Consumption–Investment problems for CARA agents

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1 Introduction

The purpose of this note is to show how to solve consumption–investment problems for CARA agents, both in discrete-time and continuous-time. The note is by no means meant to be exhaustive. Rather, I try to move directly to the models and do calculations as quickly as possible. I believe everybody who works in theoretical asset pricing should be able to perform calculations of this type. But enough about motivational issues.

Since there are many variations in the basic setup for the models, I am just going to choose the ones I prefer, and then briefly indicate how to deal with other setups. In the models of this paper we have:

- One infinitely divisible good that is used for both consumption and investment, and is treated as numeraire (money).
- A financial market with $n+1$ assets:
  - ∗ One riskless asset (the money market), with an infinitely elastic supply at a positive constant rate of return r, and gross rate of return $R = 1 + r$.
  - ∗ n risky assets (the stocks), with fixed supply S_i and which pay an exogenously determined dividend $D_{j,t}$ each period t, $j = 1, \ldots, n$.[1]
- m agents with constant absolute risk aversion (CARA), with risk aversion coefficient α_i, and time discount coefficient $\beta_i = \exp(-\rho_i)$, $i = 1, \ldots, m$.

The next two subsections introduce some notation and discuss our setup in more detail. They can be skipped by an impatient reader, but you should be aware that at least once in your life you should understand these issues thoroughly. Especially when you do calculations of this type, make sure to get your notation straight if you don't want it to later come back and haunt you.

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[1] Some authors prefer to forget about dividends and make stock prices exogenous. Then they only concentrate on solving for the optimal consumption and investment of the agents. However, I think it's more instructive not to assume this out, but rather to determine the equilibrium prices (when it can be done).

1.1 Aside on the wealth process: discrete-time

We first look at the discrete-time case, since the continuous-time setup is very similar. Trading takes place at each time $t = 0, 1, 2, \dots$ in all $n + 1$ assets. By definition, period t is the time-interval between t and $t + 1$.

At time t, denote by $P_{0,t}$ the price of the riskless asset, and by $P_{i,t}$ the *ex-dividend* price of risky asset $j = 1, \dots, n$. Define $\tilde{P}_{j,t} = P_{i,t} + D_{i,t}$ the *cum-dividend* price of stock j, and by $R_{j,t+1}$ the (cum-dividend) gross return of stock j from in period t (between t and $t + 1$):

$$R_{j,t+1} = \frac{P_{j,t+1} + D_{j,t+1}}{P_{j,t}} = \frac{\tilde{P}_{j,t+1}}{P_{j,t}}.$$

Denote by W_t the total financial wealth of an agent at t, by C_t his consumption, and by I_t his investment wealth. Denote income by Y_t (it can be non-stochastic or stochastic: see below). For $j = 0, 1, \dots, n$, denote by

$$\begin{aligned} \theta_{j,t} &= \text{number of shares of asset } j, \\ \omega_{j,t} &= \text{percentage of wealth invested in asset } j\ . \end{aligned}$$

We now analyze events at time t and between t and $t+1$. We assume that we start period t with wealth W_t and asset holdings $\theta_{j,t-1}$, $j = 0, 1, \dots, n$ (in continuous-time we don't think about initial holdings, because the time $t - dt$ doesn't quite make sense; instead we look at holdings *after* trading, which are $\theta_{j,t}$). We assume that there is no free disposal, i.e. that wealth is either consumed or invested, hence cannot be destroyed. The order of events is as follows:

- The consumption decision C_t is made. If we allow non-stochastic income, also an income Y_t is received. The investment wealth then is $I_t = W_t - C_t$ (or $I_t = W_t - C_t + Y_t$, if there is any non-stochastic income Y_t).

- The investment decision is made: trading takes place, and $\theta_{j,t} - \theta_{j,t-1}$ shares of asset i are purchased for each $j = 0, 1, \dots, n$. We have $I_t = \sum_{j=0}^{n} \theta_{j,t} P_{j,t}$.

- A stochastic dividend of $D_{j,t+1}$ per share is received between t and $t + 1$, amounting to a contribution of $\sum_{j=1}^{n} \theta_{j,t} D_{j,t+1}$ to W_{t+1}.

- If we allow stochastic income (or, equivalently, stochastic wealth shocks), an amount of $\tilde{Y}_{t+1}$ is received between t and $t + 1$ and contributes to W_{t+1}.

Denote by $\theta = (\theta_{0,t}, \dots, \theta_{n,t})$, the portfolio formed with all the assets (also called the "trading strategy"), and denote by $P_{\theta,t}$ the total price of the portfolio θ at t, and by $R_{\theta,t+1}$ its (gross) rate of return between t and $t + 1$. We have the following formulas:

$$\omega_{j,t} = \frac{\theta_{j,t} P_{i,t}}{\sum_{j=0}^{n} \theta_{j,t} P_{j,t}} = \frac{\theta_{j,t} P_{j,t}}{I_t}, \qquad \sum_{j=0}^{n} \omega_{j,t} = 1,$$

$$R_{\theta,t+1} = \sum_{j=0}^{n} \omega_{j,t} R_{j,t+1} = R + \sum_{j=1}^{n} \omega_{j,t} (R_{j,t+1} - R).$$

Assuming there is no income, either stochastic or non-stochastic, we can derive the following important expressions for the wealth process:

$$W_{t+1} = I_t R_{\theta,t+1} = (W_t - C_t)\Big(R + \sum_{j=1}^{n} \omega_{j,t}(R_{j,t+1} - R)\Big), \tag{1}$$

$$W_{t+1} = (W_t - C_t)R + \sum_{j=1}^{n} \theta_{j,t}(P_{j,t+1} - RP_{j,t} + D_{j,t+1}). \tag{2}$$

Which of these two formulas is better for our purposes? In general, the multiplicative expression (1) is used when we have agents with power or logarithmic utility (CRRA), while the additive expression (2) is used for agents with exponential utility (CARA).

1.2 Aside on the wealth process: continuous-time

A continuous-time model is just an idealization of a model in discrete-time. The only thing we have to do is to replace $t+1$ by $t + \mathrm{d}t$, and use various infinitesimal identities: $(\mathrm{d}t)^2 = 0$, $\mathrm{d}X(t) = X(t + \mathrm{d}t) - X(t)$, etc. Here we have in mind the period between t and $t + \mathrm{d}t$. In order not to confuse subscripts with taking partial derivatives, we write $X(t)$ instead of X_t. The order of events is as follows:

- The consumption decision is made: $\mathrm{d}C(t) = c(t)\,\mathrm{d}t$.[2] If we allow non-stochastic income, also an income $\mathrm{d}Y(t) = y(t)\,\mathrm{d}t$ is received.
- The investment decision is made: trading takes place at time t, and the final holdings of asset j become $\theta_j(t)$, for $j = 0, 1, \ldots, n$.
- A stochastic dividend of $\mathrm{d}D_j(t)$ per share is received between t and $t{+}\mathrm{d}t$, and contributes to $\mathrm{d}W(t)$.
- If we allow stochastic income (or, equivalently, stochastic wealth shocks), an amount of $\mathrm{d}Y(t)$ is received between t and $t + \mathrm{d}t$, and contributes to $\mathrm{d}W(t)$.

As in discrete-time, assuming we have no income stream, there are two expressions for the evolution of the wealth process

$$\mathrm{d}W(t) = rW(t)\,\mathrm{d}t - c(t)\,\mathrm{d}t + W(t)\sum_{j=1}^{n} \omega_j(t)\left(\frac{\mathrm{d}P_j(t)}{P_j(t)} - r\,\mathrm{d}t + \frac{\mathrm{d}D_j(t)}{P_j(t)}\right), \tag{3}$$

$$\mathrm{d}W(t) = rW(t)\,\mathrm{d}t - c(t)\,\mathrm{d}t + \sum_{j=1}^{n} \theta_j(t)\big(\mathrm{d}P_j(t) - rP_j(t)\,\mathrm{d}t + \mathrm{d}D_j(t)\big). \tag{4}$$

The first expression is used when stock prices are assumed to have a log-normal distribution, while the second expression is used in most other cases.

[2] One may ask why the consumption $\mathrm{d}C(t)$ is not stochastic. But then it would mean that consumption is not a control variable, known at the beginning of the period.

2 Discrete time

2.1 No income shocks

From now on we employ vector notation; by convention all vectors are column vectors. The setup is as we discussed above, but in order to completely specify the model we need to make an assumption about the dividend process. Let us first consider case when dividends are white noise

$$\boxed{D_t \;=\; \bar{D} + \varepsilon_{D,t} \;\sim\; N(\bar{D}, \Sigma_D) \quad (5)}$$

i.e. $\varepsilon_{D,t}$ are independent and identically distributed, and the distribution is multivariate normal with mean $\bar{D}$ and covariance matrix Σ_D.

We assume that agents have infinite horizon. Consider a CARA agent i whose wealth at the beginning of period t is W_t. Then each period t this agent maximizes the expected intertemporal utility of consumption, i.e. solves

$$J_t^i(W_t) \;=\; \max_{(C_s,\theta_s)_{s\geq t}} \mathsf{E}_t \left\{ -\sum_{s=t}^{\infty} \beta_i^s \exp(-\alpha_i C_s) \right\}, \tag{6}$$

subject to some condition at infinity[3], and where E_t is the expectations operator conditional on the information agents have at time t. The function J_t^i is called the *value function*[4] and it represents the maximum utility the agent can get if he consumes optimally. Then the Bellman optimality principle (see Stokey and Lucas) says that

$$\boxed{J_t^i(W_t) \;=\; \max_{C_t,\theta_t}\Big(-\beta_i^t \exp(-\alpha_i C_t) + \mathsf{E}_t J_{t+1}^i(W_{t+1}) \Big) \quad (7)}$$

where wealth W_{t+1} is given by equation (2):

$$\boxed{W_{t+1} \;=\; (W_t - C_t)R \;+\; \theta_t^\top (P_{t+1} - RP_t + D_{t+1}). \quad (8)}$$

Now we solve for the equilibrium. Since we have only CARA agents and we have assumed no shocks, we guess that the equilibrium stock price vector has to be constant. Also, we guess that the value function J_t^i is log-linear in wealth:

$$\boxed{\begin{aligned} P_t &= p & (9) \\ J_t^i(W) &= -\beta_i^t \exp(-a_i W - b_i) & (10) \end{aligned}}$$

[3]This is called the transversality condition. I am not going to talk about it, one can look it up for example in Stokey and Lucas.

[4]One can define J_t^i with β_i^{s-t} instead of β_i^s. Here we choose β_i^s for reasons that will become apparent in the continuous time case (Section 3): the definition using β_i^s leads to a slightly simpler derivation of the Hamilton-Jacobi-Bellman equation. Nevertheless, it turns out that the definition using β_i^{t-s} would lead to a time-invariant value function $I(t, W) = I(W)$.

where p is a constant n-vector, and a_i, b_i are scalars. Given the form (5) for the dividend, we can calculate $\mathsf{E}_t W_{t+1} = (W_t - C_t)R + \theta_t^\top(\bar{D} - rp)$, and $\mathsf{Var}_t W_{t+1} = \frac{1}{2}\theta_t^\top \Sigma_D \theta_t$. Now, if X is normally distributed with mean μ and variance σ^2, we know that $\mathsf{E}\exp(X) = \exp(\mu + \frac{1}{2}\sigma^2)$. Then we can calculate:

$$\begin{aligned} \mathsf{E}_t J^i_{t+1}(W_{t+1}) &= -\beta_i^{t+1}\exp\left(-a_i(W_t - C_t)R - a\theta_t^\top(\bar{D} - rp) + \tfrac{1}{2}a_i^2\theta_t^\top\Sigma_D\theta_t - b_i\right) \\ &= -\beta_i^{t+1}\exp\big(-a_i(W_t - C_t)R - b_i\big)\exp\left(-a_i\theta_t^\top(\bar{D} - rp) + \tfrac{1}{2}a_i^2\theta_t^\top\Sigma_D\theta_t\right) \end{aligned} \tag{11}$$

Because of the second equality one can notice that in the optimization problem (7) the first order condition for θ_t is independent from the first order condition for C_t.

The first order condition for θ_t is

$$\bar{D} - rp - a_i\Sigma_D\theta_t = 0$$

from which we get the optimum trading strategy θ_t^i:

$$\theta_t^i = \frac{1}{a_i}\Sigma_D^{-1}(\bar{D} - rp). \tag{12}$$

We still have to find the values for a_i and p. We will show below that

$$\boxed{a_i = \alpha_i\frac{r}{R} \quad (13)}$$

(this will be true in all portfolio problems for the CARA agent). Given that we know a_i, define the aggregate absolute risk aversion a_0 by

$$\frac{1}{a_0} = \sum_{i=1}^m \frac{1}{a_i}, \tag{14}$$

where m is the number of agents. In order to determine p, add equations (12) for all agents, and obtain the stock supply vector $S = \frac{1}{a_0}\Sigma_D^{-1}(\bar{D} - rp)$. From this we obtain

$$\boxed{p = \frac{\bar{D}}{r} - \frac{a_0}{r}\Sigma_D S. \quad (15)}$$

Note that the equilibrium price for risky asset in this model is the same as the price of a perpetuity which offers $\bar{D}$ each period, minus the risk premium $\frac{a_0}{r}\Sigma_D S$.

Substituting p in (12), we obtain

$$\boxed{\theta_t^i = \frac{\alpha_0}{\alpha_i}S \quad (16)}$$

which represents the optimal risk sharing, and is inversely proportional to the risk aversion of the agent.

Now we go back to equation (11) and define

$$\lambda = a_i(\theta_t^i)^\top(\bar{D} - rp) - \tfrac{1}{2}a_i^2(\theta_t^i)^\top\Sigma_D(\theta_t^i) = \tfrac{1}{2}(\bar{D} - rp)^\top\Sigma_D^{-1}(\bar{D} - rp) = \tfrac{1}{2}S^\top\Sigma_D S. \tag{17}$$

Since $\beta_i = \exp(-\rho_i)$, we obtain

$$\begin{aligned} \mathsf{E}_t J^i_{t+1}(W_{t+1}) &= -\beta_i^{t+1} \exp\big(-a_i(W_t - C_t)R - b_i\big) \exp(-\lambda) \\ &= -\beta_i^t \exp\big(-a_i(W_t - C_t)R - \mu_i\big), \end{aligned} \tag{18}$$

where $\mu_i = b_i + \rho_i + \lambda$. The Bellman principle (7) reads

$$\exp(-a_i W_t) = \min_{C_t} \Big(\exp(-\alpha_i C_t) + \exp\big(-a_i R(W_t - C_t) - \mu_i\big) \Big). \tag{19}$$

The first order condition for C_t is $\alpha_i \exp(-\alpha_i C_t) = a_i R \exp\big(-a_i R(W_t - C_t) - \mu_i\big)$. Taking natural logarithms on both sides, we get $\log \alpha_i - \alpha_i C_t = \log(a_i R) - a_i R(W_t - C_t) - \mu_i$. From this we obtain the optimal consumption

$$C_t^i = \frac{a_i R}{\alpha_i + a_i R} W_t + \frac{1}{\alpha_i + a_i R} \left(\mu_i + \log \frac{\alpha_i}{a_i R} \right). \tag{20}$$

We still have to find a_i and μ_i (or, equivalently, b_i). This is the most tedious part of the calculation, but let us carry it through. If we substitute for C_t^i in (19) we get

$$\begin{aligned} \exp(-a_i W_t - b_i) &= \exp\left(-\frac{\alpha_i a_i R}{\alpha_i + a_i R} W_t \right) \cdot \exp\left(-\frac{\alpha_i}{\alpha_i + a_i R} \mu_i \right) \\ &\quad \cdot \left(\exp\Big(\frac{a_i R}{\alpha_i + a_i R} \log \frac{\alpha_i}{a_i R} \Big) + \exp\Big(-\frac{\alpha_i}{\alpha_i + a_i R} \log \frac{\alpha_i}{a_i R} \Big) \right). \end{aligned}$$

Identifying the coefficients of W_t, we get $\alpha_i R = \alpha_i + a_i R$. This implies $a_i = \alpha_i \frac{r}{R}$, so formula (13) is true. We also get

$$\exp\left(\frac{r}{R} \log \frac{1}{r} \right) + \exp\left(-\frac{1}{R} \log \frac{1}{r} \right) = \exp\left(\frac{1}{R} \mu_i - b_i \right),$$

from which we can calculate

$$\mu_i - R b_i = R \log R - r \log r. \tag{21}$$

But we know that $\mu_i = b_i + \rho_i - \lambda$ and $\lambda = \frac{1}{2} S^\top \Sigma_D S$, so we solve for b_i:

$$\boxed{b_i = \frac{1}{r}\big(\rho_i + \tfrac{1}{2} S^\top \Sigma_D S\big) - \frac{r}{R} \log R + \log r.} \tag{22}$$

Going back to the formula for the optimal consumption, we get

$$\boxed{C_t^i = \frac{r}{R} W_t + \frac{1}{r\alpha_i}\big(\rho_i + \tfrac{1}{2} S^\top \Sigma_D S - \log R\big).} \tag{23}$$

With the expressions we have found for a_i and b_i we saw that the value function $J_t^i(W) = -\beta_i^t \exp(-a_i W - b_i)$ satisfies both the Bellman equation (7) and the market clearing, and we are done.

2.2 Deterministic income shocks

We assume that between t and $t+1$ agent i receives a deterministic income Y_{t+1} which is known at time t. Then the wealth process is described by

$$\boxed{W_{t+1} \;=\; (W_t - C_t)R \;+\; \theta_t^\top (P_{t+1} - RP_t + D_{t+1}) \;+\; Y_{t+1}. \quad (24)}$$

We guess again that the price is constant and the value function is log-linear in wealth. This time we must make sure the value function depends also on income:

$$\boxed{\begin{aligned} P_t &= p, & (25)\\ J_t^i(W) &= -\beta_i^t \exp(-a_i W + a_i Y_{t+1} - b_i). & (26)\end{aligned}}$$

We get the following formula for the expected optimal value at $t+1$

$$\begin{aligned} \mathsf{E}_t J_{t+1}^i(W_{t+1}) \;&=\; -\beta_i^{t+1} \exp\big(-a_i(W_t - C_t)R - a_i Y_{t+1} + a_i Y_{t+1} - b_i\big) \\ &\quad \cdot \exp\left(-a_i \theta_t^\top (\bar{D} - rp) + \tfrac{1}{2} a_i^2 \theta_t^\top \Sigma_D \theta_t\right) \end{aligned} \qquad (27)$$

which is the same formula as before (equation (11)). The other formulas stay the same.

2.3 Stochastic shocks, constant amplitude

We assume that the income received between t and $t+1$ is stochastic, and is of the form:

$$Y_{t+1} \;=\; A^\top \varepsilon_{Y,t+1} \qquad (28)$$

where the "amplitude" A is a d-dimensional constant vector, and the mean of ε_Y is zero. (We can get rid of the deterministic part of the income by incorporating it into b_i as before). The wealth process is therefore given by

$$\boxed{W_{t+1} \;=\; (W_t - C_t)R \;+\; \theta_t^\top (P_{t+1} - RP_t + D_{t+1}) \;+\; A^\top \varepsilon_{Y,t+1} \quad (29)}$$

where $\varepsilon_{Y,t+1}$ and $\varepsilon_{D,t+1}$ are correlated and bivariate normal:

$$\mathsf{E}(\varepsilon_{D,t+1} \varepsilon_{Y,t+1}^\top) \;=\; \Sigma_{DY}, \quad \mathsf{Var}(\varepsilon_{Y,t+1}) \;=\; I_Y. \qquad (30)$$

Our guesses for the price and value function are:

$$\boxed{\begin{aligned} P_t &= p, & (31)\\ J_t^i(W) &= -\beta_i^t \exp(-a_i W - b_i). & (32)\end{aligned}}$$

With these guesses,

$$\begin{aligned} \mathsf{E}_t J_{t+1}^i(W_{t+1}) \;&=\; -\beta_i^{t+1} \exp\big(-a_i(W_t - C_t)R + \tfrac{1}{2} a_i^2 A^\top \Sigma_Y A - b_i\big) \\ &\quad \cdot \exp\left(-a_i \theta_t^\top \left(\bar{D} - rp - a_i \Sigma_{DY} A\right) + \tfrac{1}{2} a_i^2 \theta_t^\top \Sigma_D \theta_t\right) \end{aligned}$$

(Note that we got the extra term $a_i^2 \theta_t^\top \Sigma_{DY} A$ coming from the correlation between the income shock and the dividend.)

The first order condition for θ_t is

$$\bar{D} - rp - a_i \Sigma_{DY} A - a_i \Sigma_D \theta_t \;=\; 0$$

from which we get the optimum trading strategy θ_t^i:

$$\theta_t^i \;=\; \frac{1}{a_i} \Sigma_D^{-1} (\bar{D} - rp) - \Sigma_D^{-1} \Sigma_{DY} A. \tag{33}$$

Adding up the demands from all m agents, we get that the total supply is

$$S \;=\; \frac{1}{a_0} \Sigma_D^{-1} (\bar{D} - rp) - m \Sigma_D^{-1} \Sigma_{DY} A.$$

From this we obtain

$$\boxed{p \;=\; \frac{\bar{D}}{r} - \frac{a_0}{r} \Sigma_D S - \frac{m a_0}{r} \Sigma_{DY} A. \quad (34)}$$

Note that the equilibrium price for risky asset in this model is the same as the price of the perpetuity, minus the risk premium $\frac{a_0}{r} \Sigma_D S$, minus a hedging premium whose sign depends on the covariance matrix Σ_{DY}. The hedging premium arises from the fact that the income shock adds to the overall risk if it is positively correlated to the dividends. Investors do not like assets want to hedge that risk.

Substitute p in (33) and get

$$\boxed{\theta_t^i \;=\; \frac{\alpha_0}{\alpha_i} S + \left(\frac{m \alpha_0}{\alpha_i} - 1 \right) \Sigma_D^{-1} \Sigma_{DY} A. \quad (35)}$$

Notice that the deviation from the typical optimal risk sharing (when there is no income) is inversely proportional to the deviation of the investor's risk aversion (α_i) from the harmonic mean of the risk aversions ($m\alpha_0$).

The rest of the derivation goes the same way, except that we have slightly different formulas for b_i and C_t^i.

2.4 Stochastic shocks, stochastic amplitude

We assume that the income received between t and $t+1$ is of the form:

$$\begin{aligned} Y_{t+1} &= A_t^\top \varepsilon_{Y,t+1}, \text{ with} \\ A_{t+1} &= a_A A_t + \varepsilon_{A,t+1}. \end{aligned}$$

The d-dimensional "amplitude" A_t follows an $AR(1)$-process, and the shocks ε_A have zero mean.

Collect the state variables into a variable denoted by Z_t, and collect the shocks into a big column variable ε_t:

$$Z_t = \begin{bmatrix} 1 \\ A_t \end{bmatrix}, \quad \varepsilon_t = \begin{bmatrix} \varepsilon_{D,t} \\ \varepsilon_{A,t} \\ \varepsilon_{Y,t} \end{bmatrix}, \quad \Sigma = \mathsf{E}(\varepsilon_t \varepsilon_t^\top).$$

We can rewrite the state equation

$$\boxed{Z_{t+1} = a_Z Z_t + b_Z \varepsilon_{t+1}} \tag{36}$$

with

$$a_Z = \begin{bmatrix} 1 & 0 \\ O_{d1} & a_A \end{bmatrix}, \quad b_Z = \begin{bmatrix} O_{1n} & O_{1d} & O_{1d} \\ O_{dn} & I_{dd} & O_{dd} \end{bmatrix},$$

where O_{mn} is the (m,n)-matrix with zero entries, and I_{dd} the d-dimensional identity matrix.

The guess for the price is linear in the state variables Z_t, while for the value function is log-linear in W_t and log-quadratic in Z_t:

$$P_t = p_0 + p_1 A_t = p \cdot Z_t, \tag{37}$$

$$J_t^i(W_t, Z_t) = -\beta_i^t \exp(-a_i W_t - \tfrac{1}{2} Z_t^\top b_i Z_t). \tag{38}$$

The other state variable is wealth, which follows the formula:

$$W_{t+1} = (W_t - C_t)R + \theta_t^\top (P_{t+1} - RP_t + D_{t+1}) + A_t^\top \varepsilon_{Y,t+1}.$$

Rewrite the state equation for W_t as:

$$W_{t+1} = (W_t - C_t)R + \tfrac{1}{2}\theta_t^\top m_{\theta\theta}\theta_t + \theta_t^\top m_{\theta Z} Z_t + \theta_t^\top m_{\theta\varepsilon}\varepsilon_{t+1} + Z_t^\top m_{Z\varepsilon}\varepsilon_{t+1} + \tfrac{1}{2} Z_t^\top m_{ZZ} Z_t$$

where (using $\bar{D} = \begin{bmatrix} \bar{D} & O_{1d} \end{bmatrix} Z_t$, $A_t = \begin{bmatrix} O_{d1} & I_{dd} \end{bmatrix} Z_t$, $\varepsilon_{D,t} = \begin{bmatrix} I_{nn} & O_{nd} & O_{nd} \end{bmatrix} \varepsilon_t$, $\varepsilon_{Y,t} = \begin{bmatrix} O_{dn} & O_{dd} & I_{dd} \end{bmatrix} \varepsilon_t$):

$$m_{\theta\theta} = O_{nn}, \quad m_{\theta Z} = pa_Z - Rp + \begin{bmatrix} O_{d1} & I_{dd} \end{bmatrix}, \quad m_{\theta\varepsilon} = pb_Z + \begin{bmatrix} I_{nn} & O_{nd} & O_{nd} \end{bmatrix},$$

$$m_{Z\varepsilon} = \begin{bmatrix} O_{1n} & O_{1d} & O_{1d} \\ O_{dn} & O_{dd} & I_{dd} \end{bmatrix}, \quad m_{ZZ} = O_{d+1,d+1} \tag{39}$$

We now need to compute

$$\mathsf{E}_t J_{t+1}^i(W_{t+1}) = -\beta_i^{t+1} \exp(-a_i W_{t+1} - \tfrac{1}{2} A_{t+1}^\top b_i A_{t+1}) = -\beta_i^{t+1} \exp(-a - b^\top \varepsilon - \tfrac{1}{2}\varepsilon^\top A \varepsilon), \tag{40}$$

where

$$\begin{aligned} \varepsilon &= \varepsilon_{t+1}, \\ a &= a_i(W_t - c_t)R + \tfrac{a_i}{2}\theta_t^\top m_{\theta\theta}\theta_t + a_i \theta_t^\top m_{\theta Z} Z_t + \tfrac{a_i}{2} Z_t^\top m_{ZZ} Z_t + \tfrac{1}{2} Z_t^\top a_Z^\top b_i a_Z Z_t, \\ b &= a_i m_{\theta\varepsilon}^\top \theta_t + \big(a_i m_{Z\varepsilon} + a_Z^\top b_i b_Z\big)^\top Z_t, \\ B &= b_Z^\top b_i b_Z. \end{aligned}$$

Now we apply the following standard lemma in multivariate normal calculus.

Lemma 1. *Let ε be a multivariate normal random variable, with zero mean and covariance matrix Σ. Let b be a constant vector, and B a constant symmetric semi-positive definite matrix. Define $\Omega = \left(B + \Sigma^{-1}\right)^{-1}$. Then*

$$\mathsf{E} \exp\left(-b^\top \varepsilon - \tfrac{1}{2}\varepsilon^\top B \varepsilon\right) = \left(\frac{|\Omega|}{|\Sigma|}\right)^{1/2} \exp\left(\tfrac{1}{2} b^\top \Omega b\right), \tag{41}$$

where $|M|$ denotes the determinant of the square matrix M.

Define

$$\delta_i = \left(\frac{|\Omega|}{|\Sigma|}\right)^{1/2}. \tag{42}$$

We finally get

$$\mathsf{E}_t J^i_{t+1} = -\delta_i \beta_i^{t+1} \exp\Big(-a_i R(W_t - C_t) - \tfrac{1}{2}(\theta_t)^\top u_{\theta\theta}\theta_t - (\theta_t)^\top u_{\theta Z} Z_t - \tfrac{1}{2} Z_t^\top u_{ZZ} Z_t\Big), \tag{43}$$

where the matrices $u_{\theta\theta}$, $u_{\theta Z}$ and u_{ZZ} are given by

$$\begin{aligned} u_{\theta\theta} &= a_i m_{\theta\theta} - \left(a_i m_{\theta\varepsilon}\right) \Omega \left(a_i m_{\theta\varepsilon}\right)^\top, \\ u_{\theta Z} &= a_i m_{\theta Z} - \left(a_i m_{\theta\varepsilon}\right) \Omega \left(a_i m_{Z\varepsilon} + a_Z^\top b_i b_Z\right)^\top, \\ u_{ZZ} &= a_i m_{ZZ} + a_Z^\top b_i a_Z - \left(a_i m_{Z\varepsilon} + a_Z^\top b_i b_Z\right) \Omega \left(a_i m_{Z\varepsilon} + a_Z^\top b_i b_Z\right)^\top. \end{aligned} \tag{44}$$

Define

$$q_i = u_{ZZ} - (u_{\theta Z})^\top (u_{\theta\theta})^{-1} u_{\theta Z}. \tag{45}$$

Define the quadratic function Φ by

$$\Phi(\theta_t, Z_t) = \tfrac{1}{2}\theta_t^\top u_{\theta\theta}\theta_t + \theta_t^\top u_{\theta Z} Z_t + \tfrac{1}{2} Z_t^\top u_{ZZ} Z_t. \tag{46}$$

Now rewrite the Bellman equation as follows

$$\begin{aligned} -\beta_i^t \exp\left(-a_i W_t - \tfrac{1}{2} Z_t^\top b_i Z_t\right) &= \\ \max_{C_t, \theta_t} \Big\{ -\beta_i^t \exp(-\alpha_i C_t) - \delta_i \beta_i^{t+1} \exp\big(-a_i R(W_t - C_t) - \Phi(\theta_t, Z_T)\big) \Big\}&. \end{aligned} \tag{47}$$

The first order conditions for C_t and θ_t are[5]

$$\begin{aligned} C_t^i &= \tfrac{1}{\alpha_i + a_i R} \ln \tfrac{\alpha_i}{\beta_i a_i R \delta_i} + \tfrac{a_i R}{\alpha_i + a_i R} W_t + \tfrac{1}{\alpha_i + a_i R} \Phi(\theta_t, Z_t), & (48) \\ \theta_t^i &= -(u_{\theta\theta})^{-1} u_{\theta Z} Z_t. & (49) \end{aligned}$$

So the optimal agent holdings are linear in the state variables Z_t. This means that the optimal portfolio must hedge against fluctuations in the state variable A_t.

The coefficient h_i can be defined by

[5] The second order condition is always satisfied for C_t, and is satisfied for θ_t if and only if $u_{\theta\theta}$ is positive definite.

$$\boxed{h_i \;=\; -(u_{\theta\theta})^{-1} u_{\theta Z}. \qquad (50)}$$

At the optimum θ_t^i we can calculate

$$\Phi(\theta_t^i, Z_t) = \Phi\big(h_i Z_t, Z_t\big) = \tfrac{1}{2} Z_t^\top \Big(u_{ZZ} - (u_{\theta Z})^\top (u_{\theta\theta})^{-1} u_{\theta Z}\Big) Z_t = \tfrac{1}{2} Z_t^\top q_i Z_t. \tag{51}$$

Substitute (48), (49) and (51) into (47), and identify the coefficients of W_t and Z_t. We obtain again the equation

$$a_i = \frac{\alpha_i\, r}{R} \tag{52}$$

and an equation for q_i

$$\boxed{b_i \;=\; \tfrac{1}{R} q_i - 2\ln\left(\tfrac{R}{r}\big(\beta_i r \delta_i\big)^{1/R}\right) E^{11}, \quad (53)}$$

where E^{11} is the $(1+d) \times (1+d)$ matrix with all entries zero, except for the top left entry, which is one. We can also substitute (52) and (51) into equation (48), and derive the optimal consumption strategy

$$C_t = \tfrac{r}{R} W_t - \tfrac{1}{\alpha_i R} \ln(\beta_i r \delta_i) + \tfrac{1}{\alpha_i R}\, \tfrac{1}{2} Z_t^\top q Z_t. \tag{54}$$

Finally, we write the market clearing equation. We normalize the total supply in each asset to one, i.e. $S = e_n$ the column n-vector with entries all equal to one. Note that S can be written as hZ, where h is the $(n, d+1)$-matrix whose every row is the vector $\begin{bmatrix} 1 & O_{1d} \end{bmatrix}$. Then the market clearing equation can be written as

$$\boxed{\sum_{i=1}^{m} h_i \;=\; h. \qquad (55)}$$

We are done as long as we show that the boxed equations resulting from verifying our guesses are satisfied by some values of p, q_i, b_i. In general, this can only be done numerically.

3 Continuous time

The notions used in this section were described in Section 1. The time interval of interest here is $[t, t + \mathrm{d}t]$. At the beginning of this interval (time t) the investor has holdings $\theta_j(t)$ in (or equivalently percentage holdings $\omega_j(t)$) in asset j. These holdings stay constant until $t + \mathrm{d}t$, when trading takes place in the amount of $\mathrm{d}\theta_j(t)$ (or $\mathrm{d}\omega_j(t)$ percentage-wise). During this interval the investor consumes $\mathrm{d}C(t) = c(t)\,\mathrm{d}t$ (deterministic consumption[6]), and receives income shock $\mathrm{d}Y(t)$. If the income shock is deterministic, it can be written as $\mathrm{d}Y(t) = y(t)\,\mathrm{d}t$.

The investment opportunity set is again formed with a risk-free asset with constant rate of return r, and n risky assets for which the dividend is an Itô process with constant coefficients. Let $\mathrm{d}B(t)$ be a standard n-dimensional Brownian motion. Then the dividend process follows

$$\mathrm{d}D(t) \;=\; \mu_D\, \mathrm{d}t + (\Sigma_D)^{1/2}\, \mathrm{d}B(t), \tag{56}$$

[6] We need deterministic consumption if we want consumption to be a control variable at t.

where the covariance matrix Σ_D is symmetric and positive definite.

We start in a partial equilibrium situation, where the prices P_j of assets are given, and we will think later about how prices are determined in equilibrium. We guess that P_j is an Itô process with constant coefficients:

$$\mathrm{d}P(t) \;=\; \mu\,\mathrm{d}t + \Sigma^{1/2}\,\mathrm{d}B(t). \tag{57}$$

The wealth process without income shocks was described in Section 1.2. It can be written in both additive and multiplicative form, although in the CARA investor case we will focus on the additive form

$$\mathrm{d}W(t) \;=\; rW(t)\,\mathrm{d}t - c(t)\,\mathrm{d}t + W(t)\sum_{j=1}^{n}\theta_j(t)\left(\frac{\mathrm{d}P_j(t)}{P_j(t)} - r\,\mathrm{d}t + \frac{\mathrm{d}D_j(t)}{P_j(t)}\right), \tag{58}$$

$$\mathrm{d}W(t) \;=\; rW(t)\,\mathrm{d}t - c(t)\,\mathrm{d}t + \sum_{j=1}^{n}\theta_j(t)\big(\mathrm{d}P_j(t) - rP_j(t)\,\mathrm{d}t + \,\mathrm{d}D_j(t)\big) \tag{59}$$

$$\;=\; rW(t)\,\mathrm{d}t - c(t)\,\mathrm{d}t + \theta(t)^\top\big(\,\mathrm{d}P(t) + \,\mathrm{d}D(t) - rP(t)\,\mathrm{d}t\big). \tag{60}$$

We assume that investor i has instantaneous CARA utility $u(c)$

$$u(c) \;=\; -\exp(-\alpha_i c)$$

and with time discount coefficient $\beta_i = \exp(-\rho_i)$. At each t the investor solves

$$J(t, W(t)) \;=\; \max_{(c_s,\theta_s)_{s\geq t}} \mathsf{E}_t\left\{\int_{s=t}^{\infty}\beta_i^s u(c_s)\,\mathrm{d}t\right\}, \tag{61}$$

In discrete time the Bellman principle of optimality says that

$$J(t, W_t) \;=\; \max_{c_t,\theta_t}\Big[\beta_i^t u(C_t) + \mathsf{E}_t J(t+1, W_{t+1})\Big].$$

In continuous time (where $t+1$ is replaced by $t+\,\mathrm{d}t$) this becomes

$$J\big(t, W(t)\big) \;=\; \max_{c(t),\theta(t)}\Big[\beta_i^t u(c(t))\,\mathrm{d}t \;+\; \mathsf{E}_t J\big(t+\,\mathrm{d}t, W(t)+\,\mathrm{d}W(t)\big)\Big].$$

Now use Itô's lemma for $J\big(t+\,\mathrm{d}t, W(t)+\,\mathrm{d}W(t)\big) = J + J_t\,\mathrm{d}t + J_W\,\mathrm{d}W + \frac{1}{2}J_{WW}\,\mathrm{d}W^2$, and subtract J from both sides to obtain the Hamilton-Jacobi-Bellman (HJB) equation

$$\boxed{\max_{c,\theta}\Big[\beta_i^t u(c)\,\mathrm{d}t + J_t\,\mathrm{d}t + J_W\mathsf{E}_t(\,\mathrm{d}W) + \tfrac{1}{2}J_{WW}\mathsf{E}_t(\,\mathrm{d}W^2)\Big] \;=\; 0,} \tag{62}$$

where the subscripts indicate partial derivatives.

3.1 No income shocks

With no income shocks, the wealth process follows (use formulas (60) and (57))

$$\mathrm{d}W \;=\; rW\,\mathrm{d}t - c\,\mathrm{d}t + \theta^\top(\,\mathrm{d}P + \,\mathrm{d}D - rP\,\mathrm{d}t). \tag{63}$$

As in the discrete case, we guess that the price is constant:

$$P \;=\; P_0 = \text{constant}. \tag{64}$$

The wealth process then follows

$$\mathrm{d}W \;=\; rW\,\mathrm{d}t - c\,\mathrm{d}t + \theta^\top\Big[(\mu_D - rP_0)\,\mathrm{d}t + \Sigma_D\,\mathrm{d}B\Big], \tag{65}$$

so we can compute

$$\mathsf{E}_t(\,\mathrm{d}W) \;=\; \big(rW - c + \theta^\top(\mu_D - rP_0)\big)\;\mathrm{d}t, \tag{66}$$

$$\mathsf{E}_t(\,\mathrm{d}W^2) \;=\; \big(\theta^\top\Sigma_D\theta\big)\;\mathrm{d}t. \tag{67}$$

The HJB equation becomes

$$\boxed{0 \;=\; \max_{c,\theta}\Big[-\beta_i^t\exp(-\alpha_i c)\,\mathrm{d}t + J_t\,\mathrm{d}t + J_W\mathsf{E}_t(\,\mathrm{d}W) + \tfrac{1}{2}J_{WW}\mathsf{E}_t(\,\mathrm{d}W^2)\Big]} \tag{68}$$

or after dividing by $\mathrm{d}t$

$$0 \;=\; \max_{c,\theta}\Big[-\beta_i^t\exp(-\alpha_i c) + J_t + J_W\left(rW - c + \theta^\top(\mu_D - rP_0)\right) + \tfrac{1}{2}J_{WW}\left(\theta^\top\Sigma_D\theta\right)\Big] \tag{69}$$

The first order conditions with respect to c and θ are

$$\alpha_i\beta_i^t\exp(-\alpha_i c^*) - J_W \;=\; 0, \tag{70}$$

$$(\mu_D - rP_0)J_W + \Sigma_D\theta^* J_{WW} \;=\; 0. \tag{71}$$

From this we obtain

$$\boxed{\begin{aligned} c^* &= \frac{\log(\alpha_i) - \rho_i t - \log(J_W)}{\alpha_i}, && (72)\\ \theta^* &= \Sigma_D^{-1}(\mu_D - rP_0)\left(-\frac{J_W}{J_{WW}}\right). && (73)\end{aligned}}$$

Substituting these equations into the HJB equation (69) (or, more simply, using the first order conditions) we get

$$0 \;=\; J_t + J_W\left[-\frac{1}{\alpha_i} + rW + \frac{1}{\alpha_i}\log\left(\frac{1}{\alpha_i}\exp(\rho_i t)J_W\right) - \frac{1}{2}\frac{J_W}{J_{WW}}(\mu_D - rP_0)^\top\Sigma_D^{-1}(\mu_D - rP_0)\right] \tag{74}$$

We can make $J_i(t, W)$ time-invariant: define[7]

$$I(W) = \exp(\rho_i t) J(t, W). \tag{75}$$

We get the HJB equation for I, which does not depend on t:

$$0 = -\rho_i I + I' \left[-\frac{1}{\alpha_i} + rW + \frac{1}{\alpha_i} \log\left(\frac{I'}{\alpha_i}\right) - \frac{1}{2}\frac{I'}{I''}(\mu_D - rP_0)^\top \Sigma_D^{-1} (\mu_D - rP_0) \right]. \tag{76}$$

Therefore this is an ODE in W. Again, we guess that that the time-invariant value function I is log-linear in wealth:

$$\boxed{I(W) = -\exp(-a_i W - b_i). \quad (77)}$$

One can compute $I'/I = -a_i$. The HJB implies

$$-\frac{\rho_i}{a_i} = -\frac{1}{\alpha_i} + rW - \frac{a_i W + b_i + \log(\alpha_i)}{\alpha_i} + \frac{1}{2a_i}(\mu_D - rP_0)^\top \Sigma_D^{-1} (\mu_D - rP_0).$$

Identify the coefficients of W:

$$\boxed{a_i = r\alpha_i. \quad (78)}$$

Identify the constant terms:

$$b_i = -\log(\alpha_i) + \frac{\rho_i - r}{r} + \frac{1}{2r}(\mu_D - rP_0)^\top \Sigma_D^{-1} (\mu_D - rP_0).$$

Notice that the optimal holdings of investor i are

$$\boxed{\theta_i = \frac{1}{r\alpha_i}\Sigma_D^{-1}\big(\mu_D - rP_0\big) \quad (79)}$$

Market clearing implies $\sum_i \theta_i = S$, the total supply vector. Denote as before $1/\alpha_0 = \sum_i 1/\alpha_i$. We obtain the equilibrium price vector

$$\boxed{P_0 = \frac{\mu_D}{r} - \alpha_0 \Sigma_D S, \quad (80)}$$

which is essentially the same formula as in the discrete case. Note that the formula for P_0 implies $\mu_D - rP_0 = \alpha_0 \Sigma_D S$, so we can recompute b_i:

$$\boxed{b_i = -\log(\alpha_i) + \frac{\rho_i - r}{r} + \frac{\alpha_0^2}{2r} S^\top \Sigma_D S. \quad (81)}$$

Use the first order condition for the optimal consumption policy c_i, and get the formula:

$$\boxed{c_i = \frac{b_i - \log(r)}{\alpha_i} + rW. \quad (82)}$$

[7]Technically, it should be defined as $I(t, W)$ and show that I does not depend on t. This can be seen from the HJB equation for I below.

3.2 Deterministic income shocks

We assume that the wealth process follows

$$\mathrm{d}W \;=\; rW\,\mathrm{d}t - c\,\mathrm{d}t + \theta^\top(\,\mathrm{d}P + \,\mathrm{d}D - rP\,\mathrm{d}t) + y_t\,\mathrm{d}t, \tag{83}$$

where y_t is known at time t, hence the income between t and $t + \,\mathrm{d}t$ is deterministic. The only term that changes in the HJB equation (68) is $\mathsf{E}_t(\,\mathrm{d}W)$ which gets an extra $y\mathrm{d}t$ term. The first order conditions are the same. The only problem is that equation (74) now gets an extra term $y_t J_W$. Nevertheless, if we allow for b_i to be time-varying, we can absorb y_t in it, so the only formula that changes is that for c_i.

3.3 Stochastic shocks, constant amplitude

Assume that the wealth process follows

$$\mathrm{d}W \;=\; rW\,\mathrm{d}t - c\,\mathrm{d}t + \theta^\top(\,\mathrm{d}P + \,\mathrm{d}D - rP\,\mathrm{d}t) + A^\top\,\mathrm{d}B_Y, \tag{84}$$

where the instantaneous covariance matrix between B_D and B_F is

$$\Sigma_{DY} \;=\; \frac{1}{\mathrm{d}t}\mathrm{cov}_t(\,\mathrm{d}B_D,\,\mathrm{d}B_Y) \;=\; \frac{1}{\mathrm{d}t}\mathsf{E}_t(\,\mathrm{d}B_D\,\mathrm{d}B_Y^\top). \tag{85}$$

As before, we guess that the price is constant

$$P \;=\; P_0 = \text{constant}.$$

We compute

$$\frac{1}{\mathrm{d}t}\mathsf{E}_t(\,\mathrm{d}W) \;=\; rW - c + \theta^\top(\mu_D - rP_0), \tag{86}$$

$$\frac{1}{\mathrm{d}t}\mathsf{E}_t(\,\mathrm{d}W^2) \;=\; \theta^\top\Sigma_D\theta + A^\top A + 2\theta^\top\Sigma_{DY}A. \tag{87}$$

The first order conditions with respect to c and θ are:

$$\alpha_i\beta_i^t\exp(-\alpha_i c) - J_W \;=\; 0, \tag{88}$$

$$(\mu_D - rP_0)J_W + (\Sigma_D\theta + \Sigma_{DY}A)J_{WW} \;=\; 0. \tag{89}$$

Again we guess

$$J(t,W) \;=\; -\beta_i^t\exp(-a_iW - b_i).$$

We compute the optimal holdings

$$\theta_i \;=\; \frac{1}{a_i}\Sigma_D^{-1}(\mu_D - rP_0) - \Sigma_D^{-1}\Sigma_{DY}A. \tag{90}$$

Denote by m the number of (CARA) investors, and by $1/\alpha_0 = \sum_i 1/\alpha_i$. We get the same formulas as in the discrete case:

$$P_0 \;=\; \frac{\mu_D}{r} - \alpha_0\Sigma_D S - m\alpha_0\Sigma_{DY}A.$$

$$\theta_i \;=\; \frac{\alpha_0}{\alpha_i}S + \left(\frac{m\alpha_0}{\alpha_i} - 1\right)\Sigma_D^{-1}\Sigma_{DY}A.$$

3.4 Stochastic shocks, stochastic amplitude

This section is very related to Merton's ICAPM with stochastic opportunity sets. We will find out that the optimal portfolio has a hedging component linear in the state variables. We assume that the wealth process follows

$$\mathrm{d}W \;=\; rW\,\mathrm{d}t - c\,\mathrm{d}t + \theta^\top(\,\mathrm{d}P + \,\mathrm{d}D - rP\,\mathrm{d}t) + A_t^\top\,\mathrm{d}B_Y, \tag{91}$$

where A_t is an Ornstein–Uhlenbeck process (the continuous time equivalent of an $AR(1)$-process) with zero mean and constant coefficients:

$$\mathrm{d}A_t \;=\; -a_A A_t\,\mathrm{d}t + \Sigma_A^{1/2}\,\mathrm{d}B_A. \tag{92}$$

Here the only state variable becomes the amplitude of the income shock, A_t.[8] We guess that the price is linear in A_t:

$$P_t \;=\; p_1 + p_A A_t, \tag{93}$$

and that the value function is log-linear in W_t and log-quadratic in A_t:

$$J(t,W,A) \;=\; \beta_i^t \exp\Big(-a_i W - b_i - A^\top u_i - \tfrac{1}{2}A^\top v_i A\Big). \tag{94}$$

We can rewrite the wealth process as follows:

$$\begin{aligned}\mathrm{d}W \;&=\; \Big(rW - c + \theta^\top\big(-p_A a_A A + \mu_D - rp_1 - rp_A A\big)\Big)\,\mathrm{d}t \\ &\quad + \theta^\top\big(p_A\Sigma_A^{1/2}\,\mathrm{d}B_A + \Sigma_D^{1/2}\,\mathrm{d}B_D\big) + A^\top\,\mathrm{d}B_Y.\end{aligned}$$

We compute

$$\frac{1}{\mathrm{d}t}\mathsf{E}_t(\,\mathrm{d}W) \;=\; rW - c + \theta^\top\big(-p_A a_A A + \mu_D - rp_1 - rp_A A\big), \tag{95}$$

$$\frac{1}{\mathrm{d}t}\mathsf{E}_t(\,\mathrm{d}W^2) \;=\; \theta^\top(p_A\Sigma_A p_A^\top + \Sigma_D + 2p_A\Sigma_{AD})\theta + A^\top A + 2\theta^\top(p_A\Sigma_{AY} + \Sigma_{DY})A. \tag{96}$$

When we have other state variables, such as A, the HJB equation is derived from the Bellman principle at time t:

$$J\big(t,W,A\big) \;=\; \max_{c,\theta}\Big[\beta_i^t u(c)\,\mathrm{d}t \;+\; \mathsf{E}_t J\big(t+\,\mathrm{d}t, W+\,\mathrm{d}W, A+\,\mathrm{d}A\big)\Big].$$

The HJB equation in this case is

$$\begin{aligned}0 \;&=\; \max_{c,\theta}\Big[-\beta_i^t\exp(-\alpha_i c)\,\mathrm{d}t \;+\; J_t\,\mathrm{d}t \;+\; J_W\mathsf{E}_t(\,\mathrm{d}W) \;+\; \tfrac{1}{2}J_{WW}\mathsf{E}_t(\,\mathrm{d}W^2) \\ &\quad + J_A\mathsf{E}_t(\,\mathrm{d}A) \;+\; \tfrac{1}{2}\mathsf{E}_t\big(\,\mathrm{d}A^\top J_{AA}\,\mathrm{d}A\big) + \mathsf{E}_t\big(\,\mathrm{d}A^\top J_{AW}\,\mathrm{d}W\big)\Big].\end{aligned}$$

[8] We could proceed as in the discrete time case and define a state variable Z_t that appends the constant 1 to A. We prefer to avoid the tedious notation from the discrete case, and instead get the formulas involving A directly.

The terms $J_A \mathsf{E}_t(\,\mathrm{d}A)$ and $\frac{1}{2}\mathsf{E}_t\big(\,\mathrm{d}A^\top J_{AA}\,\mathrm{d}A\big)$ can be omitted because they do not contain c or θ. Now we compute

$$\begin{aligned} J_A &= -\big(u_i + v_i A\big) J \qquad \Longrightarrow \qquad J_{AW} = a_i\big(u_i + v_i A\big) J, \\ J_t &= -\rho_i J, \qquad J_W = -a_i J, \qquad J_{WW} = a_i^2 J. \end{aligned}$$

One computes

$$\frac{1}{\mathrm{d}t}\mathsf{E}_t(\,\mathrm{d}W\,\mathrm{d}A^\top) = \theta^\top(p_A\Sigma_A + \Sigma_{DA}) + A^\top\Sigma_{YA},$$

and therefore (wealth W is a scalar)

$$\frac{1}{\mathrm{d}t}\mathsf{E}_t(\,\mathrm{d}A^\top J_{AW}\,\mathrm{d}W) = a_i J\Big(\theta^\top(p_A\Sigma_A + \Sigma_{DA}) + A^\top\Sigma_{YA}\Big)(u_i + v_i A). \tag{97}$$

The HJB equation becomes

$$\begin{aligned} 0 = \max_{c,\theta}\Big[& -\beta_i^t \exp(-\alpha_i c) - \rho_i J - a_i J\Big(rW - c + \theta^\top\big(-p_A a_A A + \mu_D - rp_1 - rp_A A\big)\Big) \\ & + \tfrac{1}{2}a_i^2 J\Big(\theta^\top(p_A\Sigma_A p_A^\top + \Sigma_D + 2p_A\Sigma_{AD})\theta + 2\theta^\top(p_A\Sigma_{AY} + \Sigma_{DY})A\Big) \\ & + a_i J\theta^\top(p_A\Sigma_A + \Sigma_{DA})(u_i + v_i A)\Big]. \end{aligned} \tag{98}$$

The first order conditions with respect to c and θ are

$$0 = \alpha_i\beta_i^t\exp(-\alpha_i c) + a_i J, \tag{99}$$

$$\begin{aligned} 0 = {} & \frac{1}{a_i}\Big[(\mu_D - rp_1) - p_A(a_A - r)A\Big] - \Big[\Omega_{AD}\theta + (p_A\Sigma_{AY} + \Sigma_{DY})A\Big] \\ & - \frac{1}{a_i}(p_A\Sigma_A + \Sigma_{DA})(u_i + v_i A), \end{aligned} \tag{100}$$

where we divided the second equation by $-a_i^2 J$, and defined

$$\Omega_{AD} = p_A\Sigma_A p_A^\top + \Sigma_D + 2p_A\Sigma_{AD}. \tag{101}$$

Then the optimal holdings of agent i satisfies

$$\begin{aligned} \Omega_{AD}\theta_i = {} & \frac{1}{a_i}\Big[(\mu_D - rp_1) - p_A(a_A - r)A\Big] - \frac{1}{a_i}(p_A\Sigma_A + \Sigma_{DA})(u_i + v_i A) \\ & - (p_A\Sigma_{AY} + \Sigma_{DY})A. \end{aligned} \tag{102}$$

Notice that the holdings are linear in the state variable A. We now use market clearing to compute the price P. We know that $\theta_i =$ add up to the total supply S, therefore

$$\begin{aligned} \Omega_{AD}S = {} & \frac{1}{a_0}\Big[(\mu_D - rp_1) - p_A(a_A - r)A\Big] - \sum_{i=1}^m \frac{1}{a_i}(p_A\Sigma_A + \Sigma_{DA})(u_i + v_i A) \\ & - m(p_A\Sigma_{AY} + \Sigma_{DY})A. \end{aligned}$$

One can now identify the coefficients of this linear expression in A:

$$p_A \left(\frac{a_A - r}{a_0} + \sum_{i=1}^{m} \Sigma_A \frac{v_i}{a_i} - m\Sigma_{AY} \right) \;=\; -\sum_{i=1}^{m} \Sigma_{DA} \frac{v_i}{a_i} - m\Sigma_{DY}, \tag{103}$$

$$\mu_D - rp_1 \;=\; a_0 \Omega_{AD} S + \sum_{i=1}^{m} \big(p_A \Sigma_A + \Sigma_{DA} \big) u_i \frac{a_0}{a_i}. \tag{104}$$

From these two equations we obtain p_1 and p_A as formulas of the u_i and v_i. We can then plug p_1 and p_A into the HJB equation (98) for each individual investor i, to find u_i and v_i. The equations are nonlinear in u_i, v_i.

One can simplify the formulas if one assumes that the shocks $\mathrm{d}B_A$ to the amplitude A are not correlated with either the dividend shocks $\mathrm{d}B_D$ or the income shocks $\mathrm{d}B_Y$. The equations are nevertheless still nonlinear.