{i=0}^{n} \theta_i(t) S_i(t) \quad .$$

We assume that trading occurs at times t and $t + dt$, but not in between. That means that between t and $t + dt$ the holdings θ stay constant. Therefore, if there are no incoming or outgoing cashflows from the portfolio Π, its value at $t + dt$ must be $\Pi(t + dt) = \theta(t) \cdot S(t + dt)$. The intuition behind the notion of "self-financing" portfolio is that, if trading occurs at $t + dt$, it has to be done only with the available funds, i.e. such that $\theta(t+dt) \cdot S(t+dt) = \theta(t) \cdot S(t+dt)$. Using the formula $X(t + dt) = X(t) + dX(t)$, it follows that "self-financing" is the same as $d\theta(t) \cdot S(t + dt) = 0$, or equivalently the same as $d\theta \cdot S + d\theta \cdot dS = 0$. (Note that, in contrast with ordinary calculus, $d\theta \cdot dS \neq 0$, since both θ and S are stochastic). But now observe that $d\Pi = \theta \cdot dS + d\theta \cdot S + d\theta \cdot dS = \theta \cdot dS$. We take this as the formal definition of "self-financing".

Definition 2.1. *A portfolio $\Pi = \theta \cdot S$ is said to be **self-financing** if $d\Pi = \theta \cdot dS$ on $\Omega \times [0, T)$. More generally, given a pair of stopping times α and β with $0 \leq \alpha \leq \beta \leq T$, a portfolio $\Pi(t) = \theta(t) \cdot S(t)$ is said to be **locally self-financing** on the vertical window*

$$V(\alpha, \beta) \equiv \{(\omega, t) \in \Omega \times [0, T] : \alpha(\omega) \leq t \leq \beta(\omega)\}$$

if it is self-financing there in the sense that $d\Pi = \theta \cdot dS$ on $V(\alpha, \beta)$, or more precisely

$$\Pi(t \wedge \beta) - \Pi(t \wedge \alpha) = \int_0^t \mathbf{1}_{[\alpha,\beta]}(\tau) \theta(\tau) \cdot dS(\tau) \quad .$$

It is important to realize that the notion of local self-financing is consistent in the sense that if Π is locally self-financing on $V(\alpha,\beta)$ and if $\alpha \le \alpha' \le \beta' \le \beta$, then Π is locally self-financing on $V(\alpha',\beta')$. Indeed, this is just an application of Doob's Stopping Time Theorem which guarantees that $\Pi(t\wedge\beta') - \Pi(t\wedge\alpha') = \int_0^t \mathbf{1}_{[\alpha',\beta']}(\tau)\, d\Pi(\tau)$ almost surely.

We now define the notion of riskless portfolio. Suppose Π is a portfolio as above. By construction, Π is an Itô process, so $d\Pi = u\,dt + v\cdot dW$. In the literature, u is called the drift, and $|v|^2$ is the variance.

Definition 2.2. *We say that a portfolio Π with $d\Pi = u\,dt + v\cdot dW$ is* ***riskless*** *if $v\equiv 0$. We say that Π is* ***locally riskless*** *on the vertical window $V(\alpha,\beta)$ if $v\equiv 0$ almost surely on $V(\alpha,\beta)$.*[3]

Definition 2.3. *A portfolio is said to be* ***hedging*** *if it is self-financing and riskless, and it is* ***locally hedging*** *on $V(\alpha,\beta)$ if it is locally self-financing and locally riskless there.*

We also have to clarify what we mean by "arbitrage". Intuitively, an arbitrage is an opportunity to start with zero wealth, incur at most bounded debt, and end up with no losses, and, with non-zero probability, positive net gains. More formally:

Definition 2.4. *An* ***arbitrage*** *is a self-financing portfolio Π defined on $[0,T]$, such that*

a) $\Pi(0)=0$;
b) $\exists M > -\infty$, *such that for each t, $\Pi(\omega,t)\ge M$ almost surely.*
c) $\Pi(T)\ge 0$ *almost surely and $\Pi(T)>0$ with positive probability.*

The next proposition shows that in the absence of arbitrage a self-financing portfolio cannot do better than to earn interest at the rate $r(t)$.

Lemma 2.5. *Suppose the market admits no arbitrage and money earns interest at rate $r(t)$. Then, given stopping times α and β with $0\le\alpha\le\beta\le T$, there is no portfolio Π such that*

- Π *is locally self-financing and almost surely bounded on $V(\alpha,\beta)$;*
- $\Pi\big(\omega,\beta(\omega)\big) \ge \exp\left(\int_{\alpha(\omega)}^{\beta(\omega)} r(\tau)d\tau\right)\Pi\big(\omega,\alpha(\omega)\big)$ *almost surely;*
- *The above inequality is strict with positive probability.*

Similarly, it is equally impossible for $\Pi(t)$ to have the above properties when the inequalities are reversed.

Proof. Suppose that Π were a portfolio of the sort described. To show that this is impossible, we construct an arbitrage Q by the following trading strategy: until time α don't do anything; at time α, borrow money by short-selling the bond B and buy Π from the market; at time β, sell Π and buy the bond with the proceeds (and hold it until time T). More precisely, define a portfolio $Q(t)=\theta_0(t)B(t)+\theta(t)\Pi(t)$ as follows:

- For $0\le t<\alpha$, $\theta(t)=0$ and $\theta_0(t)=0$.
- For $\alpha\le t<\beta$, $\theta(t)=1$ and $\theta_0(t) = -\frac{\Pi(\alpha)}{B(\alpha)}$.
- For $\beta\le t\le T$, $\theta(t)=0$ and $\theta_0(t)=\frac{\Pi(\beta)}{B(\beta)}-\frac{\Pi(\alpha)}{B(\alpha)}$.

Since Π is locally self-financing and a.s. bounded on $V(\alpha,\beta)$, one can easily verify that $Q(t)$ is self-financing and a.s. bounded. In addition, $Q(0)=0$, and, from the hypothesis, $Q(\beta)\ge 0$ a.s. and $Q(\beta)>0$ with positive probability. But that implies that $Q(T)\ge 0$ and $Q(T)>0$ with positive probability, and so Q violates the no arbitrage assumption. □

[3]Equivalently, Π is locally riskless on $V(\alpha,\beta)$ if and only if, for almost every ω, $\Pi(\omega,\,\cdot\,)$ is an absolutely continuous function on $\big[\alpha(\omega),\beta(\omega)\big]$.

Proposition 2.6. *Suppose the market admits no arbitrage and money earns interest at rate $r(t)$. Let Π be a portfolio given by $d\Pi = u\,dt + v\,dW$, where $u(\omega,\cdot\,)$ is continuous for almost every ω. Further, suppose that Π is locally hedging and almost surely bounded on the vertical window $V(\alpha,\beta)$. Then, for almost every ω, $u(\omega,t) = r(t)\Pi(t)$ for all $t \in \big(\alpha(\omega),\beta(\omega)\big)$. Equivalently, Π must grow at exactly the interest rate $r(t)$ during $V(\alpha,\beta)$, i.e. $d\Pi = r\Pi\,dt$ on $V(\alpha,\beta)$.*

Proof. We need to show that $P(\Gamma_+) = 0 = P(\Gamma_-)$, where $\Gamma_\pm \equiv \big\{\omega : \pm u(\omega,t) > \pm r(t)\Pi(\omega,t)\big\}$. Thus, it suffices for us to show that for every $\epsilon > 0$, $P\big(\Gamma_+(\epsilon)\big) = 0 = P\big(\Gamma_-(\epsilon)\big)$, where $\Gamma_\pm(\epsilon) \equiv \big\{\omega : \pm u(\omega,t) > \pm r(t)\Pi(\omega,t) + \epsilon\big\}$. To this end, suppose $P\big(\Gamma_+(\epsilon)\big) > 0$ for some $\epsilon > 0$. We can then define stopping times $\alpha'(\omega)$ and $\beta'(\omega)$ as follows: $\alpha'(\omega)$ is the first $t \in \big[\alpha(\omega),\beta(\omega)\big]$ for which $u(\omega,t) \ge r(t)\Pi(\omega,t) + \epsilon$; $\beta'(\omega)$ is the first $t \in [\alpha'(\omega),\beta(\omega)]$ for which $u(\omega,t) \le r(t)\Pi(\omega,t)$. (Here we mean that $\alpha' = \beta$ if either $\alpha = \beta$ or $u < r\Pi + \epsilon$ for all $t \in [\alpha,\beta]$. Similarly, $\beta' = \beta$ if either $\alpha' = \beta$ or $u > r\Pi + \epsilon$ for all $t \in [\alpha',\beta]$.) Then $V(\alpha',\beta')$ is a vertical window on which Π is uniformly bounded and hedging. Furthermore, for $\omega \in \Gamma_+(\epsilon)$,

$$\Pi\big(\omega,\beta'(\omega)\big) > \exp\left(\int_{\alpha'(\omega)}^{\beta'(\omega)} r(t)\,dt\right)\Pi\big(\omega,\alpha'(\omega)\big)\ ,$$

which, by the preceding lemma, violates the no arbitrage assumption. Obviously, the same sort of argument rules out the possibility that $P\big(\Gamma_-(\epsilon)\big) > 0$ for any $\epsilon > 0$. □

3. The Black–Scholes Formula

We restrict ourselves to a market where there is one risky asset with price process $S(t)$. As before, money earns interest at the rate $r(t)$, which is a (nonstochastic) function of t. In other words, we have a bond B with price at t equal to $B(t) = \exp\big(\int_0^t r(\tau)\,d\tau\big)$. The stock price is an Itô process which satisfies the stochastic differential equation $\frac{dS}{S} = \mu dt + \sigma dW$ with $S(0) > 0$, where $\sigma : [0,\infty) \to (0,\infty)$ and $\mu : [0,\infty) \to \mathbb{R}$ are (non-stochastic) continuous functions of t and $W(t)$ is a standard 1-dimensional Brownian motion. Our goal is to show how the absence of arbitrage determines how assets should be priced if their price is a function of time t and the stock price $S(t)$.

To be more precise, let $C(t)$ be the price at t of a European call option on S with maturity date T and strike price $K > 0$. Since K is fixed throughout this paper, we omit the dependence of C on K. We assume that the call price C at time t is a function of only t and the stock price $S(t)$ at time t. That is, $C(t) = C\big(S(t),t\big)$, where $C(s,t)$ is a function of $(s,t) \in (0,\infty)\times[0,T]$. From the definition of a call option, it follows that the value of C at the maturity T is $C(T) = \big(S(T)-K\big)^+$, which, in terms of the variables (s,t), means that $C(s,T) = (s-K)^+$ (here for $\lambda \in \mathbb{R}$ we use λ^+ to denote the non-negative part of λ, i.e. λ^+ equals λ if $\lambda \ge 0$ or 0 otherwise). Following the strategy of Black and Scholes (1973), we are going to show that, under some regularity conditions on C as a function of (s,t), no arbitrage implies that $C(s,t)$ is uniquely determined by the terminal condition $C(s,T) = (s-K)^+$ and the Black–Scholes equation (1). In what follows, we use subscripts to denote differentiation with respect to the variable in the subscript. Thus C_t is the derivative of C with respect to t, etc.

Theorem 3.1. *We assume that the price of the stock, $S = S(\omega,t)$, is an Itô process satisfying*

$$dS = S(\mu\,dt + \sigma\,dW) \quad \text{with } S(0) > 0\ ,$$

where μ and σ are bounded, nonstochastic functions of t, and σ is bounded below by a strictly positive constant. In addition, we make the following assumptions about the call price $C = C(\omega,t)$:

(i) $C(\omega,t) = C\big(S(\omega,t,),t\big)$, *where* $C : (0,\infty)\times[0,T] \to \mathbb{R}$ *is a function which is continuous everywhere, and smooth on* $(0,\infty)\times[0,T)$*;*
(ii) $\lim_{t\nearrow T} C(s,t) = C(s,T) \equiv (s-K)^+$ *for all* $s \in (0,\infty)$*;*
(iii) $\exists M > 0$ *such that* $\sup_{t\in[0,T)} |C(s,t)| \le Ms$*.*

Then, if the market admits no arbitrage, the function $C(s,t)$ *must satisfy the Black–Scholes equation on* $(0,\infty)\times[0,T)$

$$C_t + \tfrac{\sigma^2}{2} s^2 C_{ss} + rsC_s - rC = 0 \ . \tag{1}$$

Moreover, there is a unique solution of the Black–Scholes equation which satisfies conditions (i)–(iii).

Outline of Proof. Starting with a price process C which satisfies conditions (i)–(iii), we want to show that condition of no arbitrage will be violated unless C satisfies (1). The strategy which we would like to follow is Merton's variant of the argument given by Black and Scholes. That is, we would like to construct a portfolio $\Pi = aS - bC$ such that $\Pi(0) > 0$ and Π is hedging (i.e. self-financing and riskless). If such a portfolio were to exist, we could take the following steps to arrive at the desired conclusion:

• Because Π is self-financing, an application of Itô's formula yields

$$d\Pi = a\,dS - b\,dC = \sigma(a - bC_s)S\,dW + \Big(\mu(a-bC_s)S - b\big(C_t + \tfrac{1}{2}\sigma^2 S^2 C_{ss}\big)\Big)\,dt \ . \tag{2}$$

Hence, because (cf. Lemma 4.1) $S > 0$, Π is riskless if and only if $a = bC_s$. In other words, if Π is hedging, then $\Pi = b(SC_s - C)$.

• Proposition 2.6 says that if Π is hedging, the absence of arbitrage implies that $d\Pi = r\Pi\,dt$. Solving for Π, we get

$$\Pi(0)\exp\left(\int_0^t r(\tau)\,d\tau\right) = \Pi(t) = b(t)\big(SC_s(t) - C(t)\big) \ , \tag{3}$$

and so b cannot vanish.

• On the one hand, $d\Pi = r\Pi\,dt = rb\big(SC_s - C\big)\,dt$. On the other hand, from (2) and $a = bC_s$, $d\Pi = b\big(C_t + \frac{\sigma^2}{2}S^2C_{ss}\big)\,dt$. Hence, since b does not vanish, we know that

$$C_t\big(S(t),t\big) + \frac{\sigma(t)^2}{2} S(t)^2 C_{ss}\big(S(t),t\big) + rS(t)C_s\big(S(t),t\big) - rC\big(S(t),t\big) = 0 \ .$$

By Lemma 4.1, this means that C satisfies the Black–Scholes equation.

As we mentioned in the introduction, the preceding strategy runs into problems when one attempts to actually construct the hedging portfolio Π. Specifically, if there exist a and b which make $\Pi = aS - bC$ self-financing, then, as we just saw, (3) must hold. But, because $\Pi(0) > 0$, this is possible only if $SC_s - C \neq 0$. That is, if $sC_s - C = 0$ for some $(s,t) \in (0,\infty)\times[0,T)$, the preceding argument breaks down.

With this in mind, we modify the Black–Scholes–Merton strategy as follows. First, by localizing the argument just outlined, we are able to show (Lemma 4.3) that if $D(s,t) \equiv sC_s(s,t) - C(s,t) \neq 0$ at some $(s,t) \in (0,\infty)\times[0,T)$, then C must satisfy the Black–Scholes equation at (s,t). Second, we show (Lemma 4.4) that, if C satisfies equation (1) whenever $D \neq 0$ and $C(s,T) = (s-K)^+$, then C satisfies (1) throughout $(0,\infty)\times[0,T)$. As a bonus, this allows us *a posteriori* to conclude that $D > 0$ for all $t < T$ (Lemma 4.2). Uniqueness is proved in Lemma 4.2.

Because the details are somewhat technical, we have decided to put them into a separate section. □

4. Proof of theorem 3.1

From the outline given above, it is clear that Theorem 3.1 will be proved once have proved the following lemmas.

Lemma 4.1. *There exists a unique Itô process $X(t)$ satisfying $dX = uX\,dt + vX\,dW$ with initial condition $X(0) > 0$. X is given by the formula*

$$X(t) = X(0)\exp\left(\int_0^t \left(u - \tfrac{1}{2}v^2\right)(\tau)\,d\tau + \int_0^t u(\tau)\,dW(\tau)\right) \quad . \tag{4}$$

In particular, $X(t) > 0$ almost surely. Hence, if $S(t)$ is the stock price process described in Theorem 3.1, then for each t, $S(t) > 0$ almost surely. Further, for any nonempty interval $(s_1, s_2) \subset (0,\infty)$, we have $P\big(S(t) \in (s_1, s_2)\big) > 0$.

Lemma 4.2. *There exists a unique solution of the Black–Scholes equation $C_t + \frac{\sigma^2}{2}s^2C_{ss} + rsC_s - rC = 0$ satisfying conditions (i)–(iii) in Theorem (3.1). Moreover, for this unique solution C, we have $sC_s - C > 0$ everywhere in $(0,\infty)\times[0,T)$.*

Lemma 4.3. *If C satisfies conditions (i)–(iii) in Theorem (3.1), and the market admits no arbitrage, then C satisfies the Black–Scholes equation (1) on the subset of $(0,\infty)\times[0,T)$ where $sC_s - C \neq 0$.*

Lemma 4.4. *If C satisfies conditions (i)–(iii) in Theorem (3.1), and if the Black–Scholes equation (1) holds on the subset of $(0,\infty)\times[0,T)$ where $sC_s - C \neq 0$, then (1) holds on the whole set $(0,\infty)\times[0,T)$.*

Proof of Lemma 4.1. The first result is quite standard in the theory of stochastic differential equations. Indeed, let $Y(t)$ denote the right hand side of (4), note that Y solves $dY = uY\,dt + vY\,dW$ with $Y(0) = X(0)$, and use Itô's Lemma to show that $d\left(\frac{X}{Y}\right) = 0$ for any solution X.

Both parts of the last assertion follow from the preceding. Namely, by (4),

$$S(t) = S(0)\exp\left(\int_0^t \left(\mu(\tau) - \tfrac{1}{2}\sigma(\tau)^2\right)d\tau + \int_0^t \sigma(\tau)\,dW(\tau)\right) \quad .$$

Hence, since σ and r are non-stochastic, the distribution of $S(t)$ is the same as the distribution of $S(0)M(t)e^{\Sigma(t)X}$, where

$$M(t) \equiv \exp\left(\int_0^t \left(\mu(\tau) - \tfrac{1}{2}\sigma(\tau)^2\right)d\tau\right), \quad \Sigma(t) \equiv \sqrt{\int_0^t \sigma(\tau)^2\,d\tau} \quad ,$$

and X is a standard normal random variable. Since for any non-empty open interval I we have $P(X \in I) > 0$, the desired conclusion follows. □

Proof of Lemma 4.2. For technical reasons, it is preferable to make a change of variables and define

$$u(x,t) = e^{-x}C(e^x,t) \text{ for } (x,t) \in \mathbb{R}\times[0,T] \quad .$$

With this change, we can calculate: $u_t(x,t) = e^{-x}C_t(e^x,t)$, $u_x(x,t) = e^{-x}\big(e^xC_x(e^x,t) - C(e^x,t)\big)$, and $u_{xx}(x,t) = e^{-x}\big(e^{2x}C_{xx}(e^x,t) - e^xC_s(e^x,t) + C(e^x,t)\big)$. It follows that C satisfies $C_t + \frac{1}{2}\sigma^2s^2C_{ss} + rsC_s - rC = 0$ on $(0,\infty)\times[0,T)$ if and only if u satisfies

$$u_t + \tfrac{\sigma^2}{2}u_{xx} + \rho u_x = 0 \ , \text{ where } \quad \rho(t) = r(t) + \tfrac{\sigma^2(t)}{2} \quad .$$

The correspondence is given by $s = e^x$. Also, observe that conditions (i) and (iii) translate into the conditions that $u : \mathbb{R}\times[0,T]$ is bounded and continuous, and is smooth on $\mathbb{R}\times[0,T)$. Finally, the terminal condition (ii) becomes $u(x,T) = (1 - Ke^{-x})^+$. Hence, the verification of existence and uniqueness comes down to checking that there is precisely one bounded function $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ which is continuous everywhere, is smooth on $\mathbb{R} \times [0,T)$, and satisfies

$$u_t + \tfrac{\sigma^2}{2} u_{xx} + \rho u_x = 0 \text{ in } \mathbb{R} \times [0,T) \quad \text{with} \quad u(x,T) = \big(1 - Ke^{-x}\big)^+ \ . \tag{5}$$

To prove the preceding existence and uniqueness result, fix $t_0 \in [0,T)$ and define

$$X_{t_0}(x,t) = x + \int_0^t \rho(t_0+\tau)\, d\tau + \int_0^t \sigma(t_0+\tau)\, dW(\tau) \ .$$

If u satisfies (5), then an application of Itô's formula shows that $u\big(X_{t_0}(t,x), t_0+t\big)$ is a martingale for $t \in [0, T-t_0]$. In particular, this means that

$$u(x,t_0) = E\Big[u\big(X_{t_0}(x,T-t_0),T\big)\Big] \ .$$

But $X_{t_0}(x,T-t_0) - x$ has the distribution of a normal random variable with mean $m(t_0) \equiv \int_{t_0}^T \rho(\tau)\, d\tau$ and variance $V(t_0) \equiv \int_{t_0}^T \sigma^2(\tau)\, d\tau$. Hence, we have now shown that if u solves (5), then

$$u(x,t) = \frac{1}{\sqrt{2\pi V(t)}} \int_{\log K}^{\infty} \big(1 - Ke^{-y}\big) \exp\left(-\frac{\big(y - x - m(t)\big)^2}{2V(t)} \right) dy \ , \tag{6}$$

which gives the desired uniqueness. To prove existence, we need to show that the right hand side of (6) has the required properties and solves (5). This is an elementary exercise in calculus, and is left to the reader.

In addition, using (6) we can calculate

$$u_x(x,t) = \frac{K}{\sqrt{2\pi V(t)}} \int_{\log K}^{\infty} e^{-y} \exp\left(-\frac{\big(y - x - m(t)\big)^2}{2V(t)} \right) dy \ ,$$

which implies that $u_x > 0$ on $\mathbb{R} \times [0,T)$. But we know that $sC_s(s,t) - C(s,t) = e^x C_s(e^x,t) - C(e^x,t) = e^x u_x(x,t)$. This shows $sC_s - C$ is strictly positive on $(0,\infty) \times [0,T)$ when C is the unique solution to (1) satisfying (i)–(iii). □

Proof of Lemma 4.3. Define

$$D(s,t) = sC_s(s,t) - C(s,t) \ .$$

Also, if $V(\alpha,\beta)$ is a vertical window as in Definition 2.1, define its interior by

$$\operatorname{int} V(\alpha,\beta) = \big\{(\omega,t) : \alpha(\omega) < t < \beta(\omega)\big\} \ .$$

We need to show that C satisfies (1) as long as $D \neq 0$. Thus, assume that $D(s_0,t_0) \neq 0$ at some $(s_0,t_0) \in (0,\infty) \times [0,T)$. In order to show that C satisfies (1) at (s_0,t_0), we work by contradiction. Therefore, suppose

$$D(s_0,t_0) \neq 0 \quad \text{and} \quad (C_t + \tfrac{1}{2}\sigma^2 s^2 C_{ss} + rsC_s - rC)(s_0,t_0) \neq 0 \ . \tag{7}$$

On the basis of this assumption, we will construct a portfolio $\Pi = aS - bC$ and a vertical window $V(\alpha,\beta)$ in such a way that the following are true: Π is hedging and uniformly bounded on $V(\alpha,\beta)$; $\alpha < \beta$ with strictly positive probability; and either $d\Pi < r\Pi\, dt$ for all $(\omega,t) \in \operatorname{int} V(\alpha,\beta)$, or $d\Pi > r\Pi\, dt$ for all $(\omega,t) \in \operatorname{int} V(\alpha,\beta)$. Hence, in either case, Lemma 2.5 says that the no arbitrage condition is violated, thus producing a contradiction.

To see how this is done, suppose that (7) holds, and choose a $\rho > 0$, which is strictly smaller than both $\frac{t_0+T}{2}$ and $\frac{s_0}{2}$ so that for all $(s,t) \in R \equiv [s_0-\rho, s_0+\rho] \times [t_0, t_0+\rho]$ we have

$$|D(s,t)| \geq \frac{|D(s_0,t_0)|}{2} \quad \text{and} \tag{8}$$

$$\big|(C_t + \tfrac{1}{2}\sigma^2 s^2 C_{ss} + rsC_s - rC)(s,t)\big| \geq \frac{\big|(C_t + \tfrac{1}{2}\sigma^2 s^2 C_{ss} + rsC_s - rC)(s_0,t_0)\big|}{2} \ . \tag{9}$$

(Notice that ρ is chosen so that $R \subset \mathbb{R} \times [0,T)$.) Next, define the Itô process $b = b(\omega,t)$ as the solution to

$$db = \mathbf{1}_R \, b \left(\frac{\sigma^2 S^4 C_{ss}^2 \, dt}{D^2} - \frac{S \, dC_s + \sigma^2 S^2 C_{ss} \, dt}{D} \right) \quad , \tag{10}$$

with $b(0) = 1$. Here $\mathbf{1}_R(\omega,t) = \mathbf{1}_R\big(S(\omega,t),t\big)$ is given by the characteristic function of $R \subset (0,\infty) \times [0,T)$, i.e. $\mathbf{1}_R(s,t)$ equals 1 if (s,t) is in R, or is 0 otherwise. (In the preceding, it is implicitly assumed that the effect of $\mathbf{1}_R$ vanishing dominates everything else. Thus $db = 0$ when $\big(S(\omega,t),t\big) \notin R$.) By Lemma 4.1, b is strictly positive everywhere. Define also $a = a(\omega,t)$ by

$$a = bC_s \quad . \tag{11}$$

We claim that that the portfolio $\Pi = aS - bC$ is self-financing on $\operatorname{int} R$, i.e.

$$\big(S(\omega,t),t\big) \in \operatorname{int}(R) \implies d\Pi = a \, dS - b \, dC \quad . \tag{12}$$

By Itô's formula, (12) is equivalent to $S \, da - C \, db + dS \, da - dC \, db = 0$. To check that this holds, use Itô's lemma to calculate

$$\frac{dS \, db}{b} = -\frac{\sigma^2 S^3 C_{ss}}{D} \, dt \quad \text{and} \quad \frac{dC_s \, db}{b} = -\frac{\sigma^2 S^3 C_{ss}^2}{D} \, dt \quad . \tag{13}$$

Because $a = bC_s$, $da = b\,dC_s + C_s\,db + dC_s\,db$, and therefore $\frac{S\,da - C\,db}{b} = S\,dC_s + D\frac{db}{b} + S\,dC_s\frac{db}{b}$. Using (10) and (13), it follows that whenever $\big(S(\omega,t),t\big) \in \operatorname{int} R$,

$$\frac{S \, da - C \, db}{b} = -\sigma^2 S^2 C_{ss} \, dt \quad . \tag{14}$$

We now calculate $\frac{dS\,da - dC\,db}{b} = dS\,dC_s + (C_s\,dS - dC)\frac{db}{b}$. Notice that by Itô's lemma, $C_s\,dS - dC = -(C_t + \frac{1}{2}C_{ss}\sigma^2 S^2)dt$, and $dS\,dC_s = \sigma^2 S^2 C_{ss}\,dt$. This implies that on $\operatorname{int} R$ we have

$$\frac{dS \, da - dC \, db}{b} = \sigma^2 S^2 C_{ss} \, dt \quad . \tag{15}$$

Putting (14) and (15) together, we get that whenever $\big(S(\omega,t),t\big) \in \operatorname{int} R$, we have $S \, da - C \, db + dS \, da - dC \, db = 0$, and, as we already observed, this is equivalent to (12).

To complete our program, we need to define the vertical window $V(\alpha,\beta)$ on which to apply Lemma 2.5. According to Lemma 4.1, we know that $P\big(S(\omega,t_0) \in (s_0 - \rho, s_0 + \rho)\big) > 0$, so we can choose $M < \infty$ so that

$$P\Big(b(\omega,t) \le M \text{ and } \big(S(\omega,t),t\big) \in R \text{ for all } t_0 \le t \le t_0 + \rho\Big) > 0 \quad .$$

Now we define the stopping times α and β. First, set $\alpha \equiv t_0$. Second, if $b(t) \le M$ and $\big(S(t),t\big) \in \operatorname{int} R$ for all $t \in [t_0, t_0 + \rho]$, then let $\beta = t_0 + \rho$. Otherwise define

$$\beta = \inf\{t \ge t_0 : \, b(t) \ge M \text{ or } \big(S(t),t\big) \notin R\} \quad .$$

Then it is easy to see that $P(\beta > \alpha) > 0$.

In order to see that we are now in a position to apply Lemma 2.5, first observe that Π is locally hedging on $V(\alpha,\beta)$. Second, looking at the definition of b, one sees that inequality (8) together with the definition of β guarantee that Π is uniformly bounded on $V(\alpha,\beta)$. Finally, equation (2) and the fact that $a = bC_s$, we get

$$d\Pi - r\Pi \, dt = -(C_t + \tfrac{1}{2}\sigma^2 S^2 C_{ss} + rSC_s - rC)dt \quad .$$

Equation (9) implies that either $d\Pi > r\Pi\,dt$ on $\operatorname{int} V(\alpha,\beta)$, or $d\Pi < r\Pi\,dt$ on $\operatorname{int} V(\alpha,\beta)$. But $P(\beta > \alpha) > 0$, so this means that the last two hypotheses of Lemma 2.5 are satisfied, and we have arrived at a contradiction. □

Proof of Lemma 4.4. We turn now to the proof that, if C satisfies (i)–(iii) and (1) holds on the subset of $(0,\infty)\times[0,T)$ for which $sC_s - C \neq 0$, then (1) holds everywhere on the whole $(0,\infty)\times[0,T)$. Make a change of variables as in the proof of Lemma 4.2. Then we have a function $u(x,t) = e^{-x}C(e^x,t)$ defined on $\mathbb{R}\times[0,T]$, which is continuous everywhere and smooth on $\mathbb{R}\times[0,T)$. When $t=T$ we also have $u(x,T) = \big(1-Ke^{-x}\big)^+$. Now suppose that

$$u_t + \tfrac{\sigma^2}{2}u_{xx} + \rho u_x = 0 \quad \text{whenever} \quad u_x \neq 0 \ . \tag{16}$$

We need to show that u satisfies $u_t + \frac{\sigma^2}{2}u_{xx} + \rho u_x = 0$ everywhere (not only when $u_x \neq 0$).

To this end, define the following subsets of $\mathbb{R}\times[0,T)$:

$$G = \big\{(x,t)\,:\,\phi(x,t)\equiv u_t + \tfrac{\sigma^2}{2}u_{xx} + \rho u_x \neq 0\big\} \text{ and } F = \big\{(x,t)\,:\,u_x = 0\big\} \ .$$

Clearly G is open and F is closed in $\mathbb{R}\times[0,T)$. From (16), we know that $G\subseteq F$. This implies on G we have $\phi = u_t$ (since $u_x = u_{xx} = 0$ on G); also $\phi_x = 0$ (since $u_{tx} = u_{xt} = 0$ on G). Our goal is to show that G is the empty set. We first prove the following lemma: *If G contains a point $P_0 = (x_0,\tau)$, then it contains the whole horizontal line $L_\tau = \{(x,\tau)\mid x\in\mathbb{R}\}$ passing through P_0.* To see this, consider the largest interval $I\subset G\cap L_\tau$ which contains P_0. Since G is open, I is an open interval. We want to show that it is the whole line. If it is not, without loss of generality, we may assume that it has an infimum $m > -\infty$. But then $m\notin G$, and so $\phi(m,\tau) = 0$. On the other hand, because $\phi_x\equiv 0$ on G, $\phi(x,\tau) = \phi(x_0,\tau)\neq 0$ for all $x\in I$. By the continuity of ϕ in x, since m is the infimum of all x in I, it follows that $\phi(m,\tau) = \phi(x_0,\tau)\neq 0$. But this is a contradiction, because we have also shown that $\phi(m,\tau) = 0$.

Now suppose by contradiction that G has horizontal lines that get closer and closer to T. Then we know that u is constant along those lines (because $u_x = 0$ in G), and this would imply that u is constant on the horizontal L_T corresponding to $t = T$, which is false (since $u(x,T) = \big(1-Ke^{-x}\big)^+$ is not constant in x). Then it follows that either $G=\emptyset$, which is what we want, or $G\subseteq\mathbb{R}\times[0,t_0)$ for some $0<t_0<T$. Assume the latter is true. By a continuity argument of exactly the sort just given, we would then know that u is constant on L_{t_0}. Also, by the definition of G, we know that u satisfies $u_t + \frac{\sigma^2}{2}u_{xx} + \rho u_x = 0$ for $(x,t)\in\mathbb{R}\times[t_0,T)$. Summarizing, u would be a bounded continuous function on $\mathbb{R}\times[0,T]$ which is smooth on $\mathbb{R}\times[0,T)$, equal to $\big(1-Ke^{-x}\big)^+$ at $t=T$, and satisfies $u_t + \frac{\sigma^2}{2}u_{xx} + \rho u_x = 0$ in $\mathbb{R}\times[t_0,T)$. But this is impossible. Indeed, by the argument used to prove Lemma 4.2, the fact that u solves (5) in $\mathbb{R}\times[t_0,T]$ means that $u(x,t_0)$ must be given by the right hand side of (6) with $t=t_0$. In particular, this means that $u(x,t_0)$ must tend to 0 or 1 as x tends to $-\infty$ or $+\infty$, and therefore $u(\,\cdot\,,t_0)$ is certainly not constant. Hence G must be empty, and we are done. □

5. Generalization

In order to minimize the number of technical difficulties, we have been restricting our attention to situations in which the volatility σ and drift μ are deterministic functions of t. The major advantage to doing so was that the resulting stochastic differential equation for the price process $S(t)$ was trivial and the distribution of $S(t)$ was an easy transformation of a Gaussian. However, it should be recognized that our basic results apply in considerably more generality. In fact, we can prove the analog of Theorem 1 under the conditions that

- The stock price $S(t)$ satisfies a stochastic differential equation of the form

$$dS(t) = \mu(t)S(t)\,dt + \sigma\big(S(t),t\big)S(t)\,dW(t) \ ,$$

where the drift $\mu(t)$ is any bounded, adapted process and the volatility σ is a bounded, uniformly positive, smooth function on $(0,\infty)\times[0,T]$ with the property that $|s\sigma_s(s,t)|$ is uniformly bounded.

• The payoff function f is a non-negative, Lipschitz continuous function with the property that $s^{-1}f(s)$ is bounded and non-decreasing and has a strictly positive first derivative on a non-empty open interval.

The basic strategy of the proof in this generality is the same as in the case which we have already treated, and, for the most part, the necessary changes occur when we come to the verification of the results in Section 4. The first place where one encounters a problem is in the verification that $P\big(S(t)\in(s_1,s_2)\big)>0$ for all $t\in(0,T]$ and $0\le s_1<s_2$. By Girsanov's theorem, it suffices to treat the case when $\mu(t)=\frac{\sigma(S(t),t)^2}{2}$, in which case $S(t)=S(0)e^{X(t)}$, where

$$dX(t)=\sigma\big(X(t)\big)\,dW(t)\quad\text{with } X(0)=1\ .$$

Thus, it suffices to check that, with positive probability, $X(t)$ is in any given non-empty open interval. There are two ways in which this can be done. The more probabilistic approach is based on the Support Theorem (cf. [10]), which says that, because $\sigma>0$, X restricted to $[0,T]$ will, with positive probability, stay in any tubular neighborhood of any path $p\in C([0,T];\mathbb{R})$ with $p(0)=\log S(0)$. A more analytic approach is to show that the distribution of $X(t)$ is given by $g\big(\log S(0),y,t\big)\,dy$, where g is the minimal, non-negative solution to

$$\partial_t g(x,y,t)=\tfrac12\sigma(e^x,t)\partial_x^2 g(x,y,t)\quad\text{with}\quad\lim_{t\searrow0}g(\,\cdot\,,y,t)=\delta_y\ ,$$

and to then apply the strong minimum principle to conclude that $g(x,y,t)>0$ for all $(x,y,t)\in\mathbb{R}\times\mathbb{R}\times(0,\infty)$.

The questions about existence and uniqueness are best handled by considering $u(x,t)=e^{-x}C(e^x,t)$, in which case the problems come down to showing that there is one and only one bounded solution to

$$u_t=\tfrac12 a u_{xx}+bu_x\quad\text{where}\quad a(x,t)=\sigma^2\big(e^x,t\big)\text{ and } b(x,t)=r(t)+\tfrac12\sigma^2(e^x,t)\ . \tag{17}$$

By the classical theory of parabolic PDEs (cf. [3]), our conditions on σ are more than sufficient to guarantee that there is a bounded $u\in C\big(\mathbb{R}\times[0,T];\mathbb{R}\big)$ which equals f when $t=T$, is smooth in $\mathbb{R}\times[0,T)$ and satisfies (17) there. Moreover, to prove uniqueness, set $\Sigma(x,t)=\sigma\big(e^x,t\big)$ and define X_{t_0} by

$$dX_{t_0}(x,t)=\Sigma\big(X_{t_0}(x,t),t_0+t\big)\,dW(t)+b\big(X_{t_0}(x,t),t_0+t\big)\,dt\quad\text{with}\quad X_{t_0}(x,0)=x\ .$$

Then, by an application of Itô's Lemma, one sees that that

$$u(x,t_0)=E\Big[g\big(X_{t_0}(x,T-t_0)\big)\Big]\quad\text{where}\quad g(x)\equiv e^{-x}f(e^x)\ , \tag{18}$$

which is more than enough to prove uniqueness.

In addition to proving uniqueness, (18) makes it easy to verify that (cf. the conditions given above on f)

$$\lim_{x\to-\infty}u(x,t_0)=\lim_{s\searrow0}s^{-1}f(s)<\lim_{s\to\infty}s^{-1}f(s)=\lim_{x\to\infty}u(x,t_0)\ ,$$

and so we know that $u(\,\cdot\,,t_0)$ cannot be constant for any $t_0\in[0,T)$.

Finally, it remains to check that sC_s-C is positive in $(0,\infty)\times[0,T)$. Equivalently, what we must show is that $u_x>0$ on $\mathbb{R}\times[0,T)$, and perhaps the easiest way to do this is to use the representation in (18) again. Indeed, as is well-known, although $X_{t_0}(x,t)$, for each x, is only defined up to a set of probability 0, it is possible to choose $x \mapsto X_{t_0}(x,t)$ so that it is, almost surely, continuously differentiable and

$$(X_{t_0})_x(x,t) = 1 + \int_0^t \Sigma_x\big(X_{t_0}(x,\tau), t_0+\tau\big)(X_{t_0})_x(x,\tau)\, dW(\tau) + \int_0^t b_x\big(X_{t_0}(x,\tau), t_0+\tau\big)(X_{t_0})_x(x,\tau)\, d\tau \ .$$

In particular,

$$(X_{t_0})_x(x,t) = \exp\bigg(\int_0^t \Sigma_x\big(X(x,\tau), t_0+\tau\big)\, dW(\tau) + \int_0^t \Big(b_x\big(X(x,\tau), t_0+\tau\big) - \tfrac{1}{2}\Sigma_x\big(X(x,\tau), t_0+\tau\big)^2\Big)\, d\tau\bigg) > 0$$

almost surely, and, from (18), we have that

$$u_x(x,t_0) = E\Big[f'\big(X_{t_0}(x,T-t_0)\big)(X_{t_0})_x(x,T-t_0)\Big] \ ,$$

But f' is strictly positive on some non-empty interval (α,β), and by the same sort of reasoning as that alluded to above, $P\big(X_{t_0}(x,T-t_0) \in (\alpha,\beta)\big) > 0$, and we are done.

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