

# EQUIVARIANT ELLIPTIC COHOMOLOGY AND RIGIDITY

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*Abstract.* Equivariant elliptic cohomology with complex coefficients was defined axiomatically by Ginzburg, Kapranov and Vasserot and constructed by Grojnowski. We give an invariant definition of complex  $S^1$ -equivariant elliptic cohomology, and use it to give an entirely cohomological proof of the rigidity theorem of Witten for the elliptic genus. We also state and prove a rigidity theorem for families of elliptic genera.

**1. Introduction.** The classical level 2 elliptic genus is defined (see Landweber [14], p. 56) as the Hirzebruch genus with exponential series the Jacobi sine. (For a definition of the Jacobi sine  $s(x)$  see the beginning of Section 4.) It is intimately related to the mysterious field of elliptic cohomology (see Segal [19]), and to string theory (see Witten [22] and [23]). A striking property of the elliptic genus is its rigidity with respect to group actions. This was conjectured by Ochanine in [18], and by Witten in [22], where he used string theory arguments to support it.

Rigorous mathematical proofs for the rigidity of the elliptic genus were soon given by Taubes [21], Bott & Taubes [4], and Liu [15]. While Bott and Taubes' proof involved the localization formula in equivariant K-theory, Liu's proof focused on the modularity properties of the elliptic genus. The question remained, however, whether one could find a direct connection between the rigidity theorem and elliptic cohomology.

Earlier on, Atiyah and Hirzebruch [3] had used pushforwards in equivariant K-theory to prove the rigidity of the  $\hat{A}$ -genus for spin manifolds. Following this idea, H. Miller [16] interpreted the equivariant elliptic genus as a pushforward in the completed Borel equivariant cohomology, and posed the problem of developing and using a *noncompleted*  $S^1$ -equivariant elliptic cohomology to prove the rigidity theorem.

In 1994 Grojnowski [10] proposed a noncompleted equivariant elliptic cohomology theory with complex coefficients. For  $G$  a compact connected Lie group he defined  $E_G^*(-)$  as a coherent holomorphic sheaf over a certain variety  $X_G$  constructed from a given elliptic curve. Grojnowski also constructed pushforwards in this theory. At about the same time and independently, Ginzburg, Kapranov and Vasserot [9] gave an axiomatic description of equivariant elliptic cohomology.

Given Grojnowski's construction, it seemed natural to try to use  $S^1$ -equivariant elliptic cohomology to prove the rigidity theorem. In doing so, we noticed that our proof relies on a generalization of Bott and Taubes' "transfer formula" (see [4]). This generalization turns out to be essentially equivalent to the existence of a Thom class (or orientation) in  $S^1$ -equivariant elliptic cohomology.

We can generalize the results of this paper in several directions. One is to extend the rigidity theorem to families of elliptic genera, which we do in Theorem 5.6. Another would be to generalize from  $G = S^1$  to an arbitrary connected compact Lie group, or to replace complex coefficients with rational coefficients for all cohomology theories involved. Such generalizations will be treated elsewhere.

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**2. Statement of results.** All the cohomology theories involved in this paper have complex coefficients. If  $X$  is a finite  $S^1$ -CW complex,  $H_{S^1}^*(X)$  denotes its Borel  $S^1$ -equivariant cohomology with complex coefficients (see Atiyah and Bott [2]). If  $X$  is a point  $*$ ,  $H_{S^1}^*(*) \cong \mathbb{C}[u]$ .

Let  $\mathcal{E}$  be an elliptic curve over  $\mathbb{C}$ . Let  $X$  be a finite  $S^1$ -CW complex, e.g., a compact  $S^1$ -manifold. (A compact  $S^1$ -manifold always has an  $S^1$ -CW complex structure: see Alday and Puppe [1].) Then, following Grojnowski [10], we define  $E_{S^1}^*(X)$ , the  $S^1$ -equivariant elliptic cohomology of  $X$ . This is a coherent analytic sheaf of  $\mathbb{Z}_2$ -graded algebras over  $\mathcal{E}$ . We alter his definition slightly, in order to show that the definition of  $E_{S^1}^*(X)$  depends only on  $X$  and the elliptic curve  $\mathcal{E}$ . Let  $\alpha$  be a point of  $\mathcal{E}$ . We associate a subgroup  $H(\alpha)$  of  $S^1$  as follows: if  $\alpha$  is a torsion point of  $\mathcal{E}$  of exact order  $n$ ,  $H(\alpha) = \mathbb{Z}_n$ ; otherwise,  $H(\alpha) = S^1$ . We define  $X^\alpha = X^{H(\alpha)}$ , the subspace of  $X$  fixed by  $H(\alpha)$ . Then we will define a sheaf  $E_{S^1}^*(X)$  over  $\mathcal{E}$  whose stalk at  $\alpha$  is

$$E_{S^1}^*(X)_\alpha = H_{S^1}^*(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0}.$$

Here  $\mathcal{O}_{\mathbb{C},0}$  represents the local ring of germs of holomorphic functions at zero on  $\mathbb{C} = \text{Spec } \mathbb{C}[u]$ . In particular, the stalk of  $E_{S^1}^*(X)$  at zero is  $H_{S^1}^*(X) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0}$ .

**THEOREM A.**  *$E_{S^1}^*(X)$  only depends on  $X$  and the elliptic curve  $\mathcal{E}$ . It extends to an  $S^1$ -equivariant cohomology theory with values in the category of coherent analytic sheaves of  $\mathbb{Z}_2$ -graded algebras over  $\mathcal{E}$ .*

If  $f: X \rightarrow Y$  is a complex oriented map between compact  $S^1$ -manifolds, Grojnowski also defines equivariant elliptic pushforwards. They are maps of sheaves of  $\mathcal{O}_{\mathcal{E}}$ -modules  $f_!^E: E_{S^1}^*(X)^{[f]} \rightarrow E_{S^1}^*(Y)$  satisfying properties similar to those of

the usual pushforward (see Dyer [7]).  $E_{S^1}^*(X)^{[f]}$  has the same stalks as  $E_{S^1}^*(X)$ , but the gluing maps are different.

If  $Y$  is a point, then  $f_!^E(1)$  on the stalks at zero is the  $S^1$ -equivariant elliptic genus of  $X$  (which is a power series in  $u$ ). By analyzing in detail the construction of  $f_!^E$ , we obtain the following interesting result, which answers a question posed by H. Miller and answered independently by Dessai and Jung [6].

**PROPOSITION B.** *The  $S^1$ -equivariant elliptic genus of a compact  $S^1$ -manifold is the Taylor expansion at zero of a function on  $\mathbb{C}$  which is holomorphic at zero and meromorphic everywhere.*

Grojnowski's construction raises a few natural questions. First, can we say more about  $E_{S^1}^*(X)^{[f]}$ ? The answer is given in Proposition 5.7, where we show that, up to an invertible sheaf,  $E_{S^1}^*(X)^{[f]}$  is the  $S^1$ -equivariant elliptic cohomology of the Thom space of the stable normal bundle to  $f$ . (In fact, if we enlarge our category of equivariant  $CW$ -complexes to include equivariant spectra, we can show that  $E_{S^1}^*(X)^{[f]}$  is the reduced  $E_{S^1}^*$  of a Thom spectrum  $X^{-Tf}$ . See the discussion after Proposition 5.7 for details.)

This suggests looking for a Thom section (orientation) in  $E_{S^1}^*(X)^{[f]}$ . More generally, given a real oriented vector bundle  $V \rightarrow X$ , we can twist  $E_{S^1}^*(X)$  in a similar way to obtain a sheaf, which we denote by  $E_{S^1}^*(X)^{[V]}$ . For the rest of this section we regard all the sheaves not on  $\mathcal{E}$ , but on a double cover  $\tilde{\mathcal{E}}$  of  $\mathcal{E}$ . The reason for this is given in the beginning of Subsection 5.2. So when does a Thom section exist in  $E_{S^1}^*(X)^{[V]}$ ? The answer is the following key result.

**THEOREM C.** *If  $V \rightarrow X$  is a spin  $S^1$ -vector bundle over a finite  $S^1$ -CW complex, then the element 1 in the stalk of  $E_{S^1}^*(X)^{[V]}$  at zero extends to a global section, called the Thom section.*

The proof of Theorem C is essentially a generalization of Bott and Taubes' "transfer formula" (see [4]). Indeed, when we try to extend 1 to a global section, we see that the only points where we encounter difficulties are certain torsion points of  $\mathcal{E}$  which we call *special* (as defined in the beginning of Section 3). But extending our section at a special point  $\alpha$  amounts to lifting a class from  $H_{S^1}^*(X^{S^1}) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0}$  to  $H_{S^1}^*(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0}$  via the restriction map  $i^*: H_{S^1}^*(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0} \rightarrow H_{S^1}^*(X^{S^1}) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0}$ . This is not a problem, except when we have two different connected components of  $X^{S^1}$  inside one connected component of  $X^\alpha$ . Then the two natural lifts differ up to a sign, which can be shown to disappear if  $V$  is spin. This observation is due to Bott and Taubes, and is the centerpiece of their "transfer formula."

Given Theorem C, the rigidity theorem of Witten follows easily: Let  $X$  be a compact spin  $S^1$ -manifold. Then the  $S^1$ -equivariant pushforward of  $f: X \rightarrow *$  is a map of sheaves  $f_!^E: E_{S^1}^*(X)^{[f]} \rightarrow E_{S^1}^*(*)$ . From the discussion after Theorem A, we know that on the stalks at zero  $f_!^E(1)$  is the  $S^1$ -equivariant elliptic genus of  $X$ , which is a priori a power series in  $u$ . Theorem C with  $V = TX$  says that 1

extends to a global section in  $E_{S^1}^*(X)^{[f]} = E_{S^1}^*(X)^{[TX]}$ . Therefore  $f_!^E(1)$  is the germ of a global section in  $E_{S^1}^*(*) = \mathcal{O}_{\mathcal{E}}$ . But any such section is a constant, so the  $S^1$ -equivariant elliptic genus of  $X$  is a constant. This proves the rigidity of the elliptic genus (Corollary 5.5).

Now the greater level of generality of Theorem C allows us to extend the rigidity theorem to families of elliptic genera. The question of stating and proving such a theorem was posed by H. Miller in [17].

**THEOREM D. (Rigidity for families)** *Let  $\pi: E \rightarrow B$  be a spin oriented  $S^1$ -equivariant fibration. Then the elliptic genus of the family  $\pi_!^E(1)$  is constant as a rational function, i.e., when the generator  $u$  of  $H_{S^1}^*(*) = \mathbb{C}[u]$  is inverted.*

**3.  $S^1$ -equivariant elliptic cohomology.** In this section we give the construction of  $S^1$ -equivariant elliptic cohomology with complex coefficients. But in order to set up this functor, we need a few definitions.

**3.1. Definitions.** Let  $\mathcal{E}$  be an elliptic curve over  $\mathbb{C}$  with structure sheaf  $\mathcal{O}_{\mathcal{E}}$ . Let  $\theta$  be a uniformizer of  $\mathcal{E}$ , i.e., a generator of the maximal ideal of the local ring at zero  $\mathcal{O}_{\mathcal{E},0}$ . We say that  $\theta$  is an *additive uniformizer* if for all  $x, y \in V_{\theta}$  such that  $x + y \in V_{\theta}$ , we have  $\theta(x + y) = \theta(x) + \theta(y)$ . An additive uniformizer always exists, because we can take for example  $\theta$  to be the local inverse of the group map  $\mathbb{C} \rightarrow \mathcal{E}$ , where the universal cover of  $\mathcal{E}$  is identified with  $\mathbb{C}$ . Notice that any two additive uniformizers differ by a nonzero constant, because the only additive continuous functions on  $\mathbb{C}$  are multiplications by a constant.

Let  $V_{\theta}$  be a neighborhood of zero in  $\mathcal{E}$  such that  $\theta: V_{\theta} \rightarrow \mathbb{C}$  is a homeomorphism on its image. Denote by  $t_{\alpha}$  translation by  $\alpha$  on  $\mathcal{E}$ . We say that a neighborhood  $V$  of  $\alpha \in \mathcal{E}$  is *small* if  $t_{-\alpha}(V) \subseteq V_{\theta}$ .

Let  $\alpha \in \mathcal{E}$ . We say that  $\alpha$  is a torsion point of  $\mathcal{E}$  if there exists  $n > 0$  such that  $n\alpha = 0$ . The smallest  $n$  with this property is called the *exact order* of  $\alpha$ .

Let  $X$  be a finite  $S^1$ -CW complex. If  $H \subseteq S^1$  is a subgroup, denote by  $X^H$  the submanifold of  $X$  fixed by each element of  $H$ . Let  $\mathbb{Z}_n \subseteq S^1$  be the cyclic subgroup of order  $n$ . Define a subgroup  $H(\alpha)$  of  $S^1$  by:  $H(\alpha) = \mathbb{Z}_n$  if  $\alpha$  is a torsion point of exact order  $n$ ;  $H(\alpha) = S^1$  otherwise. Then denote by

$$X^{\alpha} = X^{H(\alpha)}.$$

Now suppose we are given an  $S^1$ -equivariant map of  $S^1$ -CW complexes  $f: X \rightarrow Y$ . A point  $\alpha \in \mathcal{E}$  is called *special* with respect to  $f$  if either  $X^{\alpha} \neq X^{S^1}$  or  $Y^{\alpha} \neq Y^{S^1}$ . When it is clear what  $f$  is, we simply call  $\alpha$  special. A point  $\alpha \in \mathcal{E}$  is called special with respect to  $X$  if it is special with respect to the identity function  $id: X \rightarrow X$ .

An indexed open cover  $\mathcal{U} = (U_{\alpha})_{\alpha \in \mathcal{E}}$  of  $\mathcal{E}$  is said to be *adapted* (with respect to  $f$ ) if it satisfies the following conditions:

- (1)  $U_{\alpha}$  is a small open neighborhood of  $\alpha$ ;

(2) If  $\alpha$  is not special, then  $U_\alpha$  contains no special point;

(3) If  $\alpha \neq \alpha'$  are special points,  $U_\alpha \cap U_{\alpha'} = \emptyset$ .

Notice that, if  $X$  and  $Y$  are finite  $S^1$ -CW complexes, then there exists an open cover of  $\mathcal{E}$  which is adapted to  $f$ . Indeed, the set of special points is a finite subset of  $\mathcal{E}$ .

If  $X$  is a finite  $S^1$ -CW complex, we define the holomorphic  $S^1$ -equivariant cohomology of  $X$  to be

$$HO_{S^1}^*(X) = H_{S^1}^*(X) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0}.$$

$\mathcal{O}_{\mathbb{C},0}$  is the ring of germs of holomorphic functions at zero in the variable  $u$ , or alternatively it is the subring of  $\mathbb{C}[[u]]$  of convergent power series with positive radius of convergence.

Notice that  $HO_{S^1}^*$  is not  $\mathbb{Z}$ -graded anymore, because we tensored with the inhomogenous object  $\mathcal{O}_{\mathbb{C},0}$ . However, it is  $\mathbb{Z}_2$ -graded, by the even and odd part, because  $\mathbb{C}[u]$  and  $\mathcal{O}_{\mathbb{C},0}$  are concentrated in even degrees.

**3.2. Construction of  $E_{S^1}^*$ .** We are going to define now a sheaf  $\mathcal{F} = \mathcal{F}_{\theta, \mathcal{U}}$  over  $\mathcal{E}$  whose stalk at  $\alpha \in \mathcal{E}$  is isomorphic to  $HO_{S^1}^*(X^\alpha)$ . Recall that, in order to give a sheaf  $\mathcal{F}$  over a topological space, it is enough to give an open cover  $(U_\alpha)_\alpha$  of that space, and a sheaf  $\mathcal{F}_\alpha$  on each  $U_\alpha$  together with isomorphisms of sheaves  $\phi_{\alpha\beta}: \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \longrightarrow \mathcal{F}_\beta|_{U_\alpha \cap U_\beta}$ , such that  $\phi_{\alpha\alpha}$  is the identity function, and the cocycle condition  $\phi_{\beta\gamma}\phi_{\alpha\beta} = \phi_{\alpha\gamma}$  is satisfied on  $U_\alpha \cap U_\beta \cap U_\gamma$ .

Fix  $\theta$  as additive uniformizer of  $\mathcal{E}$ . Consider an adapted open cover  $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{E}}$ .

*Definition 3.1.* Define a sheaf  $\mathcal{F}_\alpha$  on  $U_\alpha$  by declaring for any open  $U \subseteq U_\alpha$

$$\mathcal{F}_\alpha(U) := H_{S^1}^*(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_\mathcal{E}(U - \alpha).$$

The map  $\mathbb{C}[u] \rightarrow \mathcal{O}_\mathcal{E}(U - \alpha)$  is given by sending  $u$  to  $\theta$  (the germ  $\theta$  extends to  $U - \alpha$  because  $U_\alpha$  is small).  $U - \alpha$  represents the translation of  $U$  by  $-\alpha$ , and  $\mathcal{O}_\mathcal{E}(U - \alpha)$  is the ring of holomorphic functions on  $U - \alpha$ . The restriction maps of the sheaf are defined so that they come from those of the sheaf  $\mathcal{O}_\mathcal{E}$ .

First we notice that we can make  $\mathcal{F}_\alpha$  into a sheaf of  $\mathcal{O}_\mathcal{E}|_{U_\alpha}$ -modules: if  $U \subseteq U_\alpha$ , we want an action of  $f \in \mathcal{O}_\mathcal{E}(U)$  on  $\mathcal{F}_\alpha(U)$ . The translation map  $t_\alpha: U - \alpha \rightarrow U$ , which takes  $u$  to  $u + \alpha$  gives a translation  $t_\alpha^*: \mathcal{O}_\mathcal{E}(U) \rightarrow \mathcal{O}_\mathcal{E}(U - \alpha)$ , which takes  $f(u)$  to  $f(u + \alpha)$ . Then we take the result of the action of  $f \in \mathcal{O}_\mathcal{E}(U)$  on  $\mu \otimes g \in \mathcal{F}_\alpha(U) = H_{S^1}^*(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_\mathcal{E}(U - \alpha)$  to be  $\mu \otimes (t_\alpha^* f \cdot g)$ . Moreover,  $\mathcal{F}_\alpha$  is coherent because  $H_{S^1}^*(X^\alpha)$  is a finitely generated  $\mathbb{C}[u]$ -module.

Now for the second part of the definition of  $\mathcal{F}$ , we have to glue the different sheaves  $\mathcal{F}_\alpha$  we have just constructed. If  $U_\alpha \cap U_\beta \neq \emptyset$  we need to define an isomorphism of sheaves  $\phi_{\alpha\beta}: \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \longrightarrow \mathcal{F}_\beta|_{U_\alpha \cap U_\beta}$  which satisfies the cocycle

condition. Recall that we started with an adapted open cover  $(U_\alpha)_{\alpha \in \mathcal{E}}$ . Because of the condition 3 in the definition of an adapted cover,  $\alpha$  and  $\beta$  cannot be both special, so we only have to define  $\phi_{\alpha\beta}$  when, say,  $\beta$  is not special. In that case  $X^\beta = X^{S^1}$ . Consider an arbitrary open set  $U \subseteq U_\alpha \cap U_\beta$ .

*Definition 3.2.* Define  $\phi_{\alpha\beta}$  as the composite of the following maps:

$$\begin{aligned}
 (*) \quad \mathcal{F}_\alpha(U) &= H_{S^1}^*(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_\mathcal{E}(U - \alpha) \\
 &\rightarrow H_{S^1}^*(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_\mathcal{E}(U - \alpha) \\
 &\rightarrow (H^*(X^\beta) \otimes_{\mathbb{C}} \mathbb{C}[u]) \otimes_{\mathbb{C}[u]} \mathcal{O}_\mathcal{E}(U - \alpha) \\
 &\rightarrow H^*(X^\beta) \otimes_{\mathbb{C}} \mathcal{O}_\mathcal{E}(U - \alpha) \\
 &\rightarrow H^*(X^\beta) \otimes_{\mathbb{C}} \mathcal{O}_\mathcal{E}(U - \beta) \\
 &\rightarrow (H^*(X^\beta) \otimes_{\mathbb{C}} \mathbb{C}[u]) \otimes_{\mathbb{C}[u]} \mathcal{O}_\mathcal{E}(U - \beta) \\
 &\rightarrow H_{S^1}^*(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_\mathcal{E}(U - \beta) \\
 &= \mathcal{F}_\beta(U).
 \end{aligned}$$

The map on the second row is the natural map  $i^* \otimes 1$ , where  $i: X^\beta \rightarrow X^\alpha$  is the inclusion.

*LEMMA 3.3.*  $\phi_{\alpha\beta}$  is an isomorphism.

*Proof.* The second and the sixth maps are isomorphisms because  $X^\beta = X^{S^1}$ , and therefore  $H_{S^1}^*(X^\beta) \xrightarrow{\sim} H^*(X^\beta) \otimes_{\mathbb{C}} \mathbb{C}[u]$ . The properties of the tensor product imply that the third and the fifth maps are isomorphisms. The fourth map comes from translation by  $\beta - \alpha$ , so it is also an isomorphism.

Finally, the second map  $i^* \otimes 1$  is an isomorphism because:

(a) If  $\alpha$  is not special, then  $X^\alpha = X^{S^1} = X^\beta$ , so  $i^* \otimes 1$  is the identity.

(b) If  $\alpha$  is special, then  $X^\alpha \neq X^\beta$ . However, we have  $(X^\alpha)^{S^1} = X^{S^1} = X^\beta$ . Then we can use the Atiyah–Bott localization theorem in equivariant cohomology from [2]. This says that  $i^*: H_{S^1}^*(X^\alpha) \rightarrow H_{S^1}^*(X^\beta)$  is an isomorphism after inverting  $u$ . So it is enough to show that  $\theta$  is invertible in  $\mathcal{O}_\mathcal{E}(U - \alpha)$ , because this would imply that  $i^*$  becomes an isomorphism after tensoring with  $\mathcal{O}_\mathcal{E}(U - \alpha)$  over  $\mathbb{C}[u]$ . Now, because  $\alpha$  is special, the condition 2 in the definition of an adapted cover says that  $\alpha \notin U_\beta$ . But  $U \subseteq U_\alpha \cap U_\beta$ , so  $\alpha \notin U$ , hence  $0 \notin U - \alpha$ . This is equivalent to  $\theta$  being invertible in  $\mathcal{O}_\mathcal{E}(U - \alpha)$ .  $\square$

*Remark 3.4.* To simplify notation, we can describe  $\phi_{\alpha\beta}$  as the composite of the following two maps:

$$\begin{aligned}
 H_{S^1}^*(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_\mathcal{E}(U - \alpha) &\xrightarrow{i^*} H_{S^1}^*(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_\mathcal{E}(U - \alpha) \\
 &\xrightarrow{t_{\beta-\alpha}^*} H_{S^1}^*(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_\mathcal{E}(U - \beta).
 \end{aligned}$$

By the first map we really mean  $i^* \otimes 1$ . The second map is not  $1 \otimes t_{\beta-\alpha}^*$ , because  $t_{\beta-\alpha}^*$  is not a map of  $\mathbb{C}[u]$ -modules. However, we use  $t_{\beta-\alpha}^*$  as a shorthand for the corresponding composite map specified in (\*). Note that  $\phi_{\alpha\beta}$  is linear over  $\mathcal{O}_{\mathcal{E}}(U)$ , so we get a map of sheaves of  $\mathbb{Z}_2$ -graded  $\mathcal{O}_{\mathcal{E}}(U)$ -algebras.

One checks easily now that  $\phi_{\alpha\beta}$  satisfies the cocycle condition: Suppose we have three open sets  $U_\alpha$ ,  $U_\beta$  and  $U_\gamma$  such that  $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ . Because our cover was chosen to be adapted, at least two out of the three spaces  $X^\alpha$ ,  $X^\beta$  and  $X^\gamma$  are equal to  $X^{S^1}$ . Thus the cocycle condition reduces essentially to  $t_{\gamma-\beta}^* t_{\beta-\alpha}^* = t_{\gamma-\alpha}^*$ , which is clearly true.

**Definition 3.5.** Let  $\mathcal{U} = (U_\alpha)_{\alpha \in \mathcal{E}}$  be an adapted cover of  $\mathcal{E}$ , and  $\theta$  an additive uniformizer. We define a sheaf  $\mathcal{F} = \mathcal{F}_{\theta, \mathcal{U}}$  on  $\mathcal{E}$  by gluing the sheaves  $\mathcal{F}_\alpha$  from Definition 3.1 via the gluing maps  $\phi_{\alpha\beta}$  defined in 3.2.

One can check now easily that  $\mathcal{F}$  is a coherent analytic sheaf of algebras. Notice that we can remove the dependence of  $\mathcal{F}$  on the adapted cover  $\mathcal{U}$  as follows: Let  $\mathcal{U}$  and  $\mathcal{V}$  be two covers adapted to  $(X, A)$ . Then any common refinement  $\mathcal{W}$  is going to be adapted as well, and the corresponding maps of sheaves  $\mathcal{F}_{\theta, \mathcal{U}} \rightarrow \mathcal{F}_{\theta, \mathcal{W}} \leftarrow \mathcal{F}_{\theta, \mathcal{V}}$  are isomorphisms on stalks, hence isomorphisms of sheaves. Therefore we can omit the subscript  $\mathcal{U}$ , and write  $\mathcal{F} = \mathcal{F}_\theta$ . Next we want to show that  $\mathcal{F}_\theta$  is independent of the choice of the additive uniformizer  $\theta$ .

**PROPOSITION 3.6.** *If  $\theta$  and  $\theta'$  are two additive uniformizers, then there exists an isomorphism of sheaves of  $\mathcal{O}_{\mathcal{E}}$ -algebras  $f_{\theta\theta'}: \mathcal{F}_\theta \rightarrow \mathcal{F}_{\theta'}$ . If  $\theta''$  is a third additive uniformizer, then  $f_{\theta'\theta''} f_{\theta\theta'} = \pm f_{\theta\theta''}$ .*

*Proof.* We modify slightly the notations used in Definition 3.1 to indicate the dependence on  $\theta$ :  $\mathcal{F}_\alpha^\theta(U) = H_{S^1}^*(X^\alpha) \otimes_{\mathbb{C}[u]}^\theta \mathcal{O}_{\mathcal{E}}(U - \alpha)$ . Recall that  $u$  is sent to  $\theta$  via the algebra map  $\mathbb{C}[u] \rightarrow \mathcal{O}_{\mathcal{E}}(U - \alpha)$ . If  $\theta'$  is another additive uniformizer, we saw at the beginning of this section that there exists a nonzero constant  $a$  in  $\mathcal{O}_{\mathcal{E}, 0}$  such that  $\theta = a\theta'$ . Choose a square root of  $a$  and denote it by  $a^{1/2}$ . Define a map  $f_{\theta\theta', \alpha}: \mathcal{F}_\alpha^\theta(U) \rightarrow \mathcal{F}_\alpha^{\theta'}(U)$  by  $x \otimes^\theta g \mapsto a^{|x|/2} x \otimes^{\theta'} g$ . We have assumed that  $x$  is homogeneous in  $H_{S^1}^*(X^\alpha)$ , and that  $|x|$  is the homogeneous degree of  $x$ .

One can easily check that  $f_{\theta\theta', \alpha}$  is a map of sheaves of  $\mathcal{O}_{\mathcal{E}}$ -algebras. We also have  $\phi_{\alpha\beta}^{\theta'} \circ f_{\theta\theta', \alpha} = f_{\theta\theta', \beta} \circ \phi_{\alpha\beta}^\theta$ , which means that the maps  $f_{\theta\theta', \alpha}$  glue to define a map of sheaves  $f_{\theta\theta'}: \mathcal{F}_\theta \rightarrow \mathcal{F}_{\theta'}$ . The equality  $f_{\theta'\theta''} f_{\theta\theta'} = \pm f_{\theta\theta''}$  comes from  $(\theta'/\theta'')^{1/2}(\theta/\theta')^{1/2} = \pm(\theta/\theta'')^{1/2}$ .  $\square$

**Definition 3.7.** The  $S^1$ -equivariant elliptic cohomology of the finite  $S^1$ -CW complex  $X$  is the sheaf  $\mathcal{F} = \mathcal{F}_{\theta, \mathcal{U}}$  constructed above, which according to the previous results does not depend on the adapted open cover  $\mathcal{U}$  or on the additive uniformizer  $\theta$ . Denote this sheaf by  $E_{S^1}^*(X)$ .

If  $X$  is a point, one can see that  $E_{S^1}^*(X)$  is the structure sheaf  $\mathcal{O}_{\mathcal{E}}$ .

**THEOREM 3.8.**  $E_{S^1}^*(-)$  defines an  $S^1$ -equivariant cohomology theory with values in the category of coherent analytic sheaves of  $\mathbb{Z}_2$ -graded  $\mathcal{O}_{\mathcal{E}}$ -algebras.

*Proof.* For  $E_{S^1}^*(-)$  to be a cohomology theory, we need naturality. Let  $f: X \rightarrow Y$  be an  $S^1$ -equivariant map of finite  $S^1$ -CW complexes. We want to define a map of sheaves  $f^*: E_{S^1}^*(Y) \rightarrow E_{S^1}^*(X)$  with the properties that  $1_X^* = 1_{E_{S^1}^*(X)}$  and  $(fg)^* = g^*f^*$ . Choose  $\mathcal{U}$  an open cover adapted to  $f$ , and  $\theta$  an additive uniformizer of  $\mathcal{E}$ . Since  $f$  is  $S^1$ -equivariant, for each  $\alpha$  we get by restriction a map  $f_\alpha: X^\alpha \rightarrow Y^\alpha$ . This induces a map  $H_{S^1}^*(Y^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha) \xrightarrow{f_\alpha^* \otimes 1} H_{S^1}^*(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha)$ . To get our global map  $f^*$ , we only have to check that  $f_\alpha^*$  glue well, i.e., that they commute with the gluing maps  $\phi_{\alpha\beta}$ . This follows easily from the naturality of ordinary equivariant cohomology, and from the naturality in  $X$  of the isomorphism  $H_{S^1}^*(X^{S^1}) \cong H^*(X^{S^1}) \otimes_{\mathbb{C}} \mathbb{C}[u]$ .

Also, we need to define  $E_{S^1}^*$  for pairs. Let  $(X, A)$  be a pair of finite  $S^1$ -CW complexes, i.e.,  $A$  is a closed subspace of  $X$ , and the inclusion map  $A \rightarrow X$  is  $S^1$ -equivariant. We then define  $E_{S^1}^*(X, A)$  as the kernel of the map  $j^*: E_{S^1}^*(X/A) \rightarrow E_{S^1}^*(*)$ , where  $j: * = A/A \rightarrow X/A$  is the inclusion map. If  $f: (X, A) \rightarrow (Y, B)$  is a map of pairs of finite  $S^1$ -CW complexes, then  $f^*: E_{S^1}^*(Y, B) \rightarrow E_{S^1}^*(X, A)$  is defined as the unique map induced on the corresponding kernels from  $f^*: E_{S^1}^*(Y) \rightarrow E_{S^1}^*(X)$ .

Now we have to define the coboundary map  $\delta: E_{S^1}^*(A) \rightarrow E_{S^1}^{*+1}(X, A)$ . This is obtained by gluing the maps  $H_{S^1}^*(A^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha) \xrightarrow{\delta_\alpha \otimes 1} H_{S^1}^{*+1}(X^\alpha, A^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha)$ , where  $\delta_\alpha: H_{S^1}^*(A^\alpha) \rightarrow H_{S^1}^{*+1}(X^\alpha, A^\alpha)$  is the usual coboundary map. The maps  $\delta_\alpha \otimes 1$  glue well, because  $\delta_\alpha$  is natural.

To check the usual axioms of a cohomology theory: naturality, exact sequence of a pair, and excision for  $E_{S^1}^*(-)$ , recall that this sheaf was obtained by gluing the sheaves  $\mathcal{F}_\alpha$  along the maps  $\phi_{\alpha\beta}$ . Since  $\mathcal{F}_\alpha$  were defined using  $H_{S^1}^*(X^\alpha)$ , the properties of ordinary  $S^1$ -equivariant cohomology pass on to  $E_{S^1}^*(-)$ , as long as tensoring with  $\mathcal{O}_{\mathcal{E}}(U - \alpha)$  over  $\mathbb{C}[u]$  preserves exactness. But this is a classical fact: see for example the appendix of Serre [20].  $\square$

This proves Theorem A stated in Section 2.

**Remark 3.9.** Notice that we can arrange our functor  $E_{S^1}^*(-)$  to take values in the category of coherent algebraic sheaves over  $\mathcal{E}$  rather than in the category of coherent analytic sheaves. This follows from a theorem of Serre [20] which says that the categories of coherent holomorphic sheaves and coherent algebraic sheaves over a projective variety are equivalent.

**3.3. Alternative description of  $E_{S^1}^*$ .** For calculations with  $E_{S^1}^*(-)$  we want a description which involves a finite open cover of  $\mathcal{E}$ . Start with an adapted open cover  $(U_\alpha)_{\alpha \in \mathcal{E}}$ . Recall that the set of special points with respect to  $X$  is finite.



Denote this set by  $\{\alpha_1, \dots, \alpha_n\}$ . To simplify notation, denote for  $i = 1, \dots, n$

$$U_i := U_{\alpha_i}, \text{ and } U_0 := \mathcal{E} \setminus \{\alpha_1, \dots, \alpha_n\}.$$

On each  $U_i$ , with  $0 \leq i \leq n$ , we define a sheaf  $\mathcal{G}$  as follows:

(a) If  $1 \leq i \leq n$ , then  $\forall U \subseteq U_i$ ,  $\mathcal{G}_i(U) := H_{S^1}^*(X^{\alpha_i}) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha_i)$ . The map  $\mathbb{C}[u] \rightarrow \mathcal{O}_{\mathcal{E}}(U - \alpha_i)$  was described in Definition 3.1.

(b) If  $i = 0$ , then  $\forall U \subseteq U_0$ ,  $\mathcal{G}_i(U) := H^*(X^{S^1}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{E}}(U)$ .

Now glue each  $\mathcal{G}_i$  to  $\mathcal{G}_0$  via the map of sheaves  $\phi_{i0}$  defined as the composite of the following isomorphisms ( $U \subseteq U_i \cap U_0$ ):  $H_{S^1}^*(X^{\alpha_i}) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha_i) \xrightarrow{i^* \otimes 1} H_{S^1}^*(X^{S^1}) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha_i) \xrightarrow{\cong} H^*(X^{S^1}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{E}}(U - \alpha_i) \xrightarrow{t_{-\alpha_i}^*} H^*(X^{S^1}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{E}}(U)$ .

Since there cannot be three distinct  $U_i$  with nonempty intersection, there is no cocycle condition to verify.

**PROPOSITION 3.10.** *The sheaf  $\mathcal{G}$  we have just described is isomorphic to  $\mathcal{F}$ , thus allowing an alternative definition of  $E_{S^1}^*(X)$ .*

*Proof.* One notices that  $U_0 = \cup\{U_{\beta} \mid \beta \text{ nonspecial}\}$ , because of the third condition in the definition of an adapted cover. If  $U \subseteq \cup_{\beta} U_{\beta}$ , a global section in  $\mathcal{F}(U)$  is a collection of sections  $s_{\beta} \in \mathcal{F}(U \cap U_{\beta} - \beta)$  which glue, i.e.,  $t_{\beta - \beta'}^* s_{\beta} = s_{\beta'}$ . So  $t_{-\beta}^* s_{\beta} = t_{-\beta'}^* s_{\beta'}$  in  $\mathcal{G}(U \cap U_{\beta} \cap U_{\beta'})$ , which means that we get an element in  $\mathcal{G}(U)$ , since the  $U_{\beta}$ 's cover  $U$ . So  $\mathcal{F}|_{U_0} \cong \mathcal{G}|_{U_0}$ . But clearly  $\mathcal{F}|_{U_i} \cong \mathcal{G}|_{U_i}$  for  $1 \leq i \leq n$ , and the gluing maps are compatible. Therefore  $\mathcal{F} \cong \mathcal{G}$ .  $\square$

As it is the case with any coherent sheaf of  $\mathcal{O}_{\mathcal{E}}$ -modules over an elliptic curve,  $E_{S^1}^*(X)$  splits (noncanonically) into a direct sum of a locally free sheaf, i.e., the sheaf of sections of some holomorphic vector bundle, and a sum of skyscraper sheaves.

Given a particular  $X$ , we can be more specific: We know that  $H_{S^1}^*(X)$  splits noncanonically into a free and a torsion  $\mathbb{C}[u]$ -module. Given such a splitting, we can speak of the free part of  $H_{S^1}^*(X)$ . Denote it by  $H_{S^1}^*(X)_{\text{free}}$ . The map  $i^* H_{S^1}^*(X)_{\text{free}} \rightarrow H_{S^1}^*(X^{S^1})$  is an injection of finitely generated free  $\mathbb{C}[u]$ -modules of the same rank, by the localization theorem.  $\mathbb{C}[u]$  is a p.i.d., so by choosing appropriate bases in  $H_{S^1}^*(X)_{\text{free}}$  and  $H_{S^1}^*(X^{S^1})$ , the map  $i^*$  can be written as a diagonal matrix  $D(u^{n_1}, \dots, u^{n_k})$ ,  $n_i \geq 0$ . Since  $i^* 1 = 1$ , we can choose  $n_1 = 0$ .

So at the special points  $\alpha_i$ , we have the map  $i^*: H_{S^1}^*(X^{\alpha_i})_{\text{free}} \rightarrow H_{S^1}^*(X^{S^1})$ , which in appropriate bases can be written as a diagonal matrix  $D(1, u^{n_2}, \dots, u^{n_k})$ . This gives over  $U_i \cap U_0$  the transition functions  $u \mapsto D(1, u^{n_2}, \dots, u^{n_k}) \in GL(n, \mathbb{C})$ . However, we have to be careful since the basis of  $H_{S^1}^*(X^{S^1})$  changes with each  $\alpha_i$ , which means that the transition functions are diagonal only up to a (change of base) matrix. But this matrix is invertible over  $\mathbb{C}[u]$ , so we get that the free part of  $E_{S^1}^*(X)$  is a sheaf of sections of a holomorphic vector bundle.

An interesting question is what holomorphic vector bundles one gets if  $X$  varies. Recall that holomorphic vector bundles over elliptic curves were classified by Atiyah in 1957.

*Example 3.11.* Calculate  $E_{S^1}^*(X)$  for  $X = S^2(n) =$  the 2-sphere with the  $S^1$ -action which rotates  $S^2$   $n$  times around the north-south axis as we go once around  $S^1$ . If  $\alpha$  is an  $n$ -torsion point, then  $X^\alpha = X$ . Otherwise,  $X^\alpha = X^{S^1}$ , which consists of two points:  $\{P_+, P_-\}$ , the North and the South poles. Now  $H_{S^1}^*(S^2(n)) = H^*(BS^1 \vee BS^1) = \mathbb{C}[u] \times_{\mathbb{C}} \mathbb{C}[u]$ , on which  $\mathbb{C}[u]$  acts diagonally.  $i^*: H_{S^1}^*(X) \rightarrow H_{S^1}^*(X^{S^1})$  is the inclusion  $\mathbb{C}[u] \times_{\mathbb{C}} \mathbb{C}[u] \hookrightarrow \mathbb{C}[u] \times \mathbb{C}[u]$ .

Choose the bases

(a)  $\{(1, 1), (u, 0)\}$  of  $\mathbb{C}[u] \times_{\mathbb{C}} \mathbb{C}[u]$ ;

(b)  $\{(1, 1), (1, 0)\}$  of  $\mathbb{C}[u] \times \mathbb{C}[u]$ .

Then  $H_{S^1}^*(X) \xrightarrow{\sim} \mathbb{C}[u] \oplus \mathbb{C}[u]$  by  $(P(u), Q(u)) \mapsto (P, \frac{Q-P}{u})$ , and  $H_{S^1}^*(X^{S^1}) \xrightarrow{\sim} \mathbb{C}[u] \oplus \mathbb{C}[u]$  by  $(P(u), Q(u)) \mapsto (P, Q - P)$ . Hence  $i^*$  is given by the diagonal matrix  $D(1, u)$ . So  $E_{S^1}^*(X)$  looks locally like  $\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(-1 \cdot 0)$ . This happens at all the  $n$ -torsion points of  $\mathcal{E}$ , so  $E_{S^1}^*(X) \cong \mathcal{O}_{\mathcal{E}} \oplus \mathcal{O}_{\mathcal{E}}(\Delta)$ , where  $\Delta$  is the divisor which consists of all  $n$ -torsion points of  $\mathcal{E}$ , with multiplicity 1.

One can also check that the sum of all  $n$ -torsion points is zero, so by Abel's theorem the divisor  $\Delta$  is linearly equivalent to  $-n^2 \cdot 0$ . Thus  $E_{S^1}^*(S^2(n)) \cong \mathcal{O}_{\mathcal{E}} \oplus \mathcal{O}_{\mathcal{E}}(-n^2 \cdot 0)$ . We stress that the decomposition is only true as sheaves of  $\mathcal{O}_{\mathcal{E}}$ -modules, not as sheaves of  $\mathcal{O}_{\mathcal{E}}$ -algebras.

*Remark 3.12.* Notice that  $S^2(n)$  is the Thom space of the  $S^1$ -vector space  $\mathbb{C}(n)$ , where  $z$  acts on  $\mathbb{C}$  by complex multiplication with  $z^n$ . This means that the Thom isomorphism doesn't hold in  $S^1$ -equivariant elliptic cohomology, because  $E_{S^1}^*(*) = \mathcal{O}_{\mathcal{E}}$ , while the reduced  $S^1$ -equivariant elliptic cohomology of the Thom space is  $\tilde{E}_{S^1}^*(S^2(n)) = \mathcal{O}_{\mathcal{E}}(-n^2 \cdot 0)$ .

**4.  $S^1$ -equivariant elliptic pushforwards.** While the construction of  $E_{S^1}^*(X)$  depends only on the elliptic curve  $\mathcal{E}$ , the construction of the elliptic pushforward  $f_!^E$  involves extra choices, namely that of a torsion point of exact order two on  $\mathcal{E}$ , and a trivialization of the cotangent space of  $\mathcal{E}$  at zero.

**4.1. The Jacobi sine.** Let  $(\mathcal{E}, P, \mu)$  be a triple formed with a nonsingular elliptic curve  $\mathcal{E}$  over  $\mathbb{C}$ , a torsion point  $P$  on  $\mathcal{E}$  of exact order two, and a 1-form  $\mu$  which generates the cotangent space  $T_0^*\mathcal{E}$ . For example, we can take  $\mathcal{E} = \mathbb{C}/\Lambda$ , with  $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$  a lattice in  $\mathbb{C}$ ,  $P = \omega_1/2$ , and  $\mu = dz$  at zero, where  $z$  is the usual complex coordinate on  $\mathbb{C}$ .

As in Hirzerbruch, Berger and Jung ([12], Section 2.2), we can associate to this data a function  $s(z)$  on  $\mathbb{C}$  which is elliptic (doubly periodic) with respect to a sublattice  $\tilde{\Lambda}$  of index 2 in  $\Lambda$ , namely  $\tilde{\Lambda} = \mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2$ . (This leads to a double covering  $\tilde{\mathcal{E}} \rightarrow \mathcal{E}$ , and  $s$  can be regarded as a rational function on the “doubled”

elliptic curve  $\tilde{\mathcal{E}}$ .) Indeed, we can define  $s$  up to a constant by defining its divisor to be

$$D = (0) + (\omega_1/2) - (\omega_2) - (\omega_1/2 + \omega_2).$$

Then we can make  $s$  unique by requiring that  $ds = dz$  at zero. We call this  $s$  the *Jacobi sine*. It has the following properties (see [12]):

PROPOSITION 4.1.

(a)  $s(z)$  is odd, i.e.  $s(-z) = -s(z)$ . Around zero,  $s$  can be expanded as a power series  $s(z) = z + a_3 z^3 + a_5 z^5 + \cdots$ .

(b)  $s(z + \omega_1) = s(z)$ ;  $s(z + \omega_2) = -s(z)$ .

(c)  $s(z + \omega_1/2) = a/s(z)$ ,  $a \neq 0$  (this follows by looking at the divisor of  $s(z + \omega_1/2)$ ).

We now show that the construction of  $s$  is canonical, i.e., it does not depend on the identification  $\mathcal{E} \cong \mathbb{C}/\Lambda$ .

PROPOSITION 4.2. *The definition of  $s$  only depends on the triple  $(\mathcal{E}, P, \mu)$ .*

*Proof.* First, we show that the construction of  $\tilde{\mathcal{E}} = \mathbb{C}/\tilde{\Lambda}$  is canonical: Let  $\mathcal{E} \cong \mathbb{C}/\Lambda'$  be another identification of  $\mathcal{E}$ . We then have  $\Lambda' = \mathbb{Z}\omega'_1 + \mathbb{Z}\omega'_2$ , and  $P$  is identified with  $\omega'_1/2$ . Since  $\mathcal{E}$  is also identified with  $\mathbb{C}/\Lambda$ , we get a group map  $\lambda: \mathbb{C}/\Lambda \xrightarrow{\sim} \mathbb{C}/\Lambda'$ . This implies that we have a continuous group map  $\lambda: \mathbb{C} \xrightarrow{\sim} \mathbb{C}$  such that  $\lambda(\Lambda) = \Lambda'$ . Any such map must be multiplication by a nonzero constant  $\lambda \in \mathbb{C}$ . Moreover, we know that  $\lambda\omega_1/2 = \omega'_1/2$ . This implies  $\lambda\omega_1 = \omega'_1$ , and since  $\lambda$  takes  $\Lambda$  isomorphically onto  $\Lambda'$ , it follows that  $\lambda\omega_2 = \pm\omega'_2 + m\omega'_1$  for some integer  $m$ . Multiplying this by 2, we get  $\lambda \cdot 2\omega_2 = \pm 2\omega'_2 + 2m\omega'_1$ . This, together with  $\lambda\omega_1 = \omega'_1$ , implies that multiplication by  $\lambda$  descends to a group map  $\mathbb{C}/\tilde{\Lambda} \xrightarrow{\sim} \mathbb{C}/\tilde{\Lambda}'$ . But this precisely means that the construction of  $\tilde{\mathcal{E}}$  is canonical.

Notice that  $P$  can be thought of canonically as a point on the “doubled” elliptic curve  $\tilde{\mathcal{E}}$ . We denote by  $P_1$  and  $P_2$  the other two points of exact order 2 on  $\tilde{\mathcal{E}}$ . Then we form the divisor

$$D = (0) + (P) - (P_1) - (P_2).$$

Although the choice of  $P_1$  and  $P_2$  is noncanonical, the divisor  $D$  is canonical, i.e., depends only on  $P$ . Let  $s$  be an elliptic function on  $\tilde{\mathcal{E}}$  associated to the divisor  $D$ . The choice of  $s$  is well defined up to a constant which can be fixed if we require that  $ds = \pi^*\mu$  at zero, where  $\pi: \tilde{\mathcal{E}} \rightarrow \mathcal{E}$  is the projection map.  $\square$

Next, we start the construction of  $S^1$ -equivariant elliptic pushforwards. Let  $f: X \rightarrow Y$  be an equivariant map between compact  $S^1$ -manifolds such that the restrictions  $f: X^\alpha \rightarrow Y^\alpha$  are oriented maps. Then we follow Grojnowski [10] and define the pushforward of  $f$  to be a map of sheaves  $f_!^E: E_{S^1}^*(X)^{[f]} \rightarrow E_{S^1}^*(Y)$ , where  $E_{S^1}^*(X)^{[f]}$  is the sheaf  $E_{S^1}^*(X)$  twisted by a 1-cocycle to be defined later.

The main technical ingredient in the construction of the (global i.e. sheafwise) elliptic pushforward  $f_!^E: E_{S^1}^*(X)^{[f]} \rightarrow E_{S^1}^*(Y)$  is the (local i.e. stalkwise) elliptic pushforward  $f_!^E: HO_{S^1}^*(X^\alpha) \rightarrow HO_{S^1}^*(Y^\alpha)$ .

In the following subsection, we construct elliptic Thom classes and elliptic pushforwards in  $HO_{S^1}^*(-)$ . The construction is standard; the only problem is that in order to show that something belongs to  $HO_{S^1}^*(-)$ , we need some holomorphicity results on characteristic classes.

**4.2. Preliminaries on pushforwards.** Let  $\pi: V \rightarrow X$  be a  $2n$ -dimensional oriented real  $S^1$ -vector bundle over a finite  $S^1$ -CW complex  $X$ , i.e., a vector bundle with a linear action of  $S^1$ , such that  $\pi$  commutes with the  $S^1$  action. Now, for any space  $A$  with an  $S^1$  action, we can define its Borel construction  $A \times_{S^1} ES^1$ , where  $ES^1$  is the universal principal  $S^1$ -bundle. This construction is functorial, so we get a vector bundle  $V_{S^1}$  over  $X_{S^1}$ . This has a classifying map  $f_V: X_{S^1} \rightarrow BSO(2n)$ . If  $V_{univ}$  is the universal orientable vector bundle over  $BSO(2n)$ , we also have a map of pairs, also denoted by  $f_V: (DV_{S^1}, SV_{S^1}) \rightarrow (DV_{univ}, SV_{univ})$ . As usual,  $DV$  and  $SV$  represent the disc and the sphere bundle of  $V$ , respectively.

But it is known that the pair  $(DV_{univ}, SV_{univ})$  is homotopic to  $(BSO(2n), BSO(2n-1))$ . Also, we know that  $H^*BSO(2n) = \mathbb{C}[p_1, \dots, p_n, e]/(e^2 - p_n)$ , where  $p_j$  is the universal  $j$ th Pontrjagin class, and  $e$  is the universal Euler class. From the long exact sequence of the pair, it follows that  $H^*(BSO(2n), BSO(2n-1))$  can be regarded as the ideal generated by  $e$  in  $H^*BSO(2n)$ . The class  $e \in H^*(DV_{univ}, SV_{univ})$  is the universal Thom class, which we will denote by  $\phi_{univ}$ . Then the ordinary equivariant Thom class of  $V$  is defined as the pullback class  $f_V^* \phi_{univ} \in H_{S^1}^*(DV, SV)$ , and we denote it by  $\phi_{S^1}(V)$ . Denote by  $H_{S^1}^{**}(X)$  the completion of the module  $H_{S^1}^*(X)$  with respect to the ideal generated by  $u$  in  $H^*(BS^1) = \mathbb{C}[u]$ .

Consider the power series  $Q(x) = s(x)/x$ , where  $s(x)$  is the Jacobi sine. Since  $Q(x)$  is even, Definition A.6 gives a class  $\mu_Q(V)_{S^1} \in H_{S^1}^{**}(X)$ . Then we define a class in  $H_{S^1}^{**}(DV, SV)$  by  $\phi_{S^1}^E(V) = \mu_Q(V)_{S^1} \cdot \phi_{S^1}(V)$ . One can also say that  $\phi_{S^1}^E(V) = s(x_1) \cdots s(x_n)$ , while  $\phi_{S^1}(V) = x_1 \cdots x_n$ , where  $x_1, \dots, x_n$  are the equivariant Chern roots of  $V$ . We call  $\phi_{S^1}^E(V)$  the *elliptic equivariant Thom class* of  $V$ .

Also, we define  $e_{S^1}^E(V)$ , the equivariant elliptic Euler class of  $V$ , as the image of  $\phi_{S^1}^E(V)$  via the restriction map  $H_{S^1}^{**}(DV, SV) \rightarrow H_{S^1}^{**}(X)$ .

**PROPOSITION 4.3.** *If  $V \rightarrow X$  is an even dimensional real oriented  $S^1$ -vector bundle, and  $X$  is a finite  $S^1$ -CW complex, then  $\phi_{S^1}^E(V)$  actually lies in  $HO_{S^1}^*(DV, SV)$ . Cup product with the elliptic Thom class*

$$HO_{S^1}^*(X) \xrightarrow{\cup \phi_{S^1}^E(V)} HO_{S^1}^*(DV, SV),$$

*is an isomorphism, the Thom isomorphism in HO-theory.*

*Proof.* The difficult part, namely that  $\mu_Q(V)_{S^1}$  is holomorphic, is proved in the Appendix, in Proposition A.6. Consider the usual cup product, which is a map

$\cup: H_{S^1}^*(X) \otimes H_{S^1}^*(DV, SV) \rightarrow H_{S^1}^*(DV, SV)$ , and extend it by tensoring with  $\mathcal{O}_{\mathbb{C},0}$  over  $\mathbb{C}[u]$ . We obtain a map  $\cup: HO_{S^1}^*(X) \otimes HO_{S^1}^*(DV, SV) \rightarrow HO_{S^1}^*(DV, SV)$ . The equivariant elliptic Thom class of  $V$  is  $\phi_{S^1}^E(V) = \mu_Q(V)_{S^1} \cup \phi_{S^1}(V)$ , so we have to show that both these classes are holomorphic. But by Proposition A.6 in the Appendix,  $\mu_Q(V)_{S^1} \in HO_{S^1}^*(X)$ . And the ordinary Thom class  $\phi_{S^1}(V)$  belongs to  $H_{S^1}^*(DV, SV)$ , so it also belongs to the larger ring  $HO_{S^1}^*(DV, SV)$ .

Now, cup product with  $\phi_{S^1}^E(V)$  gives an isomorphism because  $Q(x) = s(x)/x$  is an invertible power series around zero.  $\square$

**COROLLARY 4.4.** *If  $f: X \rightarrow Y$  is an  $S^1$ -equivariant oriented map between compact  $S^1$ -manifolds, then there is an elliptic pushforward*

$$f_!^E: HO_{S^1}^*(X) \rightarrow HO_{S^1}^*(Y),$$

*which is a map of  $HO_{S^1}^*(Y)$ -modules. In the case when  $Y$  is a point,  $f_!^E(1)$  is the  $S^1$ -equivariant elliptic genus of  $X$ .*

*Proof.* Recall (Dyer [7]) that the ordinary pushforward is defined as the composition of three maps, two of which are Thom isomorphisms, and the third is a natural one. The existence of the elliptic pushforward follows therefore from the previous corollary. The proof that  $f_!^E$  is a map of  $HO_{S^1}^*(Y)$ -modules is the same as for the ordinary pushforward.

The last statement is an easy consequence of the topological Riemann–Roch theorem (see again [7]), and of the definition of the equivariant elliptic Thom class.  $\square$

Notice that, if  $Y$  is point,  $HO_{S^1}^*(Y) \cong \mathcal{O}_{\mathbb{C},0}$ , so the  $S^1$ -equivariant elliptic genus of  $X$  is holomorphic around zero. Also, if we replace  $HO_{S^1}^*(-) = H_{S^1}^*(-) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0}$  by  $HM_{S^1}^*(-) = H_{S^1}^*(-) \otimes_{\mathbb{C}[u]} \mathcal{M}(\mathbb{C})$ , where  $\mathcal{M}(\mathbb{C})$  is the ring of global meromorphic functions on  $\mathbb{C}$ , the same proof as above shows that the  $S^1$ -equivariant elliptic genus of  $X$  is meromorphic in  $\mathbb{C}$ . This proves the following result, which is Proposition B stated in Section 2.

**PROPOSITION 4.5.** *The  $S^1$ -equivariant elliptic genus of a compact  $S^1$ -manifold is the Taylor expansion at zero of a function on  $\mathbb{C}$  which is holomorphic at zero and meromorphic everywhere.*

**4.3. Construction of  $f_!^E$ .** The local construction of elliptic pushforwards is completed. We want now to assemble the pushforwards in a map of sheaves. Let  $f: X \rightarrow Y$  be a map of compact  $S^1$ -manifolds which commutes with the  $S^1$ -action. We assume that either  $f$  is complex oriented or spin oriented, i.e., that the stable normal bundle in the sense of Dyer [7] is complex oriented or spin oriented, respectively. (Grojnowski treats only the complex oriented case, but in order to understand rigidity we also need the spin case.)

Let  $\mathcal{U}$  be an open cover of  $\mathcal{E}$  adapted to  $f$ . Let  $\alpha, \beta \in \mathcal{E}$  be such that  $U_\alpha \cap U_\beta \neq \emptyset$ . This implies that at least one point, say  $\beta$ , is nonspecial, so  $X^\beta = X^{S^1}$  and  $Y^\beta = Y^{S^1}$ . We specify now the orientations of the maps and vector bundles involved. Since  $X^\beta = X^{S^1}$ , the normal bundle of the embedding  $X^\beta \rightarrow X^\alpha$  has a complex structure, where all the weights of the  $S^1$ -action on  $V$  are positive.

If  $f$  is complex oriented, it follows that the restriction maps  $f^\alpha: X^\alpha \rightarrow Y^\alpha$  and  $f^\beta: X^\beta \rightarrow Y^\beta$  are also complex oriented, hence oriented. If  $f$  is spin oriented, this means that the stable normal bundle  $W$  of  $f$  is spin. If  $H$  is any subgroup of  $S^1$ , we know that the vector bundle  $W^H \rightarrow X^H$  is oriented: If  $H = S^1$ ,  $W$  splits as a direct sum of  $W^H$  with a bundle corresponding to the nontrivial irreducible representations of  $S^1$ ; this latter bundle is complex, hence oriented, so the orientation of  $W$  induces one on  $W^H$ . If  $H = \mathbb{Z}_n$ , Lemma 10.3 of Bott and Taubes [4] implies that  $W^H$  is oriented. In conclusion, both maps  $f^\alpha$  and  $f^\beta$  are oriented.

According to Corollary 4.4, we can define elliptic pushforwards at the level of stalks:  $(f^\alpha)_!^E: HO_{S^1}^*(X^\alpha) \rightarrow HO_{S^1}^*(Y^\alpha)$  and  $(f^\beta)_!^E: HO_{S^1}^*(X^\beta) \rightarrow HO_{S^1}^*(Y^\beta)$ . The problem is that pushforwards do not commute with pullbacks, i.e., if  $i: X^\beta \rightarrow X^\alpha$  and  $j: Y^\beta \rightarrow Y^\alpha$  are the inclusions, then it is not true in general that  $j^*(f^\alpha)_!^E = (f^\beta)_!^E i^*$ . However, by twisting the maps with some appropriate Euler classes, the diagram becomes commutative. Denote by  $e_{S^1}^E(X^\alpha/X^\beta)$  the  $S^1$ -equivariant Euler class of the normal bundle to the embedding  $i$ , and by  $e_{S^1}^E(Y^\alpha/Y^\beta)$  the  $S^1$ -equivariant Euler class of the normal bundle to  $j$ . Denote by

$$\lambda_{\alpha\beta}^{[f]} = e_{S^1}^E(X^\alpha/X^\beta)^{-1} \cdot (f^\beta)^* e_{S^1}^E(Y^\alpha/Y^\beta).$$

A priori  $\lambda_{\alpha\beta}^{[f]}$  belongs to the ring  $HO_{S^1}^*(X^\beta) \left[ \frac{1}{e_{S^1}^E(X^\alpha/X^\beta)} \right]$ , but we will see later that we can improve this.

LEMMA 4.6. *In the ring  $HO_{S^1}^*(X^\beta) \left[ \frac{1}{u}, \frac{1}{e_{S^1}^E(X^\alpha/X^\beta)} \right]$  we have the following formula*

$$j^*(f^\alpha)_!^E \mu^\alpha = (f^\beta)_!^E (i^* \mu^\alpha \cdot \lambda_{\alpha\beta}^{[f]}),$$

*Proof.* From the hypothesis, we know that  $i^* i_!^E$  is an isomorphism, because it is multiplication by the invertible class  $e_{S^1}^E(X^\alpha/X^\beta)$ . Also, since  $u$  is invertible, the localization theorem implies that  $i^*$  is an isomorphism. Therefore  $i_!^E$  is an isomorphism. Start with a class  $\mu^\alpha$  on  $X^\alpha$ . Because  $i_!^E$  is an isomorphism,  $\mu^\alpha$  can be written as  $i_!^E \mu^\beta$ , where  $\mu^\beta$  is a class on  $X^\beta$ .

Now look at the two sides of the equation to be proved:

(1) The left-hand side =  $j^*(f^\alpha)_!^E i_!^E \mu^\beta = j^* j_!^E (f^\beta)_!^E \mu^\beta = (f^\beta)_!^E \mu^\beta \cdot e_{S^1}^E(Y^\alpha/Y^\beta)$ , because  $j^* j_!^E$  = multiplication by  $e_{S^1}^E(Y^\alpha/Y^\beta)$ .

(2) The right-hand side =  $(f^\beta)_!^E [i^* i_!^E \mu^\beta \cdot e_{S^1}^E(X^\alpha/X^\beta)^{-1} \cdot (f^\beta)^* e_{S^1}^E(Y^\alpha/Y^\beta)] = (f^\beta)_!^E [\mu^\beta \cdot (f^\beta)^* e_{S^1}^E(Y^\alpha/Y^\beta)] = (f^\beta)_!^E \mu^\beta \cdot e_{S^1}^E(Y^\alpha/Y^\beta)$ , where the last equality comes from the fact that  $(f^\beta)_!^E$  is a map of  $HO_{S^1}^*(Y^\beta)$ -modules.  $\square$

Let  $f: X \rightarrow Y$  be a complex or spin oriented  $S^1$ -map. Let  $\mathcal{U}$  be an open cover adapted to  $f$ , and  $\alpha, \beta \in \mathcal{E}$  such that  $U_\alpha \cap U_\beta \neq \emptyset$ . We know that  $\alpha$  and  $\beta$  cannot be both special, so assume  $\beta$  nonspecial. Let  $U \subseteq U_\alpha \cap U_\beta$ . Since  $\mathcal{U}$  is adapted,  $\alpha \notin U$ .

**PROPOSITION 4.7.** *With these hypotheses,  $\lambda_{\alpha\beta}^{[f]}$  belongs to  $H_{S^1}^*(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \beta)$ , and the following diagram is commutative:*

$$\begin{array}{ccc}
 H_{S^1}^*(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha) & \xrightarrow{(f^\alpha)_!^E} & H_{S^1}^*(Y^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha) \\
 \lambda_{\alpha\beta}^{[f]} \cdot i^* \downarrow & & \downarrow j^* \\
 H_{S^1}^*(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha) & \xrightarrow{(f^\beta)_!^E} & H_{S^1}^*(Y^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha) \\
 i_{\beta-\alpha}^* \downarrow & & \downarrow i_{\beta-\alpha}^* \\
 H_{S^1}^*(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \beta) & \xrightarrow{(f^\beta)_!^E} & H_{S^1}^*(Y^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \beta).
 \end{array}$$

*Proof.* Denote by  $W$  the normal bundle of the embedding  $X^\beta = X^{S^1} \rightarrow X^\alpha$ . Let us show that, if  $\alpha \notin U$ , then  $e_{S^1}^E(W)$  is invertible in  $H_{S^1}^*(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha)$ . Denote by  $w_i$  the nonequivariant Chern roots of  $W$ , and by  $m_i$  the corresponding rotation numbers of  $W$  (see Proposition A.4 in the Appendix). Since  $X^\beta = X^{S^1}$ ,  $m_i \neq 0$ . Also, the  $S^1$ -equivariant Euler class of  $W$  is given by

$$e_{S^1}(W) = (w_1 + m_1 u) \dots (w_r + m_r u) = m_1 \dots m_r (u + w_1/m_1) \dots (u + w_r/m_r).$$

But  $w_i$  are nilpotent, so  $e_{S^1}(W)$  is invertible as long as  $u$  is invertible. Now  $\alpha \notin U$  translates to  $0 \notin U - \alpha$ , which implies that the image of  $u$  via the map  $\mathbb{C}[u] \rightarrow \mathcal{O}_{\mathcal{E}}(U - \alpha)$  is indeed invertible. To deduce now that  $e_{S^1}^E(W)$ , the elliptic  $S^1$ -equivariant Euler class of  $W$ , is also invertible, recall that  $e_{S^1}^E(W)$  and  $e_{S^1}(W)$  differ by a class defined using the power series  $s(x)/x = 1 + a_3 x^2 + a_5 x^4 + \dots$ , which is invertible for  $U$  small enough.

So  $\lambda_{\alpha\beta}^{[f]}$  exists, and by the previous lemma, the upper part of our diagram is commutative. The lower part is trivially commutative.  $\square$

Now, since  $i^*$  are essentially the gluing maps in the sheaf  $\mathcal{F} = E_{S^1}^*(X)$ , we think of the maps  $\lambda_{\alpha\beta}^{[f]} \cdot i^*$  as giving the sheaf  $\mathcal{F}$  twisted by the cocycle  $\lambda_{\alpha\beta}^{[f]}$ . Recall from Definition 3.5 that  $\mathcal{F}$  was obtained by gluing the sheaves  $\mathcal{F}_\alpha$  over an adapted open cover  $(U_\alpha)_{\alpha \in \mathcal{E}}$ .

**Definition 4.8.** The twisted gluing functions  $\phi_{\alpha\beta}^{[f]}$  are defined as the composition of the following three maps:  $H_{S^1}^*(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha) \xrightarrow{i^* \otimes 1} H_{S^1}^*(X^\beta) \otimes_{\mathbb{C}[u]}$

$\mathcal{O}_{\mathcal{E}}(U - \alpha) \xrightarrow{\cdot \lambda_{\alpha\beta}^{[f]}} H_{S^1}^*(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \beta) \xrightarrow{t_{\beta-\alpha}^*} H_{S^1}^*(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \beta)$ . The third map is defined as in Remark 3.4.

As in the discussion after Remark 3.4, we can show easily that  $\phi_{\alpha\beta}^{[f]}$  satisfy the cocycle condition.

**Definition 4.9.** Let  $f: X \rightarrow Y$  be an equivariant map of compact  $S^1$ -manifolds, such that it is either complex or spin oriented. We denote by  $E_{S^1}^*(X)^{[f]}$  the sheaf obtained by gluing the sheaves  $\mathcal{F}_\alpha$  defined in 3.1, using the twisted gluing functions  $\phi_{\alpha\beta}^{[f]}$ .

Also, we define the  $S^1$ -equivariant elliptic pushforward of  $f$  to be the map of coherent sheaves over  $\mathcal{E}$

$$f_!^E: E_{S^1}^*(X)^{[f]} \rightarrow E_{S^1}^*(Y)$$

which comes from gluing the local elliptic pushforwards  $(f^\alpha)_!^E$  (as defined in Corollary 4.4). We call  $f_!^E$  the *Grojnowski pushforward*.

The fact that  $(f^\alpha)_!^E$  glue well comes from the commutativity of the diagram in Proposition 4.7. The Grojnowski pushforward is functorial: see [9] and [10].

**5. Rigidity of the elliptic genus.** In this section we discuss the rigidity phenomenon in the context of equivariant elliptic cohomology. We start with a discussion about orientations.

**5.1. Preliminaries on orientations.** Let  $V \rightarrow X$  be an even dimensional spin  $S^1$ -vector bundle over a finite  $S^1$ -CW complex  $X$  (which means that the  $S^1$ -action preserves the spin structure). Let  $n \in \mathbb{N}$ . We think of  $\mathbb{Z}_n \subset S^1$  as the ring of  $n$ th roots of unity in  $\mathbb{C}$ . The invariants of  $V$  under the actions of  $S^1$  and  $\mathbb{Z}_n$  are the  $S^1$ -vector bundles  $V^{S^1} \rightarrow X^{S^1}$  and  $V^{\mathbb{Z}_n} \rightarrow X^{\mathbb{Z}_n}$ . We have  $X^{S^1} \subseteq X^{\mathbb{Z}_n}$ .

Let  $N$  be a connected component of  $X^{S^1}$ , and  $P$  a connected component of  $X^{\mathbb{Z}_n}$  which contains  $N$ . From now on we think of  $V^{S^1}$  as a bundle over  $N$ , and  $V^{\mathbb{Z}_n}$  as a bundle over  $P$ .

Define the vector bundles  $V/V^{S^1}$  and  $V^{\mathbb{Z}_n}/V^{S^1}$  over  $N$  by

$$V|_N = V^{S^1} \oplus V/V^{S^1}; \quad V|_N^{\mathbb{Z}_n} = V^{S^1} \oplus V^{\mathbb{Z}_n}/V^{S^1}.$$

The decompositions of these two bundles come from the fact that  $S^1$  acts trivially on the base  $N$ , so fibers decompose into a trivial and nontrivial part.

Similarly, the action of  $\mathbb{Z}_n$  on  $P$  is trivial, so we get a fiberwise decomposition of  $V|_P$  by the different representations of  $\mathbb{Z}_n$ :

$$V|_P = V^{\mathbb{Z}_n} \oplus \bigoplus_{0 < k < \frac{n}{2}} V(k) \oplus V\left(\frac{n}{2}\right).$$



By convention,  $V(\frac{n}{2}) = 0$  if  $n$  is odd. Lemma 9.4 in Bott and Taubes [4] implies that  $V^{\mathbb{Z}_n}$  and  $V(\frac{n}{2})$  are even dimensional real oriented vector bundles. Denote by

$$V(K) = \bigoplus_{0 < k < \frac{n}{2}} V(k).$$

Then we have the following decompositions:

$$(1) \quad V|_P = V^{\mathbb{Z}_n} \oplus V(K) \oplus V\left(\frac{n}{2}\right).$$

$$(2) \quad V^{\mathbb{Z}_n}|_N = V^{S^1} \oplus V^{\mathbb{Z}_n}/V^{S^1}.$$

Now we define the orientations for the different bundles involved: First, if a bundle is oriented, any restriction to a smaller base gets an induced orientation.  $V$  is oriented by its spin structure.  $\mathbb{Z}_n$  preserves the spin structure of  $V$ , so we can apply Lemma 10.3 from Bott and Taubes [4], and deduce that  $V^{\mathbb{Z}_n}$  has an induced orientation. Each  $V(k)$  for  $0 < k < \frac{n}{2}$  has a complex structure, for which  $g = e^{2\pi i/n} \in \mathbb{Z}_n$  acts by complex multiplication with  $g^k$ . So  $V(K)$  has a complex orientation, too. Define the orientation on  $V(K)$  by:

- If  $V(\frac{n}{2}) \neq 0$ ,  $V(K)$  is oriented by its complex structure described above.
- If  $V(\frac{n}{2}) = 0$ , then choose the orientation on  $V(K)$  induced by the decomposition in (1):  $V|_P = V^{\mathbb{Z}_n} \oplus V(K)$ .

Then the decomposition in (1) induces an orientation on  $V(\frac{n}{2})$ .

We now orient bundles appearing in (2) as follows: Notice that  $V^{\mathbb{Z}_n}/V^{S^1}$  has only nonzero rotation numbers, so it has a complex structure for which all rotation numbers are positive. Define the orientation on  $V^{\mathbb{Z}_n}/V^{S^1}$  by:

- If  $V^{S^1} \neq 0$ ,  $V^{\mathbb{Z}_n}/V^{S^1}$  is oriented by its complex structure described above.
- If  $V^{S^1} = 0$ , then  $V^{\mathbb{Z}_n}/V^{S^1} = V^{\mathbb{Z}_n}|_N$ , so we choose this orientation, induced from that on  $V^{\mathbb{Z}_n}$  described above.

Finally, we orient  $V/V^{S^1}$  from the decomposition

$$(3) \quad V/V^{S^1} = V^{\mathbb{Z}_n}/V^{S^1} \oplus V(K)|_N \oplus V\left(\frac{n}{2}\right)|_N.$$

As a notational rule, we are going to use the subscript “*or*” to indicate the “correct” orientation on the given vector space, i.e., the orientations which we defined above. When we omit the subscript “*or*,” we assume the bundle has the correct orientation. But all bundles that appear in (3) also have a complex structure (they have nonzero rotation numbers). The subscript “*cx*” will indicate that we chose a complex structure on the given vector space. This is only intended to make calculations easier. So we choose complex structures as follows: For  $V^{\mathbb{Z}_n}/V^{S^1}$  we choose as above the complex structure where all rotation numbers are positive, and similarly for  $V(\frac{n}{2})|_N$ . Also,  $V(K)|_N$  gets an induced complex structure from the complex structure on  $V(K)$  described above. Now  $V/V^{S^1}$  gets its complex structure from the decomposition (3).

Let  $i: N \rightarrow P$  be the inclusion. Table 1 lists the bundles of interest:

Table 1.

bundle with the correct orientation	bundle with the complex orientation	sign difference between the two orientations
$(V/V^{S^1})_{or}$	$(V/V^{S^1})_{cx}$	$(-1)^\sigma$
$(V^{\mathbb{Z}_n}/V^{S^1})_{or}$	$(V^{\mathbb{Z}_n}/V^{S^1})_{cx}$	$(-1)^{\sigma(0)}$
$V(K)_{or}$	$V(K)_{cx}$	$(-1)^{\sigma(K)}$
$i^* \left( V \left( \frac{n}{2} \right)_{or} \right)$	$\left( i^* V \left( \frac{n}{2} \right) \right)_{cx}$	$(-1)^{\sigma(\frac{n}{2})}$

From the decomposition in (3) under the correct and the complex orientations, we deduce that

$$(4) \quad (-1)^{\sigma(0)}(-1)^{\sigma(K)}(-1)^{\sigma(\frac{n}{2})} = (-1)^\sigma.$$

By the splitting principle (Bott and Tu [5]), the pullback of  $V/V^{S^1}$  to the flag manifold can be thought of as a sum of complex line bundles  $L(m_j)$ ,  $j = 1, \dots, r$ . The complex structure of  $L(m_j)$  is such that  $g \in S^1$  acts on  $L(m_j)$  via complex multiplication with  $g^{m_j}$ . The numbers  $m_j \neq 0, j = 1, \dots, r$ , are the rotation numbers. By the real splitting principle, they are defined also for even dimensional real oriented vector bundles, but in that case the  $m_j$ 's are well defined only up to an even number of sign changes. We choose two systems of rotation numbers for  $V/V^{S^1}$ , one denoted by  $(m_j)_j$ , corresponding to  $(V/V^{S^1})_{or}$ ; and one denoted by  $(m_j^*)_j$ , corresponding to  $(V/V^{S^1})_{cx}$ . Of course, since the two orientations differ by the sign  $(-1)^\sigma$ , the systems  $(m_j)_j$  and  $(m_j^*)_j$  will be the same up to a permutation and a number of sign changes of the same parity with  $(-1)^\sigma$ .

For  $j = 1, \dots, r$ , we define  $q_j$  and  $r_j$  as the quotient and the remainder, respectively, of  $m_j$  modulo  $n$ . Similarly,  $q_j^*$  and  $r_j^*$  are the quotient and the remainder of  $m_j^*$  modulo  $n$ .

We define now for each  $k$  a set of indices of the corresponding rotation numbers from the decomposition in (3): if  $0 \leq k \leq \frac{n}{2}$ , define

$$I_k = \{j \in 1, \dots, r \mid r_j = k \text{ or } n - k\}.$$

Notice that for  $0 < k \leq \frac{n}{2}$ ,  $I_k$  contains exactly the indices of the rotation numbers for  $V(k)$ , and for  $k = 0$ ,  $I_0$  contains the indices of the rotation numbers corresponding to  $V^{\mathbb{Z}_n}/V^{S^1}$ . Also, define

$$I_K = \bigcup_{0 < k < \frac{n}{2}} I_k.$$

**5.2. Rigidity.** As in the beginning of Section 4, let  $\mathcal{E} = \mathbb{C}/\Lambda$  be an elliptic curve over  $\mathbb{C}$  together with a 2-torsion point and a generator of the cotangent

space to  $\mathcal{E}$  at zero. We saw that we can canonically associate to this data a double cover  $\tilde{\mathcal{E}}$  of  $\mathcal{E}$ , and the Jacobi sine function  $s: \mathcal{E} \rightarrow \mathbb{C}$ .

Let  $X$  be a compact spin  $S^1$ -manifold, i.e., a spin manifold with an  $S^1$  action which preserves the spin structure. Then the map  $\pi: X \rightarrow *$  is spin oriented, hence it satisfies the hypothesis of Definition 4.9. Therefore we get a Grojnowski pushforward  $\pi_!^E: E_{S^1}^*(X)^{[\pi]} \rightarrow E_{S^1}^*(*) = \mathcal{O}_{\mathcal{E}}$ .

We will see that the rigidity phenomenon amounts to finding a global (Thom) section in the sheaf  $E_{S^1}^*(X)^{[\pi]}$ . Since  $s(x)$  is not a well-defined function on  $\mathcal{E}$ , we cannot expect to find such a global section on  $\mathcal{E}$ . However, if we take the pullback of the sheaf  $E_{S^1}^*(X)^{[\pi]}$  along the covering map  $\tilde{\mathcal{E}} \rightarrow \mathcal{E}$ , we can show that the new sheaf has a global section.

*Convention.* From this point on, all the sheaves  $\mathcal{F}$  involved will be considered over  $\tilde{\mathcal{E}}$ , i.e., we will replace them by the pullback of  $\mathcal{F}$  via the map  $\tilde{\mathcal{E}} \rightarrow \mathcal{E}$ .

For our purposes, however, we need a more general version of  $E_{S^1}^*(X)^{[\pi]}$ , which involves a vector bundle  $V \rightarrow X$ . Consider now  $V \rightarrow X$  a spin  $S^1$ -vector bundle over a finite  $S^1$ -CW complex.

*Definition 5.1.* As in Definition 4.8, we define  $\phi_{\alpha\beta}^{[V]}$  as the composition of three maps, where the second one is multiplication by  $\lambda_{\alpha\beta}^{[V]} = e_{S^1}^E(V^\alpha/V^\beta)^{-1}$ . The bundle  $V^\alpha/V^\beta = V^{\mathbb{Z}n}/V^{S^1}$  is oriented as in the previous subsection.

We then denote by  $E_{S^1}^*(X)^{[V]}$  the sheaf obtained by gluing the sheaves  $\mathcal{F}_\alpha$  defined in 3.1, using the twisted gluing functions  $\phi_{\alpha\beta}^{[V]}$ .

Notice that, if we take the map  $f: X \rightarrow *$  and  $V = TX$ , we have  $E_{S^1}^*(X)^{[V]} = E_{S^1}^*(X)^{[f]}$ . We now proceed to proving Theorem C.

**THEOREM 5.2.** *If  $V \rightarrow X$  is a spin  $S^1$ -vector bundle over a finite  $S^1$ -CW complex, then the element 1 in the stalk of  $E_{S^1}^*(X)^{[V]}$  at zero extends to a global section on  $\tilde{\mathcal{E}}$ , called the Thom section.*

*Proof.* To simplify notation, we are going to identify  $\tilde{\mathcal{E}}$  with  $\mathbb{C}/\tilde{\Lambda}$ , where  $\tilde{\Lambda} = \mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2$  is the “doubled” lattice described in Section 4. We want now to think of points in  $\tilde{\mathcal{E}}$  as points in  $\mathbb{C}$ , and of  $E_{S^1}^*(X)$  as the pullback of  $E_{S^1}^*(X)$  on  $\mathbb{C}$  via  $\mathbb{C} \rightarrow \mathbb{C}/\tilde{\Lambda}$ . Then we call  $\alpha \in \mathbb{C}$  a torsion point if there is an integer  $n > 0$  such that  $n\alpha \in \Lambda$  (notice that torsion points are defined in terms of  $\Lambda$ , and not  $\tilde{\Lambda}$ ). The smallest such  $n$  is called the exact order of  $\alpha$ . From Proposition 4.1(b), we know that if  $a \in \Lambda$ ,  $s(x+a) = \pm s(x)$ . Since  $n\alpha \in \Lambda$ , define  $\epsilon = \pm 1$  by

$$s(x+n\alpha) = \epsilon s(x).$$

Now  $E_{S^1}^*(X)^{[V]}$  was obtained by gluing the sheaves  $\mathcal{F}_\alpha$  along the adapted open cover  $(U_\alpha)_\alpha$ . So to give a global section  $\mu$  of  $E_{S^1}^*(X)^{[V]}$  is the same as to

give global sections  $\mu_\alpha$  of  $\mathcal{F}_\alpha$  such that they glue, i.e.,  $\phi_{\alpha\beta}^{[V]}\mu_\alpha = \mu_\beta$  for any  $\alpha$  and  $\beta$  with  $U_\alpha \cap U_\beta \neq \emptyset$ . From Definition 5.1, to give  $\mu$  is the same as to give  $\mu_\alpha \in HO_{S^1}^*(X^\alpha)$  so that  $t_{\beta-\alpha}^*(i^*\mu_\alpha \cdot e_{S^1}^E(V^\alpha/V^\beta)^{-1}) = \mu_\beta$ , or  $i^*\mu_\alpha \cdot e_{S^1}^E(V^\alpha/V^\beta)^{-1} = t_{\beta-\alpha}^*\mu_\beta$  (i the inclusion  $X^\beta \hookrightarrow X^\alpha$ ). Because  $\mu$  is supposed to globalize 1, we know that  $\mu_0 = 1$ . This implies that  $\mu_\beta = t_\beta^*e_{S^1}^E(V/V^\beta)^{-1}$  for  $\beta$  in a small neighborhood of  $0 \in \mathbb{C}$ .

In fact, we can show that this formula for  $\mu_\beta$  is valid for all  $\beta \in \mathbb{C}$ , as long as  $\beta$  is not special. This means we have to check that  $\mu_\beta = t_\beta^*e_{S^1}^E(V/V^\beta)^{-1}$  exists in  $HO_{S^1}^*(X^\beta)$  as long as  $\beta$  is not special.  $\beta$  not special means  $X^\beta = X^{S^1}$ . Then consider the bundle  $V/V^{S^1}$ . We saw in the previous subsection that according to the splitting principle, when pulled back on the flag manifold,  $V/V^{S^1}$  decomposes into a direct sum of line bundles  $L(m_1) \oplus \cdots \oplus L(m_r)$ , where  $m_j$  are the rotation numbers. The complex structure on  $L(m)$  is such  $g \in S^1$  acts on  $L(m)$  by complex multiplication with  $g^m$ .

Since  $X^{S^1}$  is fixed by the  $S^1$  action, we can apply Proposition A.4 in the Appendix: Let  $x_j$  be the equivariant Chern root of  $L(m_j)$ , and  $w_j$  its usual (nonequivariant) Chern root. Then  $x_j = w_j + m_j u$ , with  $u$  the generator of  $H^*(BS^1)$ . Therefore  $t_\beta^*e_{S^1}^E(V/V^\beta) = \prod_j t_\beta^*s(x_j) = \prod_j t_\beta^*s(w_j + m_j u) = \prod_j s(w_j + m_j u + m_j \beta) = \prod_j s(x_j + m_j \beta)$ .

So we have

$$\mu_\beta = t_\beta^*e_{S^1}^E(V/V^{S^1})^{-1} = \prod_{j=1}^r s(x_j + m_j \beta)^{-1}.$$

We show that  $\mu_\beta$  belongs to  $HO_{S^1}^*(X^\beta)$  as long as  $s(m_j \beta) \neq 0$  for all  $j = 1, \dots, r$ : Since  $V/V^{S^1}$  has only nonzero rotation numbers, it has a complex structure. But changing the orientations of a vector bundle only changes the sign of the corresponding Euler class, so in the formula above we can assume that  $V/V^{S^1}$  has a complex structure, for example the one for which all  $m_j > 0$ . We group the  $m_j$  which are equal, i.e., for each  $m > 0$  we define the set of indices  $J_m = \{j \mid m_j = m\}$ . Now we get a decomposition  $V/V^{S^1} = \sum_{m>0} W(m)$ , where  $W(m)$  is the complex  $S^1$ -vector bundle on which  $g \in S^1$  acts by multiplication with  $g^m$ . (This decomposition takes place on  $X^{S^1}$ , while the decomposition into line bundles  $L(m_j)$  takes place only on the flag manifold.) Now we have to show that  $\prod_{j \in J_m} s(x_j + m\beta)^{-1}$  gives an element of  $HO_{S^1}^*(X^\beta)$ . This would follow from Proposition A.6 applied to the power series  $Q(x) = s(x + m\beta)^{-1}$  and the vector bundle  $W(m)$ , provided that  $Q(x)$  is convergent. But  $s(x + m\beta)^{-1}$  is indeed convergent, since  $s$  is meromorphic on  $\mathbb{C}$  and does not have a zero at  $m\beta$ .

Now we show that if  $\beta$  is nonspecial,  $s(m_j \beta) \neq 0$  for all  $j = 1, \dots, r$ : Suppose  $s(m_j \beta) = 0$ . Then  $m_j \beta \in \Lambda$ , so  $\beta$  is a torsion point, say of exact order  $n$ . It follows that  $n$  divides  $m_j$ , which implies  $X^{\mathbb{Z}n} \neq X^{S^1}$ . But  $X^\beta = X^{\mathbb{Z}n}$ , since  $\beta$  has exact order  $n$ , so  $X^\beta \neq X^{S^1}$  i.e.,  $\beta$  is special, a contradiction.

So we only need to analyze what happens at a special point  $\alpha \in \mathbb{C}$ , say of exact order  $n$ . We have to find a class  $\mu_\alpha \in HO_{S^1}^*(X^\alpha)$  such that  $\phi_{\alpha\beta}^{[V]}\mu_\alpha =$

$\mu_\beta$ , i.e.,  $t_{\beta-\alpha}^*(i^*\mu_\alpha \cdot e_{S^1}^E(V^\alpha/V^\beta)^{-1}) = t_\beta^*e_{S^1}^E(V/V^\beta)^{-1}$ . Equivalently, we want a class  $\mu_\alpha$  such that  $i^*\mu_\alpha = t_\alpha^*e_{S^1}^E(V/V^\beta)^{-1} \cdot e_{S^1}^E(V^\alpha/V^\beta)$ , i.e., we want to lift the class  $t_\alpha^*e_{S^1}^E(V/V^\beta)^{-1} \cdot e_{S^1}^E(V^\alpha/V^\beta)$  from  $HO_{S^1}^*(X^\beta)$  to  $HO_{S^1}^*(X^\alpha)$ . If we can do that, we are done, because the class  $(\mu_\alpha)_{\alpha \in \mathbb{C}}$  is a global section in  $E_{S^1}^*(X)^{[V]}$ , and it extends  $\mu_0 = 1$  in the stalk at zero. So it only remains to prove the following lemma, which is a generalization of the transfer formula of Bott and Taubes.  $\square$

LEMMA 5.3. *Let  $\alpha$  be a special point of exact order  $n$ , and  $V \rightarrow X$  a spin  $S^1$ -vector bundle. Let  $i: X^{S^1} \rightarrow X^{\mathbb{Z}^n}$  be the inclusion map. Then there exists a class  $\mu_\alpha \in HO_{S^1}^*(X^{\mathbb{Z}^n})$  such that*

$$i^*\mu_\alpha = t_\alpha^*e_{S^1}^E(V/V^{S^1})^{-1} \cdot e_{S^1}^E(V^{\mathbb{Z}^n}/V^{S^1}).$$

*Proof.* We first study the class  $t_\alpha^*e_{S^1}^E(V/V^{S^1})^{-1} \cdot e_{S^1}^E(V^{\mathbb{Z}^n}/V^{S^1})$  on each connected component of  $X^{S^1}$  in  $X^{\mathbb{Z}^n}$ . We will see that it lifts naturally to a class on  $X^{\mathbb{Z}^n}$ . The problem arises from the fact that we can have two connected components of  $X^{S^1}$  inside one connected component of  $X^{\mathbb{Z}^n}$ , and in that case the two lifts will differ by a sign. We then show that the sign vanishes if  $V$  has a spin structure.

As in the previous subsection, let  $N$  be a connected component of  $X^{S^1}$ , and  $P$  a connected component of  $X^{\mathbb{Z}^n}$  which contains  $N$ .

We now calculate  $t_\alpha^*e_{S^1}^E(V/V^{S^1})^{-1}$ , regarded as a class on  $N$ . From the decomposition (3)  $V/V^{S^1} = V^{\mathbb{Z}^n}/V^{S^1} \oplus V(K)_{|N} \oplus V(\frac{n}{2})_{|N}$  and from the table, we get the following formula:

$$\begin{aligned} (5) \quad t_\alpha^*e_{S^1}^E(V/V^{S^1})^{-1} &= (-1)^\sigma \cdot e_{S^1}^E(V/V^{S^1})_{cx}^{-1} \\ &= (-1)^\sigma \cdot \prod_{j \in I_0} s(x_j + m_j^* \alpha)^{-1} \\ &\quad \cdot \prod_{j \in I_K} s(x_j + m_j^* \alpha)^{-1} \cdot \prod_{j \in I_{n/2}} s(x_j + m_j^* \alpha)^{-1}. \end{aligned}$$

Before we analyze each term in the above formula, recall that we defined the number  $\epsilon = \pm 1$  by  $s(x + n\alpha) = \epsilon s(x)$ .

(a)  $j \in I_0$ : Here we choose the complex structure  $(V^{\mathbb{Z}^n}/V^{S^1})_{cx}$  such that all  $m_j^* > 0$ . Then, since  $s(x_j + m_j^* \alpha) = s(x_j + q_j^* n\alpha) = \epsilon^{q_j^*} s(x_j)$ , we have:  $\prod_{j \in I_0} s(x_j + m_j^* \alpha)^{-1} = \epsilon^{\sum_{I_0} q_j^*} \cdot \prod_{I_0} s(x_j)^{-1} = \epsilon^{\sum_{I_0} q_j^*} \cdot e_{S^1}^E(V^{\mathbb{Z}^n}/V^{S^1})_{cx}^{-1} = \epsilon^{\sum_{I_0} q_j^*} \cdot (-1)^{\sigma(0)} \cdot e_{S^1}^E(V^{\mathbb{Z}^n}/V^{S^1})_{or}^{-1}$ . So we eventually get

$$(6) \quad \prod_{j \in I_0} s(x_j + m_j^* \alpha)^{-1} = \epsilon^{\sum_{I_0} q_j^*} \cdot (-1)^{\sigma(0)} \cdot e_{S^1}^E(V^{\mathbb{Z}^n}/V^{S^1})_{or}^{-1}.$$

(b)  $j \in I_K$ , i.e.,  $j \in I_k$  for some  $0 < k < \frac{n}{2}$ . The complex structure on  $V(k)$  is such that  $g = e^{2\pi i/n} \in \mathbb{Z}_n$  acts by complex multiplication with  $g^k$ . Notice that in the previous subsection we defined the complex structure on  $V/V^{S^1}$  to come from the decomposition (3). This implies that  $m_j^* = nq_j^* + k$ , and therefore  $s(x_j + m_j^* \alpha) = s(x_j + q_j^* n \alpha + k \alpha) = \epsilon^{q_j^*} s(x_j + k \alpha)$ .

Consider  $\mu_k$  the equivariant class on  $P$  corresponding to the complex vector bundle  $V(k)$  with its chosen complex orientation, and the convergent power series  $Q(x) = s(x + k\alpha)^{-1}$ . Then  $i^* \mu_k = \prod_{I_k} s(x_j + k\alpha)^{-1}$ . Define  $\mu_K = \prod_{0 < k < \frac{n}{2}} \mu_k$ . Using the above formula for  $s(x_j + m_j^* \alpha)$  with  $j \in I_k$ , we obtain

$$(7) \quad \prod_{j \in I_K} s(x_j + m_j^* \alpha)^{-1} = \epsilon^{\sum_{I_K} q_j^*} \cdot (-\epsilon)^{\sigma(K)} \cdot i^* \mu_K.$$

(c)  $j \in I_{n/2}$ . The complex structure on  $i^* V(\frac{n}{2})$  is the one for which all  $m_j^* > 0$ . The rotation numbers satisfy  $m_j^* = q_j^* n + \frac{n}{2}$ , hence  $s(x_j + m_j^* \alpha) = \epsilon^{q_j^*} s(x_j + \frac{n}{2} \alpha)$ . Consider the power series  $Q(x) = s(x + \frac{n}{2} \alpha)^{-1}$ .  $Q(x)$  satisfies  $Q(-x) = s(-x + \frac{n}{2} \alpha)^{-1} = -s(x - \frac{n}{2} \alpha)^{-1} = -\epsilon s(x + \frac{n}{2} \alpha)^{-1} = (-\epsilon)Q(x)$ , so  $Q(x)$  is either even or odd. According to Definition A.8, since  $V(\frac{n}{2})_{or}$  is a real oriented even dimensional vector bundle,  $Q(x)$  defines a class  $\mu_{\frac{n}{2}} = \mu_Q(V(\frac{n}{2}))$ , which is a class on  $P$ . Now from the table,  $i^* V(\frac{n}{2})_{or}$  and  $(i^* V(\frac{n}{2}))_{cx}$  differ by the sign  $(-1)^{\sigma(\frac{n}{2})}$ , so Lemma A.9 (with  $\gamma = -\epsilon$ ) implies that  $i^* \mu_{\frac{n}{2}} = (-\epsilon)^{\sigma(\frac{n}{2})} \prod_{j \in I_k} s(x_j + \frac{n}{2} \alpha)^{-1}$ . Finally we obtain

$$(8) \quad \prod_{j \in I_{n/2}} s(x_j + m_j^* \alpha)^{-1} = \epsilon^{\sum_{I_{n/2}} q_j^*} \cdot (-\epsilon)^{\sigma(\frac{n}{2})} \cdot i^* \mu_{\frac{n}{2}}.$$

Now, if we put together equations (5)–(8) and (4), and define  $\mu_P := \mu_K \cdot \mu_{\frac{n}{2}}$ , we have just proved that  $t_\alpha^* e_{S^1}^E(V/V^{S^1})^{-1} = \epsilon^{\sigma(N)} \cdot e_{S^1}^E(V^{\mathbb{Z}_n}/V^{S^1})^{-1} \cdot i^* \mu_P$ , or

$$(9) \quad t_\alpha^* e_{S^1}^E(V/V^{S^1})^{-1} \cdot e_{S^1}^E(V^{\mathbb{Z}_n}/V^{S^1}) = \epsilon^{\sigma(N)} \cdot i^* \mu_P,$$

where

$$\sigma(N) = \sum_{I_0} q_j^* + \sum_{I_K} q_j^* + \sum_{I_{n/2}} q_j^* + \sigma(K) + \sigma\left(\frac{n}{2}\right).$$

Now we want to describe  $\sigma(N)$  in terms of the correct rotation numbers  $m_j$  of  $V/V^{S^1}$ . Recall that  $m_j$  are the same as  $m_j^*$  up to sign and a permutation. Denote by  $\equiv$  equality modulo 2. We have the following cases:

- (a)  $j \in I_0$ . Suppose  $m_j = -m_j^*$ . Then  $q_j = -q_j^*$ , which implies  $q_j^* \equiv q_j$ . Therefore  $\sum_{I_0} q_j^* \equiv \sum_{I_0} q_j$ .
- (b)  $j \in I_K$ . Let  $0 < k < \frac{n}{2}$ . Suppose  $m_j = -m_j^* = -q_j^* n - k = -(q_j^* + 1)n + (n - k)$ . Then  $q_j = -q_j^* - 1$ , which implies  $q_j^* + 1 \equiv q_j$ . So modulo 2, the sum  $\sum_{I_K} q_j^*$  differs

from  $\sum_{I_K} q_j$  by the number of the sign differences  $m_j = -m_j^*$ . But by definition of rotation numbers, the number of sign differences in two systems of rotation numbers is precisely the sign difference  $\sigma(K)$  between the two corresponding orientations of  $i^*V(K)$ . Therefore,  $\sum_{I_K} q_j^* + \sigma(K) \equiv \sum_{I_K} q_j$ .

(c)  $j \in I_{n/2}$ . Suppose  $m_j = -m_j^* = -q_j^*n - \frac{n}{2} = -(q_j^* + 1)n + \frac{n}{2}$ . Then this implies  $q_j^* + 1 \equiv q_j$ , so by the same reasoning as in b)  $\sum_{I_{n/2}} q_j^* + \sigma(\frac{n}{2}) \equiv \sum_{I_{n/2}} q_j$ . We finally get the following formula for  $\sigma(N)$

$$\sigma(N) \equiv \sum_{I_0} q_j + \sum_{I_K} q_j + \sum_{I_{n/2}} q_j.$$

In the next lemma we will show that, for  $N$  and  $\tilde{N}$  two different connected components of  $X^{S^1}$  inside  $P$ ,  $\sigma(N)$  and  $\sigma(\tilde{N})$  are congruent modulo 2, so the class  $\epsilon^{\sigma(N)} \cdot \mu_P$  is well defined, i.e., independent of  $N$ . Now recall that  $P$  is a connected component of  $X^{\mathbb{Z}_n}$ . Therefore  $HO_{S^1}^*(X^{\mathbb{Z}_n}) = \oplus_P HO_{S^1}^*(P)$ , so we can define

$$\mu_\alpha := \sum_P \epsilon^{\sigma(N)} \cdot \mu_P.$$

This is a well-defined class in  $HO_{S^1}^*(X^{\mathbb{Z}_n})$ , so by equation (9), Lemma 5.3 is finally proved.  $\square$

**LEMMA 5.4.** *In the conditions of the previous lemma,  $\sigma(N)$  and  $\sigma(\tilde{N})$  are congruent modulo 2.*

*Proof.* The proof follows Bott and Taubes [4]. Denote by  $S^2(n)$  the 2-sphere with the  $S^1$ -action which rotates  $S^2$   $n$  times around the north-south axis as we go once around  $S^1$ . Denote by  $N^+$  and  $N^-$  its North and South poles, respectively. Consider a path in  $P$  which connects  $N$  with  $\tilde{N}$ , and touches  $N$  or  $\tilde{N}$  only at its endpoints. By rotating this path with the  $S^1$ -action, we obtain a subspace of  $P$  which is close to being an embedded  $S^2(n)$ . Even if it is not, we can still map equivariantly  $S^2(n)$  onto this rotated path. Now we can pull back the bundles from  $P$  to  $S^2(n)$  (with their correct orientations). The rotation numbers are the same, since the North and the South poles are fixed by the  $S^1$ -action, as are the endpoints of the path.

Therefore we have translated the problem to the case when we have the 2-sphere  $S^2(n)$  and corresponding bundles over it, and we are trying to prove that  $\sigma(N^+) \equiv \sigma(N^-)$  modulo 2. The only problem would be that we are not using the whole of  $V$ , but only  $V/V^{S^1}$ . However, the difference between these two bundles is  $V^{S^1}$ , whose rotation numbers are all zero, so they do not influence the result.

Now Lemma 9.2 of [4] says that any even-dimensional oriented real vector bundle  $W$  over  $S^2(n)$  has a complex structure. In particular, the pullbacks of  $V^{S^1}$ ,  $V(K)$ , and  $V(\frac{n}{2})$  have complex structure, and the rotation numbers can be chosen to be the  $m_j$  described above. Say the rotation numbers at the South pole are  $\tilde{m}_j$

with the obvious notation conventions. Then Lemma 9.1 of [4] says that, up to a permutation,  $m_j - \tilde{m}_j = n(q_j - \tilde{q}_j)$ , and  $\sum q_j \equiv \sum \tilde{q}_j$  modulo 2. But this means that  $\sigma(N^+) \equiv \sigma(N^-)$  modulo 2, i.e.,  $\sigma(N) \equiv \sigma(\tilde{N})$  modulo 2.  $\square$

**COROLLARY 5.5.** (The Rigidity theorem of Witten) *If  $X$  is a spin manifold with an  $S^1$ -action, then the equivariant elliptic genus of  $X$  is rigid i.e., it is a constant power series.*

*Proof.* By lifting the  $S^1$ -action to a double cover of  $S^1$ , we can make the  $S^1$ -action preserve the spin structure. Then with this action  $X$  is a spin  $S^1$ -manifold.

At the beginning of this section, we say that if  $X$  is a compact spin  $S^1$ -manifold, i.e., the map  $\pi: X \rightarrow *$  is spin, then we have the Grojnowski pushforward, which is a map of sheaves

$$\pi_!^E: E_{S^1}^*(X)^{[\pi]} \rightarrow E_{S^1}^*(*) = \mathcal{O}_{\mathcal{E}}.$$

The Grojnowski pushforward  $\pi_!^E$ , if we consider it at the level of stalks at  $0 \in \mathcal{E}$ , is nothing but the elliptic pushforward in  $HO_{S^1}^*$ -theory, as described in Corollary 4.4. So consider the element 1 in the stalk at 0 of the sheaf  $E_{S^1}^*(X)^{[\pi]} = E_{S^1}^*(X)^{[TX]}$ .

From Theorem 5.2, since  $TX$  is spin, 1 extends to a global section of  $E_{S^1}^*(X)^{[TX]}$ . Denote this global section by boldface 1. Because  $\pi_!^E$  is a map of sheaves, it follows that  $\pi_!^E(1)$  is a global section of  $E_{S^1}^*(*) = \mathcal{O}_{\mathcal{E}}$ , i.e., a global holomorphic function on the elliptic curve  $\mathcal{E}$ . But any such function has to be constant. This means that  $\pi_!^E(1)$ , which is the equivariant elliptic genus of  $X$ , extends to  $\pi_!^E(1)$ , which is constant. This is precisely equivalent to the elliptic genus being rigid.  $\square$

The extra generality we had in Theorem 5.2 allows us now to extend the Rigidity theorem to families of elliptic genera. This was stated as Theorem D in Section 2.

**THEOREM 5.6.** (Rigidity for families) *Let  $F \rightarrow E \xrightarrow{\pi} B$  be an  $S^1$ -equivariant fibration such that the fibers are spin in a compatible way, i.e., the projection map  $\pi$  is spin oriented. Then the elliptic genus of the family, which is  $\pi_!^E(1) \in H_{S^1}^{**}(B)$ , is constant as a rational function in  $u$ , i.e., if we invert  $u$ .*

*Proof.* We know that the map

$$\pi_!^E: E_{S^1}^*(E)^{[\pi]} \rightarrow E_{S^1}^*(B)$$

when regarded at the level of stalks at zero is the usual equivariant elliptic pushforward in  $HO_{S^1}^*(-)$ . Now  $\pi_!^E(1) \in HO_{S^1}^*(B)$  is the elliptic genus of the family. We have  $E_{S^1}^*(E)^{[\pi]} \cong E_{S^1}^*(E)^{[\tau(F)]}$ , where  $\tau(F) \rightarrow E$  is the bundle of tangents along the fiber.

Since  $\tau(F)$  is spin, Theorem 5.2 allows us to extend 1 to the Thom section 1. Since  $\pi_!^E$  is a map of sheaves, it follows that  $\pi_!^E(1)$ , which is the elliptic genus of



the family, extends to a global section in  $E_{S^1}^*(B)$ . So, if  $i: B^{S^1} \hookrightarrow B$  is the inclusion of the fixed point submanifold in  $B$ ,  $i^* \pi_!^E(1)$  gives a global section in  $E_{S^1}^*(B^{S^1})$ . Now this latter sheaf is free as a sheaf of  $\mathcal{O}_{\mathcal{E}}$ -modules, so any global section is constant. But  $i^*: HO_{S^1}^*(B) \rightarrow HO_{S^1}^*(B^{S^1})$  is an isomorphism if we invert  $u$ .  $\square$

We saw in the previous section that, if  $f: X \rightarrow Y$  is an  $S^1$ -map of compact  $S^1$ -manifolds such that the restrictions  $f: X^\alpha \rightarrow Y^\alpha$  are oriented maps, we have the Grojnowski pushforward

$$f_!^E: E_{S^1}^*(X)^{[f]} \rightarrow E_{S^1}^*(Y).$$

Also, in some cases, for example when  $f$  is a spin  $S^1$ -fibration, we saw that  $E_{S^1}^*(X)^{[f]}$  admits a Thom section. This raises the question of whether or not we can describe  $E_{S^1}^*(X)^{[f]}$  as  $E_{S^1}^*$  of a Thom space. It turns out that, up to a line bundle over  $\mathcal{E}$  (which is itself  $E_{S^1}^*$  of a Thom space), this indeed happens: Let  $f: X \rightarrow Y$  be an  $S^1$ -map as above. Embed  $X$  into an  $S^1$ -representation  $W$ ,  $i: X \hookrightarrow W$ . ( $W$  can be also thought as an  $S^1$ -vector bundle over a point.) Look at the embedding  $f \times i: X \hookrightarrow Y \times W$ . Denote by  $V = \nu(f)$ , the normal bundle of  $X$  in this embedding (if we were not in the equivariant setup,  $\nu(f)$  would be the stable normal bundle to the map  $f$ ).

PROPOSITION 5.7. *With the previous notations,*

$$E_{S^1}^*(X)^{[f]} \cong E_{S^1}^*(DV, SV) \otimes E_{S^1}^*(DW, SW)^{-1},$$

where  $DV, SV$  are the disk and the sphere bundles of  $V$ , respectively.

*Proof.* From the embedding  $X \hookrightarrow Y \times W$ , we have the following isomorphism of vector bundles:

$$TX \oplus V \cong f^*TY \oplus W.$$

So, in terms of  $S^1$ -equivariant elliptic Euler classes we have  $e_{S^1}^E(V^\alpha/V^\beta) = e_{S^1}^E(X^\alpha/X^\beta)^{-1} \cdot f^* e_{S^1}^E(Y^\alpha/Y^\beta) \cdot e_{S^1}^E(W^\alpha/W^\beta)$ . Rewrite this as

$$\lambda_{\alpha\beta}^{[f]} = e_{S^1}^E(V^\alpha/V^\beta) \cdot e_{S^1}^E(W^\alpha/W^\beta)^{-1},$$

where  $\lambda_{\alpha\beta}^{[f]}$  is the twisted cocycle from Definition 4.8.

Notice that we can extend Definition 5.1 to virtual bundles as well. In other words, we can define  $E_{S^1}^*(X)^{[-V]}$  to be  $E_{S^1}^*(X)$  twisted by the cocycle  $\lambda_{\alpha\beta}^{[-V]} = e_{S^1}^E(V^\alpha/V^\beta)$ . The above formula then becomes

$$\lambda_{\alpha\beta}^{[f]} = \lambda_{\alpha\beta}^{[-V]} \cdot \lambda_{\alpha\beta}^{[W]},$$

which implies that

$$(10) \quad E_{S^1}^*(X)^{[f]} = E_{S^1}^*(X)^{[-V]} \otimes E_{S^1}^*(X)^{[W]}.$$

So the proposition is finished if we can show that for a general vector bundle  $V$

$$E_{S^1}^*(DV, SV) = E_{S^1}^*(X)^{[-V]}.$$

Indeed, multiplication by the equivariant elliptic Thom classes on each stalk gives the following commutative diagram, where the rows are isomorphisms:

$$\begin{array}{ccc} H_{S^1}^*(X^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha) & \xrightarrow{\cdot t_\alpha^* \phi_{S^1}^E(V^\alpha)} & H_{S^1}^*(DV^\alpha, SV^\alpha) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha) \\ e_{S^1}^E(V^\alpha/V^\beta) \cdot i^* \downarrow & & \downarrow i^* \\ H_{S^1}^*(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha) & \xrightarrow{\cdot t_\alpha^* \phi_{S^1}^E(V^\beta)} & H_{S^1}^*(DV^\beta, SV^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \alpha) \\ t_{\beta-\alpha}^* \downarrow & & \downarrow t_{\beta-\alpha}^* \\ H_{S^1}^*(X^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \beta) & \xrightarrow{\cdot t_\beta^* \phi_{S^1}^E(V^\beta)} & H_{S^1}^*(DV^\beta, SV^\beta) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathcal{E}}(U - \beta). \end{array}$$

Notice that  $E_{S^1}^*(DW, SW)$  is an invertible sheaf, because it is the same as the structure sheaf  $E_{S^1}^*(*) = \mathcal{O}_{\mathcal{E}}$  twisted by the cocycle  $\lambda_{\alpha\beta}^{[W]}$ . In fact, we can identify it by the same method we used in Proposition 3.11.  $\square$

In the language of equivariant spectra (see Chapter 8 of [13]) we can say more: With the notation we used in Proposition 5.7, we define a virtual vector bundle  $Tf$ , the tangents along the fiber, by

$$TX = Tf \oplus f^*TY.$$

Using the formula  $TX \oplus V = f^*TY \oplus W$ , it follows that  $-Tf = V \ominus W$ . From equation (10) it follows that

$$E_{S^1}^*(X)^{[f]} = \tilde{E}_{S^1}^*(X^{-Tf}),$$

where  $\tilde{E}_{S^1}^*$  is reduced cohomology, and  $X^{-Tf}$  is the  $S^1$ -equivariant spectrum obtained by the Thom space of  $V$  desuspended by  $W$ .

**Appendix A. Equivariant characteristic classes.** The results of this section are well known, with the exception of the holomorphicity result Proposition A.6.

Let  $V$  be a complex  $n$ -dimensional  $S^1$ -equivariant vector bundle over an  $S^1$ -CW complex  $X$ . Then to any power series  $Q(x) \in \mathbb{C}[[x]]$  starting with 1 we are going to associate by Hirzebruch's formalism (see [11]) a multiplicative characteristic class  $\mu_Q(V)_{S^1} \in H_{S^1}^{**}(X)$ . (Recall that  $H_{S^1}^{**}(X)$  is the completion of  $H_{S^1}^*(X)$ .)

Consider the Borel construction for both  $V$  and  $X$ :  $V_{S^1} = V \times_{S^1} ES^1 \rightarrow X \times_{S^1} ES^1 = X_{S^1}$ .  $V_{S^1} \rightarrow X_{S^1}$  is a complex vector bundle over a paracompact space, hence we have a classifying map  $f_V: X_{S^1} \rightarrow BU(n)$ . We define  $c_j(V)_{S^1}$ , the equivariant  $j$ th Chern class of  $V$ , as the image via  $f_V^*$  of the universal  $j$ th Chern class  $c_j \in H^*BU(n) = \mathbb{C}[c_1, \dots, c_n]$ . Now look at the product  $Q(x_1)Q(x_2) \cdots Q(x_n)$ . It is a power series in  $x_1, \dots, x_n$  which is symmetric under permutations of the  $x_j$ 's, hence it can be expressed as another power series in the elementary symmetric functions  $\sigma_j = \sigma_j(x_1, \dots, x_n)$ :

$$Q(x_1) \cdots Q(x_n) = P_Q(\sigma_1, \dots, \sigma_n).$$

Notice that  $P_Q(c_1, \dots, c_n)$  lies not in  $H^*BU(n)$ , but in its completion  $H^{**}BU(n)$ . The map  $f_V^*$  extends to a map  $H^{**}BU(n) \rightarrow H^{**}(X_{S^1})$ .

*Definition A.1.* Given the power series  $Q(x) \in \mathbb{C}[[x]]$  and the complex  $S^1$ -vector bundle  $V$  over  $X$ , there is a canonical complex equivariant characteristic class  $\mu_Q(V)_{S^1} \in H^{**}(X_{S^1})$ , given by

$$\mu_Q(V)_{S^1} := P_Q(c_1(V)_{S^1}, \dots, c_n(V)_{S^1}) = f_V^* P_Q(c_1, \dots, c_n).$$

*Remark A.2.* If  $T^n \hookrightarrow BU(n)$  is a maximal torus, then  $H^*BT^n = \mathbb{C}[x_1, \dots, x_n]$ , and the  $x_j$ 's are called the universal Chern roots. The map  $H^*BU(n) \rightarrow H^*BT^n$  is injective, and its image can be identified as the Weyl group invariants of  $H^*BT^n$ . The Weyl group of  $U(n)$  is the symmetric group on  $n$  letters, so  $H^*BU(n)$  can be identified as the subring of symmetric polynomials in  $\mathbb{C}[x_1, \dots, x_n]$ . Similarly,  $H^{**}BU(n)$  is the subring of symmetric power series in  $\mathbb{C}[[x_1, \dots, x_n]]$ . Under this interpretation,  $c_j = \sigma_j(x_1, \dots, x_n)$ . This allows us to identify  $Q(x_1) \cdots Q(x_n)$  with the element  $P_Q(c_1, \dots, c_n) \in H^{**}BU(n)$ .

*Definition A.3.* We can write formally  $\mu_Q(V)_{S^1} = Q(x_1) \cdots Q(x_n)$ .  $x_1, \dots, x_n$  are called the equivariant Chern roots of  $V$ .

Here is a standard result about the equivariant Chern roots:

*PROPOSITION A.4.* Let  $V(m) \rightarrow X$  be a complex  $S^1$ -vector bundle such that the action of  $S^1$  on  $X$  is trivial. Suppose that  $g \in S^1$  acts on  $V(m)$  by complex multiplication with  $g^m$ . If  $x_i$  are the equivariant Chern roots of  $V(m)$ , and  $w_i$  are its usual (nonequivariant) Chern roots, then

$$x_i = w_i + mu,$$

where  $u$  is the generator of  $H_{S^1}^*(*) = H^*BS^1$ .

We now want to show that the class we have just constructed,  $\mu_Q(V)_{S^1}$ , is holomorphic in a certain sense, provided  $Q(x)$  is the expansion of a holomorphic function around zero. But first, let us state a classical lemma in the theory of symmetric functions.

LEMMA A.5. *Suppose  $Q(y_1, \dots, y_n)$  is a holomorphic (i.e. convergent) power series, which is symmetric under permutations of the  $y_j$ 's. Then the power series  $P_Q$  such that*

$$Q(y_1, \dots, y_n) = P_Q(\sigma_1(y_1, \dots, y_n), \dots, \sigma_n(y_1, \dots, y_n)),$$

*is holomorphic.*

We have mentioned above that  $\mu_Q(V)_{S^1}$  belongs to  $H_{S^1}^{**}(X)$ . This ring is equivariant cohomology tensored with power series. It contains  $HO_{S^1}^*(X)$  as a subring, corresponding to the holomorphic power series.

PROPOSITION A.6. *If  $Q(x)$  is a convergent power series, then  $\mu_Q(V)_{S^1}$  is a holomorphic class, i.e., it belongs to the subring  $HO_{S^1}^*(X)$  of  $H_{S^1}^{**}(X)$ .*

*Proof.* We have  $\mu_Q(V)_{S^1} = P(c_1(V)_{S^1}, \dots, c_n(V)_{S^1})$ , where we write  $P$  for  $P_Q$ . Assume  $X$  has a trivial  $S^1$ -action. It is easy to see that  $H_{S^1}^*(X) = (H^0(X) \otimes_{\mathbb{C}} \mathbb{C}[u]) \oplus$  nilpotents. Hence we can write  $c_j(E)_{S^1} = f_j + \alpha_j$ , with  $f_j \in H^0(X) \otimes_{\mathbb{C}} \mathbb{C}[u]$ , and  $\alpha_j$  nilpotent in  $H_{S^1}^*(X)$ . We expand  $\mu_Q(V)_{S^1}$  in Taylor expansion in multiindex notation. We make the following notations:  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$ ,  $|\lambda| = \lambda_1 + \dots + \lambda_n$ , and  $\alpha^\lambda = \alpha_1^{\lambda_1} \dots \alpha_n^{\lambda_n}$ . Now we consider the Taylor expansion of  $\mu_Q(V)_{S^1}$  in multiindex notation:

$$\mu_Q(V)_{S^1} = P(\dots, c_j(V)_{S^1}, \dots) = \sum_{\lambda} \frac{\partial^{|\lambda|} P}{\partial c^\lambda}(\dots, f_j, \dots) \cdot \alpha^\lambda.$$

This is a finite sum, since  $\alpha_j$ 's are nilpotent. We want to show that  $\mu_Q(V)_{S^1} \in HO_{S^1}^*(X)$ .  $\alpha^\lambda$  lies in  $HO_{S^1}^*(X)$ , since it lies even in  $H_{S^1}^*(X)$ . So we only have to show that  $\frac{\partial^{|\lambda|} P}{\partial c^\lambda}(\dots, f_j, \dots)$  lies in  $HO_{S^1}^*(X)$ .

But  $f_j \in H^0(X) \otimes_{\mathbb{C}} \mathbb{C}[u] = \mathbb{C}[u] \oplus \dots \oplus \mathbb{C}[u]$ , with one  $\mathbb{C}[u]$  for each connected component of  $X$ . If we fix one such component  $N$ , then the corresponding component  $f_j^{(N)}$  lies in  $\mathbb{C}[u]$ . According to Lemma A.5,  $P$  is holomorphic around  $(0, \dots, 0)$ , hence so is  $\frac{\partial^{|\lambda|} P}{\partial c^\lambda}$ . Therefore  $\frac{\partial^{|\lambda|} P}{\partial c^\lambda}(\dots, f_j^{(N)}(u), \dots)$  is holomorphic in  $u$  around 0, i.e., it lies in  $\mathcal{O}_{\mathbb{C},0}$ . Collecting the terms for the different connected components of  $X$ , we finally get

$$\frac{\partial^{|\lambda|} P}{\partial c^\lambda}(\dots, f_j, \dots) \in \mathcal{O}_{\mathbb{C},0} \oplus \dots \oplus \mathcal{O}_{\mathbb{C},0} = H^0(X) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C},0}.$$

But  $H^0(X) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C},0} \subseteq H^*(X) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C},0} = H_{S^1}^*(X) \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0} = HO_{S^1}^*(X)$ , so we are done.

If the  $S^1$ -action on  $X$  is not trivial, look at the following exact sequence associated to the pair  $(X, X^{S^1})$ :

$$0 \rightarrow T \hookrightarrow H_{S^1}^*(X) \xrightarrow{i^*} H_{S^1}^*(X^{S^1}) \xrightarrow{\delta} H_{S^1}^{**+1}(X, X^{S^1}),$$

where  $T$  is the torsion submodule of  $H_{S^1}^*(X)$ . (The fact that  $T = \ker i^*$  follows from the following arguments: on the one hand,  $\ker i^*$  is torsion, because of the localization theorem; on the other hand,  $H_{S^1}^*(X^{S^1})$  is free, hence all torsion in  $H_{S^1}^*(X)$  maps to zero via  $i^*$ .) Also, since  $T$  is a direct sum of torsion modules of the form  $\mathbb{C}[u]/(u^n)$

$$T \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{C},0} \cong T \cong T \otimes_{\mathbb{C}[u]} \mathbb{C}[[u]].$$

Now tensor the above exact sequence with  $\mathcal{O}_{\mathbb{C},0}$  and  $\mathbb{C}[[u]]$  over  $\mathbb{C}[u]$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & T & \hookrightarrow & HO_{S^1}^*(X) & \xrightarrow{i^*} & HO_{S^1}^*(X^{S^1}) \xrightarrow{\delta} HO_{S^1}^{**+1}(X, X^{S^1}) \\ & & \parallel & & \downarrow s & & \downarrow t \\ 0 & \longrightarrow & T & \hookrightarrow & H_{S^1}^{**}(X) & \xrightarrow{i^*} & H_{S^1}^{**}(X^{S^1}) \xrightarrow{\delta} H_{S^1}^{**+1}(X, X^{S^1}) . \end{array}$$

We know  $\alpha := \mu_Q(V)_{S^1} \in H_{S^1}^{**}(X)$ . Then  $\beta := i^* \mu_Q(V)_{S^1} = i^* \alpha$  was shown previously to be in the image of  $t$ , i.e.  $\beta = t\tilde{\beta}$ .  $\delta\beta = \delta i^* \alpha = 0$ , so  $\delta t\tilde{\beta} = 0$ , hence  $\delta\tilde{\beta} = 0$ . Thus  $\tilde{\beta} \in \text{Im } i^*$ , so there is an  $\tilde{\alpha} \in HO_{S^1}^*(X)$  such that  $\tilde{\beta} = i^* \tilde{\alpha}$ .  $s\tilde{\alpha}$  might not equal  $\alpha$ , but  $i^*(\alpha - \tilde{\alpha}) = 0$ , so  $\alpha - \tilde{\alpha} \in T$ . Now,  $\tilde{\alpha} + (\alpha - \tilde{\alpha}) \in HO_{S^1}^*(X)$ , and  $s(\tilde{\alpha} + (\alpha - \tilde{\alpha})) = \alpha$ , which shows that indeed  $\alpha \in \text{Im } s = HO_{S^1}^*(X)$ .  $\square$

There is a similar story when  $V$  is an oriented  $2n$ -dimensional real  $S^1$ -vector bundle over a finite  $S^1$ -CW complex  $X$ . We classify  $V_{S^1} \rightarrow X_{S^1}$  by a map  $f_V: X_{S^1} \rightarrow BSO(2n)$ .  $H^*BSO(2n) = \mathbb{C}[p_1, \dots, p_n]/(e^2 - p_n)$ , where  $p_j$  and  $e$  are the universal Pontrjagin and Euler classes, respectively. The only problem now is that in order to define characteristic classes over  $BSO(2n)$  we need the initial power series  $Q(x) \in \mathbb{C}[[x]]$  to be either even or odd:

*Remark A.7.* As in Remark A.2, if  $T^n \hookrightarrow BSO(2n)$  is a maximal torus, then the map  $H^*BSO(2n) \rightarrow H^*BT^n$  is injective, and its image can be identified as the Weyl group invariants of  $H^*BT^n$ . Therefore  $H^*BSO(2n)$  can be thought of as the subring of symmetric polynomials in  $\mathbb{C}[x_1, \dots, x_n]$  which are invariant under an even number of sign changes of the  $x_j$ 's. A similar statement holds for  $H^{**}BSO(2n)$ . Under this interpretation,  $p_j = \sigma_j(x_1^2, \dots, x_n^2)$  and  $e = x_1 \cdots x_n$ .

So, if we want  $Q(x_1) \cdots Q(x_n)$  to be interpreted as an element of  $H^{**}BSO(2n)$ , we need to make it invariant under an even number of sign changes. But this is clearly true if  $Q(x)$  is either an even or an odd power series.

Let us be more precise:

(a)  $Q(x)$  is even, i.e.,  $Q(-x) = Q(x)$ . Then there is another power series  $S(x)$  such that  $Q(x) = S(x^2)$ , so  $Q(x_1) \cdots Q(x_n) = S(x_1^2) \cdots S(x_n^2) = P_S(\dots, \sigma_j(x_1^2, \dots, x_n^2), \dots) = P_S(\dots, p_j, \dots)$ .

(b)  $Q(x)$  is odd, i.e.,  $Q(-x) = -Q(x)$ . Then there is another power series  $R(x)$  such that  $Q(x) = xT(x^2)$ , so  $Q(x_1) \cdots Q(x_n) = x_1 \cdots x_n T(x_1^2) \cdots T(x_n^2) = x_1 \cdots x_n P_T(\dots, \sigma_j(x_1^2, \dots, x_n^2), \dots) = e \cdot P_T(\dots, p_j, \dots)$ .

**Definition A.8.** Given the power series  $Q(x) \in \mathbb{C}[[x]]$  which is either even or odd, and the real oriented  $S^1$ -vector bundle  $V$  over  $X$ , there is a canonical real equivariant characteristic class  $\mu_Q(V)_{S^1} \in H_{S^1}^{**}(X)$ , defined by pulling back the element  $Q(x_1) \cdots Q(x_n) \in H^{**}BSO(2n)$  via the classifying map  $f_V: X_{S^1} \rightarrow BSO(2n)$ .

Proposition A.6 can be adapted to show that, if  $Q(x)$  is a convergent power series,  $\mu_Q(V)_{S^1}$  actually lies in  $HO_{S^1}^*(X)$ .

The next result is used in the proof of Lemma 5.3.

**LEMMA A.9.** *Let  $V$  be an orientable  $S^1$ -equivariant even dimensional real vector bundle over  $X$ . Suppose we are given two orientations of  $V$ , which we denote by  $V_{or_1}$  and  $V_{or_2}$ . Define  $\sigma = 0$  if  $V_{or_1} = V_{or_2}$ , and  $\sigma = 1$  otherwise. Suppose  $Q(x)$  is a power series such that  $Q(-x) = \gamma Q(x)$ , where  $\gamma = \pm 1$ . Then*

$$\mu_Q(V_{or_1}) = \gamma^\sigma \mu_Q(V_{or_2}).$$

*Proof.* (a) If  $Q(-x) = Q(x)$ ,  $\mu_Q(V)$  is a power series in the equivariant Pontrjagin classes  $p_j(V)_{S^1}$ . But Pontrjagin classes are independent of the orientation, so  $\mu_Q(V_{or_1}) = \mu_Q(V_{or_2})$ .

(b) If  $Q(-x) = -Q(x)$ , then  $Q(x) = x\tilde{Q}(x)$ , with  $\tilde{Q}(-x) = \tilde{Q}(x)$ . Hence  $\mu_Q(V) = e_{S^1}(V) \cdot \mu_{\tilde{Q}}(V)$ .  $e(V)_{S^1}$  changes sign when orientation changes sign, while  $\mu_{\tilde{Q}}(V)$  is invariant, because of a).  $\square$

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